CHAPTER 2

# ANALYTIC CONTINUATION

## 1. Why Riemann's Existence Theorem?

We start with two different definitions of algebraic functions. An imprecise version of Riemann's Existence Theorem is that these describe the same set of functions. Chap. 2 has two goals. First: To define and show the relevance of *analytic continuation* in defining algebraic functions. Second: To illustrate points about Riemann's Existence Theorem in elementary situations supporting the main ideas. Our examples are *abelian* algebraic functions. They come from analytic continuation of a branch of the log function. This also shows how integration relates algebraic functions to crucial functions that aren't algebraic. These examples depend only on homology classes, rather than homotopy classes, of paths. The slow treatment here quickens in Chap. 4 to show how Riemann's approach organized algebraic functions without intellectual inundation.

**1.1. Introduction to algebraic functions.** The complex numbers are $\mathbb{C}$, the nonzero complex numbers $\mathbb{C}^*$ and the reals $\mathbb{R}$. We start with analytic (more generally, meromorphic) functions defined on an open connected set $D$, a *domain* on $\mathbb{P}^1_z = \mathbb{C} \cup \{\infty\}$, the Riemann sphere: §4.6 defines *analytic* and *meromorphic*. The standard complex variable is $z$. When $D$ is a disk, a function $f(z)$ analytic on $D$ has a presentation as a convergent power series about the center $z_0$ of $D$. The first part of the book describes *algebraic functions* (of $z$). Let $D$ be any domain in $\mathbb{P}^1_z$ and $z_0, z' \in D$. Denote (continuous) paths beginning at $z_0$ and ending at $z'$ by $\Pi_1(D, z_0, z')$ (§2.2.2). Use $\Pi_1(D, z_0)$ for closed paths in $D$ based at $z_0$. For any finite set $\boldsymbol{z} = \{z_1, \dots, z_r\} \subset \mathbb{P}^1_z$ denote $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ by $U_{\boldsymbol{z}}$.

1.1.1. *Riemann's definition of algebraic functions.* Suppose $f(z)$ is analytic in a neighborhood of $z_0$. Call $f$ *algebraic* if some finite set $\boldsymbol{z} \subset \mathbb{P}^1_z$ has these properties.

(1.1a) An *analytic continuation* (Def. 4.1) of $f(z)$ along each $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ exists. Call this $f_\lambda(z)$. Let $\mathcal{A}_f(U_{\boldsymbol{z}})$ be the collection $\{f_\lambda\}_{\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)}$.

(1.1b) The set $\mathcal{A}_f(U_{\boldsymbol{z}})$ is finite.

(1.1c) For $z' \in \boldsymbol{z}$, limit values of $f_\lambda$ along $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0, z')$ is a finite set.

1.1.2. *Standard definition of algebraic functions.* There is another definition of algebraic function (of $z$). Suppose $f(z)$ is analytic on a disk $D$. It is algebraic if some polynomial $m(z, w) \in \mathbb{C}[z, w]$ (nonconstant in $w$) satisfies

(1.2) $m(z, f(z)) \equiv 0$ for all $z \in D$.

This chapter explains (1.1) and its equivalence with (1.2) (Prop. 7.3).

Simple examples illustrate (1.1) and (1.2). These often appear briefly in a first course in complex variables. Though they give only algebraic functions with abelian monodromy group, they hint how Chap. 4 *lists* all algebraic functions.

We review elementary field theory as it applies to $f(z)$ satisfying (1.2). With no loss assume $m(z,w)$ in (1.2) is irreducible in the ring $\mathbb{C}[z,w]$ ([9.8]) and $f(z)$ satisfies (1.2). Any graduate algebra book is proper for this review, including [**Lan71**], [**Jac85**] and [**Isa94**]. The latter, with the best treatment of permutation representations and group theory, will be our basic reference. [**Isa94**, Chap. 17] contains material supporting the comments of §1.2.

**1.2. Equivalence of algebraic functions of $z$.** Let $\mathbb{C}(z)$ be the field of the rational functions in $z$. Its elements $u(z)$ consist of ratios $P_1(z)/P_2(z)$ with $P_1, P_2 \in \mathbb{C}[z]$. Standard notation denotes the greatest common divisor of $P_1$ and $P_2$ as $(P_1, P_2)$. Suppose $P_1$ and $P_2$ have no common nonconstant factor: Write this as $(P_1, P_2) = 1$. Then the integer *degree* of $u(z)$, $\deg(u)$, is $\max(\deg(P_1), \deg(P_2))$. The Euclidean algorithm finds the greatest common divisor of $P_1$ and $P_2$. Factor this out to compute $\deg(u)$. This degree is also the *degree* of the field extension $\mathbb{C}(z)$ over $\mathbb{C}(u(z))$: $[\mathbb{C}(z) : \mathbb{C}(u(z))]$ [9.3].

Suppose $L$ and $K$ are fields with $K \subset L$. The degree $[L : K]$ of $L/K$ is the dimension of $L$ as a vector space over $K$. Assume $L = K(\alpha)$ for some $\alpha \in L$. Then, $[L : K]$ is the maximal number of linearly independent powers of $\alpha$ over $K$: the *degree* of $\alpha$ over $K$. This degree is also the minimal positive degree of an irreducible polynomial $f_\alpha(w) \in K[w]$ having $\alpha$ as a zero. Up to multiplication by a nonzero element of $K$, $f_\alpha(w)$ is unique. If $L/K$ is a field extension, $\alpha \in L$ is *algebraic* over $K$ if $[K(\alpha) : K] < \infty$.

1.2.1. *The degree of* $\mathbb{C}(z)/\mathbb{C}(u(z))$. Introduce variables $z'$ and $w'$. Write $u(w')$ as $P_1(w')/P_2(w')$ with $(P_1, P_2) = 1$, and

$$m(z', w') = P_1(w') - z'P_2(w') \in \mathbb{C}[z', w'].$$

Then, $m(z', w')$ is irreducible of degree $n = \max(\deg(P_1), \deg(P_2))$ [9.3]. Consider $m(z', w')$ as a polynomial in $w'$ with coefficients in the field $\mathbb{C}(z')$. Let $w''$ be a zero of this polynomial in *some* algebraic closure of $\mathbb{C}(z') = K$. Then, $L = \mathbb{C}(z')(w'') = \mathbb{C}(w'')$ is the quotient field of the integral domain $R = K[w']/(m(z', w'))$. It is a degree $n$ extension of $\mathbb{C}(z')$. Now $\mathbb{C}(z')$ is isomorphic to $\mathbb{C}(u(z))$: map $z'$ to $u(z)$. Map $w''$ to $z$ to extend this to an isomorphism of $L$ with $\mathbb{C}(z)$.

1.2.2. *Degree of function fields over* $\mathbb{C}(z)$. §1.2.1 uses Cauchy's abstract production of $\mathbb{C}(z')(w'')$ with $w''$ a zero of $m(z', w')$ [**Isa94**, Lem. 17.18]. It, however, explicitly identifies $w''$ with $z$ and $z'$ with $u(z)$. Putting $L$ in $\mathbb{C}(z)$, a *genus* 0 or *pure transcendental* field over $\mathbb{C}$, is convenient for seeing the algebraic relation between functions —like $z'$ and $w''$.

Now assume $f(z)$ is any algebraic function according to (1.2). Similarly construct $L = \mathbb{C}(z, f(z))$, a degree $\deg_w(m(z,w))$ field extension of the rational functions $\mathbb{C}(z)$. This is the *algebraic function field* of $m$ (or of $f$). Call any $f^* \in L$ with $L = \mathbb{C}(z, f^*)$ a *primitive generator* of $L/\mathbb{C}(z)$. (Or, $f$ is just a primitive generator when reference to $z$ is clear.)

1.2.3. *Equivalence of presentations of* $L/\mathbb{C}(z)$. Infinitely many algebraic functions $f$ gives the same field $L$ up to isomorphism as an extension of $\mathbb{C}(z)$. Within a fixed algebraic closure of $\mathbb{C}(z)$ it is abstractly easy to list all primitive generators of $L$. They have the form $f^* = g(z, f_k)$ with $f_k$ any other zero of $m(z,w)$ and $g(z,u) \in \mathbb{C}(z)[u]$. To assure $\mathbb{C}(z, f^*) = L$ add that $[\mathbb{C}(z, f^*) : \mathbb{C}(z)] = [L : \mathbb{C}(z)]$. Riemann's Existence Theorem lists algebraic extensions of $\mathbb{C}(z)$ efficiently by listing the isomorphism class of extensions $L/\mathbb{C}(z)$ and not specific algebraic functions.

Suppose $\mathbb{C}(f(z))$ contains $z$. Then, $L = \mathbb{C}(f(z))$ is *pure transcendental*. So, it is easy to list (without repetition) generating algebraic functions. Even, however, when the total degree of $m$ is as small as 3, $L$ usually is not pure transcendental field [9.10g]. While listing generating functions of $L$ is then harder, it isn't our main problem. To identify when two function field extensions $L_1/\mathbb{C}(z)$ and $L_2/\mathbb{C}(z)$ are (or are not) isomorphic is more important. Two questions arise: Is $L_1$ isomorphic to $L_2$? If so, does the isomorphism leave $\mathbb{C}(z)$ fixed?

Abel handled these questions for cubic equations. His results would have been easy if $L$ was pure transcendental. This book includes applying Riemann's extension of Abel's Theorems. Riemann's Existence Theorem is the start of this extension.

Riemann's Existence Theorem foregoes having all algebraic functions within one convenient algebraic closure. There may be no unique algebraic closure of $\mathbb{C}(z)$ so useful as $\mathbb{C}$. §1.3 introduces the infinite collection of incompatible algebraically closed fields appearing in Riemann's Existence Theorem. Every algebraic function $f(z)$ appears in each of them.

**1.3. Puiseux expansions.** Consider the *Laurent field* $\mathcal{L}_{z'}$ consisting of series $f(z) = \sum_{n=N}^{\infty} a_n(z-z')^n$, with $N$ any integer, possibly negative, where $f(z)(z-z')^{-N}$ is convergent in some disk about $z'$. Elements of $\mathcal{L}_{z'}$ define functions meromorphic at $z'$. Then, $\mathcal{L}_{z'}$ is a field, containing $\mathbb{C}(z)$ and we are familiar with it. It isn't, however, algebraically closed. To remedy that, for any positive integer $e$ form $\mathcal{P}_{z',e}$, convergent series in a variable $u_e$. Think of $u_e$ as $(z-z')^{1/e}$: $u_e^e = z-z'$.

For $e \mid e^*$ let $t = e^*/e$. Map $\mathcal{P}_{z',e}$ into $\mathcal{P}_{z',e^*}$ by substituting $u_{e^*}^t$ for $u_e$. Regard the union $\cup_{e=1}^{\infty}\mathcal{P}_{z',e} = \mathcal{P}_{z'}$ as a field, the *direct limit* of the fields $\cup_{e=1}^{\infty}\mathcal{P}_{z',e}$ with its set of compatible generators $\{u_e\}_{e=1}^{\infty}$. Details on the following are in [9.9].

LEMMA 1.1. *Suppose $\mathcal{P}^*/\mathcal{L}_{z'}$ is any field extension generated by a sequence of elements $\{u_e^*\}_{e=1}^{\infty}$ with these properties.*

(1.3a) *$u_e^*$ is a solution of the equation $u^e = z - z'$.*

(1.3b) *$(u_{ee'}^*)^{e'} = u_e^*$ for all positive integers $e, e'$: compatibility condition.*

*Then, $u_e \mapsto u_e^*$ gives a canonical isomorphism between $\mathcal{P}^*$ and $\mathcal{P}_{z'}$ that is the identity on $\mathcal{L}_{z'}$. In particular, automorphisms of the Galois extension $\mathcal{P}_{z'}/\mathcal{L}_{z'}$ correspond one-one with compatible systems of roots of 1.*

The field $\mathcal{P}_{z'}$ of *Puiseux expansions* around $z'$ provides an explicit algebraically closed field extension of $\mathbb{C}(z)$. It is clear fractional exponents are necessary for an algebraic closure. It is harder to see they give an algebraically closed field (Cor. 7.5). The fields $\mathcal{P}_{z'}$ and $\mathcal{P}_{z''}$ are isomorphic. Such an isomorphism, however, restricts to mapping $\mathbb{C}(z) \to \mathbb{C}(z)$ by $z \mapsto z - (z'' - z')$. For comparing all algebraic functions of $z$ we usually must regard these algebraically closed fields as distinct. Each, in its own way contains the field of algebraic functions (using either (1.1) or (1.2)).

Comparing expressions for a given algebraic function embedded in different Puiseux fields leads to our precise version of Riemann's Existence Theorem.

**1.4. Monodromy groups and the genus.** Both definitions (1.1) and (1.2) readily attach a group $G_f$ to any algebraic function $f(z)$. Using an irreducible $m(z,w)$ from (1.2) (with $m(z,f(z)) \equiv 0$) it is the group of the splitting field of $m(z,w)$ over $\mathbb{C}(z)$ ([**Isa94**, p. 267] and [9.5]). The order of this group is the degree of the splitting field extension over $\mathbb{C}(z)$. Efficient use of group theory gives more structured information than describing field extensions. Knowing something about the Galois group is usually better information than comes from looking at polynomial coefficients.

§4.4.1 gives a geometric construction for $G_f$. Chap. 4 has this group as its main theme. This group reveals $\mathcal{A}_f(D)$ from (1.1) as the complete set of zeros $w$ of $m(z,w)$ (Prop. 6.4). Then, $G_f$ acts through analytic continuation. This representation of $G_f$ on $\mathcal{A}_f(D)$ (of degree $\deg_w(m(z,w))$ is discrete data from $f$. *Discrete* here means the group $G_f$ does not change with continuous changes in $\boldsymbol{z}$.

Every algebraic function $f$ has another integer attached to it, the *genus* of its function field (Chap. 4). If $L = \mathbb{C}(z, f(z))$ is isomorphic to $\mathbb{C}(t)$ for some $t \in L$, it has *genus 0* as above. This means all genus 0 function fields are abstractly isomorphic. Note: The integer $[L : \mathbb{C}(z)]$ is rarely a good clue for computing the genus [9.3]. Abel's results allow viewing genus 1 function fields as similar to genus 0 function fields, though that similarity has limits. Crucial: Unlike genus 0 fields, there are many isomorphism classes of genus 1 function fields (over $\mathbb{C}$).

Abel's results allow listing isomorphism classes of genus 1 function fields, exactly as we list points of $\mathbb{P}^1_z$. That is, with a classical parameter $j$ replacing $z$, finite values of $j$ correspond one-one to isomorphism classes of genus 1 function fields. As with $\mathbb{P}^1_z$ the value $j = \infty$ requires special consideration. Even if $L$ has genus 1, we don't easily find where its corresponding $j$ value is in this list. Still, for many problems this is a satisfactory theory.

Riemann generalized much of Abel's Theorem to function fields of all genuses. Most difficult was his analog, for genus greater than 1, of a parameter space for isomorphism classes of fields. Variants on its study continue today, and this book is an example.

**1.5. Advantages of Riemann's definition.** Defining branches of $z^{\frac{1}{e}}$ (§8.3) on any disk $D$ in $\mathbb{C} \setminus \{0\}$ gives a practical introduction to analytic continuation. This gives the simplest algebraic functions. Still, how would we have located $w = f(z)$ satisfying $f(z)^5 - 2zf(z) + 1 = 0$ by a similar definition? The field $\mathbb{C}(z, f(z))$, like $\mathbb{C}(z^{\frac{1}{e}})$, is pure transcendental [9.3]. Yet, this is not obvious from a Puiseux expansion of $f(z)$ around some point.

Suppose $f(z)$ is a convergent power series satisfying (1.1). Can we expect to find data appropriate to its description?: The set $\boldsymbol{z}$ of exceptional values, and the finite group expressing there are but finitely many analytic continuations around closed paths. Excluding elementary examples, the Riemann's Existence Theorem approach suggests it doesn't pay to give functions by their power series. *Elliptic functions* (Chap. 4 §6.1) are a good example where the functions are explicit, though power series don't give their definition. Riemann's Existence Theorem uses **group data** to replace power series information about $f(z)$.

This is practical, computable information about algebraic equations making Riemann's approach useful to the rest of mathematics. Especially it gives a way to track complete it gives a way to track complete collections of related algebraic functions. This is the story of *moduli* of families of covers. Abel used the modular function that classical texts call $j(\tau)$ where $\tau$ is a complex number in the upper half plane. We refine and generalize this theme.

## 2. Paths

We assume elementary properties of the *complete fields*, the *real numbers* $\mathbb{R}$ and the *complex numbers* $\mathbb{C}$ as in [**Rud76**, Chap. 1], [**Ahl79**, §1.1-1.3].

**2.1. Notation from calculus.** For each positive integer $n$, let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the set of ordered $n$-tuples $\boldsymbol{x} = (x_1, \dots, x_n)$ (resp. $\boldsymbol{z} = (z_1, \dots, z_n)$) of real (resp. complex) numbers. The set $\mathbb{R}^n$ is a *vector space* over $\mathbb{R}$: addition of $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ gives $(x_1 + y_1, \dots, x_n + y_n)$; and scalar multiplication of $\boldsymbol{x}$ by $r \in \mathbb{R}$ gives $r\boldsymbol{x} = (rx_1, \dots, rx_n)$. The zero element (*origin*) of $\mathbb{R}^n$ is $\mathbf{0} = (0, \dots, 0)$. The *inner product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^n x_i y_i$. The *law of cosines* (from high school trigonometry) interprets the dot product $\cdot$ to give the expression $|\boldsymbol{x}||\boldsymbol{y}|\cos(\theta)$ where $\theta$ is the (counter clockwise) angle from the side from $\mathbf{0}$ to $\boldsymbol{x}$ to the side from $\mathbf{0}$ to $\boldsymbol{y}$ in (a/the) plane containing $\mathbf{0}$, $\boldsymbol{x}$, $\boldsymbol{y}$. Define the *distance between* points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ to be

$$|\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}.$$

Here are simple properties of the distance function.

(2.1a) $|\boldsymbol{x}| \ge 0$ and $|\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.

(2.1b) $|\boldsymbol{x} - \boldsymbol{z}| \le |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$ for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$: the *triangle inequality*.

Thus, the distance function gives a *metric* on $\mathbb{R}^n$.

**2.2. Elementary properties and paths.** Multiplication of complex numbers is crucial, especially that each nonzero complex number has a multiplicative inverse. Still, vector calculus often appears in the study of analytic functions using the topological identification of $\mathbb{R}^2$ with $\mathbb{C}$. In standard coordinates: $(x, y) \in \mathbb{R}^2 \mapsto x + i\,y = z \in \mathbb{C}$. Rephrase multiplication of complex numbers on elements of $\mathbb{R}^2$: $z_1 \leftrightarrow (x_1, y_1)$ and $z_2 \leftrightarrow (x_2, y_2)$ gives the association $z_1 z_2 \leftrightarrow (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$. Beyond these properties we gradually introduce statements from a one semester graduate course in complex variables. Paths and integration, however, are so important, we pause for notation around integration of 1-forms and Riemannian metrics.

For $a, b \in \mathbb{R}, a < b$, $[a, b]$ denotes the closed interval of $\mathbb{R}$ with $a$ and $b$ as end points. A path in $\mathbb{R}^n$ consists of a *continuous* map $\gamma : [a, b] \to \mathbb{R}^n$ for some choice of $a$ and $b$ with $a < b$. That is, for each $t \in [a, b]$, there is a *range* value $\gamma(t)$, the point on the path at time $t$.

Integration around paths turns computations into first year calculus integrals or derivatives. Such integration extends to manifolds (Chap. 3) because they are pieces of $\mathbb{R}^n$ tied together. Since $\gamma(t)$ is a point of $\mathbb{R}^n$, it has coordinates. One standard notation for these coordinates is $(f_1(t), \dots, f_n(t))$ ($f$ is for function). Another possible notation is $(x_1(t), \dots, x_n(t))$. We prefer $(\gamma_1(t), \dots, \gamma_n(t))$. The points $\gamma(a)$ and $\gamma(b)$ are, respectively, the *initial* and *end* points of the path. The path $\gamma$ is *closed* if $\gamma(a) = \gamma(b)$.

2.2.1. *Derivatives of a path.* Call $\gamma$ differentiable if

$$\frac{d\gamma(t)}{dt} = \Big(\frac{d\gamma_1(t)}{dt}, \dots, \frac{d\gamma_n(t)}{dt}\Big),$$

the *tangent vector* to $\gamma$ at $t$, exists and is continuous for each $t \in [a, b]$. (Use one-sided limits at the end points.) Reminder: $\frac{d\gamma(t)}{dt}$ is a point in $\mathbb{R}^n$. Interpret it as giving a direction and speed (length of the vector $\frac{d\gamma(t)}{dt}$) of travel along the path $\gamma$ at time $t$. We always insist $\gamma$ is continuous (to be a path).

Definition 2.1. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a path. For $a \le a' < b' \le b$ denote the restriction of $\gamma$ to $[a', b']$ by $\gamma_{[a',b']}$. Call $\gamma$ *simplicial* if for some integer $m$ there exist

$t_0 = a < t_1 < \cdots < t_{m-1} < t_m = b$ with $\gamma_{|_{[t_i,t_{i+1}]}}$ differentiable, $i = 0, \ldots, m-1$. This includes $\gamma_{|_{[t_i,t_{i+1}]}}$ having a one-sided derivative at the end points.

2.2.2. *Paths and connectedness.* The notation $\Pi_1(X, x_0, x_1)$ denotes the collection of (continuous) paths in a topological space $X$, starting at $x_0$ and ending $x_1$. Write $\Pi_1(X, x_0)$ when $x_0 = x_1$. We often need paths in integrals to be simplicial. When necessary, the text assumes this implicitly for $\gamma$, though we may merely write $\gamma \in \Pi_1(X, x_0, x_1)$. For analytic continuation, or integrating meromorphic differentials, simplicialness is necessary only for paths satisfying explicit conditions as in (Rem. 4.4). One subtle use of simplicial paths is to give *classical generators* of the fundamental group of $U_{\boldsymbol{z}}$ (Chap. 4).

If $\Pi_1(X, x_0, x_1)$ is nonempty, then $x_1$ is *path-connected* to $x_0$. This is an equivalence relation, and the equivalence classes are the path-connected components of $X$. For subspaces of manifolds (Chap. 3; in particular, subspaces of $\mathbb{R}^n$), the path-connected components are the same as the connected components. Further, for our examples, using simplicial paths would define the same components. [**Ahl79**, p. 54-58] discusses connectedness at greater length.

**2.3. Integrals along a simplicial path.** Using simplicial paths guarantees existence of various integrals, including *arc length* and *line integrals* along $\gamma$. We explain this. Let $\gamma$ be a simplicial path in $\mathbb{R}^n$. Consider $T_\gamma : [a, b] \to \mathbb{R}^{2n}$ defined by $t \mapsto (\gamma(t), \frac{d\gamma(t)}{dt})$. Suppose $F = F(\boldsymbol{x}, \boldsymbol{y}) = F(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is defined and continuous on an open set containing the range of $T$. The integral

$$\int_\gamma F \stackrel{\text{def}}{=} \int_a^b F \circ T_\gamma \, dt \tag{2.2}$$

exists, though $\frac{d}{dt}(\gamma_i(t))$ may be undefined for finitely many $t$ [**Rud76**, p. 126]. Here are two traditional cases.

(2.3a) $F = \sqrt{\boldsymbol{y} \cdot Q(\boldsymbol{x})(\boldsymbol{y})}$ with $Q(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{y} \mapsto Q(\boldsymbol{x})(\boldsymbol{y})$ linear in $\boldsymbol{y}$, where $Q(\boldsymbol{x})$ is a symmetric and positive definite matrix for each $\boldsymbol{x}$.

(2.3b) $F = G(\boldsymbol{x}) \cdot \boldsymbol{y}$ with $G = (G_1(\boldsymbol{x}), \ldots, G_n(\boldsymbol{x})) : \mathbb{R}^n \to \mathbb{R}^n$ a continuous function (*vector field*) defined on the range of $\gamma$.

Definition 2.2. Suppose $\gamma$ is a one-one function onto its range. Case (2.3a) of (2.2) is the *arc length* of $\gamma$ relative to the *infinitesimal metric* $Q(\boldsymbol{x})$ at $\boldsymbol{x}$. [9.19] explains the value of *tensor* form for metrics. In case (2.3b), (2.2) is the line integral of the *differential one form* $G \cdot d\boldsymbol{x} = \sum_{i=1}^n G_i(\boldsymbol{x})\, dx_i$ along $\gamma$.

Here is the crucial point of these examples. Suppose we change $\gamma$ to another parameterization $\gamma^*$ of the same set. Then, (2.2) doesn't change modulo these conditions: $\gamma^*$ is one-one in case (2.3a); and $\gamma^*$ has the same beginning and end points as $\gamma$ in case (2.3b). Proving this uses Lemma 2.3 [9.19b].

Recall from vector calculus, the physical meaning of (2.3b). It is the *work* done in moving a particle along the path parametrized by $\gamma$ against the force field $G$. Here is the formula for computing integrals of such differential expressions along $\gamma$:

$$\int_\gamma \sum_{i=1}^n G_i(\boldsymbol{x})\, dx_i \stackrel{\text{def}}{=} \sum_{i=1}^n \int_a^b G_i(\gamma_1, \ldots, \gamma_n) \frac{d\gamma_i}{dt}\, dt. \tag{2.4}$$

Tensor form of a metric defines distance along $\gamma$ from an integral of positive functions [9.19]. The triangle inequality is automatic: $\int_a^b f(t)\, dt + \int_b^c f(t)\, dt \ge \int_a^c f(t)\, dt$ if $f(t) \ge 0$ for $t \in [a, c]$.

