Q1

a) $$f(x,y) = (5y^3 + 2x^2y)^8$$

$$\partial_x f(x,y) = 8(5y^3+2x^2y)^7 \cdot (2\cdot(2x)\cdot y)$$
$$= 32(5y^3+2x^2y)^7 \cdot xy$$

$$\partial_y f(x,y) = 8(5y^3+2x^2y)^7(15y^2+2x^2)$$

$$\Rightarrow \nabla f(x,y) = 8(5y^3+2x^2y)^7 \cdot \left[4xy\,\hat{i} + (15y^2+2x^2)\hat{j}\right] \quad \square$$

b) $$F(\alpha,\beta) = \alpha^2 \ln \alpha^2 + \beta^2$$

$$\partial_\alpha F(\alpha,\beta) = 2\alpha \ln \alpha^2 + \alpha^2 \frac{1}{\alpha^2} \cdot 2\alpha$$
$$= 2\alpha(\ln \alpha^2 + 1)$$

$$\partial_\beta F(\alpha,\beta) = 2\beta$$

$$\Rightarrow \nabla F(\alpha,\beta) = 2\alpha(\ln\alpha^2+1)\,\hat{i} + 2\beta\,\hat{j} \quad \square$$

c) $$G(x,y,z) = e^{xz}\sin(y/z)$$

$$\partial_x G(x,y,z) = z e^{xz}\sin(y/z)$$

$$\partial_y G(x,y,z) = \frac{1}{z}\cos(y/z)\cdot e^{xz}$$

$$\partial_z G(x,y,z) = x e^{xz} \sin(y/z) + e^{xz} \cdot \cos(y/z) \cdot \left(-\frac{y}{z^2}\right)$$

$$= e^{xz}\left(x \sin(y/z) + \cos(y/z)\left(-\frac{y}{z^2}\right)\right)$$

$$\Rightarrow \nabla G(x,y,z) = e^{xz}\left[z \sin(y/z)\, \hat{i} + \cos(y/z) \frac{1}{z} \hat{j} + \left(x \sin(y/z) - \cos(y/z) \frac{y}{z^2}\right) \hat{k} \right]$$

D

d) $$S(u,v,w) = u \arctan(u\sqrt{w})$$

$$\partial_u S(u,v,w) = \arctan(u\sqrt{w}) + u \cdot \frac{1}{(u\sqrt{w})^2+1} \cdot \sqrt{w}$$

$$\partial_v S(u,v,w) = 0$$

$$\partial_w S(u,v,w) = u \cdot \frac{1}{(u\sqrt{w})^2+1} \cdot u \cdot \frac{1}{2\sqrt{w}}$$

$$= \frac{u^2}{2\sqrt{w}\left((u\sqrt{w})^2+1\right)}$$

$$\Rightarrow \nabla S(u,v,w) = \left[\arctan(u\sqrt{w}) + \frac{u\sqrt{w}}{(u\sqrt{w})^2+1}\right] \hat{i} + \frac{u^2}{2\sqrt{w}\left((u\sqrt{w})^2+1\right)} \hat{k}$$

D

Q3/ Let find the linear approximation,

$$f(x,y,z) \approx f(x_0,y_0,z_0) + \partial_x f(x_0,y_0,z_0)(x-x_0)$$
$$+ \partial_y f(x_0,y_0,z_0)(y-y_0)$$
$$+ \partial_z f(x_0,y_0,z_0)(z-z_0) = L(x,y,z)$$

$\Rightarrow$ we need to compute the partial derivatives. at $(2,3,4) = (x_0,y_0,z_0)$.

$$\Rightarrow \quad \partial_x f(x,y,z) = 3x^2\sqrt{y^2+z^2}$$
$$\partial_y f(x,y,z) = x^3 \frac{y}{\sqrt{y^2+z^2}}$$
$$\partial_z f(x,y,z) = x^3 \frac{z}{\sqrt{y^2+z^2}}$$

$$\Rightarrow \quad \partial_x f(2,3,4) = 3 \cdot 4 \cdot \sqrt{9+16} = 3 \cdot 4 \cdot 5 = 60$$
$$\partial_y f(2,3,4) = 8 \cdot \frac{3}{\sqrt{9+16}} = \frac{24}{5}$$
$$\partial_z f(2,3,4) = 8 \cdot \frac{4}{\sqrt{9+16}} = \frac{32}{5}$$

and $f(2,3,4) = 8 \cdot \sqrt{9+16} = 8 \cdot 5 = 40$

then $L(x,y,z) = 40 + 60(x-2) + \frac{24}{5}(y-3) + \frac{35}{5}(z-4).$

Q4 Let $z = y + f(x^2 - y^2)$

we want to show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x. \quad (1)$$

to do so we compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$; and we evaluate the left-hand side in (1).

