Connected Components of Sphere Covers

12:10 PM, April 21, 2015: Mike Fried, Emeritus UC Irvine [conn-comp]

- **Goal**: Extend Serre's "ℓ-adic representations" book [Se68] and his O(pen)I(mage)T(heorem) beyond *modular curves*.
- Each ℓ-adic representation has ∞-ly many pieces. This talk applies the pieces that appear in Serre's book.
- The extension works by recognizing the *starting – level 0 – piece* as driven by a finite group that is *not* abelian.
- *Two* types of ℓ-adic representations in the **OIT**: CM and GL_2.
- *I turn it around*: Solving two distinct problems on *rational functions* will come through connecting to those two types.

Rational function f: Analytic map between complex spheres

$$f : \mathbb{P}^1_w = \mathbb{C}_w \cup \{\infty\} \to \mathbb{P}^1_z = \mathbb{C}_z \cup \{\infty\} \text{ (degree } n).$$

Then, f has finitely many (branch) points, z': $< n$ distinct ws over z'. Denote by $\{z_1, \dots, z_r\} = \boldsymbol{z}$ ($r = 4$ in our examples).

For $z_0 \in \mathbb{P}^1_z \setminus \boldsymbol{z} \stackrel{\text{def}}{=} U_{\boldsymbol{z}}$, $n = \deg(f)$
label points of $\mathbb{P}^1_w$ over z_0 as $\{1', \dots, n'\}$.

Classical generators $\{P_1, \dots, P_r\} = \boldsymbol{P}$ of $\pi_1(U_{\boldsymbol{z}}, z_0)$ and *unique path lifting* produce *branch cycles* $(g_1, \dots, g_r) = \boldsymbol{g}$.

- $\boldsymbol{g}$ satisfies these properties:
  1. *Product-one*: $g_1 \cdots g_r = 1$ (the order of $\boldsymbol{P}$ emanating from z_0)
  2. *Generation*: $\langle g_1, \dots, g_r \rangle = G_f$, *geometric monodromy* (Galois closure) group of f (has $\deg(f)$ rep. $T_f : G_f \to S_n$).
- $\boldsymbol{g}$ defines *Conjugacy* Classes $\mathbf{C}$ in G_f.

Properties of spaces of solutions of Problem P

- General method for producing spaces $\mathcal{H}_P$ whose points correspond to solutions of Problem P. Understanding solutions starts by computing two $\mathcal{H}_P$ properties:
  1. Connected components and their fields of definition.
  2. Geometric monodromy groups as covers of the natural configuration space (we stick with the j-line).^{*0}
- Consider $f : W \to \mathbb{P}^1_z$: compact Riemann surface, f nonconstant, with $r > 3$ branch points.
  Local path connected topology on pairs (W, f) uses *dragging the branch points* of f based on *branch cycles*.^{*1}
- **Connectedness**: When can you drag (W_1, f_1) to (W_2, f_2)?

Conversely: Given $\boldsymbol{P}$, $\boldsymbol{g}$ (satisfying product-one) and an embedding $G = \langle \boldsymbol{g} \rangle \mapsto S_n$ defines $f_{\boldsymbol{g}} : W_{\boldsymbol{g}} \to \mathbb{P}^1_z$, with $G = G_{f_{\boldsymbol{g}}}$.

1. Labelings of $f_{\boldsymbol{g}}^{-1}(z_0) = \{1', \dots, n'\}$ matter.
   Equivalences of covers:
   Absolute if two such labelings differ by an element of S_n.*2
2. When is $W_{\boldsymbol{g}}$ connected? When given by a rational function?*3

Ingredients for connectedness:

3. If (f_1, W_1) connects to (f_2, W_2), they have the same $(G, \mathbf{C})$.
4. Sometimes one connected component has all covers with $(G, \mathbf{C})$.
   Sometimes we need higher invariants to describe components.

Two problems on Rational Functions

Consider pairs: (f, K), f a rational function defined over a number field K, with ring of integers $\mathcal{O}_K$. Find all (f, K) with

1. **Prob$_1$**: *Schur* cover property: For ∞-ly many primes $\boldsymbol{p}$ of $\mathcal{O}_K$,

   $f : w \mapsto f(w) = z \in \mathcal{O}_K/\boldsymbol{p} \cup \{\infty\}$ is 1-1 for $w \in \mathcal{O}_K/\boldsymbol{p} \cup \{\infty\}$.

2. **Prob$_2$**: f indecomposable over K, but decomposes as $f_1 \circ f_2$ with rational $\deg(f_i) > 1$, $i = 1, 2$ over $\mathbb{C} \Leftrightarrow T_f$ *imprimitive*.*4
3. Are there ∞-ly many f, $\deg(f) = n$ giving (f, K) solving problem Prob$_i$, $i = 1, 2$.
4. If yes, find components of spaces, $\mathcal{H}_{n,i}$, of rational functions so *reduced equivalence class**5 of (f, K) gives a K point on $\mathcal{H}_{n,i}$.

Fiber Product

For $f : W \to Z$ over K, form the (normalization of the) *fiber product*: $W_2 = W \times_Z W$.

- $\hat{K}_{W_2}$: Galois closure of definition field of (absolutely irr.) components of W_2:

$$G_2 = G(\hat{K}_{W_2}/K) \text{ permutes the components.}$$

Thm (Characterization of Schur/Exceptional Covers[Fr5, Prop. 2.8])

f exceptional over K iff some $\sigma \in G_2$ fixes no component of $W_2 \setminus \Delta$.