LEMMA 2.3 (Change of Variable Formula). *Let $\gamma : [c,d] \to \mathbb{R}$ be a simplicial path. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, defined on the range of $\gamma$ and $a = \gamma(c)$, $b = \gamma(d)$. Then,*

$$\int_a^b f(x)\,dx = \int_c^d f(\gamma(t))\,\frac{d}{dt}(\gamma(t))dt.$$

PROOF. This is a variant on [**Apo57**, p. 216]. Let $F(x) = \int_a^x f(t)\,dt$ for $x$ in the range of $\gamma$, and $H(x) = \int_c^x f(\gamma(t))\frac{d}{dt}(\gamma(t))\,dt$. The functions $F(\gamma(x))$ and $H(x)$ are both continuous. Excluding finitely many $x$, the chain rule shows they have the same derivatives. Thus $H(x) - F(\gamma(x))$ is a constant evaluated by taking $x = c$:

$$H(c) - F(\gamma(c)) = H(c) - F(a) = 0 - 0 = 0.$$

The formula follows by taking $x = d$. □

Apostol notes: "Many texts prove the preceding theorem under the added hypothesis that $\frac{d\gamma(t)}{dt}$ is never zero on $[c,d]$. The interval joining $a$ to $b$ need not be the image of $[c,d]$ under $\gamma$."

**2.4. Relation between integrals and analytic functions.** Integration theory is the heart of complex variables. Equations, algebraic or differential, with coefficients analytic on a *domain* $D$, define the classical functions of complex variables. By a domain we mean an open connected topological subspace of a given topological space. The first examples of the subject are domains in $\mathbb{C}$, the complex plane. As we use them, we will remind of most basics from a first semester graduate complex variables course.

This chapter refers to basic material of [**Ahl79**]. The notation $\mathcal{H}(D)$ denotes the ring (integral domain [9.8a]) of functions analytic (equivalently, *holomorphic*) on $D$. With $R$ any ring, let $R[w]$ be polynomials in $w$ with coefficients in $R$.

2.4.1. *Analytic Functions.* The definition of analytic function reflects how the chain rule works for a composition of an analytic function and a path. Assume $\lambda : [a,b] \to D$ is any differentiable path: $t \mapsto \lambda_1(t) + i\lambda_2(t)$ has $\lambda_1 = \Re(\lambda)$ and $\lambda_2 = \Im(\lambda)$, differentiable on the interval $[a,b]$.

DEFINITION 2.4. Suppose $z_0 \in D$, $t_0 \in [a,b]$ and $\lambda : [a,b] \to D$ is any path, differentiable at $t_0$, for which $\lambda(t_0) = z_0$. Then, $f(z)$ defined on $D$ is analytic at $z_0$ if there exists a complex number $M + iN$ dependent only on $f$ and $z_0$ with

$$\frac{d}{dt}(f \circ \lambda)(t_0) = (M + iN)\frac{d\lambda}{dt}(t_0). \tag{2.5}$$

To compute the derivative on the left, assume $f(z) = u(x,y) + iv(x,y)$ has partial derivatives (not necessarily continuous) and use the chain rule.

Apply (2.5) to $t \mapsto z_0 + (t - t_0)\boldsymbol{v}$ in two cases: $\boldsymbol{v} = 1$ and $\boldsymbol{v} = i$. This produces *two* expressions for each of $M$ and $N$. That $M$ and $N$ could satisfy both expressions is equivalent to the Cauchy-Riemann equations:

$$M = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } N = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2.6}$$

with each expression evaluated at $\lambda(t_0)$.

2.4.2. *The notation* $f'(z)$. To accentuate that the expression $M + iN$ comes from $f$ alone, denote it by $f'(z)$ or $\frac{df}{dz}$. It only, however, *exists* for functions satisfying the Cauchy-Riemann equations. Here are ways it is like a derivative.

(2.7a) It fits in the chain rule for $\frac{d}{dt}$ of $f(\lambda(t))$ like a derivative.

(2.7b) Directional derivative $D_{\boldsymbol{v}}$ of $f(z)$ in the direction $\boldsymbol{v}$ works as does the gradient for a general function $F : \mathbb{R}^2 \to \mathbb{R}^2$: $D_{\boldsymbol{v}}(f)(z_0) = f'(z_0)v$ is $\frac{d}{dt}(f(z_0 + tv))(0)$. Check equivalence of this with being analytic!

(2.7c) Analytic composites $\mathbb{C}\xrightarrow{h}\mathbb{C}\xrightarrow{g}\mathbb{C}$ have a simple chain rule [**Con78**, p. 35]:

$$\frac{d}{dz}(g \circ h)(z) = \frac{dg}{dw}(w)_{|w=h(z)}\frac{dh}{dz}(z).$$

(2.7d) $f'(z)\,dz$ acts like the differential 1-form $h'(x)\,dx$ in first year calculus.

**2.5. More explanation of differential forms.** First, consider (2.7d) in more detail. The fundamental theorem of calculus says $\int_a^b h'(x)\,dx = h(b) - h(a)$. A partial analog for integration on $\mathbb{C}$ considers $f'(z)\,dz$, with $f$ analytic. We say $f$ is a *primitive* (or antiderivative) of $f'$. The outcome is the same. Let $z_a$ and $z_b$ be two points in $D$. Then, let $\lambda : [a, b] \to D$ be a piecewise differentiable path from $z_a$ to $z_b$. [**Con78**, Ch. IV, Th. 1.18]:

$$\int_\lambda f'(z)\,dz = \int_a^b f'(\lambda(t))\frac{d}{dt}(\lambda(t))\,dt = f(z_b) - f(z_a). \tag{2.8}$$

Definition 2.5 (Differential forms). Suppose $m, n : \mathbb{C} \to \mathbb{C}$ are continuous on $D$, though maybe not analytic. The symbol $m(z)\,dx + n(z)\,dy$ is a differential (complex 1-form) on $D$. *Closed*, *locally exact* and *exact* differentials appear later.

A differential 1-form is analytic (or *holomorphic*) if on each disc in $D$ it has the form $f(z)\,dz$ with $f(z)$ analytic. We also use *meromorphic* differentials: $f$ is meromorphic on $D$. [**Con78**, p. 63] introduces only the differential 1-forms $m(z)\,dz$, ($m(z)$ may not be analytic). It often uses $\int_\lambda f$ to substitute for $\int_\lambda f\,dz$. These have the form above: Write $dz$ as $dx + idy$. They don't, however, include all differential 1-forms $m(z)\,dx + n(z)\,dy$.

It is convenient to change variables from $(x, y)$ to $(z, \bar{z})$ to write differentials in the form $u(z)\,dz + n(z)\,d\bar{z}$ with $\bar{z} = x - iy$ (and $d\bar{z} = dx - idy$). Chap. 3 Lem. 5.6 formulates the several complex variable version of the next lemma. Call a function *anti-holomorphic* if about each point it has a power series expression in $\bar{z}$.

Lemma 2.6. *The operator* $\frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ *maps* $z$ *to 1 and* $\bar{z}$ *to 0. So, it extends the action of* $\frac{\partial}{\partial z}$ *on holomorphic functions, and it kills anti-holomorphic functions. Similarly,* $\frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ *extends the action of* $\frac{\partial}{\partial \bar{z}}$ *from anti-holomorphic functions to all differentiable functions.*

*If* $f$ *is a differentiable function, the expression for the total differential* $df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ *is the same as* $\frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,dz$.

Proof. Everything is from the definitions. The sums defining $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ act on differentiable functions. For the last equality in differentials, check that $\frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,dz$, written in $x$ and $y$, gives $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$. □

## 3. Branch of $\log(z)$ along a path

Let $D$ be a domain in $\mathbb{C}^*$. Denote a path $\gamma : [a, b] \to D$ by just $\gamma$. A power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ defines the *exponential function* $e^z$.

**3.1. How $e^z$ defines branches of $\log(z)$.** The exponential has properties so valuable for explicit computation that many parts of mathematics find functions generalizing it. This chapter practices with the exponential function how that works. Here are basic properties of $e^z$.

(3.1a) $e^0 = 1$ and $e^{z_1+z_2} = e^{z_1}e^{z_2}$: $e^z$ gives a homomorphism $\mathbb{C} \to \mathbb{C}^*$.
(3.1b) $e^{x+i\,y} = e^x(\cos(y) + i\,\sin(y))$.

In particular, the exact values $w \in \mathbb{C}$ with $e^w = 1$ are in the set $\{n2\pi i \mid n \in \mathbb{Z}\}$. Variants of the following definition appear throughout this chapter.

Definition 3.1. Suppose $h(t)$ is a continuous function defined on $[a, b]$ satisfying $e^{h(t)} = \gamma(t)$. Call $h$ a *branch of* $\log(z)$ (or, of log) along $\gamma$.

For $z_0 \in D$, let $\gamma : [a, b] \to z_0$ be the constant path. Suppose $w = w_0$ is one solution of $e^w = z_0$. Then, all solutions are $\{w_0 + n2\pi i\}$: possible values of a branch of $\log h(z)$ at $z_0$. An easier definition is of a branch of log on the domain $D$. This is a continuous function $H : D \to \mathbb{C}$ satisfying $e^{H(z)} = z$ for all $z \in D$: a right inverse to the exponential function. It is necessary to assure $0 \notin D$; $e^{H(0)} = 0$ has no solution $H(0)$ because $e^z$ never equals 0.

**3.2. Questions about branches of log.** The two definitions raise the following questions. Variants apply to the general topic of analytic continuation.

(3.2a) What is the relation between Def. 3.1 and the definition of $H$?
(3.2b) When does a branch of log exist along $\gamma$, and if it exists how many such branches are there?
(3.2c) How does Def. 3.1 give a simple criterion for the existence of $H$ (on $D$)?
(3.2d) What integrals naturally associate with interpreting existence of $H(z)$?
(3.2e) What natural geometric relation between $\mathbb{C}^*$ and $\mathbb{C}$ codifies the answers to the previous questions?

Prop. 3.2 answers questions (3.2a), (3.2b) and (3.2c). Then, Prop. 3.5 answers those remaining. These arguments motivate the theory of Riemann surface covers and their moduli. We never use classical language referring to *branch cuts* (except in a simple example for its historical utility). In the proposition, unless otherwise said, assume $[a, b]$ is the domain of any path.

Proposition 3.2. *Suppose $H(z)$ is a branch of* log *on $D$. Fix $z_0 \in D$. Then, $h^\dagger(t) = H(\gamma(t))$ is a branch of* log *along $\gamma$. Further, suppose $h(t)$ is a branch of* log *along $\gamma$. Then, for $t_0 \in [a, b]$ there is a branch $H$ of* log *on a neighborhood of $\gamma(t_0)$ with $H(\gamma(t)) = h(t)$ for $t$ close to $t_0$.*

*Even if there is no branch of log on $D$, the following hold.*

(3.3a) *There is always a branch $h(t)$ of* log *along $\gamma$.*
(3.3b) *For $h^*(t)$ any branch of* log *along $\gamma$, $h(t) - h^*(t)$ is constant on $[a, b]$.*
(3.3c) *$h(t) + 2\pi i m$, $m \in \mathbb{Z}$, gives the complete set of branches of log along $\gamma$.*
(3.3d) *There is a branch $H(z)$ of* log *on $D$ precisely if for each $\gamma \in \Pi_1(D, z_0)$, $h(b) = h(a)$ for $h$ some branch of* log *along $\gamma$.*

**3.3. Proof of Prop. 3.2.** If $e^{H(z)} \equiv z$ for $z \in D$, then $e^{H(\gamma(t))} \equiv \gamma(t)$ for $t \in [a,b]$ as in the proposition statement. Thus, $h^{\dagger}(\gamma(t))$ is a branch of log along $\gamma$.

Now suppose $h^*(t)$ is any branch of log along $\gamma$. Then,

$$e^{h(t)}/e^{h^*(t)} = e^{h(t)-h^*(t)} = \gamma(t)/\gamma(t) \equiv 1$$

for $t \in [a,b]$. So, the continuous function $F(t) = h(t) - h^*(t)$ maps the connected set $[a,b]$ into the topological subspace $2\pi i\,\mathbb{Z}$ of $i\,\mathbb{R}$. The range of a connected set under a continuous function is connected. This shows the range of $F(t)$ is a single point; $F(t)$ is constant on $[a,b]$.

Suppose $z_0 \in \mathbb{C}$ satisfies $e^{z_0} = \gamma(a)$. The rest of the proof has three parts, corresponding to patching pieces of branches of log along $\gamma$.

3.3.1. *Extending a branch of log on a subpath.* Suppose $[a',b'] \subset [a,b]$. Then, restriction of $\gamma$ to $[a',b']$ produces a new path, $\gamma_{[a',b']}$. Let $h_{t_0}(t)$ be a branch of log along $\gamma_{[a,t_0]}$ for $t_0 \in [a,b]$ with $t_0 < b$.

A classical construction produces a branch $H(z)$ of log in any sector

$$S_{\theta_1,\theta_2} = \{re^{i\,\theta} \mid \theta_1 < \theta < \theta_2\} \text{ with } \theta_2 - \theta_1 \le 2\pi \text{ [9.7}a].$$

Any disk in $\mathbb{C}^*$ is in some sector. Restrict $H$ to a disk around $\gamma(t_0) = z_0$ and translate it by an integer multiple of $2\pi i$ to assume $H(z_0) = h(t_0)$. From above, $H(\gamma(t))$ is a branch of log along $\gamma$ restricted to $[t_0-\epsilon, t_0+\epsilon]$ for $\epsilon > 0$ small. Since $H(z_0) = h(t_0)$, these two branches of log are equal on $\gamma_{[t_0-\epsilon,t_0]}$. If $t_0+\epsilon \le b$, this defines a branch of log along $\gamma_{[a,t_0+\epsilon]}$:

$$h_{t_0+\epsilon}(t) = \begin{cases} h_{t_0}(t) & \text{for } t \in [a,t_0] \\ H(\gamma(t)) & \text{for } t \in [t_0, t_0+\epsilon]. \end{cases} \tag{3.4}$$

We say $h_{t_0+\epsilon}$ *extends* $h_{t_0}$.

3.3.2. *Sequences of extensions of branch of log.* Suppose $t_0 < t_1 < \cdots < b$ and $h_i(t)$ is a branch of log along $\gamma_{[a,t_i]}$, with $h_i(a) = z_0$ for each $i$. Then, from the first part of the proof, $h_{i+1}$ extends $h_i$. As the $t_i$ s are increasing and bounded, they have a limit point, $t^*$. Define $h_{t^*}$ by this formula: for $t < t^*$, $h_{t^*}(t) = h_i(t)$ where $t < t_i$; and $h_{t^*}(t^*) = \lim_i h_i(t_i)$. Note: The left side is independent of $i$. The right side has a limit because it is a Cauchy sequence.

3.3.3. *Completing existence of branch of log.* §3.3.2 shows there is a maximal $t'$ having a branch of log $h_{t'}$ along $\gamma_{[a,t']}$. Then, if $t' < b$, §3.3.1 gives an extension to $\gamma_{[a,t'+\epsilon]}$ for some $\epsilon > 0$. Thus, $t' = b$. That completes proving existence of the extension. Criterion (3.3d) for a branch of log on a domain is a special case of Lemma 4.12. This depends only on the notion of multiplying paths.

Suppose, as in Prop. 3.2, $h$ is a branch of log along $\gamma$. For $t \in [a,b]$ there is a neighborhood $D_t$ of $\gamma(t)$ and a branch $H_t(z)$ of log on $D_t$ satisfying this property.

(3.5) $H(\gamma(t')) = h(t')$ for $t'$ close to $t$.

This matches Def. 4.1: There is an analytic continuation of $H_a(z)$ along $\gamma$.

Example 3.3 (Branch of log along a circle). The function $t \mapsto e^{2\pi i t} = \gamma(t)$, $t \in [0,1]$, parametrizes the counterclockwise unit circle. Let $\epsilon > 0$ be small. As in [9.7a], $H_\epsilon(re^{2\pi i t}) = \ln(|r|) + 2\pi t$ is a branch of log for all $z$ of form $re^{2\pi i t}$, $0 \le t \le 1-\epsilon$. So, $h_\epsilon(t) = 2\pi i t$ is a branch of log along $\gamma_{[0,1-\epsilon]}$. Like the proof of Prop. 3.2, $h(t) = 2\pi i t$ extends $h_\epsilon$ to be a branch of log along $\gamma$.

**3.4. Branch of log as a primitive.** Let $g : D \to D'$ by $w \mapsto g(w)$ be continuous. Assume $g(w_0) = z_0$ with $w_0 \in D$ and $\gamma : [a,b] \to D'$ has $\gamma(a) = z_0$.

DEFINITION 3.4. Consider $\gamma^* : [a,b] \to D$ with $\gamma^*(a) = w_0$. Call it a *lift* (relative to $g$) of $\gamma$ (based at $w_0$) if $g(\gamma^*(t)) = \gamma(t)$ for all $t \in [a,b]$.

§4.4 has explicit notation for multiplying paths, as in $\gamma \cdot \gamma'$. Let $D$ be a domain in $\mathbb{C}^*$; $f(z) = 1/z$ is analytic in $D$. Suppose $\gamma \in \Pi_1(D, z_0, z')$ and $\Delta_{z'}$ is a disc in $D$ about $z'$. For $z \in \Delta_{z'}$ define $F_1(z)$ as $\int_{\gamma\cdot\gamma'} \frac{dz}{z}$ where $\gamma'$ is any path from $z'$ to $z$ in $\Delta_{z'}$. The discussion before Def. 5.1 has the precise definition of winding number.

PROPOSITION 3.5. *Given $\gamma$, $F_1(z) = F_{1,\gamma}(z)$ depends only on the end point of $\gamma'$. Also, $\frac{dF_1}{dz} = \frac{1}{z}$ for all $z \in \Delta_{z'}$. In particular, $F_1(z)$ differs by a constant from a branch of* log *along $\gamma \cdot \gamma'$. Suppose $\gamma_1$ and $\gamma_2$ have the same end points. Then, $F_{1,\gamma_1} - F_{1,\gamma_2} = 2\pi i m$ with $m$ the winding number of $\gamma_1 \cdot \gamma_2^{-1}$ about the origin.*

*Consider $\psi : \mathbb{C} \to \mathbb{C}^*$ by $w \mapsto e^w$. Suppose $\gamma : [a,b] \to \mathbb{C}^*$ has beginning point $z_0$ with $e^{w_0} = z_0$. Then, a branch of* log *along $\gamma$ (with initial value $w_0$) is a lift of $\gamma$ (starting at $w_0$; relative to $\psi$). Let $D^*$ be the connected component of $\psi^{-1}(D)$ through $w_0$. Then, there is a branch of* log *on $D$ with value $w_0$ at $z_0$ exactly when $\psi$ is one-one to $D$ on $D^*$.*

The first part requires Cauchy's Theorem ([**Ahl79**, p. 141, Cor. 1], [**Con78**, p. 84]). This typifies how integration of analytic functions arises. Abel and Riemann based information on differentials; in Riemann's Existence Theorem they are a substantial subplot.

PROPOSITION 3.6 (Cauchy's Theorem on a disk). *Suppose $D$ is a domain in $\mathbb{P}^1_z$ and $f(z)$ is analytic on $D$. Further, assume $D$ is either analytically isomorphic to $\mathbb{C}$ or to a disk. Then, $\int_\gamma f(z)\,dz = 0$ for each closed path in $D$.*

PROOF OF PROP. 3.5. Integration of $f(z) = 1/z$ along paths in $\mathbb{C}^*$ analytically continues a primitive for $f$ at the initial point. Thus, to prove $F_1(z)$ is independent of $\gamma'$ only requires showing the integral is 0 for any closed path $\gamma'$ in $\Delta_{z'}$. This, follows from Prop. 3.6. The remainder follows by plugging in a lift $\gamma^*$ of $\gamma$: $e^{\gamma^*(t)} = \gamma(t)$ for $t \in [a,b]$. By definition $\gamma^*$ gives a branch of log along $\gamma$. □

## 4. Analytic continuation along a path

Suppose $f(z)$ is a branch of log on a domain $D \subset \mathbb{C}^*$. Since $e^z$ is analytic on $\mathbb{C}$, Def. 3.1 provides *analytic continuation* of $f(z)$ along any path in $\mathbb{C}^*$. It does so using an equation $e^w = z$ to force the desired extension. The following generalizes Def. 3.1 (see §6.1). It requires *no* equation for extending an analytic function.

**4.1. Definition of analytic continuation.** Suppose $f$ is analytic in a neighborhood $U_{z_0} \subset D$ of $z_0$ and $\gamma : [a,b] \to D$ is a path in $D$ based at $z_0$.

DEFINITION 4.1 (Analytic continuation of $f$ along $\gamma$). Let $f^* : [a,b] \to \mathbb{C}$ be a continuous function with the following properties.

(4.1a) $f^*(t) = f(\gamma(t))$ for $t$ close to $a$ (in $[a,b]$).
(4.1b) For each $t' \in [a,b]$, there is a function $h_{t'}(z)$ analytic on a disk $D_{t'}$ about $\gamma(t')$ with $h_{t'}(\gamma(t)) = f^*(t)$ for $t$ near $t'$ (in $[a,b]$).

If such an $f^*$ exists, this definition produces $h_{t'}(z)$. This is the analytic continuation of $f$ *to* $t'$. It is an analytic function in some neighborhood of $\gamma(t')$. Usually, however, the important reference is to the *end* function $h_b(z)$, analytic in a neighborhood of $\gamma(b)$. This we call $f_\gamma(z) = f_\gamma$, analytic continuation of $f$ (along $\gamma$).

Note: $f^*(t)$ determines all data for an analytic continuation. It is unique: its difference from another function suiting (4.1) must be constant (hint of [9.8a]). Again, there is a related definition.

Suppose $\hat{f} : D \to \mathbb{C}$ satisfies $\hat{f}(z) = f(z)$ for all $z \in U_{z_0}$. We call $\hat{f}$ an analytic continuation or *extension* of $f$ to $D$.

REMARK 4.2. Let $\gamma : [a, b] \to \mathbb{P}^1_z$ be a nonconstant path. Here is an example of a function analytic at $\gamma(a)$ with no analytic continuation along $\gamma$. Assume $\gamma(t') \neq \gamma(a)$ for $t'$ close to $a$ and let $f$ be a branch of $\log(z - \gamma(t'))$ about $\gamma(a)$. Algebraic functions, and others, like branches of log, analytically continue along any path missing some finite set $\boldsymbol{z}$ of points on $\mathbb{P}^1_z$. Def. 4.5 introduces $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$, analytic functions around $z_0$ that are *extensible* if we avoid $\boldsymbol{z}$.

**4.2. Practical analytic continuation.** Analytic functions have a power series expression around each point of their domain. This converges in any disc not containing a singularity of the analytic function [**Ahl79**, p. 179, Thm. 3].

4.2.1. *Using disks of convergence.* In Def. 4.1, for example, consider $\gamma$ with range a segment of the real axis. Assume also $f^*$ is real-valued along $\gamma$ with continuous derivatives of all order. Then, an analytic function restricts to $f^*$ along $\gamma$ if and only if $f^*$ has a Taylor series around each point. This gives a practical alternative definition of analytic continuation using *polygonal paths* like $\gamma^*$ in the next lemma. Notation is from Def. 4.1.

LEMMA 4.3. *The following is equivalent to $f$ having an analytic continuation along $\gamma$. There exists a partition $a = t_0 < t_0^* < t_1 < t_1^* < \cdots < t_{n-1}^* < t_n = b$ of $[a, b]$, disks $D_i$ centered about $\gamma(t_i)$ and $f_i \in \mathcal{H}(D_i)$ with these properties.*

(4.2a) *$D_i \cap D_{i+1} \neq \emptyset$ and $f_i(z) = f_{i+1}(z)$ for $z \in D_i \cap D_{i+1}$.*
(4.2b) *$\gamma(t) \in D_i$ for $t \in [t_i, t_i^*]$, $\gamma(t) \in D_{i+1}$ for $t \in [t_i^*, t_{i+1}]$, $i = 0, \dots, n-1$.*
(4.2c) *$f_0(z) = f(z)$ for $z \in D_0$.*

*Further, let $\gamma^*$ be the path following consecutive line segments $\gamma(t_i)$ to $\gamma(t_i^*)$, then $\gamma(t_i^*)$ to $\gamma(t_{i+1})$, $i = 0, \dots, n-1$. Then, $f_{\gamma^*} = f_\gamma$.*

PROOF. Suppose we have the pairs $(D_i, f_i)$, $i = 1, \dots, n$, and the partition of $[a, b]$. This gives an analytic continuation of $f$ along $\gamma$ by the following formula:

$$f^*(t) = \begin{cases} f_i(\gamma(t)) & \text{for } t \in [t_i, t_i^*] \\ f_{i+1}(\gamma(t)) & \text{for } t \in [t_i^*, t_{i+1}]. \end{cases}$$

Then, $f^*(t)$ provides an analytic continuation from Def. 4.1.

Follow notation of §3.3.1. Inductively consider analytic continuation of $f$ to the end point of $\gamma_{[a,t_i]}$ (and $\gamma_{[a,t_i^*]}$). Set up the induction by showing this is analytic continuation of $f$ to the end point of $\gamma^*_{[a,t_i]}$ (and $\gamma^*_{[a,t_i^*]}$). The essential point is $f_i$ exists on a disk containing the range of $\gamma$ on $[t_i, t_i^*]$. So, $f_i$ in a neighborhood of $\gamma(t_i^*)$ analytically continues $f_i$ (from a neighborhood of $\gamma(t_i)$) along *any* path entirely within $D_i$. Then, at the end points of $\gamma$ and $\gamma^*$, $f_{\gamma^*} = f_\gamma$.

Now assume we have an analytic continuation of $f$ along $\gamma$. Completing the lemma requires creating $(D_i, f_i)$ for a corresponding partition of $[a, b]$. Since the range of $\gamma$ is compact, the distance between $\gamma(t)$ and $\gamma(t')$ is a uniformly continuous function of $(t, t')$. So, for $d' > 0$ there exists $d > 0$ with $|\gamma(t) - \gamma(t')| < d'$ if $|t - t'| < d$. Choose $d'$ with the following property.

(4.3) For each $t' \in [a, b]$, there is a disk of radius no more than $d'$ around $\gamma(t')$ supporting analytic $h_{t'}(z)$ as in Def. 4.1.

