Alternating Groups and lift invariants

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- Start with computing Geometric Monodromy of moduli space covers using the $(G, \mathbf{C})$ *collection cover*, $G_{\rm col}$.
- Given a braid orbit O on $\mathrm{Ni}(G, \mathbf{C})$ $\exists$ minimal $G_{{\rm col},O} \to G$ satisfying these properties:
  1. All classes of $\mathbf{C}$ lift uniquely:
  2. One cover in $\mathrm{Ni}(G_{{\rm col},O}, \mathbf{C})$ maps to all covers in O over $\boldsymbol{z}_0$.
  3. *Monodromy Automorphism*: Each $q \in H_r$ acts as an automorphism $A_q \in \mathrm{Aut}(G_{{\rm col},O})$ on $\{\boldsymbol{g} \in O\}$.

$$\text{Geometric Monodromy of } \Phi_{G,\mathbf{C}} \text{ is } \{A_q \mid q \in H_r\}/G_{{\rm col},O}.$$

- Example: Collection cover for $(D_p, \mathbf{C}_{3^4})$ is

$G_{{\rm col},p} = (\mathbb{Z}/p)^2 \times^s \mathbb{Z}/2 : \mathrm{GL}_2(\mathbb{Z}/p)$ preserves Nielsen classes.

Geometric monodromy of $\bar{\mathcal{H}}(D_p, \mathbf{C}_{2^4})^{\mathrm{abs,rd}} \to \mathbb{P}^1_j$

- Genus of deg p^2 covers in $\mathrm{Ni}^{\mathrm{abs}} = \mathrm{Ni}(G_{\mathrm{col}}, \mathbf{C}_{2^4})/\mathrm{GL}_2(\mathbb{Z}/p)$:
$$\mathbf{g}_p : 2(p^2 + \mathbf{g}_p - 1) = 4(p^2-1)/2 \text{ or } \mathbf{g}_p = 0.$$
- Geometric monodromy of $\bar{\mathcal{H}}(G_{\mathrm{col},p}, \mathbf{C}_{2^4})^{\mathrm{in,rd}} \to \mathbb{P}^1_j$: Check automorphism action, A_q of braids. With $\langle \mathbf{v}_2, \mathbf{v}_3\rangle = (\mathbb{Z}/p)^2$, what do these braids do to $M(\mathbf{v}_2, \mathbf{v}_3)$?:[*0]
- Nielsen classes for $G_{\mathrm{col}.p}$ by substituting $\mathbf{v}$ s in $(\mathbb{Z}/p)^2$ for b s:

$$M(\mathbf{v}_2,\mathbf{v}_3) \stackrel{\mathrm{def}}{=} \left(\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_2} \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_3} \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_4} \\ 0 & 1\end{pmatrix}\right)$$
$$\langle \mathbf{v}_2, \mathbf{v}_3\rangle = (\mathbb{Z}/p)^2, \mathbf{v}_4 = \mathbf{v}_3 - \mathbf{v}_2.^{*1}$$
$$A_{q_2} : \left(\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & 2\mathbf{v_2}-\mathbf{v_3} \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_2} \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_3}-\mathbf{v_2} \\ 0 & 1\end{pmatrix}\right)).$$
$$A_{\mathrm{sh}} : ((\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_3}-\mathbf{v_2} \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_3}-2\mathbf{v_2} \\ 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & \mathbf{v_2} \\ 0 & 1\end{pmatrix}))).$$

Components of $\mathcal{H}(G_{\mathrm{col},p}, \mathbf{C}_{2^4})^{\mathrm{in,rd}} = X(p)$ minus cusps

- Find $A_{q_2} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, A_{\mathbf{sh}} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}: \langle A_{\mathbf{sh}}, A_{q_2} \rangle = \mathrm{SL}_2(\mathbb{Z}/p)$.
- Conjugate $M(\boldsymbol{v}_2, \boldsymbol{v}_3)$ by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to get $M(-\boldsymbol{v}_2, -\boldsymbol{v}_3)$.
- Geometric monodromy of $\mathcal{H}(G_p, \mathbf{C}_{2^4})^{\mathrm{in,rd}} \to \mathbb{P}^1_j : \mathrm{SL}_2(\mathbb{Z}/p)/\langle \pm 1 \rangle$.
- Conclude: $\mathcal{H}(G_p, \mathbf{C}_{2^4})^{\mathrm{in,rd}}$ has $(p-1)$ components given by *unbraidable outer automorphisms*.
- These components are conjugate over $\mathbb{Q}(\zeta_p)$. Traditional interpretation is the Weil pairing; Another interpretation comes with Hurwitz spaces.*2

Inner vs Absolute spaces

$K, L, K', \hat{K}, \ldots,$ number fields; X a variety, $X(K)$ are K points.

- Inner spaces: Equivalence Galois covers (X, ψ) with group G. $\psi : \mathrm{Aut}(X \to \mathbb{P}^1_z) \equiv G$ mod inner automorphism.

