What Gauss told Riemann about Abel's Theorem

Riemann sphere: $\mathbb{P}^1_z = \mathbb{C}_z \cup \{\infty\}$. For a finite set $\boldsymbol{z} = z_1, \dots, z_r$ on $\mathbb{P}^1_z$, denote complement by $U_{\boldsymbol{z}}$. We use the fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$ and related groups.

Abel's work on elliptic integrals motivated approaches to algebraic equations not easily connected to him through just the published papers. A student runs quickly into many difficulties that have their seeds from the 1850s.

- How to divide the vast finite group area between those that are nilpotent and those that are simple?

- How to meld existence results from complex variables with manipulative needs of algebraic equations?

Algebra manipulations

Problem 1. Relate general substitutions in integrals of a Riemann surface and algebraic manipulations from functions on that surface.

Abel's example allowed the following operations as acceptable.

Iterating compositions of rational functions in z. Selecting from known *elementary functions*. The conceptual addition of functional inverse.

Only two elementary analytic functions: z and $\log(z)$. Consider an integral locally as an antiderivatives in z.

Example 2. $\cos(z) = (e^{iz} + e^{-iz})/2$

$$z^{1/k} = e^{\log(z)/k}$$

Abel's Case

A branch of primitive ($c, d \in \mathbb{C}$):

$$f(z) = f_{c,d}(z) = \text{Int}(\frac{dz}{(z^3 + cz + d)^{\frac{1}{2}}})_\gamma \qquad (1)$$

For $g(z)$ a branch of $\log(z)$ near z_0, $e^{g(z)} = z$, and $e^{g(f(z))/k}$ is a branch of $f(z)^{1/k}$ on *any disk* (or on any simply connnected set) avoiding the zeros and poles of f. There is a branch of $f(z)^{1/k}$ along *any path* in $U_{\boldsymbol{z}}$ ($\boldsymbol{z}$ containing zeros and poles of f).

Apply chain rule to $f(g(u)) = u$:

- $\frac{dg(u)}{du} = (g(u)^3 + cg(u) + d)^{\frac{1}{2}}$, a parametrization of the algebraic curve from the equation $w^2 = z^3 + cz + d$.

Analytic continuation

Let $\boldsymbol{z} = \boldsymbol{z}_{c,d} = \{z_1, z_2, z_3, \infty\}$, the three (distinct) zeros of $z^3 + cz + d$ and ∞: V^0 the set of such $(c,d) \in \mathbb{C}^2$; $X^0_{c,d}$ the points (z,w) on () over $U_{\boldsymbol{z}}$. It has a unique complex manifold compactification $X_{c,d}$. Analytically continue $f_{c,d}\,dz$ along any path in $X_{c,d}$ (holomorphic integral).

Collection of analytic functions around z_0:

$$\mathcal{A}_f(U_{\boldsymbol{z}}, z_0) = \{\mathrm{Int}(h_{c,d}(z)\,dz)_\gamma\}_{\gamma\in\pi_1(U_{\boldsymbol{z}},z_0)}.$$

The related subset is $\mathcal{A}_f(X^0_{c,d}, x_0)$: Analytic continuations along the image of closed paths from $X^0_{c,d}$. Then, $u \in \mathbb{C}_u \mapsto (g_{c,d}(u), \frac{dg_{c,d}}{du}(u))$ is one-one up to translation by elements of

$$L_{c,d} = \{s_\gamma = \int_\gamma h_{c,d}(z)\,dz, \gamma \in \pi_1(X^0_{c,d}, x_0)\}.$$

Two paths $\gamma_1, \gamma_2 \in \pi_1(U_{\boldsymbol{z}}, z_0)$ lift to generators of $H_1(X_{c,d}, \mathbb{Z})$: $f_{\gamma_i} = s_i$, $i = 1, 2$, independent over $\mathbb{R}$, generate $L_{c,d}$.

Substitutions versus field operations

A substitution $w(z)$ in z is a composition:

$z \mapsto f_{a,b}(w(z))$. Rewrite $f \circ g(u) = u$ as

$$f \circ w \circ w^{-1} \circ g(u) = u.$$

Substitution $f \circ w(z)$ equivalent to composition $w^{-1} \circ g(u)$.

Important case: w^{-1} is a rational function. Get a field $M_{c,d} = \mathbb{C}(g_{c,d}(u), \frac{dg_{c,d}(u)}{du})$: Up to translation, $g_{c,d}$ has a unique pole of order 2 at $u = 0$ (no residue, so is *even*).

Problem 3. For what pairs (c, d) and (c', d') is $g_{c',d'}$ an element of $M_{c,d}$?

For each (c, d): $g_{c,d}(u)$ is to e^u as $f_{c,d}(z)$ is to a branch of $\log(z)$.

Functions on a surface

Describing elements of $M_{c,d}$ is the same as describing analytic maps

$$\varphi : \mathbb{C}/L_{c,d} \to \mathbb{P}^1_w :$$

φ has as many zeros $D_0(\varphi) = \{a_1, \dots, a_n\}$ (with multiplicity) as it has poles $D_\infty(\varphi) = \{b_1, \dots, b_n\}$.