Compactness of the range of $\gamma$ produces such a $d'$. Use $d$ from the above comment. Partition $[a,b]$ so $|t_i - t_i^*|$ and $|t_i^* - t_{i+1}|$ are at most $d$. Then, inductively show this partition has the desired properties. □

REMARK 4.4 (Nonsimplicial paths). §4.6 extends Lemma 4.3 to $D \subset \mathbb{P}^1_z$. There *geodesic paths* on $\mathbb{P}^1_z$ might replace polygonal paths: its pieces are arcs on longitudinal circles. The proof extends with no change.

Lem. 4.3 makes no assumption paths are simplicial. Chap. 3 applies the lemma to general continuous paths. A simplicial assumption allows integrating general differential 1-forms or for computing arc length. Still, suppose $\omega = f(z)\,dz$ is an analytic 1-form in a neighborhood of $z_0$ and $\gamma : [a,b] \to \mathbb{C}$ is a (continuous, not necessarily simplicial) path with beginning point $z_0$.

Let $D$ be any domain containing the range of $\gamma$ in which $f$ extends analytically along each path. Lemma 4.3 produces a simplicial (or polygonal) path $\gamma^*$ in $D$ (notice $D$ contains no potential poles of $f$) along which integration of $f$ is defined. Let $F(z)$ be an antiderivative of $f(z)$. Analytic continuation of $F(z)$ along $\gamma^*$ allows defining $\int_\gamma \omega$ to be $F(\gamma(b)) - F(\gamma(a))$.

4.2.2. *The word monodromy*. *Monodromy* isn't in Webster's dictionary. It is in [**Ahl79**, p. 295] and [**Con78**, p. 219] in the statement of the *Monodromy Theorem* (§8.2 and Chap. 3 Prop. 6.11). The Oxford English Dictionary references exactly the same theorem. It gives it the following meaning:

> The characteristic property: If the argument returns by any path to its original value, the function also returns to its original value.

We extend that to include regions where a function may not return to its original value. For this we add group data that accounts for the nonreturn. The loose name for that structure is *monodromy action*, though we often drop the last word.

The simplest setup for discussing monodromy starts with these elements:

(4.4a) a domain $D$ and $z_0 \in D$
(4.4b) a closed path $\lambda$ based at $z_0$
(4.4c) $f(z)$ analytic in a neighborhood of $z_0$
(4.4d) $f$ has an analytic continuation around $\lambda$

Then, analytic continuation around $\lambda$ produces a (possibly) new function, $f_\lambda$ analytic in a neighborhood of $z_0$.

DEFINITION 4.5 (Extensibility). Assume the setup of (4.4) for *every* closed path in $D$. Call such an $f$ *extensible* in $D$: $(f,D) = (f,D,z_0)$ is extensible. This is a neologism, differing from the notion $f$ has an extension (is extendible) to $D$. Denote the complete set of extensible functions in $D$ (based at $z_0$) by $\mathcal{E}(D,z_0)$.

By assumption $\mathcal{E}(D,z_0) \subset \mathcal{L}_{z_0}$. So, field operations like multiplication and taking ratios make sense. Suppose $f,g \in \mathcal{E}(D,z_0)$. Recall the notation $\mathbb{C}[z,u,v]$ for polynomials in $z,u,v$. Define $\mathbb{C}[z,f,g] = R$ to be $\{\alpha(z,f,g)$ with $\alpha \in \mathbb{C}[z,u,v]\}$.

LEMMA 4.6. *With the above assumptions, the ring $R$ consists of extensible functions. For any $\lambda \in \Pi_1(D,z_0)$, $\alpha_\lambda = \alpha(z,f_\lambda,g_\lambda)$.*

*Assume $f \in \mathcal{E}(D,z_0)$ and $D$ is analytically isomorphic to a disk (or to $\mathbb{C}$). Then, $f$ is extendible (restriction of an analytic function) on $D$.*

PROOF. For the first part, show the last result for $f+g$ and $fg$. Every element in $R$ is built from such algebraic operations. Now consider the case $D$ is a disk. Cauchy's Integral formula for an analytic function says a power series for an analytic function converges up to a singularity on its boundary of convergence. Consider $f \in \mathcal{E}(D, z_0)$ with $z_0$ the center of the disk $D$.

Suppose the power series for $f$ converges only on a disk of radius smaller than $D$. Then, analytic continuation of $f$ to some singular boundary point fails. This is contrary to $f \in \mathcal{E}(D, z_0)$.

More generally, let $\beta : D \to \Delta$ be an analytic isomorphism of $D$ with a disk. Then, $(f \circ \beta^{-1}, \Delta, \beta(z_0))$ extends to $F(z)$, and $F(\beta(z))$ extends $f$. □

REMARK 4.7. Webster's dictionary defines extensible to mean capable of being extended, whether in length or breadth; susceptible of enlargement. That agrees with our definition. Still, it has extendible as a synonym of extensible, whereas we distinguish between the two words.

4.2.3. *Meromorphic extensibility.* It simplifies many discussions to allow meromorphic functions in $\mathcal{E}(D, z_0)$. Even on $U_{\boldsymbol{z}}$, in considering $f \in \mathcal{E}(D, z_0)$, we eventually remove $z'$ from $\boldsymbol{z}$ if it is only a pole of $f$. The simplest way is to allow in $\mathcal{E}(D, z_0)$ functions $f$ having for each path $\gamma$ some $g \in \mathbb{C}(z)$ with $g(z)f(z)$ extensible along $\gamma$ as in Def. 4.5. Technical proofs would use extensibility of $g(z)f(z)$ and analytic continuation to the end point of $\gamma$ would be $(g(z)f(z))_\gamma/g(z)$. The result, of course, could have a pole at the end of the path.

In Def. 4.1 there is an auxiliary function $f^* : [a, b] \to \mathbb{C}$: $f^*(t) = f(\gamma(t))$, the values of $f$ along $\gamma$. Extending $f^*$ to allow poles requires allowing maps into $\mathbb{P}^1_z$.

For example: If $g(z)$ is a branch of log at $z_0 = 1$, we allow $g(z)/(z-1)$ in $\mathcal{E}(\mathbb{C}^*, 1)$. Unless there is a reason to be careful about poles, most discussions will proceed as with extensibility of analytic functions. Integrals and primitives of a function require such care (§4.3). Occasions may need extending this definition to include infinitely many poles.

4.2.4. *Conjugates of $f$.* Assume $f \in \mathcal{E}(D, z_0)$. Even if $\lambda$ isn't closed, $f_\lambda$ has meaning for any path $\lambda$ in $D$ based at $z_0$. This produces *conjugates* of $f$ (in $D$) or the *monodromy range* of $(f, D, z_0)$:

$$\mathcal{A}_f(D, z_0) = \mathcal{A}_f(D) = \{f_\lambda(z)\}_{\lambda \in \Pi_1(D, z_0)}.$$

Regard $f_{\lambda_1}, f_{\lambda_2} \in \mathcal{A}_f(D)$ as equal if are the same function near $z_0$. As in [9.8a], $f_{\lambda_1}$ and $f_{\lambda_2}$ are then equal in any neighborhood of $z_0$ where they are meromorphic. Prop. 7.3 implies conjugate here is exactly as in basic Galois Theory. Suppose $h \in K[x]$ an irreducible polynomial over a field $K$ and $h(\alpha) = 0$. Then, the full collection of zeros of $h$ are the *conjugates of* $\alpha$.

Recall the *Laurent series* field $\mathcal{L}_{z_0}$ (about $z_0$). This consists of ratios of power series convergent around $z_0$. The ring $\mathcal{A}_f(D, z_0)$ is in $\mathcal{L}_{z_0}$. So we may form the composite field $\mathbb{C}(\mathcal{A}_f(D, z_0))$ these functions generate. Still, not all elements of $\mathbb{C}(\mathcal{A}_f(D, z_0))$ are in $\mathcal{E}(D, z_0)$ unless $f$ is algebraic.

LEMMA 4.8. *If $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ is algebraic (as in* (1.2)*), then $1/f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. So, the field $\mathbb{C}(z, f)$ that $z$ and $f$ generate in $\mathcal{L}_{z_0}$ is in $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$.*

PROOF. This requires showing extensions of $f$ have only finitely many zeros. Suppose $f$ satisfies an equation $m(z, f(z))$ with $m \in \mathbb{C}[z, w]$. Then, $\deg_z(m)$ bounds the number of solutions of $m(z, 0) = 0$. That shows $f(z)$ has only finitely many zeros among its analytic continuations, so $1/f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. □

Prop. 7.3, showing equivalence of (1.1) and (1.2), lets Lem. 4.8 apply without reservation to algebraic functions.

**4.3. A branch of a primitive.** Continue notation from §4.1. Suppose $F(z)$ is a primitive of $f(z)$ in $U_{z_0}$: $\frac{dF}{dz} = f(z)$. This discussion does require care on extensibility of meromorphic functions as in §4.6. If $f$ is meromorphic in $D$, and $z' \in D$, write $f$ as $h_1(z) + f_1(z)$ with these properties.

(4.5a) $f_1$ is analytic in a neighborhood of $z'$.
(4.5b) $h_1(z) = \frac{1}{z-z'} m_{z'}(\frac{1}{z-z'})$ with $m_{z'}(z) \in \mathbb{C}[z]$ ($\equiv 0$ for $f$ analytic at $z'$).

Then, the *residue* of $f$ at $z' \in D$ is $m_{z'}(0)$.

DEFINITION 4.9. Consider $f \in \mathcal{E}(D, z_0)$, $z' \in D$ and a path $\gamma : [a, b] \to D$ based at $z_0$. Denote the restriction $\gamma_{[a,t]}$ to $[a, t]$ by $\gamma_t$. We say $f$ has no residue along $\gamma$ if $f_{\gamma_t}$ has no residue for each $t \in [a, b]$.

A (branch of) primitive of $f(z)$ along $\lambda : [a, b] \to D$ is an analytic continuation $\hat{F}_\lambda$ of $F(z)$ along $\lambda$. We also label it by $\hat{F} : [a, b] \to D$.

LEMMA 4.10. *Assume $f \in \mathcal{E}(D, z_0)$. Then, $f$ has a primitive in a neighborhood of $z_0$ when it has no residue at $z_0$. Let $\gamma : [a, b] \to D$ be a path in $D$ along which $f$ has no residue. Then there exists a primitive $\hat{F} : [a, b] \to \mathbb{C}$ of $f$ along $\gamma$. Further, for $c \in \mathbb{C}$, there is a unique such $\hat{F}$ with $\hat{F}(a) = c$.*

PROOF. Get a primitive for $f$ in a neighborhood of $z_0$ from a primitive for each term in the Laurent series for $f$ around $z_0$. The function $z^k$ has a primitive $\frac{1}{k+1} z^{k+1}$ if $k \neq -1$. The discussion from §3.4 has done overkill on showing $z^{-1}$ has no primitive. That is, $f$ must have 0 as residue at $z_0$ to have a primitive. Further, by assumption every analytic continuation of $f$ (in $D$) has this property.

Let $D_0$ be a disk centered at $z_0$ and contained in $D$. By assumption $f(z)$ has no residue along any path in $D$. So, it has a primitive $F(z) = F_0(z)$ in this disk; integrate the power series for $f(z)$ term by term. The primitive is unique up to addition of a constant.

Now apply the notation of Lemma 4.3. Similarly, there exists $F_i(z)$, a primitive of $f_i(z)$ in $D_i$, $i = 1, \dots, n$. Since $f_i = f_{i+1}$ in $D_i \cap D_{i+1}$, $F_i(z)$ and $F_{i+1}$ have equal derivatives on this intersection. Thus, $F_i - F_{i+1}$ is a constant on $D_i \cap D_{i+1}$. This sets up for an induction. Assume $k$ is an integer for which $F_0(z), \dots, F_k(z)$ give an analytic continuation of $F(z)$ along $\gamma_{[a,t_k]}$. Let $F_{k+1}$ be the function we just produced, where $F_k - F_{k+1} = b$ for $z \in D_k \cap D_{k+1}$. Now replace $F_{k+1}$ by $F_{k+1} + b$. Continue inductively on $k$ to conclude the result. □

**4.4. Continuation along products of paths.** Let $\lambda_1 : [a, b] \to D$ be a path where $\lambda_1(a) = z_0$ and $\lambda_1(b) = z_1$. Assume $\lambda_2 : [a^*, b^*] \to D$ is another path and $\lambda_1(b) = \lambda_2(a^*)$. Create a new path $\lambda_1 \cdot \lambda_2 \stackrel{\text{def}}{=} \lambda^\dagger : [a, b + b^* - a^*] \to D$:

$$\lambda^\dagger = \begin{cases} \lambda_1(t) & \text{for } t \in [a, b] \\ \lambda_2(t + a^* - b) & \text{for } t \in [b, b + b^* - a^*]. \end{cases} \tag{4.6}$$

The proof of Lemma 4.12 includes detailed notation for a sequence of analytic continuations. Use that notation for details of the following lemma. Given a path $\lambda$, denote the path $t \mapsto \lambda(b - t + a)$, $t \in [a, b]$, by $\lambda^{-1}$, the *inverse of* $\lambda$. If $\lambda$ is simplicial so is $\lambda^{-1}$. Continue notation for the function $f$ and let $f_1 = f_{\lambda_1}$ be analytic continuation of $f$ along a path $\lambda_1$.

LEMMA 4.11. *For paths $\lambda_1$, $\lambda_2$ and $\lambda_3$, assume the end point of $\lambda_i$ equals the beginning point of $\lambda_{i+1}$, $i = 1, 2$. Analytic continuation of $f_1$ along $\lambda_2$, $f_2 = (f_1)_{\lambda_2}$,* *is the analytic continuation* $f_{\lambda_1\cdot\lambda_2}$ *of* $f$ *along* $\lambda_1\cdot\lambda_2$. *Then,* $f_{(\lambda_1\cdot\lambda_2)\cdot\lambda_3} = f_{\lambda_1\cdot(\lambda_2\cdot\lambda_3)}$ *giving unambiguous meaning to* $f_{\lambda_1\cdot\lambda_2\cdot\lambda_3}$. *Also,* $f_{\lambda\cdot\lambda^{-1}} = f$.

*As in §2.3,* $\int_{\lambda_1\cdot\lambda_2} F\,dz = \int_{\lambda_1} F\,dz + \int_{\lambda_2} F\,dz$, *Further,* $\int_{\lambda\cdot\lambda^{-1}} F\,dz = 0$.

While $\lambda\cdot\lambda^{-1}$ isn't the constant path (at $\lambda(a)$), Lemma 4.11 lists situations where it acts as if it is.

LEMMA 4.12. *Suppose* $f \in \mathcal{E}(D, z_0)$. *Let* $\lambda^*$ *be any path with beginning point* $z_0$ *and end point* $z_1$. *Let* $f_1 = f_{\lambda^*}$. *There is a one-one map between* $\mathcal{A}_f(D, z_0)$ *and* $\mathcal{A}_{f_1}(D, z_1)$. *Also,* $f$ *is extendible to* $D$ *if and only if* $f_\lambda = f$ *for each* $\lambda \in \Pi_1(D, z_0)$.

§4.5 has the proof of Lemma 4.12. It says there is an analytic function $\hat{f}$ on $D$ restricting to $f$ around $z_0$ exactly when $\mathcal{A}_f(D, z_0)$ has a single element. Then, monodromy action on $(f, D)$, or (if $D$ is clear, on $f$) is trivial.

4.4.1. *A permutation representation.* For $f \in \mathcal{E}(D, z_0)$ and $\lambda \in \Pi_1(D, z_0)$, Lemma 4.11 gives a permutation of $\mathcal{A}_f(D, z_0)$ by $h \mapsto h_\lambda$ for $h \in \mathcal{A}_f(D, z_0)$. Denote $h_\lambda$ by $(h)T(\lambda)$ to distinguish $T(\lambda)$ as a permutation of the set $\mathcal{A}_f(D, z_0)$. According to Lemma 4.11,

$$((h)T(\lambda_1))T(\lambda_2) = (h)T(\lambda_1) \circ T(\lambda_2) = (h)T(\lambda_1\cdot\lambda_2), \tag{4.7}$$

for $\lambda_1, \lambda_2 \in \Pi_1(D, z_0)$.

That is, analytic continuation gives a homomorphism from the semi-group (set with multiplication) $\Pi_1(D, z_0)$ to permutations on $\mathcal{A}_f(D, z_0)$. From Lem. 4.11, the permutation $T(\lambda)$ has $T(\lambda^{-1})$ as its inverse permutation. So, the image set of permutations is a group. Call it the *monodromy group* $G_{f,D}$ of $(f, D)$.

Chap. 3 puts an equivalence relation, *homotopy*, on $\Pi_1(D, z_0)$ to produce the *fundamental group* $\pi_1(D, z_0)$. In particular, from those results $T$ produces a permutation representation of $\pi_1(D, z_0)$. This chapter's elementary examples depend only on *homology classes* of $\Pi_1(D, z_0)$ (§5 and [9.12]; Chap. 3 §6.2 has the comparison).

**4.5. Proof of Lemma 4.12.** We show unique analytic continuation to the end points of each closed path implies $f$ extends analytically to $D$. First, we construct the map between $\mathcal{A}_f(D, z_0)$ and $\mathcal{A}_{f_1}(D, z_1)$ based on $\lambda^*$ as in the lemma. Then, $\mathcal{A}_f(D, z_0)$ consists of a single element if and only if $\mathcal{A}_{f_1}(D, z_1)$ does. Then, we construct $F$, the extension of $f$.

4.5.1. *Identifying* $\mathcal{A}_f(D, z_0)$ *and* $\mathcal{A}_{f_1}(D, z_1)$. Given $h = f_\lambda \in \mathcal{A}_f(D, z_0)$, apply Lemma 4.11 several times to produce this chain:

$$\begin{aligned} h_{\lambda^*} = {}& f_{\lambda\cdot\lambda^*} = \\ f_{\lambda^*\cdot(\lambda^*)^{-1}\cdot\lambda\cdot\lambda^*} = {}& (f_1)_{(\lambda^*)^{-1}\cdot\lambda\cdot\lambda^*}, \end{aligned} \tag{4.8}$$

since $(\lambda^*)^{-1}\cdot\lambda\cdot\lambda^* \in \Pi_1(D, z_1)$. This gives a map from $\mathcal{A}_f(D, z_0)$ to $\mathcal{A}_{f_1}(D, z_1)$: *Conjugating* paths based at $z_0$ by $\lambda^*$.

Map in the other direction by conjugating by $(\lambda^*)^{-1}$. These maps between $\mathcal{A}_f(D, z_0)$ and $\mathcal{A}_{f_1}(D, z_1)$ are inverse to each other. That is, conjugating $\mathcal{A}_f(D, z_0)$ by $\lambda^*\cdot(\lambda^*)^{-1}$ acts trivially on $\mathcal{A}_f(D, z_0)$ (from in Lemma 4.11). Conclude: Monodromy action on $f$ (in $D$) is trivial if and only the same holds for $f_{\lambda^*}$.

4.5.2. *Extending* $f$ *to be analytic on* $D$. We prove the last statement of the lemma. Suppose $f$ extends to $\hat{f}$ analytic on $D$. Then uniqueness of analytic continuation shows $f_\lambda(\lambda(t)) = \hat{f}(\lambda(t))$ for each $t$ near $b$ ($\lambda \in \Pi_1(D, z_0)$).

Now suppose $f_\lambda = f$ for each $\lambda \in \Pi_1(D, z_0)$. For $z' \in D$, assume $z$ is in a disk neighborhood about $z'$ entirely contained in $D$. Set $\hat{f}(z)$ equal to $f_\lambda(z)$

with $\lambda : [a, b] \to D$ a path where $\lambda(a) = z_0$ and $\lambda(b) = z'$. Lem. 4.6 says $f_\lambda$ extends to be analytic in the whole disk neighborhood. So this defines $f_\lambda(z)$. Let $\lambda^* : [a^*, b^*] \to D$ be another such path with the same end points. We have only to show $f_{\lambda^*}(z) = f_\lambda(z)$.

Then, $\lambda^\dagger = \lambda^{-1} \cdot \lambda^*$ is a closed path based at $\lambda(b)$. From §4.5.1, analytic continuation of $f_\lambda$ around $\lambda^\dagger$ equals $f_\lambda(z)$. It also equals analytic continuation of $f_\lambda$ along $\lambda^{-1}$ followed by analytic continuation of $f$ along $\lambda^*$. The result of these analytic continuations is $f_{\lambda^*}$. This proves the desired equalities.

**4.6. Extending analytic continuation to $\mathbb{P}^1_z$.** Similar definitions work for meromorphic functions in a domain, including analytically continuing meromorphic functions. It simplifies results of Chap. 3 to systematically extend paths into $\mathbb{P}^1_z$. Recall: A neighborhood basis of open sets around each point gives the topology on a space. Around $\infty$ the neighborhood basis consists of sets of form $N \cup \{\infty\}$ where $N$ is the complement of any closed set in $\mathbb{C}$.

EXAMPLE 4.13 (Meromorphic functions). Suppose for some disc $\Delta_{z_0}$ about $z_0$, $D \cap \Delta_{z_0} = \Delta_{z_0} \setminus \{z_0\}$. That is, $z_0$ is an *isolated boundary point* of a domain $D$. Further, assume $f$ is analytic on $D$ and it extends to a meromorphic function at $z_0$. That means $\lim_{z \mapsto z_0} (z - z_0)^{n+1} f(z) = 0$ for some $n \in \mathbb{Z}$ [**Con78**, p. 109]. The minimal such $n$ allows expressing $f(z)$ as $(z - z_0)^n h(z)$ with $h$ holomorphic and nonzero in a neighborhood of $z_0$. If the minimal $n$ is negative, then $f$ has a *pole* of order $n$. Define $F : D \cup \{z_0\} \to \mathbb{P}^1_z$ by this formula:

$$F(z) = \begin{cases} f(z) & \text{for } z \in D \\ \infty & \text{for } z = z_0. \end{cases} \tag{4.9}$$

Continuity of $F$ is equivalent to continuity of $z \mapsto 1/F(z)$ around $z_0$. This function is continuous at $z_0$ (taking the value 0). So it is continuous around $z_0$.

DEFINITION 4.14 (Analytic maps to $\mathbb{P}^1_z$). Suppose $f : D \to \mathbb{P}^1_z$ is analytic. Assume $z_0$ is an isolated boundary point of $D$ and $f$ extends to be meromorphic in a neighborhood of $z_0$. Then, we say the extension $F : D \to \mathbb{P}^1_z$ is analytic. If $f(z_0) = \infty$, this means $z \mapsto 1/f(z)$ (with $z_0 \mapsto 0$) is analytic in a neighborhood of $z_0$. Also, suppose $\infty$ is an isolated boundary point of $D$ on $\mathbb{P}^1_z$. Let $D'$ be the image of $D$ under $z \mapsto 1/z$. Then, $f$ extends analytically to $F : D \cup \{\infty\} \to \mathbb{P}^1_z$ if $g(z) = f(1/z)$ extends analytically to $D' \cup \{0\}$ in a neighborhood of 0.

Those functions $f : \mathbb{P}^1_z \to \mathbb{P}^1_z$ analytic everywhere are the rational functions $\mathbb{C}(z)$ in $z$ [9.3f]. Extending Lem. 4.10 to allow any $D$ in $\mathbb{P}^1_z$ only requires clarifying what will be the residue at $\infty$. This allows integrations of analytic functions $f : D \to \mathbb{P}^1_z$ along paths for any domain $D$ in $\mathbb{P}^1_z$.

DEFINITION 4.15. By definition a function $f(z)$ meromorphic in a neighborhood of $\infty$ is in $\mathcal{L}_\infty$, Laurent series in $1/z$: $f(z) = g(1/z)$ with $g \in \mathcal{L}_0$. The residue at $\infty$ is the coefficient of $z$ in $\frac{-g(z)}{z^2}$.

For example, $f(z) = 1/z$ has residue $-1$ at $\infty$. So, it has no primitive at $\infty$.

This chapter's examples explicitly compute conjugates of special functions $f$. Riemann's Existence Theorem turns this around when $D$ is $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$. Running over all algebraic $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$, Chap. 4 describes all possible permutations of the sets $\mathcal{A}_f(U_{\boldsymbol{z}}, z_0)$. The goal will be to recognize $f$ by the permutations that come from applying $\Pi_1(U_{\boldsymbol{z}}, z_0)$. Then Riemann's Existence Theorem produces (algebraic) $f$ realizing a given labeling. It doesn't, however, give $f$ explicitly; it only exists.

Given such an $f$, suppose $g \in \mathbb{C}(z,f)$ and $\mathbb{C}(z,g) = \mathbb{C}(z,f)$: $f$ and $g$ are *primitive generators* of this field (over $z$; §1.2.2). §1.2 gives $u(w), v(w) \in \mathbb{C}(z)[w]$ with $g = u(f)$ and $f = v(g)$. Here is a particular case of Lem. 4.6.

LEMMA 4.16. *For* $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$, $g_\lambda = u(f_\lambda)$ *and* $f_\lambda = v(g_\lambda)$.

## 5. Winding numbers and homology

Winding numbers appear in §3.4. Here is the formal definition for the winding number of the closed path $\gamma$ (in $\mathbb{C}$, not passing through $z'$) about $z'$:

$$n_{z'}(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z'}.$$

This definition alone would justify complex variables; it defines this winding for any path avoiding $z'$.

DEFINITION 5.1. Suppose $D$ is a domain in $\mathbb{C}$, $z_0 \in D$ and $\gamma_1, \gamma_2 \in \Pi_1(D, z_0)$ have the same winding numbers about each point in $\mathbb{C} \setminus D$. We say they are *homologous* (in $D$). A path is homologous to 0 if all winding numbers for points in $\mathbb{C} \setminus D$ are 0. It is obvious this forms an equivalence relation on $\Pi_1(D, z_0)$. Denote the equivalence classes by $H_1(D)$: the (first) *homology* group of $D$.