Thm (Fried-Voelklein 1991)

Natural map $\Psi_{\mathrm{in,abs}} : \mathcal{H}(G, \mathbf{C})^{\mathrm{in}} \to \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}$.
Restrict to component $\mathcal{H}' \le \mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ *gives Galois cover:*

$$\mathcal{H}' \to \mathcal{H}'' \text{ group } \le N_{S_n}(G)/G.^{*3}$$

- $\boldsymbol{p} \in \mathcal{H}''(K) \leftrightarrow W_{\boldsymbol{p}} \to \mathbb{P}^1_z$ has geometric (resp. arithmetic) monodromy G (resp. G^{ar}, $G^{\mathrm{ar}}/G \le \mathrm{Aut}(\mathcal{H}' \to \mathcal{H}'')$): a (G^{ar}, G) *realization*.*4

CM and GL_2 fibers

- CM: $f_{\boldsymbol{p}}$ a solution of Prob_1 over $K \leftrightarrow \boldsymbol{p} \in \mathcal{H}^{\mathrm{abs}}(D_p, \mathbf{C}_{2^4})(K)$. But not a (D_p, D_p) (or *involution*) realization over K.[*4]
- $\mathcal{H}(G_{\mathrm{col},p}, \mathbf{C}_{2^4})(K)^{\mathrm{abs}} = U_j = \mathbb{P}^1_j \setminus \{\infty\}$
  Arithmetic monodromy of
  $\mathcal{H}(G_{\mathrm{col},p}, \mathbf{C}_{2^4})^{\mathrm{in}} \to \mathcal{H}(G_{\mathrm{col},p}, \mathbf{C}_{2^4})^{\mathrm{abs}}$

  $$\text{over } \mathbb{Q} \text{ is } \mathrm{GL}_2(\mathbb{Z}/p)/\langle \pm 1 \rangle.$$
- GL_2: $f_{\boldsymbol{p}}$ a solution of Prob_2 over $K \leftrightarrow \boldsymbol{p} \in U_j(K)$ an (M_p, G_p) realization with $\langle M_p, G_p \rangle \le S_{p^2}$ primitive (M_p irreducible on $(\mathbb{Z}/p)^2$).[*5,[Fr05, §6.3]]

Enriched Comments on CM and GL_2 fibers

- CM $\leftrightarrow$ complex quadratic j-invariant. Minimal fields over which an elliptic curve can have a degree p isogeny.*6
- $\boldsymbol{p} \in \mathcal{H}(D_{p^{k+1}}, \mathbf{C}_{2^{2u}})^{\mathrm{in}}(K) \leftrightarrow p^{k+1}$ *cyclotomic points* on *Hyperelliptic Jacobians*: *involution realization* of p^{k+1}. None known beyond $p^{k+1} = 7$ (requires $r \geq 6$).[DFr94, §5]
- Exceptional (f, K), f indecomposable rational function:
  1. f cyclic, Chebychev, Redei, CM;
  2. $\mathrm{GL}_2 \leftrightarrow \exists M \in M_p/G_p$ with $\langle M, G_p\rangle$ primitive $\leftrightarrow f$ exceptional.*7,[Fr05,Prop. 6.6] (equivalence for degree p^2 as solutions for $\mathbf{Prob}_1$ and $\mathbf{Prob}_2$); or
  3. $\deg(f) \leq N'$, an explicit integer.[GMS03]
- Exceptional primes for exceptional (f, K): Euler ($\mathbb{F}_q^*$ cyclic) $\implies$ in union of arithmetic progressions, except for the GL_2 case.*8,[Fr05, §6.3]

Schur multipliers and Lift Invariants

Denote the conjugacy class of $g \in G$ by $\mathrm{C}_g = \mathrm{C}_{G,g}$.

- *Frattini cover* $G^* \to G$: Group cover with restriction to any proper subgroup of G^* not a cover.
- *Schur-Zassenhaus*: If $g \in G$ and $(|g|, |\ker(G^*/G)|) = 1$, then C_g lifts uniquely to $\mathrm{C}_{\hat{g}}$, $\hat{g} \in G^* \mapsto g \in G$ with $|g| = |\hat{g}|$.
- *Central Frattini extension*: $\psi : R \to G$: $\ker(R \to G)$ is a quotient of the Schur multiplier, SM_G, of G. *Representation cover* if $\ker(R \to G) = \mathrm{SM}_g$.
- *Lift invariant* if $\mathbf{C}$ is $|\mathrm{SM}_G|'$: *9 For $\boldsymbol{g} \in \mathcal{H}(G, \mathbf{C})$,

$$s_{R/G}(\boldsymbol{g}) = \prod_{i=1}^{r} \hat{g}_i \in \ker(R/G).$$

Special case: R/G is Spin_n/A_n

- Spin_n^+: (unique) nonsplit degree 2 cover of

$$O_n^+ = \ker(\text{Det} : O_n \to \mathbb{R}).$$

- Regard $S_n < O_n$ (orthogonal group); $A_n < O_n^+$. Spin_n: Pullback of A_n to Spin_n^+: $\ker(\text{Spin}_n \to A_n) = \{\pm 1\}$.[*10]
- $g \in S_n$ with $|g|$ odd, then $g \in A_n$. Our case:

C is $|\text{Spin}_n/A_n|' = 2'$: **C** has only odd order elements.