Then using the chain rule

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\cdot 2x$$

$$\frac{\partial z}{\partial y} = 1 + f'(x^2-y^2)\cdot(-2y)$$

then $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$ is equal to.

$$y\left(f'(x^2-y^2)\cdot 2x\right) + x\left(1 - f'(x^2-y^2)\,2y\right)$$

$$= f'(x^2-y^2)\underbrace{\left(2xy - 2xy\right)}_{=0} + x = x$$

then z satisfies (1)

□

Q6/ let $u = x^2 y^3 + z^4$ ①

and
②
$$\begin{cases} x = p + 3p^2 \\ y = pe^p \\ z = p \sin p \end{cases}$$

in this case we can replace x, y, z in ① and then compute a messy expression, or we can use the chain rule.

to compute $\frac{du}{dp}$ we need to realize that ② is a parametrization depending on p; i.e. $\underline{r}(p)$ then in this case we have that $u(p) = u(\underline{r}(p))$ then by chain rule.

$$\frac{du}{dp} = \nabla u \cdot \frac{d\underline{r}(p)}{dp}.$$

where ∇u is evaluated at $\underline{r}(p)$

in this case

$$\nabla u = 2xy^3 \hat{\imath} + 3x^2y^2 \hat{\jmath} + 4z^3 \hat{k}$$

and $$\frac{d\underline{r}(p)}{dp} = (1 + 6p)\hat{\imath} + (e^p + pe^p)\hat{\jmath} + (\sin p + p\cos p)\hat{k}$$

now we need to evaluate ∇u at $\underline{r}(p)$

i.e. $$\nabla u(\underline{r}(p)) = 2(p+3p^2)(pe^p)^3\,\hat{i} + 3(p+3p^2)^2(pe^p)^2\,\hat{j} + 4(p\sin p)^3\,\hat{k}$$

and perform the dot product.

$$\nabla u(\underline{r}(p))\frac{d\underline{r}(p)}{dp} = 2(p+3p^2)(pe^p)^3(1+6p) + 3(p+3p^2)^2(pe^p)^2(e^p+pe^p) + 4(p\sin p)^3(\sin p + p\cos p).$$

Q7 We know that the direction of maximum rate of change is parallel to the gradient of a function; and the maximum rate of change is the magnitude of the gradient.

Then we need to compute the gradient,

$$f(x,y) = x^2 y + \sqrt{y}$$

then $\nabla f(x,y) = 2xy\,\hat{i} + \left(x^2 + \frac{1}{2\sqrt{y}}\right)\hat{j}$

then $\nabla f(2,1) = 4\hat{i} + \left(4 + \frac{1}{2\sqrt{1}}\right)\hat{j}$

$$= 4\hat{i} + \frac{9}{2}\hat{j}$$

$\Rightarrow$ the direction of maximum rate of change is parallel to $4\hat{i} + \frac{9}{2}\hat{j}$ and the maximum rate of change is

$$|\nabla f(2,1)| = \sqrt{16 + \frac{81}{4}}.$$

$$= \sqrt{\frac{64+81}{4}}$$

$$= \sqrt{\frac{145}{4}} = \sqrt{\frac{29.5}{4}} =$$

Q10/

We transform the expression

$$\cos(xyz) = 1 + x^2y^2 + z^2$$

to the form $F(x,y,z) = 0$ (1)

where $F(x,y,z) = 1 + x^2y^2 + z^2 - \cos(xyz)$

in this case we have that we can differentiate (1) with respect to ∂x and we obtain

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

and $\frac{\partial x}{\partial x} = 1$ & $\frac{\partial y}{\partial x} = 0$ (y and x are independent).

then we have. $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$

or

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

in a similar fashion we can deduce.

Then we can just use the formula to evaluate the partial derivatives.

$$\partial_x F(x,y,z) = 2xy^2 + \sin(xyz) \cdot yz$$

$$\partial_y F(x,y,z) = 2yx^2 + \sin(xyz)\, xz$$

$$\partial_z F(x,y,z) = 2z + \sin(xyz)\, xy$$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial z}{\partial x} &= - \frac{2xy^2 + \sin(xyz) \cdot yz}{2z + \sin(xyz)\, xy} \\ \frac{\partial z}{\partial y} &= - \frac{2yx^2 + \sin(xyz)\, xz}{2z + \sin(xyz) \cdot xy} \end{aligned}}$$