**5.1. Extending Def. 5.1.** Suppose $\gamma_1, \gamma_2 \in \Pi_1(D, z_0, z_1)$. Extend the definition of homologous paths: $\gamma_1$ and $\gamma_2$ are homologous if the closed path $\gamma = \gamma_1 \cdot \gamma_2^{-1}$ is homologous to 0. Suppose $\gamma$ is a closed path in $\mathbb{C}$. Use the notation $\mathbb{P}^1_z \setminus \gamma$ for the complement of the range of $\gamma$ in $\mathbb{P}^1_z$. If $z' \in \mathbb{C} \setminus \gamma$, we have a winding number $n_{z'}(\gamma)$ of $\gamma$ about $z'$. If $\gamma_1, \gamma_2 \in \Pi_1(D, z_0)$, then $\gamma_1 \cdot \gamma_2$ is homologous to $\gamma_2 \cdot \gamma_1$. This is because all winding numbers are from computations of integrals in Lem. 4.11. For $\gamma$ a closed path in $\mathbb{P}^1_z$ denote the complement of the range of $\gamma$ by $\mathbb{P}^1_z \setminus \gamma$.

LEMMA 5.2. *In the previous notation, let* $U_1, \dots, U_{r'}$ *be the connected components of* $\mathbb{P}^1_z \setminus \gamma$. *One of these, say* $U_{r'}$ *includes* $\infty$. *Then* $n_{z'}(\gamma)$ *is a constant function of* $z'$ *(with* $\gamma$ *fixed) as* $z'$ *runs over a connected component of* $\mathbb{P}^1_z \setminus \gamma$. *So, if* $z' \in U_{r'} \setminus \{\infty\}$, *then* $n_{z'}(\gamma) = 0$.

*Let* $n_i(\gamma)$ *be the winding number of* $\gamma$ *around any point in* $U_i$, $i = 1, \dots, r'$. *Suppose* $D \subset \mathbb{C}$ *is any domain containing the range of* $\gamma$. *Any connected component of* $\mathbb{C} \setminus D$ *is in one of the* $U_i$ *s. Denote the set of integers* $i$ *with* $U_i$ *containing a component of* $\mathbb{C} \setminus D$ *by* $I_D$. *Then, the function* $i \in I_D \mapsto n_i(\gamma)$ *determines the homology class of* $\gamma$ *in* $D$.

PROOF. This follows immediately by noticing $g(z') = \int_\gamma \frac{dz}{z-z'}$ is an analytic (and therefore continuous) function on $\mathbb{P}^1_z \setminus \gamma$. Its values are in $2\pi i\mathbb{Z}$, a discrete set. So, it is constant on each connected component of $\mathbb{P}^1_z \setminus \gamma$ (proof of Prop. 3.2).

Now suppose $z' \in U_r \setminus \{\infty\}$. Then, some big disc $\Delta'$ contains all of (the range of) $\gamma$. Let $z''$ be any other point in $U_r \setminus \{\infty\}$ outside $\Delta'$. A previous observation shows $n_{z'}(\gamma) = n_{z''}(\gamma)$. Further, $g(z) = 1/(z-z'')$ is analytic in $\Delta'$. Apply Cauchy's Theorem 3.6 to conclude $n_{z''}(\gamma) = 0$.

Finally, consider the function $i \in I_D \mapsto n_i(\gamma)$. This determines the winding numbers of $\gamma$ on each connected component of $\mathbb{C} \setminus D$. This, in turn determines the homology class of $\gamma$. □

Denote the image of $\gamma$ in $H_1(D)$ by $[\gamma]_h$. We understand that a tuple of integers from Lemma 5.2 may be our best interpretation. Further, additivity of winding numbers gives $[\gamma_1 \cdot \gamma_2]_h = [\gamma_1]_h + [\gamma_2]_h$.

**5.2. Homology for domains including $\infty$.** Def. 5.1 doesn't include defining homologous paths if a domain in $\mathbb{P}^1_z$ includes $\infty$. (This includes allowing the paths to go through $\infty$.) Several adjustments allow extending the definition. Chap. 3 has a general approach, one that will not put $\infty$ in a special place. Here we follow implications from a standard complex variables course.

5.2.1. *Use linear transformations.* If $z' \in \mathbb{P}^1_z \setminus D$ and $\infty \in D$, choose a linear (fractional) transformation $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ mapping $z'$ to $\infty$ [9.14]. Since $\gamma_1, \gamma_2$ are paths in $D$, $\alpha \circ \gamma_1$ and $\alpha \circ \gamma_2$ don't go through $\infty$. Now, apply Def. 5.1 to $\alpha \circ \gamma_1$ and $\alpha \circ \gamma_2$ relative to $\alpha(D)$. To justify this, check that $\alpha \circ \gamma_1 \cdot (\alpha \circ \gamma_2)^{-1}$ being homologous to 0 doesn't depend on $\alpha$ [9.14e]. If $D = \mathbb{P}^1_z$, declare all closed paths to be homologous to 0.

There is one obvious problem. Suppose $\psi_{D_1,D_2} : D_1 \subset D_2$ is the inclusion map. Yet, you have already chosen points $z'_i \in \mathbb{C} \setminus D_i$ for reverting homology to a winding number computation, with $z'_1 \neq z'_2$. Then, we lose having an explicit map $\bar\psi_{D_1,D_2} : H_1(D_1) \to H_1(D_2)$ induced from paths in $D_1$ also being paths in $D_2$.

5.2.2. *Excising* $\infty$. Assume $\infty \in D$, $z_0 \in D \setminus \{\infty\}$ and $\Delta_\infty$ is some closed disk about $\infty$ lying entirely in $D$. Regard $\mathbb{P}^1_z$ as an actual sphere (in $\mathbb{R}^3$). Assume the radius of $\Delta_\infty$ is one unit (see §5.4.1). Let $\Delta_{\infty,s}$ be the closed disk about $\infty$ of radius $s$, $0 < s \le 1$. Let $D_\infty = D \setminus \{\infty\}$. Now, $H_1(D_\infty)$ has meaning from Def. 5.1.

Let $U_1, \dots, U_r$ be the connected components of $\mathbb{C} \setminus D$. Each defines a winding number for $\gamma \in \Pi_1(D_\infty, z_0)$. Use notation from Lemma 5.2:

$$\gamma \in \Pi_1(D_\infty, z_0) \mapsto [\gamma]_h = (n_1(\gamma), \dots, n_r(\gamma)) \in \mathbb{Z}^r.$$

Define $H_1(D)$ by extending $[\gamma]_h$ to paths in $\Pi_1(D_\infty, z_0)$ going through $\infty$. For this, consider the submodule $M_r$ of $\mathbb{Z}^r$ that $\boldsymbol{v}_r = (1, 1, \dots, 1) \in \mathbb{Z}^r$ generates.

Suppose $\gamma \in \Pi_1(D, z_0)$ goes through $\infty$. Apply Lemma 4.3 to replace $\gamma$ by a geodesic path $\gamma^*$ in $D$ (Rem. 4.4) with these properties.

(5.1a) $\gamma$ and $\gamma^*$ have the same end points.
(5.1b) If $f \in \mathcal{E}(D, z_0)$, then $f_\gamma = f_{\gamma^*}$.

If $\gamma^*$ doesn't go through $\infty$, precede as below. Otherwise, If $\gamma^*$ goes through $\infty$ then it does so only finitely many times. It is the product of a finite number of paths $\gamma'$ with the property there is a neighborhood of $\infty$, $\Delta_{s_0} \subset D$, which $\gamma'$ returns to and leaves just once. With no loss assume there exists $a < t_1 < t_2 < b$ with $\gamma(t) \in \Delta_{s_0}$ for $t \in [t_1, t_2]$ and $\gamma(t) \notin \Delta_{s_0}$ for $t$ outside this interval. Therefore, $\gamma(t_1)$ and $\gamma(t_2)$ are on the boundary $\partial\Delta_{s_0}$ of $\Delta_{s_0}$. There are two paths on $\partial\Delta_{s_0}$ going at constant speed from $\gamma(t_1)$ to $\gamma(t_2)$. Let $\tau$ be one of these. Form a new path, $\gamma^*$ from $\gamma$ using this formula:

$$\gamma^*(t) = \begin{cases} \gamma(t) & \text{for } t \in [a, t_1] \\ \tau(t) & \text{for } t \in [t_1, t_2] \\ \gamma(t) & \text{for } t \in [t_2, b]. \end{cases} \tag{5.2}$$

Then, $[\gamma^*]_h \in H_1(D_\infty)$.

Definition 5.3. In the above, when $\infty \in D$, define $H(D)$ to be $H_1(D_\infty)/M_r$. Denote the canonical map $H_1(D_\infty) \to H(D)$ by $\psi$. Extend to $[\gamma]_h$: Take $\psi([\gamma^*]_h)$ to be its image in $H(D)$. Prop. 5.4 completes why this is well defined.

**5.3. Computing $H_1(D)$ for explicit domains.** The word *explicit* has only subjective meaning. It depends on personally interpreting what it means to *know data*. Still, consider $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ for some set of $r$ points $\boldsymbol{z}$. Then, giving $\boldsymbol{z}$ explicitly has comfortable interpretation from experience.

This generalizes to when $\mathbb{P}^1_z \setminus D$ has $r$ connected components, $C_1, \dots, C_r$. Our treatment tacitly assumes $r$ is finite. Then, interpret *giving $D$ explicitly* as knowing simple closed paths bounding each of the $C_i$ s. Such paths might be circles or polygons with explicit beginning and end points. Given these conditions, computing the homology class of an *explicit* path in $D$ uses calculations within our experience.

The next proposition specializes a statement in Chap. 4 with *homotopy classes* replacing homology classes. It gives $\infty$ a special status, that Chap. 4 will not. Simple examples, like [9.10], illustrate having $\infty$ play a special role.

Suppose $D$ is a domain in $\mathbb{P}^1_z$ whose complement $C(D)$ in $\mathbb{P}^1_z$ has $r > 0$ connected components $C_1, \dots, C_r = C(D)_1, \dots, C(D)_r$. Denote this ordering of the components as $J_D$ with the proviso $C_r = C_\infty$ is the component containing $\infty$ if $D \subset \mathbb{C}$. If $\infty \in D$, add $C_\infty$ by including the empty set $\emptyset$ as the last position. Write $D_\infty$ for $D \setminus \infty$. As in §5.2.1, consider an inclusion map $\psi_{D_1,D_2} : D_1 \subset D_2$.

Each connected component of $\mathbb{P}^1_z \setminus D_2$ is in some connected component of $\mathbb{P}^1_z \setminus D_1$. (If $\infty \in D_2$ regard $\emptyset$ as $C(D_2)_\infty$.) This induces a map $\psi^\dagger_{D_1,D_2} : J_{D_2} \to J_{D_1}$. The module $M_r$ is from §5.2.2. Recall the definition of a residue of a meromorphic function $f$ at a point $z' \in D$ from (4.5).

PROPOSITION 5.4. *Suppose $D$ is a domain in $\mathbb{P}^1_z$ where $C(D)$ has $r$ connected components. Then, $H_1(D)$ is isomorphic to $\mathbb{Z}^{r-1}$. If $\infty \in C(D)$, then $\gamma \in \Pi_1(D, z_0) \mapsto [\gamma]_h$ of Prop. 5.2 and Def. 5.3 gives this isomorphism explicitly. If $\infty \in D$, this identifies $H_1(D)$ with $\mathbb{Z}^r/M_r$ (isomorphic to $H_1(D_\infty)/M_r$), also isomorphic to $\mathbb{Z}^{r-1}$.*

*Suppose $C(D')$ has $r'$ components and $D \subset D'$, with $\infty \in C(D')$. Then, these isomorphisms induce $\mathbb{Z}^{r-1} \to \mathbb{Z}^{r'-1}$ where $n_1, \dots, n_{r-1} \mapsto m_1, \dots, m_{r'-1}$ by*

$$m_j = \sum_{i \in J_{D'}, \psi^\dagger_{D,D'}(i) = j} n_i.$$

*Assume $f$ is meromorphic in $D$ and $\gamma \in \Pi_1(D, z_0)$ passes through no residue of $f$. Then, $\int_\gamma f(z)\,dz$ depends only on $[\gamma]_h$ and the residues of $f$ at points in $D$.*

**5.4. Proof of Prop. 5.4.** Let $z_0 \in D$. As above, denote the $r$ connected components of $\mathbb{P}^1_z \setminus D$ by $C_1, \dots, C_r$. First assume $\infty \in C_r$. For each $i$, $1 \le i \le r-1$, there is a closed path $\gamma_i = \delta_i \cdot \bar{\gamma}_i \cdot \delta_i^{-1} \in \Pi_1(D, z_0)$ with the following description.

(5.3a) $\delta_i : [0,1] \to D$ and $\bar{\gamma}_i : [0,1] \to D$ are paths with $\bar{\gamma}_i$ closed.
(5.3b) $\delta_i(0) = z_0$ and $\delta_i(1) = \bar{\gamma}_i(0)$.
(5.3c) $\bar{\gamma}_i$ has winding number 1 around each point in $C_i$.
(5.3d) $\bar{\gamma}_i$ has winding number 0 around each point in $C_j$, $j \ne i$.

5.4.1. *Construction of $\bar{\gamma}_i$.* Our construction of $\gamma_i$ is similar to that of [**Ahl79**, p. 140]. Again use the metric topology on $\mathbb{P}^1_z$ identifying it with a sphere in $\mathbb{R}^3$ with coordinates $(r, u, v)$. So, $z_0 \in \mathbb{P}^1_z$ corresponds to $(r_0, u_0, v_0) \in \mathbb{R}^3$. Each point of the sphere has a vector pointing *outward*, perpendicular to the tangent plane to the sphere at $(r_0, u_0, v_0)$. Further, in any disk on the sphere around $(r_0, u_0, v_0)$, the boundary of this disk has a well-defined orientation around $(r_0, u_0, v_0)$. We take it counterclockwise around the outward normal to the disk at its center. This orientation applies to any simple closed path in the disk [9.17].

Components of $C(D)$ are closed, disjoint (and bounded). Let $d(z_i, z_j)$ be the distance (along the minor arc) between $z_i \in C_i$ and $z_j \in C_j$. The function $1/d(z_i, z_j)$ has a minimum on $C_i \times C_j$. Running over all $i$ and $j$ let $\delta$ be at most $1/\sqrt{2}$ times the smallest of these minimums. Form a grid on $\mathbb{P}^1_z$ of equally spaced longitudes and latitudes, with spacing at most $\delta$. The closed (spherical) squares (and triangles) of this grid each meet at most one component of $C(D)$.

Let $Q$ be one of the closed grid squares. Its boundary orientation is counter clockwise around any outward normal to an interior point of $Q$ [9.17e]. Define $\bar{Q}_i$ to be the union of all $Q$s meeting $C_i$. Such a $Q$ meets none of the $C_j$s with $j \ne i$. Let $\bar{\gamma}_i$ be the topological boundary of $\bar{Q}_i$. This is the union of bounding sides — oriented counter clockwise from the paths bounding the $Q$s — to squares of $\bar{Q}_i$. Also, $\bar{Q}_i$ includes only sides appearing in exactly one $Q$. Such a side has three (or two, if the grid element is by chance a triangle) other sides of grid squares meeting each vertex. Exactly one side is in $D$ and on another square in $\bar{Q}_i$. So, each vertex has an adjoining segment of $\bar{\gamma}_i$; $\bar{\gamma}_i$ is a simple closed (oriented) path.

5.4.2. *Winding numbers of* $\bar{\gamma}_i$. Choose any square $Q^*$ in $\bar{Q}_i$ and any point $z' \in Q^* \cap C_i$. The winding number of $\bar{\gamma}_i$ about $z'$ is

$$n_i(\bar{\gamma}_i) = n_{z'}(\bar{\gamma}_i) = \sum_{Q \in \bar{Q}_i} n_{z'}(\partial Q) = n_{z'}(\partial Q^*) = 1.$$

Similarly, $n_j(\bar{\gamma}_i) = 0$ for $j \ne i$. Winding numbers of the path $\gamma_i$ with respect to the $C_j$s are the same as for $\bar{\gamma}_i$. This is from their definition as an integral (5.3); the integral along $\delta_i$ cancels with the integral along $\delta_i^{-1}$.

Suppose $\infty \in C(D)$. Let $\gamma$ be any closed path in $D$. To $\gamma$ associate the $r$-tuple $(n_1(\gamma), \dots, n_r(\gamma)) \in \mathbb{Z}^r$. Then, the path $\prod_{i=1}^r \gamma_i^{n_i}$ is homologous to $\gamma$. Thus, the winding number map is onto $\mathbb{Z}^{r-1}$. This completes Prop. 5.4 for $\infty \in C(D)$.

5.4.3. *The case* $\infty \in D$. Consider the map $H_1(D_\infty) \to H_1(D_\infty)/M_r = H_1(D)$. The latter is the definition of $H_1(D)$. So we comment only on why the image of $\gamma \in \Pi_1(D, z_0)$ depends only on the path $\gamma^*$ from (5.2). There were two stages to forming $\gamma^*$. The first replaced $\gamma$ by a geodesic path where (5.1) gives its relation to $\gamma$. Suppose $\gamma_1$ and $\gamma_2$ are two such choices. Then, $f_{\gamma_1} = f_{\gamma_2}$ for any $f$ extensible to all of $D$. In particular, this applies to $f$ a branch of $\log(\frac{z-z_i}{z-z_j})$ with $z_i \in C_i$. Its analytic continuations around $\gamma_1$ and $\gamma_2$ are the same. Therefore, if neither $\gamma_1$ nor $\gamma_2$ go through $\infty$, the winding numbers of $\gamma_1\gamma_2^{-1}$ with respect to all components of the complement of $D$ are the same.

Then, we adjusted the geodesic path to a new path $\gamma^*$ which for certain did not go through $\infty$. There were, however, two such choices for $\gamma^*$. Label these $\gamma_1^*$ and $\gamma_2^*$. Let $\delta$ be the parametrized boundary $\partial\Delta_{s_0}$ of $\Delta_{s_0}$. Then $\delta = \tau_1 \cdot \tau_2$ with $\tau_1$ going from $\gamma(t_1)$ to $\gamma(t_2)$ and $\tau_2$ going (in the same direction) from $\gamma(t_2)$ to $\gamma(t_1)$. For simplicity assume $\delta$ goes clockwise around $\infty$ (as in §5.4.1).

Then, $\gamma_1^* = \gamma_{[a,t_1]} \cdot \tau_1 \cdot \gamma_{[t_2,b]}$ and $\gamma_2^* = \gamma_{[a,t_1]} \cdot \tau_2^{-1} \cdot \gamma_{[t_2,b]}$. Integrals determine homology classes in $H_1(D_\infty)$. From Lemma 4.11, $\gamma_2^*$ and

$$\gamma_2' = \gamma_{[a,t_1]} \cdot \tau_2^{-1} \cdot \tau_1^{-1} \cdot \tau_1 \cdot \gamma_{[t_2,b]}$$

have the same homology class. So, $[\gamma_2^*]_h - [\gamma_1^*]_h$ is $[\tau_1 \cdot \tau_2]_h$. From Cauchy's Theorem 3.6, $[\tau_1 \cdot \tau_2]_h$ is independent of $s_0$. On the other hand, $\delta$ bounds the disk complement of $\Delta_{s_0}$ in the counter clockwise direction. By assumption that disk contains all components of $C(D)$. So, $n_{z'}(\delta) = 1$ as $z'$ runs over points in all components of $C(D)$: $[\gamma_2^*]_h - [\gamma_1^*]_h = (1, \dots, 1)$. This shows the images of $[\gamma_1^*]_h$ and $[\gamma_2^*]_h$ in $H_1(D)$ are the same. That is, $M_r$ measures exactly the discrepancy in substituting $\gamma^*$ for the original path.

5.4.4. *Integrals along homologically trivial paths.* Now assume $f$ is meromorphic in $D$. It suffices to show the following. If $\gamma_1, \gamma_2 \in \Pi_1(D, z_0)$, and $\gamma = \gamma_1 \cdot \gamma_2^{-1}$ is homologous to 0, then $\int_{\gamma_1} f\,dz - \int_{\gamma_2} f\,dz = \int_\gamma f\,dz$ depends only on the residues of $f$ in $D$. Let $R_f$ be the poles of $f$ for which $f$ has nonzero residues. If $\infty \in C(D)$, and $\gamma \in \Pi_1(D, z_0)$ is homologically trivial, then Cauchy's Residue Theorem ([**Ahl79**, p. 149] or [**Con78**, p. 112]) says $\int_\gamma f\,dz$ is $\sum_{z' \in R_f} n_{z'}(\gamma)\mathrm{Res}'_z(f)$. This is the result we want, at least if $\infty \in C(D)$. We won't need to consider the possibility of $f$ having infinitely many nonzero residues.

A reduction of the Residue Theorem to the case $f$ is analytic in $D$ is algebraic. Cauchy's Theorem in this case may be the most important result from first year complex variables. We state it and a generalization for use later.

DEFINITION 5.5. Suppose $u, v : D \to \mathbb{C}$ are continuous (though maybe not analytic). The differential 1-form $\omega = u(z)\,dx + v(z)\,dy$ is *locally exact* if for each $z_0 \in D$, there exists $F_{z_0}(z) = F(z)$ in a neighborhood of $z_0$ with these properties.

(5.4a) $F(z)$ has continuous partial derivatives.
(5.4b) $\frac{\partial F}{\partial x} = u(z)$ and $\frac{\partial F}{\partial y} = v(z)$.

THEOREM 5.6. *Suppose $f$ is analytic in $D$, and $\gamma \in \Pi_1(D, z_0)$ is homologous to 0 in $D$. Then, $\int_\gamma f\,dz = 0$. More generally, this holds with any locally exact differential $\omega$ on $D$ replacing $f\,dz$* [**Ahl79**, p. 144, Thm. 16].

Thm. 5.6 holds even if $\infty \in D$ [9.13a]. If we only assume $f \in \mathcal{E}(D, z_0)$, then $\int_\gamma f\,dz$, $\gamma \in \Pi_1(D, z_0)$, usually depends on more than the residues of $f$ and $[\gamma]_h \in H_1(D)$ [9.13d].

## 6. Branch of solutions of $m(z, w) = 0$

This section discusses the *implicit function* theorem. It is the key ingredient for showing a function satisfying (1.2) satisfies (1.1),

### 6.1. Branch of inverse of $f(z)$.

Suppose $f(z)$ is meromorphic on $D$ and has range $D'$. A branch of (right) *inverse* of $f(z)$ on $D'$ is a continuous function $g : D' \to D$ with $f \circ g(z) = z$ for $z \in D'$.

DEFINITION 6.1 (Branch of inverse of $f$ along a path). Let $\gamma : [a, b] \to D$ be a path and $f \in \mathcal{E}(D, z_0)$. Let $g(z)$ be a branch of inverse of $f(z)$ in a neighborhood of $z_0$. Then a branch of (right) inverse of $f$ along $\gamma$ is an analytic continuation of $g(z)$ along $\gamma$.

We now change the variable $z$ to $w$, and discuss functions analytic in $w$. This sets notation for the full implicit function theorem. Suppose $f(w)$ is analytic in a neighborhood $\Delta_{w_0}$ of $w_0$, and $f(w_0) = z_0$. For a given fixed $z$, assume $\partial\Delta_{w_0}$ passes through no zero or pole of $f(w) - z$ (as a function of $w$). Then,

$$(6.1) \qquad n_z = \frac{1}{2\pi i}\int_{\partial\Delta_{w_0}} \frac{f'(w)\,dw}{f(w) - z} \quad \text{and} \quad g(z) = \frac{1}{2\pi i}\int_{\partial\Delta_{w_0}} \frac{wf'(w)\,dw}{f(w) - z}$$

count the number $n_z$ (resp. the sum $g(z)$) of zeros of $f(w)-z$ in $\Delta_{w_0}$. By Leibniz's theorem, compute the derivative of $g(z)$ by applying $\frac{\partial}{\partial z}$ under the integral sign (see §7.1). So, $g(z)$ is analytic in $z$ for $z$ close to $z_0$.

LEMMA 6.2. *Suppose $f(w)-z_0$ has exactly one zero (and no poles) in a neighborhood $\Delta_{w_0}$ of $w_0$. For $z$ sufficiently close to $z_0$, $f(w)-z$ also has only one zero (and no poles). Thus, the second expression of* (6.1) *defines a branch $g(z)$ of the inverse of $f(z)$ locally.*

The proof of the implicit function theorem in §6.2 includes the proof of Lemma 6.2.

6.1.1. *Branch of $f(z)^{\frac{1}{e}}$ along a path.* For $e$ a positive integer, we use the inverse of the $e$th power map in a general form. This returns to branch of log.

Suppose $f$ is meromorphic in a domain $D$. Let $\gamma:[a,b]\to D$ be any path whose range misses all zeros and poles of $f(z)$. Then, define a branch of $\log(f(z))$ along $\gamma$ to be a continuous function $h(t)$, for which $e^{h(t)}=f(\gamma(t))$, $t\in[a,b]$. Existence of a branch of $\log(f(z))$ along such $\gamma$ follows from Prop. 3.2. It is the same as a branch of log along the path $f\circ\gamma:[a,b]\to f(D)$.

Define a branch of $f(z)^{\frac{1}{e}}$ along $\gamma$ using $h(t)$ a branch of $\log(f(z))$ along $\gamma$:

$$e^{h(t)/e}\overset{\text{def}}{=}\mathrm{Br}((f(z))^{\frac{1}{e}})(\gamma(t)). \tag{6.2}$$

The left side has a clear meaning. Define the right side to be the value of the branch at $\gamma(t)$. Check: The left of (6.2) to the $e$th power is $f(\gamma(t))$, as expected. As before, there are $e$ such branches.

Applying Prop. 3.2 gives a unique branch $h(t)$ having a specific value $h(a)$ equal to one of the $e$th roots of $f(\gamma(a))$.