Thm (Prob$_1$: $\deg(f) = p > 2$ prime)

For f exceptional, $G_f \le \mathbb{Z}/p \times^s (\mathbb{Z}/p)^*$, r is at most 4. If $r = 4$:

$$G_f = D_p \stackrel{\text{def}}{=} \{ \begin{pmatrix} \pm 1 & b \\ 0 & 1 \end{pmatrix} \} \mid b \in \mathbb{Z}/p \}.^{*6}$$

Solution spaces from Hurwitz monodromy

- *Action on Branch cycles*: For $i = 1, \dots, r-1$, there is a

  $$q_i : (g_1, \dots, g_r) \mapsto (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_r).$$

- $H_r \stackrel{\text{def}}{=} \langle q_2, \mathbf{sh} \rangle$ acts on where $\mathbf{sh} = q_1 \cdots q_{r-1}$ has the effect

  $$\mathbf{sh} = \boldsymbol{g} \mapsto (g_2, g_3, \dots, g_r, g_1).$$

- Branch cycles for $(G, \mathbf{C})$ are called *Nielsen classes* $\mathrm{Ni}(G, \mathbf{C})$:

  $H_r = \Pi_1(U_r, \boldsymbol{z}_0)$ acts on Ni: $\boldsymbol{z}_0$, base point for dragging a cover.

  1. Space of distinct unordered branch points $\stackrel{\text{def}}{=} U_r$.
  2. $U_r = \mathbb{P}^r$ minus its *discriminant locus*, D_r.
  3. $U_r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\text{def}}{=} J_r$; $\qquad J_4 = \mathbb{P}^1_j \setminus \{\infty\}$.

Prob$_1$: Ni$(G, \mathbf{C})$ for $r = 4$ exceptionals, $deg(f) = p$, p odd

$$\mathrm{Ni}(D_p, \mathbf{C}_{4^2})^{\mathrm{abs}} = \{(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & b_2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & b_3 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & b_4 \\ 0 & 1 \end{smallmatrix}\right))$$

$$\mid b_2 - b_3 + b_4 \equiv 0 \mod p, \text{not all } b_i \text{ s are } 0\}/\mathbb{Z}/p \times^s (\mathbb{Z}/p)^*.^{*7a}$$

- H_4 acting on $\mathrm{Ni}(D_p, \mathbf{C}_{3^4})^{\mathrm{abs}} \Leftrightarrow$ (unramified) cover,

$$\mathcal{H}(D_p, \mathbf{C}_{3^4}) \to U_r.$$

- Then, reduced classes of solutions of Prob$_1$ are in

$$\mathcal{H}_1 \stackrel{\mathrm{def}}{=} \mathcal{H}(D_p, \mathbf{C}_{3^4})/\mathrm{PGL}_2(\mathbb{C}),$$

  a (ramified) cover of J_4. That cover is $X_0(p) \setminus \{\text{cusps}\}$.
- To be exceptional a cover corresponding to $\boldsymbol{p} \in X_0(p)(K)$ must have the *right* **arithmetic** monodromy?^{*7b}

$\mathcal{H}(G, \mathbf{C})^{*,\mathrm{rd}}$ is an upper half-plane quotient.

Compactified $\Phi_{G,\mathbf{C}}: \bar{\mathcal{H}}(G, \mathbf{C})^{*,\mathrm{rd}} \to \mathbb{P}^1_j$:

$$\text{ramified at } j = 0,1,\infty.^{*8,[\mathrm{BFr02},\S 3.7]}$$

1. Action of the H_4 normal subgroup

$$\mathcal{Q}'' = \langle q_1 q_3^{-1}, \mathbf{sh}^2 \rangle.^{*9}$$

2. Genus of $\bar{\mathcal{H}}(G, \mathbf{C})^{*,\mathrm{rd}}$. From branch cycles for $\Phi_{G,\mathbf{C}}$: through $(\gamma_1 = q_1 q_2, \gamma_2 = q_1 q_2 q_1, \gamma_3 = q_2)$ acting on

$$H_4 \text{ orbits on } \mathrm{Ni}(G, \mathbf{C})^* / \mathcal{Q}'' = \mathrm{Ni}(G, \mathbf{C})^{*,\mathrm{rd}}.$$

3. Geometric and arithmetic monodromy of space corresponding to each H_4 orbit in (2) under $\Phi_{G,\mathbf{C}}$.

Bibliography

This talk and the beginning of the Geometry I talk are based on §6.1 and §6.2 of [Fr05]. At the Institute for Advanced Study (1967–1969) I interacted with the two amensuenses of Serre's book as they wrote it. Only in [Fr77, §3] did I mention Serre's result before [Fr05]: in a natural way, modular curves are Hurwitz spaces. I developed new applications for each of the people I trained in later years. Only in 1995 could I expand the context of Serre's result as Geometry Talk II does, but many applications naturally came before this one.

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- [Fr77] *Galois groups and Complex Multiplication*, T.A.M.S. **235** (1978), 141–162.
- [Fr05] *The place of exceptional covers among all diophantine relations*, J. Finite Fields 11 (2005) 367–433, arXiv:0910.3331v1 [math.NT]