A Formula for the Spin-Lift Invariant

For odd order $g \in A_n$, let $w(g)$ be the number disjoint cycles of length l in g with $\frac{l^2-1}{8} \equiv 1 \mod 2$.

Thm (Fried-Serre)

If $\phi : W \to \mathbb{P}^1_z$ in $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$ ($\deg(\phi) = n$ and $\boldsymbol{g}_W = 0$), then $s_{Spin_n/A_n}(\phi) = (-1)^{n-1}$.
For $\boldsymbol{g} = (g_1, \dots, g_r) \in \mathrm{Ni}(G, \mathbf{C})$, genus 0 Nielsen class, $\mathbf{C}$ is 2':

$$s_{Spin_n/A_n}(\boldsymbol{g}) = (-1)^{\sum_{i=1}^r w(g_i)}.^{*11,\text{[Ser90a] ,[Fr10, Cor. 2.3]}}$$

Meaning of Spin lift invariant

- For $\phi : W \to \mathbb{P}^1_z$, $\deg(\phi) = n$ and odd order branching ; $\hat{\phi} : \hat{W} \to \mathbb{P}^1_z$ its geometric Galois closure. Then:

  $s_{\mathrm{Spin}_n/A_n}(\phi) = 1 \implies \exists \mu : Y \to \hat{W}$ unramified : $\phi \circ \mu$ is Galois with group Spin_n.

- Exercise: Genus 0 assumption doesn't apply to
  $\boldsymbol{g}_1 = ((1\,2\,3)^{(3)}, (1\,4\,5)^{(3)})$, or to
  $\boldsymbol{g}_2 = ((1\,2\,3)^{(3)}, (1\,3\,4), (1\,4\,5), (1\,5\,3))$,
  Yet, easily compute $s(\boldsymbol{g}_1) = 1, s(\boldsymbol{g}_2) = -1$.

Constellation of spaces $\mathcal{H}(A_n, \mathbf{C}_{3^r})^*$, *=abs/in
$\oplus$ (resp. $\ominus$) for a component $\leftrightarrow$ lift inv. +1 (resp. -1).

$g\geq 1 \rightarrow$	$\ominus\oplus$	$\ominus\oplus$	$\dots$	$\ominus\oplus$	$\ominus\oplus$	$\leftarrow 1\leq g$,[Fr10,Th.1.3]
$g=0 \rightarrow$	$\ominus$	$\oplus$	$\dots$	$\ominus$	$\oplus$	$\leftarrow 0=g$,[Fr10,Th.1.2]
$n \geq 4$	$n = 4$	$n = 5$	$\dots$	n even	n odd	$4 \leq n$

1. The lift invariant is a braid invariant.
2. $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}} \rightarrow \mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$, degree 2, both irreducible.*12a
3. $n = 4$: $\mathbf{C}_{\pm 3^2} = \mathbf{C}_{3^4}$, two each of classes of (1 2 3) and (3 2 1).
4. $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})$ contains $\boldsymbol{g}_{4,+} = ((1\,3\,4), (1\,4\,3), (1\,2\,3), (1\,3\,2))$ (H-M rep. *12b) and $\boldsymbol{g}_{4,-} = ((1\,2\,3), (1\,3\,4), (1\,2\,4), (1\,2\,4))$.

- [DFr94] P. Dèbes and M.D. Fried, *Nonrigid situations in constructive Galois theory*, Pacific J Math, 1994, 163: 81–122.
- [Fr10] M. Fried, *Alternating groups and moduli space lifting invariants*, Arxiv textbf#0611591v4. Israel J. Math. 179 (2010), 57-125 (DOI 10.1007/s11856-010-0073-2).
- [FrJ] M. Fried and M. Jarden, *Field Arithmetic*, Springer Ergebnisse der Math. III, 11, Springer, Heidelberg, 1986; 2nd ed. 2004, 780 pps. ISBN 3-540-22811-x.
- [GMS03] R. Guralnick, P. M§ller and J. Saxl, *The rational function analoque of a question of Schur and exceptionality of permutations representations*, Memoirs of the AMS **162** 773 (2003), ISBN 0065-9266.
- [LOs09] F. Liu and B. Osserman, *The Irreducibility of Certain Pure-cycle Hurwitz Spaces*, Amer. J. Math. # 6, vol. **130** (2008).
- [Ser90] J.-P. Serre, *Relèvements dans* $\tilde{A}_n$, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 8, 477–482. MRMR1076476 (91m:20010)