6.1.2. *Local inverses of rational functions.* Suppose $f=f_1/f_2\in\mathbb{C}(w)$ with $(f_1,f_2)=1$. Consider the set $X_f=\{(z,w)\in\mathbb{P}^1_z\times\mathbb{P}^1_w \mid f(w)-z=0\}$. Each point $(z_0,w_0)$ on $\mathbb{P}^1_z\times\mathbb{P}^1_w$ has a basis of open sets; each set in the basis is the product of an open set around $z_0$ and an open set around $w_0$. Intersect those open sets with $X_f$ to get neighborhoods of points of $X_f$. We discuss for which $(z_0,w_0)$ there exists $g(z)$ analytic in a neighborhood of $z_0$ satisfying

(6.3) $g(z_0)=w_0$ and $f(g(z))=z$.

That is, $g$ produces a local parametrization of a neighborhood of $(f(w_0),w_0)$ by $z\mapsto(z,g(z))$: $(z,g(z))$ is on $X_f$ because $f(g(z))-z\equiv0$.

There is a global parametrization of $X_f$ by $w\mapsto(f(w),w)$: $f(w)-f(w)\equiv0$. This parametrization, however, isn't as a function of $z$. It is insistent reference to $z$ as the parameter that gives coherent information about the algebraic function $g(z)$.

Lemma 6.2 says points $(z_0,w_0)$ with a multiplicity one zero $w_0$ of $f(w)-z_0$ have neighborhoods projecting one-one to the $z$-line: $(z,g(z))\mapsto z$. Assume $z_0\ne\infty$. Then, $w_0$ is a multiplicity one zero of $f_1(w)-z_0f_2(w)$. If this doesn't hold, then $w_0$ is a zero of $f_1(w)-z_0f_2(w)$ and its derivative $f_1'(w)-z_0f_2'(w)$ in $w$. Call it a *critical* value. Eliminate $z_0$.

(6.4) Critical values of $w_0$ are zeros of $f_1(w)f_2'(w)-f_2(w)f_1'(w)$.

In particular, there are at most $\deg(f_1)+\deg(f_2)-1$ critical values of $w_0$ (or of $z_0$). [9.4] precisely defines critical values when $w_0$ is a pole of $f$.

6.1.3. *Abel's application.* Apply the chain rule to $f(g(z))\equiv z$:

$$\frac{df}{dw}_{|_{w=g(z)}}\frac{dg}{dz}=1. \tag{6.5}$$

Therefore, $\frac{dg}{dz} = 1/\frac{df}{dw}_{|w=g(z)}$. This is the complex variable variant of how first year calculus computes an antiderivative of inverse trigonometric functions. Abel applied this to a (right) inverse of a branch of primitive from the following integral

$$\int_\gamma \frac{dz}{(z^3+cz+d)^{\frac{1}{2}}} \tag{6.6}$$

with $c, d \in \mathbb{C}$ (Chap. 4 §6.1). Use (6.2) to interpret $h(z)\,dz = \frac{dz}{(z^3+cz+d)^{\frac{1}{2}}}$ around some base point $z_0$: $h(z)$ is a branch of $(z^3+cz+d)^{-\frac{1}{2}}$. Let $f(z)$ be a primitive for $h(z)\,dz$. Apply (6.5) to $f(g(z)) = z$ (special case of (7.3)):

$$\frac{dg(z)}{dz} = (g(z)^3 + cg(z) + d)^{\frac{1}{2}}. \tag{6.7}$$

Let $\boldsymbol{z}\{z_1, z_2, z_3, \infty\}$, the three zeros of $z^3+cz+d$ and $\infty$. Analytic continuation of $(z^3+cz+d)^{-\frac{1}{2}}$ and its primitive $f(z) = f(z; c, d)$ produce the collection $\mathcal{A}_f(U_{\boldsymbol{z}})$. First year calculus computes the inverse of a primitive of $h_1(z) = (z^2+cz+d)^{-\frac{1}{2}}$, recognizing it from the trigonometric function $\sin(z)$. This has a unique analytic continuation everywhere in $\mathbb{C}$. Abel discovered the same was true for the inverse $g(z) = g(z; c, d)$ of $f(z; c, d)$; it extends everywhere in $\mathbb{C}$. Many conclusions follow.

This example will inspire later topics. For example, dependence of $g(z) = g(z; c, d)$ on $(c, d)$ usefully distinguishes between algebraic curves defined by $w^2 - z^3 + cz + d$ as a function of $(c, d)$ (Chap. 4 §6.1). For each $(c, d)$, $g(z; c, d)$ is to the exponential function as (6.6) is to a branch of $\log(z)$.

**6.2. Implicit function theorem.** Consider $m(z, w) \in \mathcal{H}(D)[w]$ (a polynomial in $w$ with coefficients in $\mathcal{H}(D)$). Suppose $g(z)$ is analytic on $D$ and $m(z, g(z)) \equiv 0$. We discuss paths $\gamma \to D$ along which there is an analytic continuation of $g(z)$. Such paths should exclude $z'$ having a $w'$ with

(6.8) $m(z', w') = 0$ and $\frac{\partial m}{\partial w}(z', w') = 0$.

Riemann's Existence Theorem produces the *Riemann surface* attached to $g(z)$ (Chap. 4). Data for the Riemann surface include information about all embeddings of $\mathbb{C}(z, g(z))$ in Puiseux fields. This important, though lesser data, is available from the proof that Puiseux fields are algebraically closed (§7.3). Given a polynomial $m(z, w)$ it is theoretically possible, though not always practical, to compute exactly the Puiseux embeddings of $\mathbb{C}(z, g(z))$ from $m$.

6.2.1. *Branch and critical points.* A branch of *solutions to* $m(z, w)$ along $\gamma$ is an analytic continuation of $g(z)$ along $\gamma$. Such analytic continuations avoid points $z'$ having $w'$ satisfying (6.8). Prop. 6.4 references $h_0 \in \mathbb{C}[z]$ in the expression

$$m(z, w) = h_0(z)w^n + h_1(z)w^{n-1} + \cdots + h_n(z). \tag{6.9}$$

If $z'$ is a zero of $h_0$, $m(z', w)$ has degree lower than $n$ in $w$.

Definition 6.3 (Branch point of $(m, w)$). A point $(z', w')$ is *critical* for $(m, w)$ if it satisfies (6.8). Call $z' \in \mathbb{C}$ a *branch* point of $(m, w)$ if either there exists $w'$ with $(z', w')$ a critical point or $\deg(m(z', w)) < \deg_w(m(z, w)) = n$.

Suppose $z'$ is not a branch point of $(m, w)$. Then, there are exactly $n$ distinct values $w'$ with $m(z', w') = 0$. The substitutions $z \mapsto 1/z$ and/or $w \mapsto 1/w$ allows extending the definition of critical points of $m(z, w)$ to include $z'$ and/or $w'$ equal to $\infty$ (see [9.4] and [9.11]). Use the notation of (6.9) and $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \boldsymbol{z}$.

6.2.2. *Algebraic according to* (1.2) *implies* (1.1). Now we see that algebraic by the equation definition implies algebraic by the analytic continuation definition.

PROPOSITION 6.4. *Suppose $\boldsymbol{z}$ includes $\infty$ and all branch points of $(m, w)$. Assume $(z_0, w_0)$ satisfies the first equation of* (6.8)*, but $z_0 \notin \boldsymbol{z}$. Then, there is a $g(z)$ analytic near $z_0$ with $m(z, g(z)) \equiv 0$ and $g(z_0) = w_0$. For $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$, $g(z)$ analytically continues along $\gamma$ and $m(z, g_\gamma(z)) \equiv 0$ (near the end point of $\gamma$).*

*If $m(z, w) \in \mathbb{C}[z, w]$ is irreducible, then $\boldsymbol{z}$ is a finite set. There are exactly $n$ branches of solutions of $m(z, w)$ along any $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ (and exactly $n$ elements of $\mathcal{A}_g(U_{\boldsymbol{z}})$). Conclude: $X_m = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid m(z, w) = 0, z \in U_{\boldsymbol{z}}\}$ is connected and $g$ is algebraic according to* (1.1).

The proof takes up §7.1. Then we get complete equivalence between (1.1) and (1.2).

## 7. Equivalence of the two definitions of algebraic

We show $(m, w)$ has only finitely many branch points if $m \in \mathbb{C}[z, w]$.

LEMMA 7.1. *Assume $m \in \mathcal{H}(D)[w]$ and $\deg_w(m) = n > 0$. Suppose there is no domain $D' \subset D$ in which all $z' \in D'$ are branch points. Then, the branch points of $(m, w)$ have no accumulation point in $D$. Further, if $m \in \mathbb{C}[z, w]$, either $m$ and $\frac{\partial m}{\partial w}$ have a common factor, or $(m, w)$ has only finitely many branch points.*

PROOF. Suppose the lemma is false, and $z'$ is such an accumulation point. Let $\Delta_{z'} \subset D$ be a disk around $z'$. So, in this disk there is a sequence of pairs $(z_j, w_j)$, $j = 1, 2, \ldots$ with these properties:

(7.1) $w_j$ is a multiple zero of $m(z_j, w)$ and $\lim_{j \mapsto \infty} z_j = z'$.

Let $\mathcal{R}_{z'}$ be the ring of power series in $z$ convergent in a neighborhood of $z'$. Then, $\mathcal{R}_{z'}$ is a principle ideal domain.

Regard $m$ and $\frac{\partial m}{\partial w}$ as polynomials in $w$ with coefficients in $\mathcal{R}_{z'}$. Apply the Euclidean algorithm [9.11]. It produces the greatest common divisor $m_1(w)$ of $m$ and $\frac{\partial m}{\partial w}$ in the form $a(z, w)m + b(z, w)\frac{\partial m}{\partial w} = m_1(z, w)$, a nonzero polynomial. These polynomials in $w$ have coefficients in $\mathcal{H}(D')$ with $D'$ a neighborhood of $z'$.

If $\deg_w(m_1) \geq 1$ for each $z' \in D'$, a zero $w'$ of $m_1(z', w)$ gives a common zero of $m(z', w)$ and $\frac{\partial m}{\partial w}(z', w)$. This is contrary to our assumption. So $\deg_w(m_1) = 0$ and the $z_j$ s are zeros of $m_1$, an analytic function of $z$, accumulating at $z'$. So, $m_1$ is identically zero contrary to a previous observation.

Apply the Euclidean algorithm to the case $m \in \mathbb{C}[z, w]$. Conclude: If $m$ and $\frac{\partial m}{\partial w}$ have no common factor, then $m_1$ is a polynomial in $z$, and all branch points are zeros of it. Thus, there are only finitely many such zeros. □

**7.1. Proof of Prop. 6.4.** Assume $(z_0, w_0)$ is not a critical point of $(m, w)$. Let $g(z)$ be $\frac{1}{2\pi i} \int_C w \frac{\partial m}{\partial w}(z, w)\, dw/m(z, w)$ for each $z$ close to $z_0$ with $C$ a counter clockwise circle suitably close to $w_0$. We show there are neighborhoods, $U_{z_0}$ of $z_0$ and $U_{w_0}$ of $w_0$, with $U_{z_0} \times U_{w_0}$ free of critical points of $(m, w)$.

To do this, extend Lemma 7.1. Simplify notation by taking $z_0 = 0$ and $w_0 = 0$. Then, $m(0, w) \neq 0$ for $0 < |w| < r_1$. As $z \mapsto 0$, $m(z, w) \mapsto m(0, w)$ uniformly with respect to $w$. So, there exists $r < r_2 < r_1$ with $|m(z, w) - m(0, w)| < |m(0, w)|$ for $|z| < r_2$ and $|w| < r$. By Rouche's Theorem [**Con78**, p. 125], $m(z, w)$ and $m(0, w)$ have the same number of zeros in $|w| < r$. So, $m(z, w)$ has a single zero in this region and $g(z)$ gives it.

With $C$ fixed and $z$ close to (but not equal) $z_0$, apply $\frac{\partial}{\partial z}$ under the integral giving $g(z)$ to compute its derivative. The partial derivative of $w\frac{\partial m}{\partial w}(z,w)/m(z,w)$ exists and is continuous. Thus, Leibniz's rule [**Con78**, p. 68] says this gives $\frac{dg}{dz}$, showing it is analytic.

Now consider analytic continuation of $g(z)$ along any path in $U_{\boldsymbol{z}}$. This is the same as the proof of Prop. 3.2 starting at §3.3.1. The key ingredient was analytically continuing $g(z)$ beyond the end point of any given path. We have the tools now for that. If $\gamma : [a,b] \to U_{\boldsymbol{z}}$ is any path, there is a neighborhood of $\gamma(b)$ and $g_1(z)$ analytic in this neighborhood with $g_1(\gamma(b))$ the value of the extension of $g(z)$ to the end point. As in that proof, since $m(\gamma(t), g_1(\gamma(t))) \equiv 0$ for $t$ close to $b$, $m(z, g_1(z)) \equiv 0$ for all $z$ with $g_1(z)$ defined.

This leaves showing that as $\gamma$ runs over $\Pi_1(U_{\boldsymbol{z}}, z_0)$, $g_\gamma$ runs over all $n$ branches $g_1, \dots, g_n$ of solutions of $m(z,w)$ around $z_0$. Suppose, however, it runs over only the subset $g_1, \dots, g_t$ with $t < n$. Consider

$$M(z,w) \stackrel{\text{def}}{=} \prod_{i=1}^t (w - g_i(z)) = w^t - G_1(z)w^{t-1} + G_2(z)w^{t-2} + \cdots + (-1)^t G_t(z). \tag{7.2}$$

Each $G_i(z)$ is a symmetric polynomial $S_i(w_1, \dots, w_t)$ in $w_1, \dots, w_t$ evaluated at $(g_1, \dots, g_t)$. So, $G_i \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ (Lem. 4.6).

By assumption, for $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$, $g_{1,\gamma}, \dots, g_{t,\gamma}$ is a permutation of $g_1, \dots, g_t$. Thus, $G_{i,\gamma} = S_i(g_{1,\gamma}, \dots, g_{t,\gamma}) = S_i(g_1, \dots, g_t)$ (Lem. 4.6). So, $\mathcal{A}_{G_i}(U_{\boldsymbol{z}})$ contains a single element, $i = 1, \dots, t$. Apply Riemann's removable singularity theorem [**Ahl79**, p. 124] exactly as in the proof of Cor. 7.5. Conclude: Singularities of $G_i$ in $\mathbb{P}^1_z$ are at worst poles. So $G_i$ is a rational function in $z$: $M(z,w) \in \mathbb{C}(z)[w]$.

Plug in $g_1(z) = g(z)$, $M(z, g(z)) \equiv 0$. Therefore, $M$ is an irreducible polynomial for $g(z)$ over $\mathbb{C}(z)$ of degree $t < n$. This is contrary to the function field being of degree $n$. This contradiction proves the transitivity statement and concludes the proof of Prop. 6.4. The $n$ elements of $\mathcal{A}_g(U_{\boldsymbol{z}})$ give the $n$ values $w'$ satisfying $m(z_0, w) = 0$. So, as $\lambda$ runs over closed paths for which $g_\lambda(z_0) = w'$, this connects all the points of $X_m$ lying over $z_0$. Therefore, analytic continuation along the connected set $U_{\boldsymbol{z}}$ connects all the points of $X_m$. For future use, here is the lemma hidden in this argument.

LEMMA 7.2. *Suppose $f(z)$ is analytic in a neighborhood of $z_0 \notin \boldsymbol{z}$ with $\boldsymbol{z}$ the branch points of $m(z,w) \in \mathbb{C}[z,w]$ and $m(z, f(z)) \equiv 0$. Let $g \in \mathbb{C}(z, f(z))$ and assume $g_\lambda = g$ for each $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$. Then, $g \in \mathbb{C}(z)$.*

**7.2. The converse and integrals along paths.** Assume $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. If $f$ satisfies (1.1) we see it satisfies a nontrivial polynomial equation. Let $f_1, \dots, f_n$ be the conjugates of $f$. Apply to $f_1, \dots, f_n$ the argument in (7.2) for $g_1, \dots, g_t$.

PROPOSITION 7.3. *The definitions* (1.1) *and* (1.2) *are equivalent.*

Assume $m(z, g(z)) \equiv 0$, as in Prop. 6.4. Analytic continuation of $g(z)$ along $\gamma : [a,b] \to U_{\boldsymbol{z}}$ produces $t \mapsto h(t)$, continuous; $h(t)$ is one of the $n$ distinct values $w'$ of $m(\gamma(t), w') = 0$. For $n_1, n_2 \in \mathbb{C}[z,w]$, let $n_1(z,w)/n_2(z,w) = n(z,w)$. Define the integral of $n(z, g(z))$ along $\gamma$:

$$\int_\gamma n(z, g(z))\, dz \stackrel{\text{def}}{=} \int_a^b n(\gamma(t), h(t))\, dt. \tag{7.3}$$

Avoid paths through zeros of $n_2$ to assure the integral exists.

**7.3. $\mathcal{P}_{z'}$ is algebraically closed.** Let $\Delta_{z'}$ be a closed disk in $\mathbb{P}^1_z$ centered at $z'$. Denote $\Delta_{z'} \setminus \{z'\}$ by $\Delta^0_{z'}$. We show analytic continuations of $f(z) \in \mathcal{E}(\Delta^0_{z'}, z_0)$ depend only on analytic continuation of $f$ on a circle about $z'$. This will show $\mathcal{P}_{z'}$ is algebraically closed. Let $\delta$ be the counter clockwise circle about $z'$ through $z_0$.

PROPOSITION 7.4. *If $\lambda \in \Pi_1(\Delta^0_{z'}, z_0)$ has winding number $n_{z'}(\lambda) = e(\lambda)$, then $f_\lambda = f_{\delta^{e(\lambda)}}$.*

Prop. 7.4 gives the complete theory of Riemann surface covers of a punctured disk (in Chap. 3). The proof of Prop 7.4 is in §7.4.

COROLLARY 7.5. *As in Prop. 7.4, assume $f \in \mathcal{E}(\Delta^0_{z'}, z_0)$ is algebraic over $\mathcal{L}_{z'}$. Let $e = e_f$ be the minimal positive integer with $f_{\delta^e}(z) = f(z)$ (near $z_0$). Then, $f \in \mathcal{P}_{z',e}$ and $\mathcal{L}_{z'}(f)/\mathcal{L}_{z'}$ is isomorphic to $\mathcal{P}_{z',e}/\mathcal{L}_{z'}$. In particular, the Puiseux expansion field $\mathcal{P}_{z'}$ is algebraically closed. Algebraic functions in $\mathcal{P}_{z',e}$ consist of composites $h(\alpha(z))$ with $h$ algebraic in $\mathcal{L}_{z'}$ and $\alpha(z)$ in the set $\{(z-z')^{1/e}\}_{e=1}^{\infty}$.*

PROOF. If $f(z)$ is algebraic over $\mathcal{P}_{z'}$, then it satisfies an equation of degree $n$ with coefficients in $\mathcal{P}_{z'}$. There are only a finite number of coefficients. With no loss assume these are in $\mathcal{P}_{z',e'}$ for some $e'$; $f$ is algebraic over $\mathcal{P}_{z',e'}$. We want to show $f \in \mathcal{P}_{z',e'e}$ for some $e$.

Replace $u_{e'} = (z-z')^{1/e'}$ by $z-z'$ everywhere in the equation for $f(z)$ to revert this to where $f$ is algebraic over $\mathcal{P}_{z'}$. Or, use this usual algebra observation: If $f$ is algebraic over $\mathcal{P}_{z',e'}$, since $\mathcal{P}_{z',e'}$ is algebraic over $\mathcal{L}_{z'}$, the degree of $f$ is finite over $\mathcal{L}_{z'}$, equal to $[\mathcal{P}_{z',e'}(f) : \mathcal{P}_{z',e'}][\mathcal{P}_{z',e'} : \mathcal{P}_{z'}]$ (§1.2).

Suppose $f \in \mathcal{E}(\Delta_{z'}, z_0)$. Also, $m(f(z)) \equiv 0$ for $z \in \Delta_{z'}$ with $m(w) \in \mathcal{L}_{z'}[w]$ and $\lambda \in \Pi_1(\Delta_{z'}, z_0)$. Then, $f_\lambda$ is another zero of $m(w)$ [9.8c]. Let $\deg_w(m(w)) = n$. Then $f_{\lambda^e} = f$ for some integer $e \le n$. Choose $e$ minimal. Then, use $\delta$ as in Prop. 7.4. It shows $e$ is the minimal integer with $f_{\delta^e} = f$.

For simplicity, assume $z' = 0$ ($\Delta_{z'} = \Delta_0$) with $w_0$ a solution of $w_0^e = z_0$. Let $\Delta_1$ be the preimage of $\Delta_0$ by the map $\psi : u \to u^e$: $\Delta^0_1$ the preimage of $\Delta^0_0$. Finally, let $\delta_1$ be the counter clockwise circle through $w_0$ around 0 in $\Delta^0_1$. Then,

$$f \circ \psi_{\delta_1}(u) = f_{\delta^e}(\psi(u)) = f(\psi(u)).$$

Apply Prop. 7.4 to $(f\circ\psi, \Delta^0_1, w_0)$ to conclude $f\circ\psi_\gamma = f\circ\psi$ for $\gamma \in \Pi_1(\Delta^0_1, w_0)$. Lemma 4.12 implies $f \circ \psi$ is analytic in $\Delta^0_1$. Replace $z$ by $u^e$ in the coefficients of $m(w)$. Let $\mathcal{L}_{0,u}$ be convergent Laurent series in $u$ around $u = 0$. This gives $m_1(w) \in \mathcal{L}_{0,u}[w]$ and $m_1(f \circ \psi(u)) \equiv 0$. So, as $u \mapsto 0$, $f \circ \psi(u)$ goes to one of finitely many values on the Riemann sphere.

Apply Riemann's removable singularity theorem [**Ahl79**, p. 124]: $f\circ\psi$ extends to an analytic function $\Delta_1 \to \mathbb{C} \cup \{\infty\}$. That is, $f \circ \psi$ is analytic in $u$ with $u^e = z$. As in [9.9g], this embeds the function field $\mathbb{C}(z, f(z))$ into $\mathcal{P}_{z',e}$. As $f(z)$ has $e$ conjugates over $\mathcal{L}_{z'}$, $[\mathcal{L}_{z'}(f(z)) : \mathcal{L}_{z'}]$ is at least $e$. As $\mathcal{L}_{z'}(f(z))$ is a subfield of $\mathcal{P}_{z',e}$, with $[\mathcal{P}_{z',e} : \mathcal{L}_{z'}] = e$, the two fields are equal. This concludes the proof. □

**7.4. Proof of Prop. 7.4.** Let $\lambda \in \Pi_1(\Delta^0_{z'}, z_0)$ have winding number $n_{z'}(\lambda)$ around $z'$. The proof is in parts for later use. They consist of preliminary notation and description; explicit contraction of $\lambda$ to a path having range the points of $\delta$; and an observation on analytic continuation around such a path. Lemma 4.3 assures $f_\lambda = f_{\lambda^*}$ with $\lambda^*$ a polygonal path. So, with no loss assume $\lambda$ is polygonal.

7.4.1. *Notational simplifications.* The range of $\lambda$ is compact, and it does not include $z'$. So, there is a minimal distance $r_0$ between $z'$ and the range of $\lambda$. Let $A$ be an annulus around $z'$ with inner radius $r' < r_0$ and outer radius $R'$ giving the boundary of $\Delta_{z'}$. For simplicity assume $z' \neq \infty$ and the disk $\Delta_{z'}$ is in the complex plane, rather than on the Riemann sphere. Since circles go to circles by stereographic projection, the only adjustment to use the Riemann sphere would be to compose the description of the sets here with stereographic projection. Also, for simplicity, assume $z_0 - z' = r_0 e^{2\pi\theta_0}$ has $\theta_0 = 0$.

7.4.2. *Description of $A$.* The point $z_v = z' + r_0 e^{2\pi i v}$ lies on $\delta$. We also use $z_v^- = z' + r' e^{2\pi i v}$ and $z_v^+ = z' + R' e^{2\pi i v}$. The points of the line segment cut by a ray from $z'$ to $z_v^+$ meet $A$ in the set

$$L_v = \{z_v - s(z_v^- - z_v) \mid s \in [-1, 0]\} \cup \{z_v + s(z_v^+ - z_v) \mid s \in [0, 1]\}.$$

Thus the annulus is the union of the points on $L_v$, $v \in [0, 1]$. Reference the point on $L_v$ corresponding to $s \in [-1, 1]$ by $L_v(s)$.

7.4.3. *Contraction of $A$ to $\delta$.* Define $\Gamma : A \times [0, 1] \to A$ by

$$\Gamma(L_v(s), u) = \begin{cases} z_v - (1-u)s(z_v^- - z_v) & \text{for } s \in [-1, 0] \\ z_v + (1-u)s(z_v^+ - z_v) & \text{for } s \in [0, 1]. \end{cases}$$

Finally, for each $u \in [0, 1]$ we have a path $\gamma_u : [a, b] \to A$:

$$t \mapsto \gamma_u(t) = \Gamma(\gamma(t), u).$$

Note: $\gamma_0(t) = \gamma(t)$ and $\gamma_1(t)$ has range in the points of $\delta$. Further, $\gamma_1(t)$, being the contraction of a polygonal path to $\delta$ changes direction but finitely many times. Take $f$ as in the statement of Prop. 7.4. Conclude easily: $f_{\gamma_1} = f_{\delta^{e_1}}$ with $e_1$ the winding number of $\gamma_1$ around $z'$.

7.4.4. *$f_{\gamma_u}$ constant in $u \in [0, 1]$.* For $u \in [0, 1]$ consider the continuous function $f_u^*(t)$ giving analytic continuation (according to Def. 4.1) along $\gamma_u$. Let $h_{u,t}$ be the analytic function with restriction to $\gamma_u(t')$ giving $f_u^*(t')$ for $t'$ close to $t$.

Lemma 4.3 says for $(u', t')$ close to $(u, t)$, $h_{u,t}$ restricts to $\gamma_{u'}(t')$ to give $f_{u'}^*(t')$. Since $f_{u'}^*(t')$ is a composition of two continuous functions $\gamma_{u'}(t')$ and $h_{u,t}$, it is continuous. Thus, $f_u^*(b)$ is a continuous function of $u$. As $f_u^*(b)$ is in the discrete set of end values of the analytic continuations of $f$ in $\Delta_{z'}^0$, it is constant in $u$.

