Math 112A final exam practice problems

ATTENTION: The final exam will contain problems from the first and the second parts of the course. Here I only present problems from the second half of the course (after the midterm). Use the midterm preparation materials as well as this sheet to prepare for the final. Most problems will be from the second half of the course. I included solutions to most of the problems.

1. Consider a wave equation on an infinite line,

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^4}\frac{\partial^2 u}{\partial x^2} = 0.$$

Find the characteristics though the point $(0, 3)$. Draw the domains of dependence and influence of the point $(0, 3)$ (for $t \geq 0$).
Solution: The equations of characteristics are $t = \pm x^3/3 + 3$. The domain of dependence is below the two curves. The domain of influence is above.

2. Is the equation

$$\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial t} - \frac{9}{4}\frac{\partial^2 u}{\partial x^2} = 0$$

hyperbolic, elliptic or parabolic (explain)? Find the general equations for characteristics if possible.
Solution: The equation is hyperbolic because $A^2 - 4BC = 25 > 0$. The characteristics are $\xi = 2x + t$, $\eta = 2x - 9t$.

3. Consider the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

defined in the inside of the circle, $x^2 + y^2 = 25$. The boundary condition is given by $u(x, y) = x$ for such (x, y) that $x^2 + y^2 = 25$. Can we have $u(x, y) = 10$ somewhere inside the circle? (Hint: use the maximum principle, p. 56.)

Solution. According to the maximum principle, the value of the function $u(x, t)$ in the whole domain must be contained between m and M, where m

is the minimum value on the boundary and M is the maximum value on the boundary. For this problem, the max value on the boundary is 5. To see this, draw the domain (a circle centered at zero with radius 5) ans see that the max value of x is 5. Therefore, the value of $u(x, y)$ in the interior cannot exceed 5, and thus it cannot be equal to 10.

4. Are the functions $\phi_1(x) = \sin x$ and $\phi_2(x) = \cos x$ orthogonal on the interval $[0, \pi]$? Are they orthogonal on the interval $[-\pi, \pi]$? Take $\rho(x) = 1$. Prove.

Solution. Two functions are orthogonal if $\int_a^b \phi_1(x)\phi_2(x)\rho(x)\,dx = 0$. We have $\int_0^\pi \sin x \cos s\,dx = 2\int_0^\pi \sin(2x)\,dx = 0$, and $\int_{-\pi}^\pi \sin x \cos s\,dx = 2\int_{-\pi}^\pi \sin(2x)\,dx = 0$, so they are orthogonal on both intervals.

5. Suppose that the function $f(x) = x$ is defined for $0 \le x \le \pi$. Find the sine series for $f(x)$. In other words, find the coefficients b_n in the expansion of $f(x)$ given by $\sum_{n=1}^\infty c_n \varphi_n(x)$, where $\varphi_n(x) = \sin(nx)$.

Solution.

(a) To calculate b_n, we write

$$b_n = \frac{\int_0^\pi x\sin(nx)\,dx}{\int_0^\pi \sin^2(nx)\,dx}.$$

We have

$$\int_0^\pi x\sin(nx)\,dx = \frac{-\pi\cos(n\pi)}{n} + 0 = -\frac{\pi(-1)^n}{n},$$

and

$$\int_0^\pi \sin^2(nx)\,dx = \frac{\pi}{2},$$

therefore the answer is $b_n = -2(-1)^n/n$, with $n = 1, 2, \ldots$.

6. Consider the initial-boundary value problem:

$$\frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \pi, \tag{1}$$

$$u(0,t) = u(\pi,t) = 0, \tag{2}$$

$$u(x,0) = x(\pi - x). \tag{3}$$

Solve the problem by the method of separation of variables. (a) Present $u(x,t) = X(x)T(t)$, and formulate the ordinary differential equations and boundary conditions for X and T. (b) Solve the eigenvalue problem and find the partial solutions, $u_n(x,t)$. (c) Which of the equations (1-3) do these solutions satisfy? (d) Write down the solution of the full problem as an infinite series.

Solution. (a) We present it as $u(x,t) = X(x)T(t)$ and obtain

$$T'X - kTX'' = 0 \quad \Rightarrow \frac{X''}{X} = \frac{T}{kT} = -\lambda.$$

This gives rise to two ordinary differential equations. For X we have the following boundary value problem,

$$X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0,$$

and for T we have

$$T' + 2\lambda T = 0.$$

(b) The problem for X is an eigenvalue problem, it only has nontrivial solutions if $\lambda = n^2$ for integer values of n, and the solutions are

$$X_n(x) = \sin nx.$$

The problem for T becomes $T' + 2n^2T = 0$, and its general solution is $T(t) = Ce^{-2n^2t}$. The partial solutions of the whole problem for $u(x,t)$ are

$$u_n(x,t) = \sin nx\, e^{-2n^2t}.$$

(c) These partial solutions satisfy equations (1) and (2), but not (3).

(d) **This part will not be tested.** To satisfy the remaining boundary condition, equation (3), we present the solution as an infinite sum,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-2n^2t}.$$

From $u(x,0) = x(\pi - x)$ we obtain

$$\sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x).$$

Then b_n are found as the Fourier sine coefficients of $f(x) = x(\pi - x)$,

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx = \frac{4(1-\cos n\pi)}{n^3\pi} = \begin{cases} 8/(\pi n^3), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore we have

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^3}\sin nx\,e^{-2n^2t}.$$

Alternatively, you can write it as

$$u(x,t) = \frac{8}{\pi}\sum_{n\text{Odd}}^{\infty}\frac{\sin nx}{n^3}e^{-2n^2t}.$$

Another way to write the same thing is

$$u(x,t) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sin[(2k-1)x]}{(2k-1)^3}e^{-2(2k-1)^2t}.$$