Since $z_0$ is not a branch point of the algebraic function $f$, the end value $f_u^*(b)$ determines $f_{\gamma_u}$. So, $f_{\gamma_1} = f_\gamma$, to conclude the proof of the proposition.

**7.5. Ramification indices, branch cycles and inertia groups.** Consider $L/\mathbb{C}(z)$, a finite extension. Let $z' \in \mathbb{P}_z^1$ and let $\mu : L \to \mathcal{P}_{z'}$ be an embedding of $L$ into Puiseux expansions about $z'$. As in [9.9], let $\zeta_e = e^{2\pi i/e}$ for $e \geq 1$ an integer.

Definition 7.6. The *ramification index* of $(L, z', \mu)$ is the minimal integer $e = e(L, z', \mu)$ for which $\mathcal{P}_{z',e}$ contains $\mu(L)$.

7.5.1. *A crucial automorphism.* Let $\hat{L}$ be the Galois closure of $L/\mathbb{C}(z)$. Cor. 7.5 says there is an integer $\hat{e}$ giving an embedding $\psi : \hat{L} \to \mathcal{P}_{z',e}$ fixed on $\mathbb{C}(z)$. Here is how $\psi$ produces a conjugacy class in $G(\hat{L}/\mathbb{C}(z))$ depending only on $z'$. Let $g_{z'}$ be the automorphism of $\mathcal{P}_{z',\hat{e}}$ mapping $(z - z')^{1/\hat{e}}$ to $\zeta_{\hat{e}}^{-1}(z - z')^{1/\hat{e}}$. This is restriction of a topological generator of the group of the whole algebraic closure.

Denote invertible integers modulo $e$ by $(\mathbb{Z}/e)^*$. Consider *compatible* sequences of integers $m_e \in \mathbb{Z}/e^*$, $e \geq 1$: $m_{ee'} \bmod e = m_e$ for all integers $e, e'$. Denote this collection $\hat{\mathbb{Z}}^*$. Similarly, $\hat{\mathbb{Z}}$ is the compatible collection of $m_e \in \mathbb{Z}/e$. Then, $\hat{\mathbb{Z}}$ is a topological ring whose (multiplicative) units are $\mathbb{Z}^*$ [**FJ86**, Chap. 1].

REMARK 7.7 (Use of the $p$-adics). Here is a reminder of the algebra for writing elements of $\hat{\mathbb{Z}}^*$. First: Consider only $e$ that are powers of a particular prime $p$. Then, the compatible sequences $\{m'_{p^k}\}_{k=1}^{\infty}$ analogous to $\hat{\mathbb{Z}}$ is $\mathbb{Z}_p$, the *p-adic integers*. These satisfy $m'_k \in \mathbb{Z}/p^k$, with $m'_{k+1} = m'_k \bmod p^k$ with $k = 1, \dots$. The direct product of the $\mathbb{Z}_p$ s over primes $p$ is $\hat{\mathbb{Z}}$. The direct product of the units $\hat{\mathbb{Z}}_p^*$ of $\hat{\mathbb{Z}}$ is $\hat{\mathbb{Z}}^*$. Symbolically write elements of $\hat{\mathbb{Z}}_p^*$ as series $a_0 + a_1 p + a_2 p^2 + \cdots$. Here $1 \le a_0 \le p-1$ and $0 \le a_i \le p-1$ are arbitrary. Without this procedure, excluding 1 and -1, it might be hard to list *any* elements of $\hat{\mathbb{Z}}^*$.

LEMMA 7.8. *The automorphism $g_{z'}$ maps $\mathcal{P}_{z',e}$ into itself for each $e$. Its effect on $\mathcal{P}_{z',ee'}$ extends its effect on $\mathcal{P}_{z',e}$.*

*Let $\sigma$ be any automorphism of $\mathcal{P}_{z'}$ fixed on $\mathcal{L}_{z'}$. The effect of $\sigma$ on $\mathcal{P}_{z',e}$ is the same as $g_{z'}^{m_e}$ for some $m_e \in (\mathbb{Z}/e)^*$. So, $\sigma$ corresponds to an element of $\hat{\mathbb{Z}}^*$.*

PROOF. This requires checking the effect of $g_{z'}$ on generators of the field extensions. By definition, $g_{z'}(z-z')^{1/ee'} = \zeta_{ee'}^{-1}(z-z')^{1/ee'}$. Put both sides to the power $e'$ and then apply $g_{z'}$. As $g_{z'}$ is a field automorphism,

$$(g_{z'}((z-z')^{1/ee'}))^{e'} = g_{z'}(((z-z')^{1/ee'})^{e'}) = g_{z'}((z-z')^{1/e}).$$

Yet, $(g_{z'}((z-z')^{1/ee'}))^{e'} = (\zeta_{ee'}^{-1}(z-z')^{1/ee'})^{e'}$. As $\zeta_{ee'}^{e'} = \zeta_e$ (by definition), this concludes the first part.

Powers of $g_{z'}$ give the group of the degree $e$ extension $\mathcal{P}_{z',e}/\mathcal{L}_{z'}$ [9.9d]. So, $\sigma$ restricted to $\mathcal{P}_{z',e}$ equals $g_{z'}^{m_e}$ for some $m_e \in (\mathbb{Z}/e)^*$. Let $\sigma_e$ be restriction of $\sigma$ to $\mathcal{P}_{z',e}$. Compatibility of these $m_e$ s is from $\sigma_e$ being restriction of $\sigma_{ee'}$ to $\mathcal{P}_{z',e}$. □

7.5.2. *Embeddings and branch cycles.* Continue the discussion starting §7.5.1. Restrict $g_{z'}$ to $\hat{L}$. Since $\hat{L}/\mathbb{C}(z)$ is Galois and $g_{z'}$ fixes $\mathbb{C}(z)$, this gives an automorphism $g_{z',\psi}$ of $\hat{L}$. Denote this element of $G(\hat{L}/\mathbb{C}(z)) = G$ by $g_{z',\psi}$. It depends on $\psi$, the choice of the embedding. Call it the *branch cycle* attached to the pair $(z', \psi)$.

LEMMA 7.9. *For $z' \in \mathbb{P}^1_z$, $[\hat{L} : \mathbb{C}(z)]$ distinct embeddings $\psi : \hat{L} \to \mathcal{P}_{z',\hat{e}}$ leave $\mathbb{C}(z)$ fixed. As $\psi$ runs over such embeddings, $g_{z',\psi}$ runs over a conjugacy class in $G$. Suppose $f(z)$, meromorphic about a nonbranch point $z_0$, satisfies $m(z, f(z)) \equiv 0$, $m \in \mathbb{C}[z,w]$. So, $z' \in \mathbb{P}^1_z$ produces a conjugacy class $\mathrm{C}_{z'}$ of $G = G(\hat{L}/\mathbb{C}(z))$. With $\boldsymbol{z}$ the branch points of $(m, w)$, for each $z' \notin \boldsymbol{z}$, $\mathrm{C}_{z'} = \{1\}$.*

*Let $\delta$ be a clockwise (closed) circle around $z' \in \boldsymbol{z}$ bounding a closed disk $\Delta_{z'}$. Assume $\Delta_{z'}$ (excluding possibly $z'$) contains no other branch point of $(m, z)$ and $z_0 \in \Delta_{z'}$. Let $f_1, \dots, f_n$ be a complete list of conjugates of $f$. Denote analytic continuation of $f_j$ around $\delta$ by $f_{j,\delta}$. Then, for some choice of $\psi$, $g_{z',\psi}$ maps this set to $f_{1,\delta}, \dots, f_{n,\delta}$.*

PROOF. Cor. 7.5 produces one embedding, $\psi : \hat{L} \to \mathcal{P}_{z',\hat{e}}$. Let $\alpha$ run over the automorphisms of $\hat{L}$ fixed on $\mathbb{C}(z)$. Then, $\psi \circ \alpha : \hat{L} \to \mathcal{P}_{z',\hat{e}}$ runs over $[\hat{L} : \mathbb{C}(z)]$ embeddings of $\hat{L}$ into the algebraic closure of $\mathcal{L}_{z'}$ fixed on $\mathbb{C}(z)$. Galois theory says this is the exact number of embeddings possible. So we have listed them all.

Consider the effect on $g_{z',\psi}$ of composing $\psi$ with $\alpha$. The new automorphism is

$$g_{z',\psi\circ\alpha} = (\psi \circ \alpha)^{-1} \circ g_{z'} \circ (\psi \circ \alpha) = \alpha^{-1} g_{z',\psi} \alpha.$$

That is, $g_{z',\psi\circ\alpha}$ runs over the conjugacy class of $g_{z',\psi}$ in $G$ as $\alpha$ runs over $G$.

Regard elements $f_1,\dots,f_n$ as in $\mathcal{L}_{z_0}$. Let $h(z)$ be a branch of $(z-z')^{1/\hat{e}}$ defined in this neighborhood of $z_0$. Giving an embedding of $\hat{L}$ (fixed on $\mathbb{C}(z)$) into $\mathcal{P}_{z',\hat{e}}$ is equivalent to giving an embedding of $\hat{L}$ mapping $f_1,\dots,f_n$ into power series $g_1(h(z)),\dots,g_n(h(z))$ in $h(z)$, $g_1,\dots,g_n \in \mathcal{L}_0$. Analytic continuation of $g_1(h(z)),\dots,g_n(h(z))$ around $\delta$ maps $g_i(h(z))$ to $g_i(\zeta_{\hat{e}}^{-1}h(z))$. This is the effect of restriction of $g_{z'}$ on the embedding of the $f_i$ s in the Puiseux expansions. □

7.5.3. *Branch cycles and inertia groups.* Choosing $\zeta_{\hat{e}}^{-1}$ (rather than $\zeta_{\hat{e}}$) in the definition of $g_{z'}$ is convenient (later). This assures $\delta$ in Lem. 7.9 is a clockwise path. The conjugacy class $\mathrm{C}_{z'}$ in Lem. 7.9 is crucial to precise formulations of Riemann's Existence Theorem. This is the branch cycle conjugacy class attached to $z'$. Using $G \le S_n$, disjoint cycle data (Chap. 3 §7.1) for elements of $\mathrm{C}_{z'}$ is sufficient for some applications, though not for the more serious.

Definition 7.10 (Inertia groups). The branch cycle $g_{z',\psi}$ in Lem. 7.9 generates a group, $I_{z',\psi}$ of $G(\hat{L}/\mathbb{C}(z))$. This is the *inertia group* attached to the embedding $\psi$. The notation $I_{z'}$ refers to any choice of the groups conjugate to $I_{z',\psi}$. Points $z' \in \mathbb{P}^1_z$ for which $I_{z'}$ is nontrivial are the *branch points* of $L/\mathbb{C}(z)$.

7.5.4. *Two definitions of branch points.* There are now two definitions of branch points. Def. 7.10 gives it for the function field $L/\mathbb{C}(z)$ and §6.2 for the pair $(m,z)$. They are related though they may not be equal [9.11].

Proposition 7.11. *Suppose $m(z,f(z)) \equiv 0$ and $L = \mathbb{C}(z,f(z))$. If $z' \in \mathbb{C}$ is a branch point of $L/\mathbb{C}(z)$, then it is also a branch point of $(m,z)$.*

Proof. Suppose $z'$ is a branch point of $L/\mathbb{C}(z)$. Then, there is an embedding $\psi : \mathbb{C}(z,f(z)) \to \mathcal{P}_{z',e}$ where the image of $f$ is not in $\mathcal{L}_{z'}$. In particular, the power series $\psi(f)$ and $g_{z'}(\psi(f))$ in $(z-z')^{1/e}$ have the same value after substituting 0 for $(z-z')^{1/e}$. Since $(w-\psi(f(z))(w-g_{z'}(\psi(f)))$ divides $m(z,w)$ (in $\mathcal{P}_{z'}[w]$), this shows $m(z',w)$ has multiple zeros. □

## 8. Abelian functions from branch of log

A branch of log isn't an algebraic function. Still, it allows explicit construction of all the algebraic functions we call *abelian*, the topic of this subsection.

**8.1. Further notation around extensible functions.** Let $\mathcal{E}(U_{\boldsymbol{z}},z_0)$ be the extensible (meromorphic) functions on $U_{\boldsymbol{z}}$ (as in Def. 4.5; given by elements of $\mathcal{L}_{z_0}$). Denote algebraic elements of $\mathcal{E}(U_{\boldsymbol{z}},z_0)$ (as in Def. 1.1) by $\mathcal{E}(U_{\boldsymbol{z}},z_0)^{\mathrm{alg}}$.

Definition 8.1. Let $G$ be a finite group having a specific property $P^*$. Say an element $f \in \mathcal{E}(U_{\boldsymbol{z}},z_0)^{\mathrm{alg}}$ has property $P^*$ if its monodromy group $G_f$ (§4.4.1) has this property. This allows referring to abelian, *nilpotent* ($G_f$ is a product of its $p$-Sylow subgroups), solvable or primitive functions.

Example: Suppose $[\mathbb{C}(z,f):\mathbb{C}(z)] = n$. Then, $f$ is *primitive* if $G_f$ is a primitive subgroup of $S_n$ (Chap. 3 Def. 7.9). Equivalently, by the Galois correspondence, there is no field properly between $\mathbb{C}(z)$ and $\mathbb{C}(z,f)$ [9.5]. Later chapters show this is a very important concept. Unfortunately, the word *primitive* appears in many guises in mathematics (already in this chapter). It has even more meanings in the Webster's dictionary. The closest to our meaning here is this: not derived; as a primitive verb in grammar. So, $\mathbb{C}(z,f)$ is an extension not (even partially) derived from any other proper extension of $\mathbb{C}(z)$. Note that this is different in English than it being generated by a single element over $\mathbb{C}(z)$ (primitive generator). Denote the abelian (resp. nilpotent) functions in $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$ by $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{ab}}$ (resp. $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{nil}}$).

**8.2. Abelian monodromy.** For $e \in \mathbb{Z}$ and $\gamma : [a, b] \to D$ a path whose range misses all zeros and poles of $f(z)$, (6.2) defines branch of $f(z)^{\frac{1}{e}}$ along $\gamma$.

Here is data for abelian functions of index $e$:

- distinct points $\boldsymbol{z} = z_1, \dots, z_r$ in $\mathbb{P}^1_z$: *branch points*
- $\Delta_{z_0}$, a disk neighborhood of $z_0$: *base point*
- an integer $e$: *index*
- a branch $g_{i,j}$ of $\log(\frac{z-z_i}{z-z_j})$ in $\Delta_{z_0}$, $1 \le i < j \le r$

Denote the field $\mathbb{C}(z, e^{g_{i,j}/e}, 1 \le i < j \le r)$ by $L_{e,\boldsymbol{z}}$: The field of *abelian functions* (on $\mathbb{P}^1_z$) ramified over $\boldsymbol{z}$ of index dividing $e$. It is a subfield of $\mathcal{L}_{z_0}$. Any $f \in L_{e,\boldsymbol{z}}$ defines an analytic $f : \Delta_{z_0} \to \mathbb{P}^1_z$ according to notation of §4.6. If some $z_i = \infty$ replace $z - z_i$ by 1 in the definition. In particular, when $z_r = \infty$, $g_{i,r}$ is a branch of $\log(z - z_i)$, $i = 1, \dots, r-1$. This definition includes all algebraic functions having abelian monodromy group. It will give a valuable comparison in Chap. 4. There is a similar definition of algebraic functions on $D$ with any domain $D$ replacing $\mathbb{P}^1_z$.

8.2.1. *Galois group of* $L_{e,\boldsymbol{z}}$. A complete description of $L_{e,\boldsymbol{z}}$ depends only on homology classes of paths in $\Pi_1(U_{\boldsymbol{z}}, z_0)$.

COROLLARY 8.2. *Assume $\gamma_1, \gamma_2 \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ are homologous and $f$ is an algebraic abelian function on $U_{\boldsymbol{z}}$ corresponding to the data* (8.2). *Then, the analytic continuations $f_{\gamma_1}$ and $f_{\gamma_2}$ (back to $z_0$) are equal. Monodromy from $\Pi_1(U_{\boldsymbol{z}}, z_0)$ induces a faithful action of $H_1(U_{\boldsymbol{z}})/eH_1(U_{\boldsymbol{z}})$ on $L_{e,\boldsymbol{z}}$ and therefore on $\mathbb{C}(z, \mathcal{A}_f(U_{\boldsymbol{z}}, z_0))$ (§4.2.2). In particular, $L_{e,\boldsymbol{z}}/\mathbb{C}(z)$ is Galois with group $H_1(U_{\boldsymbol{z}})/eH_1(U_{\boldsymbol{z}})$. For $f \in L_{e,\boldsymbol{z}}/\mathbb{C}(z)$, $\mathbb{C}(z, \mathcal{A}_f(U_{\boldsymbol{z}}, z_0))/\mathbb{C}(z)$ is Galois with group a quotient of this group.*

PROOF. For simplicity assume $z_r = \infty$. Take $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ and

$$f(z) = m_1(e^{g_{1,\gamma}(z)/e}, \dots, e^{g_{r-1,\gamma}(z)/e})/m_2(e^{g_{1,\gamma}(z)/e}, \dots, e^{g_{r-1,\gamma}(z)/e}),$$

where $g_{j,\gamma}$ denotes analytic continuation of $g_j$ around $\gamma$. Let $m_j$ be the winding number of $\gamma$ about $z_j$. Analytic continuation of $g_j$ around $\gamma$ adds $2\pi i m_j$ to $g_j$ (Prop. 3.5). Since $\gamma_1$ and $\gamma_2$ have the same winding numbers around each $z_j$, this proves the effect of their analytic continuations on $f$ are the same.

Note that $L_{e,\boldsymbol{z}}/\mathbb{C}(z)$ is the composite of the field extensions $\mathbb{C}(z, e^{g_j(z)/e})/\mathbb{C}(z)$. Apply [9.9] using $(e^{g_j(z)/e})^e = z - z_j$. Conclude: $\mathbb{C}(z, e^{g_j(z)/e})\mathbb{C}(z)$ is Galois with group $\mathbb{Z}/(e)$. From [9.5d], the composite of these fields is Galois, with group a subgroup of $\mathbb{Z}/(e) \times \cdots \times \mathbb{Z}/(e)$. The image of $H_1(U_{\boldsymbol{z}})/eH_1(U_{\boldsymbol{z}})$ produces field automorphisms of $L_{e,\boldsymbol{z}}$. We know these explicitly. Let a closed path $\lambda$ have respective winding numbers $(a_1, \dots, a_{r-1})$ around $(z_1, \dots, z_{r-1})$. If $e$ does not divide $a_j$, then monodromy action of $\lambda$ on $g_j$ is nontrivial. So the automorphism group is all of $(\mathbb{Z}/e)^{r-1}$. This shows the result. □

**8.3. Deeper into the Monodromy Theorem.** Consider $m \in \mathbb{C}[z, w]$ and $D$ a domain in $\mathbb{P}^1_z$. It is a fundamental to decide when some branch of solutions of $m(z, w) = 0$ is a meromorphic function on all of $D$. Riemann's Existence Theorem gives a satisfactory answer to versions of this question.

8.3.1. *Simple connectedness.* Call a domain in $\mathbb{C}$ *simply connected* if there is at most one connected component in $\mathbb{P}^1_z \setminus D$. Chap. 3 has the usual definition of a simply connected topological space. For open subsets of $\mathbb{P}^1_z$ these definitions describe the same sets. The following is an application of Cauchy's Residue Theorem for later comparison with the general Monodromy Theorem.

THEOREM 8.3 (Monodromy Theorem). *Suppose $D \subset \mathbb{C} \setminus \{z_1, \dots, z_r\}$ is simply connected. Assume $f$ has no residues in $D$. Then $f(z)$ has a primitive (antiderivative; §2.5) $F(z)$ on $D$. Suppose $\boldsymbol{z}$ contains the zeros and poles of $f(z)$. Apply this to $\frac{df}{dz}/f$ to conclude there is a branch of $\log(f(z))$) on $D$.*

8.3.2. *Homological triviality versus simple connectedness.* Being simply connected has another characterization: the winding number of any closed path in $D$ relative to any point $z'$ outside of $D$ is 0. That is, $D$ is simply connected if all paths in $D$ are homologous to 0. Beware! If $D$ is not simply connected, some paths may be homologous to 0, though not trivial for our applications. For example, any function that isn't abelian has a nontrivial analytic continuation around some path homologous to 0. For, however, abelian functions, most questions use just the Monodromy Theorem in Prop. 7.4. For example, suppose $m(z, g(z)) \equiv 0$, and $\mathbb{C}(z, g(z))/\mathbb{C}$ is an abelian extension ($g$ is *abelian*). Then, we can characterize those $D$ that aren't simply connected on which $g$ is extendible. It is tougher to be so precise about antiderivatives for even abelian functions $g$ along paths in $D$.

**8.4. Primitive tangential base points.** Let $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\rm alg}$ and $z' \in \boldsymbol{z}$. Suppose $\lambda$ in $U_{\boldsymbol{z}}$ goes from $z_0$ to $z_1$. Analytic continuation of $f$ produces $f_\lambda \in \mathcal{E}(U_{\boldsymbol{z}}, z_1)$. Consider $\lambda$ a *restriction* map. Applying $\lambda$ restricts $f$ to $f_\lambda \in \mathcal{L}_{z_1}$.

How about using a path to *restrict* $f$ to a function around $z'$? That is, let $\lambda$ be a path with end point close to $z'$. Can we consider $f_\lambda$ restriction of $g \in \mathcal{P}_{z'}$? The simple answer is No!, unless $f_\lambda$ extends to an analytic function around $z'$. It is valuable, however, to add data to $\mathcal{P}_{z'}$ so the answer will be Yes!

Choose an open disk $D'$ in $U_{\boldsymbol{z}}$, with $z'$ on its boundary. Let $g_e(z)$ be a branch of $(z - z')^{1/e}$ on $D'$, one for each positive integer $e$. This always exists from (6.2). Further, we ask the system of these be *compatible*:

(8.1) For all integers $(e, e', e'')$ satisfying $ee' = e''$, $g_{e''}(z)^{e'} = g_e(z)$.

Call this collection $\{g_e\}_{e=1}^{\infty} = \mathcal{G}(D', z')$ a *system of branches* on $(D', z')$. The following is a slight enhancement of Lem. 7.8.

PROPOSITION 8.4. *Given $\mathcal{G}(D', z')$, any system of branches on $(D', z')$ corresponds one-one with elements of $\hat{\mathbb{Z}}$ (§7.5.1). Precisely: $\{m_e\} \in \hat{\mathbb{Z}} \mapsto \{\zeta_e^{m_e} g_e(z)\}_{e=1}^{\infty}$.*

*Let $D'' \subset D'$ be any (open) disk tangent to $z'$. Restriction of $\mathcal{G}(D', z')$ to $D''$ defines a system of branches $\mathcal{G}(D'', z')$. Let $\boldsymbol{v}$ be the direction from $z'$ along a geodesic on $U_{\boldsymbol{z}}$ toward the center of $D'$. (Consider $U_{\boldsymbol{z}}$ a subset of the sphere with its metric; geodesics being great circles.) Containment orders disks tangent to $z'$ with $\boldsymbol{v}$ pointed into the disk. There is a maximal element*

$$\mathcal{G}(\boldsymbol{v}, z', U_{\boldsymbol{z}}) = \mathcal{G}(\boldsymbol{v}, z') = \mathcal{G}(D_{\boldsymbol{v}}, z') :$$

*Take $D_{\boldsymbol{v}}$ the largest disk in $U_{\boldsymbol{z}}$ having radius along $\boldsymbol{v}$ and tangent to $z'$.*

So, the set of branch systems satisfying (8.1) is a *homogeneous space* for $\hat{\mathbb{Z}}$. That is, an action of the group $\hat{\mathbb{Z}}$ on one of them gives all. You still, however, need one choice $\mathcal{G}(D', z')$ to get the process going.

Definition 8.5. Call $\mathcal{G}(\boldsymbol{v}, z') = \hat{\boldsymbol{v}}$ a *primitive* (or *naive*) tangential base point: $\hat{\boldsymbol{v}}$ has an underlying point $z'$, direction $\boldsymbol{v}$ and system of branches on $D_{\boldsymbol{v}}$.

From Cor. 7.5, elements in $\mathcal{P}_{z',e}$ have the form $f^* = h((z-z')^{1/e})$ with $h \in \mathcal{L}_{z'}$. Define $\text{rest}_{\hat{\boldsymbol{v}}}(f^*)$ to be $h(g_e(z))$. For any simply connected subspace $Y$ of $U_{\boldsymbol{z}}$, denote paths in $U_{\boldsymbol{z}}$ from $z_0$ with endpoint in $Y$ by $\Pi_1(z_0, Y)$.

Proposition 8.6 (Tangential Base Point Restriction). *Assume* $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ *and* $\gamma \in \Pi_1(z_0, D_{\boldsymbol{v}})$. *There is a unique* $f^* \in \mathcal{P}_{z'}$ *with* $\text{rest}_{\hat{\boldsymbol{v}}}(f^*) = f_\lambda$.

Proof. Uniqueness of $f^*$ is clear. Existence is from Cor. 7.5. Here are details. Let $\delta$ be a clockwise circle bounding a disk $\Delta_{z'}$ with center $z'$ with $\Delta_{z'} \setminus \{z'\} \subset U_{\boldsymbol{z}}$. Assume $\delta$ meets $D_{\boldsymbol{v}}$. Connect the end point of $\lambda$ to some point on $\delta$ by a path lying entirely in $D_{\boldsymbol{v}}$. From Cauchy's Theorem (Prop. 3.6), there is a unique function $g$ defined by a power series on $D_{\boldsymbol{v}}$ that restricts to $f_\lambda$. So, any analytic continuation of $f_\lambda$ along a path in $D_{\boldsymbol{v}}$ equals $g$. Thus it depends only on the end point of this path. Assume with no loss $\lambda$ ends on $\delta$.

Let $e = e_f$ be the order of the monodromy action of $\delta$ on $f_\lambda$. Then, Cor. 7.5 says $f_\lambda$ is $f^* = h(g_{e_f}(z))$ with $h$ holomorphic in the disk $\delta$ bounds. □

Example 8.7 (Deligne tangential base points). Take $z' = 0$ and $\boldsymbol{v}$ any direction $0 \le \theta < 2\pi$ on $\mathbb{C}_z$ represented by $e^{i\theta}$. Define $g_e(z)$ to be $e^{i\theta/e}$ times the unique branch of $(e^{-i\theta}z)^{1/e}$ taking positive real values along the direction $\boldsymbol{v}$ from 0: [**De89**, §15] or [**Ihar91**, p. 103].

**8.5. Describing all algebraic abelian functions.** Suppose $f(z)$ is algebraic and $\mathbb{C}(z,f)/\mathbb{C}(z)$ is a Galois extension with abelian Galois group $G$. Assume $\boldsymbol{z}$ contains the branch points of $f$ and the ramification indices at all points of $\boldsymbol{z}$ divide some integer $e$. Each $z' \in \boldsymbol{z}$ produces an inertia group $I_{z'}$ (Def. 7.10). More explicitly it produces a well defined conjugacy class $\mathrm{C}_{z'}$ in $G$ (Lem. 7.9). Since, however, $G$ is abelian, this conjugacy class is an element $g_{f,z'} \in G$.

Theorem 8.8. *Under the above hypotheses,* $g_{f,z'}$*, as* $z'$ *runs over* $\boldsymbol{z}$*, determines the field extension* $\mathbb{C}(z,f)$*. Further, two other properties hold.*

- $\langle g_{f,z'}, z' \in \boldsymbol{z}\rangle = G$*: generation*
- $\prod_{z' \in \boldsymbol{z}} g_{f,z'} = 1$ *: product-one condition*

*Conversely, suppose given* $G$ *and elements* $g_{z'} \in G$ *for each* $z' \in \boldsymbol{z}$ *satisfying* (8.8). *Then, there exists algebraic* $f$ *(given as above by branches of log) satisfying* $g_{f,z'} = g_{z'}$ *for* $z' \in \boldsymbol{z}$. *Another algebraic function* $f^*$ *produces the same data if and only if* $\mathbb{C}(z,f^*) = \mathbb{C}(z,f)$.

Proof. There is a standard reduction for showing the field is determined by the data $g_{f,z'}, z' \in \boldsymbol{z}$. Write $G$ as $\prod_{i=1}^{u} G_i$ where $G_i$ is cyclic of some prime power order. Every finite abelian group has this form ([**Isa94**, p. 90], see [9.15]). Then, $\mathbb{C}(z,f)$ is the composite of field extensions $L_i/\mathbb{C}(z)$ with group $G_i$, $i = 1, \dots, u$. Further, any subextension $\mathbb{C}(z) < M < L_i$ is Galois with group a quotient of $G_i$. So, it is cyclic of prime power order. So, with no loss assume $\mathbb{C}(z,f)/\mathbb{C}(z)$ is Galois with group isomorphic to $\mathbb{Z}/p^t$ for some integer $t$ and prime $p$. List $\boldsymbol{z}$ as $z_1, \dots, z_r$, then list the group data as $(g_1, \dots, g_r)$ with $g_i = g_{f,i}$ attached to $z_i$. Since $G = \mathbb{Z}/p^t$, identify $g_i$ with an integer $n_i \in \mathbb{Z}/p^t$.

It is easy to produce a cyclic extension that has exactly this attached data. For simplicity, assume $z_r = \infty$. Then, for any $z_0$ not in $\boldsymbol{z}$, let $h(z) = \prod_{i=1}^{r-1} h_i(z)^{n_i}$

with $h_i$ a branch of $(z-z_i)^{\frac{1}{p^t}}$ in a neighborhood of $z_0$. The lemma is done if $\mathbb{C}(z,h(z)) = \mathbb{C}(z,f(z))$. Both fields embed in $\mathcal{P}_{z_i}$ and the action of $g_{z_i}$ restricts to both fields the same way. Any function in the fixed field of all the $g_i$ s is extensible over the whole Riemann sphere, as in §7.1. So such a function is a rational function in $z$. Therefore, the fixed field of $\langle g_1,\dots,g_r\rangle$ in $\mathbb{C}(z,f(z))$ is trivial. Apply [9.5d] to the composite of the two fields and conclude they are equal.

Consider the generation condition. Assume $\langle g_{f,z'}, z' \in \boldsymbol{z}\rangle = H$ is a proper subgroup of $G$. If $f_1 \in \mathbb{C}(z,f)$ is in the fixed field of $H$, then $f_{1,\lambda} = f_1$ for all $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$. Lem. 7.2 implies $f_1 \in \mathbb{C}(z)$. So $\mathbb{C}(z)$ is the exact fixed field of $H$ and $H = G$. The product-one condition appears by recognizing $g_{f,z'}$ as restriction of the $g_{e,z'}$ for the field $L_{e,\boldsymbol{z}}$. Apply the product of the $g_{e,z'}$ to generating functions in $L_{e,\boldsymbol{z}} = \mathbb{C}(z, e^{g_{i,j}/e}, 1 \le i < j \le r)$ (from (8.2)). It comes to showing $g_{e,z_j} g_{e,z_i}(e^{g_{i,j}/e}) = e^{g_{i,j}/e}$. With no loss take $z_i = 0$ and $z_j = \infty$ [9.10a]. □

The full version of Riemann's Existence Theorem generalizes the generation and product-one conditions (8.8) to $\mathbb{C}(z,f(z))$ where $f$ is any algebraic function. When $G$ is abelian, the product-one condition is independent of the order of the elements $g_{f,z'}$. Keep your eye on the analysis that goes into tracking the order of elements appearing in the product-one condition when $G$ is not abelian. This is what produces the significant action of the *Hurwitz monodromy group* in Chap. 5. Further, the converse holds in generality. Without, however, the abelian condition producing the algebraic function $f$ is more mysterious.

Suppose $G$ and $G^*$ are abelian groups and $\boldsymbol{g}_{\boldsymbol{z}}$ and $\boldsymbol{g}^*_{\boldsymbol{z}^*}$ satisfy the conditions of (8.8). Consider two triples $\mathcal{G} = (G, \boldsymbol{z}, \boldsymbol{g}_{\boldsymbol{z}})$ and $\mathcal{G}^* = (G^*, \boldsymbol{z}^*, \boldsymbol{g}^*_{\boldsymbol{z}^*})$ as in Thm. 8.8. Assume $\boldsymbol{z}$ is a subset of $\boldsymbol{z}^*$. For this discussion, if $z' \in \boldsymbol{z}^* \setminus \boldsymbol{z}$ regard $\boldsymbol{g}_{\boldsymbol{z}}$ as having the identity element at $z'$. Also, assume there is a homomorphism $\alpha : G^* \to G$ taking $g^*_{z'}$ to $g_{z'}$ for $z' \in \boldsymbol{z}^*$. Regard $\alpha = \alpha_{\mathcal{G}^*,\mathcal{G}}$ as a map from $\mathcal{G}^*$ to $\mathcal{G}$.

Corollary 8.9. *The projective system $\{\mathcal{G}, \alpha_{\mathcal{G}^*,\mathcal{G}}\}$ of triples with maps has a limit consisting of a group $\mathcal{G}^{\mathrm{ab}}$ and elements $g^{\mathrm{ab}}_{z'}$ running over $z' \in \mathbb{P}^1_z$. Then, $\mathcal{G}^{\mathrm{ab}}$ identifies with the maximal abelian quotient of the absolute Galois group of $\mathbb{C}(z)$. Also, $g^{\mathrm{ab}}_{z'}$ acts trivially on any abelian algebraic function in $\mathcal{L}_{z'}$ and identifies with a generator of the automorphisms of $\mathcal{P}_{z'}/\mathcal{L}_{z'}$ in its restriction to the abelian algebraic functions in $\mathcal{P}_{z'}$ (Cor. 7.5).*

Call the group $\mathcal{G}^{\mathrm{ab}}$, the Galois group of the *maximal abelian extension* of $\mathbb{C}(z)$. A collection $\{g^{\mathrm{ab}}_{z'}\}_{z' \in \mathbb{P}^1_z}$ will be a *canonical system of generators* of $\mathcal{G}^{\mathrm{ab}}$. Any $g \in \mathcal{G}^{\mathrm{ab}}$ acts on the abelian algebraic functions in $\mathcal{P}_{z'}$ for any $z'$. This action is also the restriction of an automorphism of $\mathcal{P}_{z'}/\mathcal{L}_{z'}$. So monodromy action on a branch of $\log(z-z')$ determines this restriction element as a multiple of $g^{\mathrm{ab}}_{z'} \in \hat{\mathbb{Z}}$.

## 9. Exercises

Some exercises remind of basic Galois Theory. Use $\mathrm{char}(K)$ to denote the characteristic of a field $K$: The minimal positive integer $n$ for which $n$ times the identity in $K$ is 0 (if such an integer exists, or 0 otherwise).

**9.1. Substitutions and the chain rule.** Consider more on (2.7c) as the defining property of analyticity.

(9.1a) For a path $\lambda : [a,b] \to \mathbb{C}$, compose it with any analytic function $h : \mathbb{C} \to \mathbb{C}$ to give $h \circ \lambda : [a,b] \to \mathbb{C}$, another path. If $g$ and $h$ satisfy (2.7c), show

$$\begin{aligned}\tfrac{d}{dt}(g \circ h)(\lambda(t_0)) = \tfrac{d}{dt}(g(h(\lambda)))(t_0) &= \tfrac{dg}{dw}_{|w=h(\lambda(t_0))} \tfrac{d}{dt}(h \circ \lambda)_{|t=t_0} \\ &= \tfrac{dg}{dw}(h \circ \lambda_{|t=t_0}) \tfrac{dh}{dz}(\lambda(t_0)) \tfrac{d\lambda}{dt}(t_0).\end{aligned}$$

(9.1b) Show: Existence of $f'(z_0)$ requires only checking (2.5) for $\lambda : [-1,1] \to D$ by $t \mapsto z_0 + tv$ with $v \neq 0$. That is, check directional derivative rule (2.7b).

(9.1c) Conclude, if in (2.7c) two of $g \circ h, g, h$ are analytic, then so is the third.

With $m(z,w) = w^k - h(z)$ and $w(t)$ and $z(t)$ (nonconstant) rational functions with $w(t)^k \equiv h(z(t))$ for all $t$, consider indefinite integrals for $I(z) = \int h(z)^{\frac{1}{k}}\, dz$.

(9.2a) Substitute $z(t)$ for $t$. Rewrite $I(z)$ as an antiderivative for $\frac{dz(t)}{dt}/w(t)$. Apply this with $k = 2$ and $h(z) = z^2 + az + b$ using [9.3d].

(9.2b) Ex. [9.10f] shows [9.2a] won't work often, not even with $k = 2$ and $\deg(h) = 3$ having no repeated roots. Show it does work for any $h$ with at most two distinct zeros, but arbitrary degree.

(9.2c) Calculus uses a different substitution: $w(t)$ and $z(t)$ are trigonometric in $t$ with $w(t)^2 = z(t)^2 + az(t) + b$. Result: The square root expression disappears; replaced by a function. Why choose transcendental over rational functions? Hint: Consider the antiderivative as a function of $z$.

**9.2. Rational functions and field theory.** Suppose $K$ is any field. Consider $u(z) = P_1(z)/P_2(z)$ in $K(z)$. Follow the notation of §1.2.1.

(9.3a) Show $P_1(w) - zP_2(w)$ is irreducible. Hint: Factor it as $m_1(z,w)m_2(z,w)$. Then compute the degree in $z$ of each factor.

(9.3b) Suppose $m \in K[z,w]$, $\deg_z(m) = 1$ and $m(z,f(z)) \equiv 0$ for some $f(z)$ analytic on a domain $D$. Show $K(z,f(z)) = K(f(z))$.

(9.3c) If $M \le L_1 \le L_2$ is a chain of fields, *transitivity for degrees* says $[L_2 : M] = [L_1 : M][L_2 : L_1]$. Use it to show $\deg(u_1(u_2(z))) = \deg(u_1)\deg(u_2)$ for $u_1, u_2 \in K(z) \setminus \{0\}$.

(9.3d) Suppose $M$ is a field and $\mathrm{char}(K) \neq 2$. Assume $m(z,w) \in K[z,w]$ of total degree 2 is irreducible, $z_0, w_0 \in K$, $m(z_0,w_0) = 0$ and $w'$ is a zero of $m(z,w)$ in $\overline{K(z)}$. Show $K(z)(w')$ is isomorphic to $K(t)$ for some $t \in K(z)(w')$. Hint: With $t$ and $s$ variables, let $z_0 + s = z$ and $w' = w_0 + ts$. Solve for $s$ as a function of $t$ in $m(z,'w) = 0$.

(9.3e) Show $z_0, w_0 \in K$ is necessary for the existence of $t$ in (9.3d).

(9.3f) The fundamental theorem of algebra follows from knowing a function $f(z)$ bounded and analytic on $\mathbb{C}$ is constant. How does this imply every analytic function $P : \mathbb{P}^1_z \to \mathbb{P}^1_w$ (§4.6) by $z \mapsto P(z)$ is an element of $\mathbb{C}(z)$?

Now consider parametrizations by rational function curves. Use §6.1.2 with $f = f_1/f_2 \in \mathbb{C}(w)$ and $(f_1,f_2) = 1$. Parametrize $X_f$ near $(z_0,w_0)$ if $w_0$ is not a zero of the *Wronskian* $f_1(w)f_2'(w) - f_2(w)f_1'(w)$ of $f_1, f_2$ and $f_2(w_0) \neq 0$.

(9.4a) Use Def. 4.14 to show this includes when $w_0$ is a zero of $f_2$ ($z_0 = \infty$).

(9.4b) Extend a) to $w_0 = \infty$. Show an analytic parametrization of a neighborhood by $(z, g(z))$ exists if and only if $|\deg(f_1) - \deg(f_2)| \le 1$.

(9.4c) Suppose $f(g(z)) \equiv z$ for $g(z)$ analytic in a neighborhood of $z_0$. With these extensions, show the maximal number of branch points for $\mathbb{C}(z, g(z))$ (§6.2) is $2(\deg(f) - 1)$ with equality occurring for some rational functions $f$ of degree $n$ for any positive integer $n$.

(9.4d) Suppose $w_0$ is a zero of $f_1(w)f_2'(w) - f_2(w)f_1'(w)$ of multiplicity $e_{w_0} - 1$ and $f(w_0) = z_0$. Apply the Cor. 7.5 proof to find $e_{w_0}$ distinct functions $g(u)$ analytic around 0 with $g(0) = w_0$ and $f(g(u)) - z_0 - u^{e_{w_0}} \equiv 0$?

(9.4e) Extend d) to have either $z_0$ or $w_0$ is $\infty$. Conclude for $f \in \mathbb{C}(z) \setminus \mathbb{C}$:

$$2(\deg(f) - 1) = \sum e_{w_0} - 1.$$

**9.3. Galois theory of composite fields and using group theory.** Suppose $L_1/K$ and $L_2/K$ are two field extensions. Given a field $L$ containing both $L_1$ and $L_2$, there is an immediate *minimal field* $L_1 \cdot L_2$ in $L$ containing them both [**Isa94**, Chap. 18].

(9.5a) Suppose $M/K$ is *Galois*: Its group of automorphisms $G(M/K) = G$ fixed on $K$ has order $[M : K]$. Consider $K < L < M$, a chain of fields. Suppose $L = L_1, \dots, L_n$ are the fields conjugate to $L/K$. Show $L_1 \cdot L_i = L_1$, $i = 1, \dots, n$, if and only if $L/K$ is Galois ($G(M/L)$ is a normal subgroup; closed under conjugation from $G$).

(9.5b) Let $T : G \to S_n$ be the permutation representation of $G$ on cosets of $G(M/L)$ (as in a). Show there is $j \neq 1$ with $L_1 = L_1 \cdot L_j$ if and only if $(1)T(g) = 1 \Leftrightarrow (j)T(g) = j$ for each $g \in G$.

(9.5c) The following notation holds for the next two subexercises. Suppose $M_i/K$ is Galois with group $G_i$, $i = 1, 2$. Consider the group $G$ defined as follows:

$$\{g = (g_1, g_2) \in G_1 \times G_2 \mid g_1(\alpha) = g_2(\alpha), \alpha \in M_1 \cap M_2\}.$$

Show $G$ acts as automorphisms of $M_1 \cdot M_2$.

(9.5d) Show $|G| = [M_1 \cdot M_2 : K]$, and so $M_1 \cdot M_2/K$ is Galois with group $G$. Hint: Apply the Fundamental Theorem of Galois Theory [**Isa94**, Thm. 18.21] to the fixed field of $G$.

(9.5e) Conclude $M_1 \cdot M_2$ doesn't depend (up to isomorphism over $K$) on what field they both sit inside if both extensions are Galois.

(9.5f) Assume $\mathrm{char}(K)$ is $p$ (a prime or 0). Suppose $K$ has at most one extension of degree $n$ for any integer $n > 0$ (or if $p > 0$, prime to $p$). Show extensions of $K$ of degree prime to $p$ are Galois with cyclic group.

We warmup in interpreting field theory with group theory. Let $K = \mathbb{C}(z)$. If $f$ is algebraic over $K$ denote $K(f)$ by $L_f$, and the Galois closure of $L_f/K$ by $\hat{L}_f$. Suppose $m_i \in \mathbb{C}[z, w]$, of degree $n_i$ in $w$, is the irreducible polynomial for a function $f_i$ (algebraic according to (1.2)) over $K$, $i = 1, 2$. Denote $G(\hat{L}_{f_i}/K)$ by $G_i$, $i = 1, 2$. As in [9.5d], regard $G \overset{\text{def}}{=} G(\hat{L}_{f_1} \cdot \hat{L}_{f_2}/K)$ as a subgroup of $S_{n_1} \times S_{n_2}$. Let $\pi_i : G_1 \times G_2 \to G_i$ be projection on the $i$th factor.

(9.6a) For $H$ a subgroup of $G_1 \times G_2$, let $\ker(\pi_i(H))$ be the kernel of projection of $H$ on $G_i$. For $H \leq G_1 \times G_2$ with $\pi_i(H) = G_i$, $i = 1, 2$, let $A_H$ be $\langle \ker(\pi_1(H)), \ker(\pi_2(H)) \rangle$. Show $H = \{(g_1, g_2) \mid \psi_1(g_1) = \psi_2(g_2)\}$ with $\psi_i : G_i \to G_1 \times G_2/A_H = G_H$: $H$ is the *fiber product* of $\psi_1$ and $\psi_2$.

(9.6b) Consider $L/F$ and $F/M$ algebraic field extensions, with $\psi : F \to \bar{M}$ an embedding of $F$ in the algebraic closure of $M$. Galois theory depends on the *Extension Theorem* [**Isa94**, Thm. 17.30]: There exists an embedding $\psi' : L \to \bar{M}$ extending $\psi$. Explain why this shows $\pi_i(G) = G_i$, $i = 1, 2$.

(9.6c) Let $G_2(1) = G(\hat{L}_{f_2}/L_{f_2})$. Consider $\pi_2^{-1}(G_2(1))$, the biggest subgroup of $G$ projecting to $G_2(1)$. Show $m_1$ is irreducible over $L_{f_2}$ if and only if $\pi_1(\pi_2^{-1}(G_2(1)))$ is transitive.

(9.6d) Let $f^{(1)}, \dots, f^{(n)}$ be the conjugates of $f^{(1)} = f$ with $f$ algebraic over $K$ of degree $n$. Denote $G(\hat{L}_f/K(f^{(i)}))$ by $G(i)$. Show: $K(f^{(1)})$ contains $f^{(i)}$ if and only if $G(1) = G(i)$.

**9.4. Branch of** log **and Puiseux expansions.** Assume $D \subset \mathbb{C}^*$ is a domain.

(9.7a) A classical domain $D$ supporting a branch of log on $D$ is any (subdomain of a) sector: $S_{\theta_1,\theta_2} = \{re^{i\theta} \mid \theta_1 < \theta < \theta_2\}$ under the condition $\theta_2 - \theta_1 \le 2\pi$. Give the branches of log on $S_{\theta_1,\theta_2}$.

(9.7b) If $H_1(z)$ and $H_2(z)$ are two branches of log in $D$ and $H_1(z_0) = H_2(z_0)$ for $z_0 \in D$, show $H_1(z) = H_2(z)$ for $z \in D$.

(9.7c) Prop. 3.2 shows there exists a branch $g_\lambda$ of log along any path in $D$. If for any $\lambda \in \Pi_1(D, z_0)$, $g_\lambda(1) = g_\lambda(0)$, show there is a branch of log on $D$. Hint: Let $G(z)$ be $g_\lambda(b)$ with $\lambda : [a,b] \to D$ so $\lambda(a) = z_0$, $\lambda(b) = z$ and $g_\lambda$ is a branch of log along $\lambda$ with $g_\lambda(a) = w_0$ (fixed). Apply Lem. 4.11.

(9.7d) Show there is a branch of log in a domain $D$ if and only if each closed path in $D$ has winding number 0 about the origin.

(9.7e) Consider $\gamma_1, \gamma_2; [0,1] \to \mathbb{P}^1_z$ with these properties: $\gamma_1(0) = \gamma_2(0) = 0$, $\gamma_1(1) = \gamma_2(1) = \infty$, and for $t \in (0,1)$ $\gamma_1(t) \ne \gamma_2(t)$, and $\gamma_i(t) \in \mathbb{C}^*$, $i = 1, 2$. Let $D$ be any component ([9.17e]: there are two) of $\mathbb{C}^* \setminus \{\gamma_1, \gamma_2\}$. Show there is a branch of log in $D$.

Assume $f(z)$ is analytic near $z_0$ and algebraic according to (1.2): $m(z, f(z)) \equiv 0$ for some nonzero $m \in \mathbb{C}[z, w]$.

(9.8a) Why can we assume $m(z, w)$ is *irreducible* in the ring $\mathbb{C}[z, w]$? How does this same observation show the ring of analytic functions on a (connected) domain $D$ is an integral domain. Hint: $h(z)$ analytic on $D$ and zero at a set with a limit point in $D$ is identically zero [**Ahl79**, p. 127].

(9.8b) Assume $(f, D, z_0)$ is extensible. As in (1.1), why does $h(z) \in \mathcal{A}_f(D)$ also satisfy $m(z, h(z)) \equiv 0$. Conclude: $f(z)$ satisfies (1.1b).

(9.8c) Note in b) for given $D$, the conclusion requires only that $m(z, w)$ has coefficients meromorphic on $D$ (not necessarily on $\mathbb{P}^1_z$).

(9.8d) Use §6.1 to complete showing $f(z)$ satisfies (1.1).

(9.8e) Suppose $f(z)$ is a branch of log on $D$. Show it satisfies neither of the properties (1.1a) or (1.1b). Yet, it does satisfy (1.1c).

(9.8f) If $g : D_1 \to D$ is analytic and $f(g(z)) \equiv z$, show $g(z)$ satisfies (1.2).

(9.8g) Suppose $f \in \mathcal{H}(\mathbb{C})$. Let $\boldsymbol{z} = \{\infty\}$. Then, $f$ satisfies (1.1a) and (1.1b). Suppose $f$ is not a polynomial function. Show it doesn't satisfy (1.1c). Hint: Apply the Caseroti-Weierstrass theorem [**Con78**, p. 109].

Consider how branches of log closely tie to Puiseux expansions. Use notation of §1.3 for the field $\mathcal{L}_{z'}$ around $z'$. For integer $e > 1$ create a copy $\mathcal{P}_{z',e}$ of $\mathcal{L}_{z'}$ by replacing $z - z'$ by a new variable $u_e$. Set $e^{2\pi i/e} = \zeta_e$.

(9.9a) Why is $\mathcal{L}_{z'}$ a field?

(9.9b) Suppose $e \mid e^*$: $t = e^*/e$. Map $\mathcal{P}_{z',e}$ to $\mathcal{P}_{z',e^*}$ by substituting $u_{e^*}^t$ for $u_e$. Show this map extends to a field homomorphism.

(9.9c) Identify $\mathcal{P}_{z',e}$ with its image in $\mathcal{P}_{z',e^*}$. Form the union, the ring of *Puiseux expansions* $\mathcal{P}_{z'}$, over all $e$. Why is it a field?

(9.9d) Show $\mathcal{P}_{z',e}$ is a Galois extension of $\mathcal{L}_{z'}$ with group $\mathbb{Z}/(e)$. Hint: A generator acts by $u_e \mapsto \zeta_e u_e$.

(9.9e) Suppose $z_0 \neq z'$. Let $h(z)$ be a branch of $\log(z - z')$ in a neighborhood $D$ of $z_0$. Show $f_e(z) = e^{h(z)/e}$ is a branch of solutions of $w^e = z - z'$. So $f(z)$ is an algebraic function.

(9.9f) If $e > 1$, show $f_e(z)$ is not the analytic continuation of a function in $\mathcal{L}_{z'}$.

(9.9g) Consider $\varphi : \mathbb{P}^1_w \to \mathbb{P}^1_z$ by $w \mapsto w^e + z'$. Form $g(w) = f_e \circ \varphi$ and show it is an analytic continuation of some function (of $w$) around 0.

We may equally consider Puiseux expansions at $\infty$. Denote the Laurent series around $\infty$ by $\mathcal{L}_\infty$: expressions $(1/z)^n h(1/z)$ with $n$ an integer and $h(z)$ convergent near $z = 0$. As in [9.9], form a copy $\mathcal{P}_{\infty,e}$ of $\mathcal{L}_\infty$ by replacing $1/z$ by $u_e$.

(9.10a) Follow [9.9] to form $\mathcal{P}_\infty$, the analog of $\mathcal{P}_{z'}$. Analytically continue a branch of $z^{1/e}$ *counterclockwise* on a circle around $\infty$. Hint: Apply $z \mapsto 1/z$; it is the same as continuing $z^{-1/e}$ clockwise around the origin.

(9.10b) For $f(w) \in \mathbb{C}[w]$ of degree $n$ with leading coefficient 1, write $f(w) = w^n + a_{n-1}w^{n-1} + \cdots + a_0$, let $m(z,w) = f(w) - z$. Show there is $g(z) \in \mathcal{P}_\infty$ of form $z^{\frac{1}{n}} + \sum_{j=0}^{\infty} b_j z^{-\frac{j}{n}}$ with $f(g(z)) \equiv z$.

(9.10c) Let $L_f$ be $\mathbb{C}(z, g(z))$, $g$ from b). Let $\hat{L}_f/\mathbb{C}(z)$ be the splitting field of $L_f/\mathbb{C}(z)$. Show there is $g \in G(\hat{L}_f/\mathbb{C}(z))$ acting as an $n$-cycle on conjugates of $g(z)$. Hint: Apply $1/z^{\frac{1}{n}} \mapsto \zeta_n 1/z^{\frac{1}{n}}$.

(9.10d) Consider $f, h \in \mathbb{C}[w]$ with $\deg(h) = m$. Apply [9.6c] to $\hat{L}_f$ and $\hat{L}_h$. Show the group of $\hat{L}_f \cdot \hat{L}_h/\mathbb{C}(z)$ contains $\sigma$ of order $nm/\gcd(n,m)$ with restriction of $\sigma$ to $\hat{L}_f$ an $n$-cycle and its restriction to $\hat{L}_h$ an $m$-cycle.

(9.10e) If $(\deg(f), \deg(h)) = 1$, show $f(w) - h(u)$ is irreducible. Hint: Irreducibility is equivalent to $[K(w) : K] = \deg(w)$ with $K = \mathbb{C}(u)$. Use that d) shows $[K'(w) : K'] = \deg(w)$ with $K' = \mathbb{C}((1/u))$.

(9.10f) Suppose in d) (with $(\deg(f), \deg(h)) = 1$), $L_f \cdot L_h$ is pure transcendental (equals $\mathbb{C}(t)$). Show for some choice of $t$ there are polynomials $g(t), k(t)$ of respective degrees $m$ and $n$ with $f(g(t)) = h(k(t))$.

(9.10g) Apply f) to $f(w) = w^2$ and $h(u) = u^3 - au - b$ where $h$ has distinct zeros. Show $L_f \cdot L_h$ is not pure transcendental. Hint: Zeros of $g(t)^2$ are multiple.

Critical points over $z \in \mathbb{C}$ appear in (6.8). Now consider $z = \infty$. With $m \in \mathbb{C}[z,w]$ of degree $n$ and $m = h_0(z)w^n + h_1(z)w^{n-1} + \cdots + h_n(z)$, assume $h_0$ has $z_0$ as multiplicity $t$ zero. When $h_0$ is constant call $m$ *integral* (over $z$).

(9.11a) Write $t = kn + t_0$ with $0 \le t_0 < n$. Show there is an integral polynomial $m_1(z,w) \in \mathbb{C}[z,w]$ satisfying $m_1(z,(z - z_0)^{k+1}w) \equiv (z - z_0)^{n-t_0} m(z,w)$.

(9.11b) Suppose $K$ is a field and $P_1, P_2 \in K[w]$. The *Euclidean algorithm* gives the greatest common divisor of $P_1$ and $P_2$. Write $P_1 = R_0, P_2 = R_1$. Form the remainder $R_2$ of the division $R_1\overline{)R_0}$. Inductively form successive remainders, $R_3, \ldots, R_u$, until the next stage remainder is 0. Do an induction to produce $A(w), B(w) \in K[w]$ with $A(w)P_1(w) + B(w)P_2(w) = R_u(w)$.

(9.11c) Continue b): Use that $\mathbb{C}[z]$ has unique factorization to clear denominators on $A(w)P_1(w) + B(w)P_2(w) = R_u(w)$. Suppose $P_i = P_i(z,w) \in \mathbb{C}[z,w]$, $i = 1, 2$, have no common factor in $w$. Find $A(z,w), B(z,w) \in \mathbb{C}[z,w]$ and $M(z) \in \mathbb{C}[z] \setminus \{0\}$ with $A(z,w)P_1(z,w) + B(z,w)P_2(z,w) = M(z)$.

(9.11d) Result c) applies with any unique factorization domain replacing $\mathbb{C}[z]$. Comment on how it applies to $K = \mathcal{L}_{z'}$.

(9.11e) We outline examples where critical points of $(m, w)$ $(m(z, f(z)) \equiv 0)$ properly contain critical points of $\mathbb{C}(z, f)/\mathbb{C}(z)$. Let $g_{z_0}$ be the conjugacy class of the branch cycle for $m$ at $z_0$. Suppose $e = e_{z_0}$ is the order of $g_{z_0}$. Show, if $m(u^e + z_0, w) = m_1(u, w)$ is irreducible, then $u = 0$ is a branch point of $m_1(u, w)$ but not a branch point of $\mathbb{C}(u, f(u^e + z_0))$.

(9.11f) Apply [9.10e] to give examples of e) by taking $h \in \mathbb{C}[w]$ of degree prime to $e$, so $h(w) - u^e$ is irreducible.

**9.5. Elementary permutations from $\Pi_1(D, z_0)$.** Let $\Delta_{z'}$ be a disk about $z'$ and $\Delta^0_{z'} = \Delta_{z'} \setminus \{z'\}$. Choose $z_0 \in \Delta^0_{z'}$.

(9.12a) Suppose $h(t)$ is a branch of $\log(z - z')$ along $\lambda : [a, b] \to \mathbb{C} - \setminus\{z'\}$. Then, what path is $h(t)$ a branch of log along?

(9.12b) Suppose $f(z) = (z - z')h(z)$ is analytic in $\Delta_{z'}$ with $h(z) \neq 0$ for any point in $\Delta_{z'}$. Show a branch $F(z)$ of $f(z)^{\frac{1}{e}}$ exists at any point in $\Delta^*_{z'}$. Further, show there is an embedding of the field $\mathbb{C}(z, F(z))$ into $\mathcal{P}_{z',e}$.

(9.12c) Let $g_j(z)$ be a branch of $(z - z_j)^{1/e_j}$, $j = 1, \ldots, r$ analytic in a neighborhood of $z_0$. With $f(z) = \prod_{j=1}^r g_j$ and $\lambda : [a, b] \to \mathbb{C}$ a path with winding number $m_j$ around $z_j$, explicitly relate $f(z)$ and $f_\lambda(z)$.

Consider how analytic continuation easily forces us into groups that are not abelian. Follow Thm. 5.6 notation.

(9.13a) Show the conclusion of the case $\infty \in D$ as in §5.4.4 follows.

(9.13b) Recall the semi-direct product $M \times^s H$ of groups of $H$ and $M$ with $\psi : H \to \mathrm{Aut}(M)$ a homomorphism into the automorphisms of $M$. Then, $(m, h) \cdot (m', h') \stackrel{\text{def}}{=} (m \cdot \psi(h)(m), h \cdot h')$ defines multiplication on $M \times H$. Consider $M_0 = \mathbb{Z}^3$, and $H_0 = \mathbb{Z}/3$ where $1 \in H_0$ maps $(m_1, m_2, m_3) \in M_0$ to $(m_2, m_3, m_1)$. Show $M_0 \times^s H_0$ is not abelian.

(9.13c) Let $f(z)$ be a branch of $z^{1/3}$ around $z_0 \neq 0$. For $a \notin \{0, \infty, z_0\}$, consider $h = \frac{1}{f(z)^2(f(z)-a)} \in \mathbb{C}(z, f(z))$. Find $\boldsymbol{z} \subset \mathbb{P}^1_z$ so $(h, U_{\boldsymbol{z}})$ is extensible. Find the image of the permutation representation of $\Pi_1(U_{\boldsymbol{z}}, z_0)$ on $\mathcal{A}_h(U_{\boldsymbol{z}})$.

(9.13d) Let $H(z)$ be a primitive for $h$ (in d)) around $z_0$. Show the image of the permutation representation of $\Pi_1(U_{\boldsymbol{z}}, z_0)$ on $\mathcal{A}_H(U_{\boldsymbol{z}})$ is $M_0 \times^s H_0$ from b). Hint: Substitute $w$ with $w^3 = z$.

**9.6. Fractional transformations and the elementary divisor theorem.** Recall: For any ring $R$ and integer $n \geq 1$, $\mathrm{PGL}_n(R)$ is $\mathrm{GL}_n(R)/\langle R^* I_n \rangle$ and $\mathrm{PSL}_n(R) = \mathrm{SL}_n(R)/\mathrm{SL}_n(R) \cap \langle R^* I_n \rangle$. Several nonabelian subgroups of $\mathrm{PGL}_2(\mathbb{C})$, like $\mathrm{PGL}(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{Z})$ appear often in complex variables. We contrast their different appearances. Let $\mathcal{T}$ be the translations $\{\alpha \in \mathrm{PGL}_2(\mathbb{C}) \mid \alpha(z) = z + a,\ a \in \mathbb{C}\}$. Let $\mathcal{M}$ be the multiplications $\{\alpha \in \mathrm{PGL}_2(\mathbb{C}) \mid \alpha(z) = bz,\ a \in \mathbb{C}^*\}$. Finally, consider $\tau : z \mapsto 1/z$.

(9.14a) Show each $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ has is one of $a'(z - z_1)$, $a'(z - z_1)/(z - z_2) = a'(1 + (z_2 - z_1)/(z - z_2))$, or $a'/(z - z_2)$. Why is $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ a composition of elements from $\mathcal{M}$, $\mathcal{T}$ and $\tau$: $\mathcal{M}$, $\mathcal{T}$ and $\gamma$ generate $\mathrm{PGL}_2(\mathbb{C})$.

(9.14b) Give an $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ mapping $\mathbb{R}$ to the boundary of the unit circle.

(9.14c) Elements of $\mathrm{PGL}_2(\mathbb{C})$ mapping $\mathbb{R} \cup \{\infty\}$ to itself are in $\mathrm{PGL}_2(\mathbb{R})$. What is the subgroup of these mapping the upper half plane $\mathbb{H}$ (Chap. 3 §3.2.2) into itself? Hint: $z \mapsto 1/z$ does not.

(9.14d) Combine with b) to describe elements of $\mathrm{PGL}_2(\mathbb{C})$ mapping $\mathbb{R} \cup \{\infty\}$ to the unit circle. Which map $\mathbb{H}$ to the inside of the circle?

(9.14e) Which $f \in \mathbb{C}(z)$ map the unit circle into the unit circle. Hint: $f \in \mathbb{C}(z)$ mapping $\mathbb{R} \to \mathbb{R}$ has zero and pole set closed under complex conjugation.

Let $R$ be a principal ideal domain, $M$ a finitely generated free $R$ module, and $N$ an $R$ submodule of $M$. The Elementary Divisor Theorem (EDT [**Jac85**, p. 192]): There is a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ of $M$ and elements $a_1, \dots, a_m \in R$ with nonzero elements of $a_1\boldsymbol{v}_1, \dots, a_m\boldsymbol{v}_m$ a basis of $N$. If $a_1, \dots, a_t$ are the nonzero $a_i$ s, then we may choose $a_1, \dots, a_t$ so $a_i | a_{i+1}$, $i = 1, \dots, t$.

(9.15a) Consider an abelian group quotient $A$ of $\mathbb{Z}^n$. Apply EDT to show $A$ is isomorphic to $\oplus_{i=1}^n \mathbb{Z}/(a_i)$ for some integers $a_1, \dots, a_n \in \mathbb{Z}$.

(9.15b) Show in a), if $A$ is a finite group and $a_1 | a_2 | \cdots | a_m$ are positive integers, then the $a_1, \dots, a_n$ are unique.

(9.15c) $\mathrm{SL}_2(\mathbb{Z})$ ($2 \times 2$ matrices over $\mathbb{Z}$ of determinant 1) acts on $M_2 = \mathbb{Z}^2$ taking one basis to another. If $N$ is a subgroup of $M_2$ of index $n$, then $\mathrm{SL}_2(\mathbb{Z})$ maps it in an orbit of index $n$ subgroups. Apply EDT to count $N \le M_2$ of index $n = p^k$ ($p$ a prime). Hint: Start with $N$ for which $M/N$ is cyclic.

(9.15d) Each $N$ from c) defines a subgroup $\Gamma_N$ of $\mathrm{PSL}_2(\mathbb{Z})$: the image of the stabilizer in $\mathrm{SL}_2(\mathbb{Z})$ of $N$. If $n = p$ is a prime, and $U$ is the biggest normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ in $\Gamma_N$, show $\mathrm{PSL}_2(\mathbb{Z})/U = \mathrm{PSL}_2(\mathbb{Z}/p)$.

Let $\Delta$ be the open unit circle. Denote the linear fractional transformations that map $\Delta \to \Delta$ by $\mathrm{PGL}_2(\Delta)$. Form

$$(w_3 - w_1)(w - w_2)/(w_2 - w_1)(w - w_3) = L(w) = L(w_1, w_2, w_3, w)$$

for $w_1, w_2, w_3 \in \mathbb{C}$. This problem follows a treatment from [**Spr57**, §9.2]

(9.16a) Use [9.14]. Show $\mathrm{PGL}_2(\mathbb{C})$ fixes $L(w)$:

$$L(w_1, w_2, w_3, w) = L(\alpha(w_1), \alpha(w_2), \alpha(w_3), \alpha(w)), \text{ for } \alpha \in \mathrm{PGL}_2(\mathbb{C}).$$

(9.16b) Suppose $w_1, \dots, w_4 \in \mathbb{C}$ are on a circle in that order. Show: $L(w_4) > 1$. Conclude: With $w_1, w_2, w_3$ fixed, $w \mapsto L(w)$ maps the interior of the disk bounded counterclockwise by $w_1, w_2, w_3$ to the upper half plane $\mathbb{H}$.

(9.16c) Suppose $w_2, w_3 \in \Delta$. Let $C_{w_2,w_3}$ be the unique circle containing $w_2$ and $w_3$ meeting the unit circle at right angles (at two points). Why is $C_{w_2,w_3}$ unique? Hint: Use $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ taking the unit circle to the real line.

(9.16d) Let $w_1$ be the point on $C_{w_2,w_3} \cap \partial\Delta$ closest to $w_2$. Similarly, $w_4$ is the other point of intersection closest to $w_3$. Define the distance $d(w_2, w_3)$ to be $\frac{1}{2}\log(L(w_1, w_2, w_3, w_4))$. When $w_2 = 0$ and $w_3 = re^{i\theta}$ express this as a function of $r$.

(9.16e) Notice $\beta_{w_2}(w) = \frac{w - w_2}{1 - \bar{w}_2 w}$ is in $\mathrm{PGL}_2(\Delta)$ and it maps $w_2 \mapsto 0$. Use this to express $d(w_2, w_3)$ as $\frac{1}{2}\log\left(\frac{1+|\beta_{w_2}(w_3)|}{1-|\beta_{w_2}(w_3)|}\right)$.

**9.7. Metrics on $\mathbb{P}^1_z$, $\Delta$ and more generally.** The metric topology on $\mathbb{P}^1_z$ identifies it with the sphere around the origin in $\mathbb{R}^3$. Use coordinates $(r, u, v)$: $z_0 \in \mathbb{P}^1_z \mapsto (r_0, u_0, v_0) \in \mathbb{R}^3$. The unit sphere has this analytical description: $\{(r, u, v) \mid r^2 + u^2 + v^2 = 1\} = S$.

(9.17a) From vector calculus, this *implicit* description of $S$ gives a unit normal direction to $S$ at $(r_0, u_0, v_0)$. It is a unit vector $\mathbb{N}_{(r_0,u_0,v_0)}$, (from the origin) in the direction of the gradient of $f(r,u,v) = r^2+u^2+v^2$. Compute two such vectors. Which suits the definition of *outward* normal vector?

(9.17b) Let $\mathbb{T}_{(r_0,u_0,v_0)}$ be points on the plane through $(r_0,u_0,v_0)$ tangent to the sphere. There are two possible definitions of $\mathbb{T}_{(r_0,u_0,v_0)}$. Suppose the range of $(x,y) \mapsto (r(x,y),u(x,y),v(x,y)) = H(x,y)$ is a neighborhood of $(r_0,u_0,v_0)$; $H$ is differentiable in a neighborhood of the origin and $H(0,0) = (r_0,u_0,v_0)$, and $\frac{\partial H}{\partial x}(0,0)$ and $\frac{\partial H}{\partial y}(0,0)$ are linearly independent vectors in $\mathbb{R}^3$. Apply the chain rule to show

$$\mathbb{T}^{\dagger}_{(r_0,u_0,v_0)} \stackrel{\text{def}}{=} \{(r_0,u_0,v_0) + x\frac{\partial H}{\partial x}(0,0) + y\frac{\partial H}{\partial y}(0,0) \mid (x,y) \in \mathbb{R}^2\}$$

is independent of the choice of $H$.

(9.17c) The second definition of $\mathbb{T}_{(r_0,u_0,v_0)}$ is

$$\mathbb{T}^{\dagger\dagger}_{(r_0,u_0,v_0)} \stackrel{\text{def}}{=} \{(r,u,v) \mid ((r,u,v)-(r_0,u_0,v_0)) \cdot \mathbb{N}_{(r_0,u_0,v_0)} = 0\}.$$

Use the expression $f(H(x,y)) \equiv 0$ to show $\mathbb{T}^{\dagger\dagger}_{(r_0,u_0,v_0)} = \mathbb{T}^{\dagger}_{(r_0,u_0,v_0)}$.

(9.17d) Let $\gamma : [a,b] \to S$ be a simple closed path. Suppose $\frac{d\gamma}{dt}$ exists and is nonzero at $t_0 \in [a,b]$. Define the direction to the *left* of $\gamma$ at $t_0$ to be the unit vector $\boldsymbol{u}_1$ for which $\det(\boldsymbol{u}_1 \mid \mathbb{N}_{\gamma(t_0)} \mid \frac{d\gamma}{dt}(t_0))$ is positive.

(9.17e) The complement $S \setminus \gamma$ of a simple closed path has two components $U_1$ and $U_2$: The *Jordan curve Theorem*. For simplicial $\gamma$ this is easy (Chap. 4 [10.3]). Assume $t_0$ as in d). Give meaning to this: $\gamma$ has positive orientation relative to $U_1$. Hint: Interpret $\boldsymbol{u}_1$ being parallel to $U_1$.

We explore $d(w_2,w_3)$ from [9.16], to prove the triangle inequality and to find its differential distance tensor. Use $U(z) = \frac{1+|z|}{1-|z|}$.

(9.18a) Use [9.16e] and find $\beta(w) \in \mathrm{PGL}_2(\Delta)$ with $\beta(w_2) = 0$, $\beta(w_1) = a > 0$ to reduce $d(w_1,w_3) \le d(w_1,w_2) + d(w_2,w_3)$, $w_1,w_2,w_3 \in \Delta$ to showing $U(\frac{z-a}{1-az}) \le U(a) \cdot U(z)$ with $a \in [0,1)$ and $z \in \Delta$.

(9.18b) Write $z = be^{i\theta}$. Show $U(\frac{z-a}{1-az})$ is maximum in $\theta$ when $z$ is real. Conclude the inequality of a). Hint: $U(w)$ is increasing in $|w|$ and $\frac{z-a}{1-az}$ maps the circle of radius $b$ on a circle with real center.

(9.18c) Use [9.16e] to compute the differential distance $S(x,y,dx,dy)$ by considering $w_1 = x + i\,y$ close to $w_2$. Show $S(x,y,dx,dy)$ to be $|\frac{dx+idy}{1-(x^2+y^2)}|$.

(9.18d) Apply $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ mapping the upper half plane $\mathbb{H}$ to $\Delta$. Define a distance on $\mathbb{H}$ by pulling back two points and using the value of the distance on $\Delta$. Show this depend on the particular choice of $\alpha$. Show geodesics on $\mathbb{H}$ are half-circles perpendicular to the real axis.

(9.18e) Use d) to show the metric on $\mathbb{H}$ has differential distance element $\frac{|dx+idy|}{y}$.

Consider [9.18] from the differential distance tensor view:

$$F_\Delta = |\frac{dx+idy}{1-(x^2+y^2)}| = h(x,y)\sqrt{dx^2+dy^2}$$

with $h(x,y) = |1-(x^2+y^2)|^{-1/2}$. Recover this metric's geodesics, circles perpendicular to the boundary of $\Delta$, by applying the *Euler-Lagrange variational principle* from f). Consider $F^2 = \boldsymbol{y} \cdot Q(\boldsymbol{x})(\boldsymbol{y})$ in (2.3a): $Q(\boldsymbol{x})$ is an $n \times n$ positive definite symmetric matrix. Tensor notation replaces $\boldsymbol{y}$ by $dx_1,\dots,dx_n$. Classically, $F^2 = \sum_{1\le i,j\le n} q_{i,j}(\boldsymbol{x})\, dx_i \otimes dx_j$ (with $q_{i,j} = q_{j,i}$) for a 2-tensor.

(9.19a) Suppose $\gamma$ and $\lambda$ are a pair of paths with $\gamma(t_0)=\lambda_2(t_0)=\boldsymbol{x}_0$. Define:

$$F^2(\frac{d\gamma}{dt}(t_0),\frac{d\lambda}{dt}(t_0))=\sum_{i,j}q_{i,j}(\boldsymbol{x}_0)\,\frac{d\gamma_i}{dt}\frac{d\lambda_j}{dt}.$$

Show $\frac{F^2(\frac{d\gamma}{dt}(t_0),\frac{d\lambda}{dt}(t_0))}{F(\frac{d\gamma}{dt}(t_0),\frac{d\gamma}{dt}(t_0))F(\frac{d\lambda}{dt}(t_0),\frac{d\lambda}{dt}(t_0))}$ has absolute value at most 1. So, it has the form $\cos(\theta(\gamma,\lambda))$. Show $\theta(\gamma,\lambda)$, the *angle* between $\gamma$ and $\lambda$ at $\boldsymbol{x}_0$, is independent of their parametrizations.

(9.19b) Apply Ex. 9.1. Show $\sum_{i,j}\int_a^b\sqrt{q_{i,j}(\gamma(t))\,\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}}\,dt$ is independent of how we parametrize the range of $\gamma$ assuming $\gamma:[a,b]\to\mathbb{R}^n$ is one-one.

(9.19c) Let $H:\mathbb{R}^2\to\mathbb{R}^n$ by $(u_1,u_2)\mapsto(h_1(u_1,u_2),\dots,h_n(u_1,u_2))=\boldsymbol{h}(\boldsymbol{u})$ be a one-one (differentiable) map. Define $H^*(F^2)$, pullback of $F^2$ on the range of $H$, as $\sum_{1\le i,j\le n}q_{i,j}(\boldsymbol{h}(\boldsymbol{u}))\,dh_i\otimes dh_j$: $dh_i(\boldsymbol{u})=\frac{\partial h_i}{\partial u_1}\,du_1+\frac{\partial h_i}{\partial u_2}\,du_2$. Suppose $\gamma:[a,b]\to H(\mathbb{R}^2)$. Show $\int_\gamma F=\int_{H^{-1}\circ\gamma}\sqrt{H^*(F^2)}$ from b).

(9.19d) Consider $H^*(F^2)$ in c) when $n=2$. Call $H$ *isothermal coordinates* if $H^*(F^2)$ is $h(u_1,u_2)(du_1\otimes du_1+du_2\otimes du_2)$. Use $n=2$ to factor $F^2$ to

$$(A(\boldsymbol{x})\,dx_1+B(\boldsymbol{x})\,dx_2)\otimes(A(\boldsymbol{x})\,dx_1+\bar{B}(\boldsymbol{x})\,dx_2)$$

($\bar{B}(\boldsymbol{x})$ is the complex conjugation of $B(\boldsymbol{x})$). Suppose $k(\boldsymbol{x})$ (complex valued) gives $k(\boldsymbol{x})(A(\boldsymbol{x})\,dx_1+B(\boldsymbol{x})\,dx_2)$ with the form $du_1+idu_2$. Show $(u_1(\boldsymbol{x}),u_2(\boldsymbol{x}))$ gives isothermal coordinates.

(9.19e) Produce $k(\boldsymbol{x})$ near any $(x_1^0,x_2^0)$, as in c). Outline: Take real and imaginary parts. Rewrite: $du_i=\frac{\partial u_i}{\partial x_1}\,dx_1+\frac{\partial u_i}{\partial x_2}\,dx_2$. Finding $k$ comes to this. Suppose $M_1(\boldsymbol{x}),M_2(\boldsymbol{x})$ are real valued and differentiable. Then, there is $k_1(\boldsymbol{x})$ and $M^*(\boldsymbol{x})$ with $k_1(M_1(\boldsymbol{x})\,dx_1+M_2(\boldsymbol{x})\,dx_2)$ of form $dM^*(\boldsymbol{x})$. Then, $M_1(\boldsymbol{x})\,dx_1+M_2(\boldsymbol{x})\,dx_2=0$ defines $\{(x_1,x_2\mid M^*(x_1,x_2)=0\}$, an implicit surface, near $(x_1^0,x_2^0)$. Find $k_1$.

(9.19f) We assume the situation of [9.18]. Let $\gamma=\gamma_1+i\,\gamma_2:[0,1]\to\Delta$ be a path from $z_0$ to $z_0'$. Minimize $\int_\gamma F_\Delta=\int_0^1 S(\gamma_1(t),\gamma_2(t),\frac{d\gamma_1}{dt},\frac{d\gamma_2}{dt})\,dt$ over all such $\gamma$. The Euler-Lagrange variation produces two partial differential equations, one for $x$, $\frac{d}{dt}\frac{\partial S}{\partial\dot{x}}=\frac{\partial S}{\partial x}$, and a similar one for $y$. Solve to show $F_\Delta$ geodesics are circles perpendicular to the boundary of $\Delta$.