Describing when $M_{c',d'} \subset Mc, d$ equivalent to describing analytic maps

$$\psi_{(c,d),(c',d')} : \mathbb{C}/L_{c,d} \to \mathbb{C}/L_{c',d'}.$$

Abel's equivalence on $\psi_{(c,d),(c',d')}$ for prime degree p: Relates the j-*invariant* $j(\boldsymbol{z}_{c,d})$ of $\boldsymbol{z}_{c,d}$ to that of $\boldsymbol{z}_{c',d'}$; the *modular curve* $X_0(p)$.

Let $\varphi : X \to \mathbb{P}^1_z$ be a meromorphic function on a compact Riemann surface. Up to constant $D_0(\varphi)$ and $D_\infty(\varphi)$ determine φ.

Functions from zeros and poles

Suppose $\varphi : \mathbb{P}^1_u \to \mathbb{P}^1_z$. Modulo $\mathbb{C}^*$,

$$\varphi(u) = \frac{\prod_{i=1}^n (u - a_i)}{\prod_{i=1}^n (u - b_i)}.$$

Ingredients: u is an odd function (one zero, multiplicity one) whose translations craft $\varphi(u)$ having the right zeros and poles.

For $\varphi : \mathbb{C}/L_{c,d} = X_{c,d} \to \mathbb{P}^1_w$: $D^0 = \varphi^{-1}(0)$ and $D^\infty = \varphi^{-1}(\infty)$; branch points $\boldsymbol{w} = \{w_1, \dots, w_{r'}\}$. Tricky notation: Denote subset of $X_{c,d}$ over $U_{\boldsymbol{w}}$ by $X^{w,0}_{c,d}$. Take $\gamma \in \pi_1(U_{\boldsymbol{w}}, 0, \infty)$: γ_i be the unique left to $X^{w,0}_{c,d}$ of γ starting at a_i (it will end in D^∞). Works even if D^0 has multiplicity.

Proposition 4 (Abel's necessary condition). $\sum_{i=1}^n \int_{\gamma_i} h_{c,d}\, dz = 0$: *$\varphi$ requires paths $\{\gamma'_i\}_{i=1}^\infty$ on $X_{c,d}$ with initial points D^0, end points D^∞ and*

$$\sum_{i=1}^n \int_{\gamma'_i} h_{c,d}\, dz = 0.$$

Imitating the genus 0 case

Find odd function, zero of multiplicity one, whose translates can craft all functions. No such function on $\mathbb{C}/L_{c,d}$. Notation:

$$u \in \mathbb{C}_u \mapsto [u] \in \mathbb{C}/L_{c,d}.$$

Proposition 5. *The odd homolomorphic function $\sigma_{c,d}(u)$ with a unique (multiplicity one zero; modulo $L_{c,d}$) at $u = 0$, with*

$$\sigma_{c,d}(u+s) = e^{a_s u}\sigma_{c,d}(u) = \sigma(u), a_s \in \mathbb{C}^*, s \in L_{c,d}.$$

Derivative of $\frac{d(\log(\sigma(u))}{du}$ is $g_{a,b}(u)$:

$$\varphi(u) = \frac{\prod_{i=1}^n \sigma(u - a_i)}{\prod_{i=1}^n \sigma(u - b_i)}$$

is invariant by translations in $L_{c,d}$ if and only if $\sum_{i=1}^n [a_i] - \sum_{i=1}^n [b_i]$ is 0 in $\mathbb{C}/L_{c,d}$.

Denote $L_{a,b} \setminus \{0\}$ by $L^*_{a,b}$:

$$\sigma_{c,d}(u) = u \prod_{s \in L^*_{a,b}} \left(1 - \frac{u}{s}\right) e^{u/s + \frac{1}{2}(u/s)^2}. \qquad (2)$$

Puzzles from Abel's Theorem

- General $\varphi : \mathbb{C}/L_{c,d} \to \mathbb{P}^1_w$ has no branch of log description (Galois). How to picture?

- How to relate beginning/end points of allowable paths lifts from D^0 to D^∞ in Prop. 4?

All functions on 1-dim. complex torii from one:

$$\sigma : ((c,d),u) \in V^0 \times \mathbb{C}_u \mapsto \sigma_{c,d}(u), \qquad (3)$$

Description depends on function's zeros and poles. Perfect for forming abelian covers of a complex torus similar to using branches of log. Knowing only the generators of a function field like $M_{c,d}$ ineffective for properties of elements in the field [# 6].

Compact surfaces from cuts

Product-one condition: r elements $\boldsymbol{g} = (g_1, \dots, g_r)$ in S_n with $g_1 \cdots g_r = 1$. Paths: r paths $\gamma_1, \dots, \gamma_r : [0,1] \to \mathbb{P}^1_z$:

Nonintersecting: Each has beginning point at z_0; the only common point to the paths. Clockwise order: Order in leaving z_0 is the order of their numbering. End respectively at $z_1, \dots, z_r$.

Let $\mathbb{P}^1_i$, $i = 1, \dots, r$, be copies of $\mathbb{P}^1_z$: remove points labeled $z_0, z_1, \dots, z_r$ to get $\mathbb{P}_j$. Form a pre-manifold $\mathbb{P}^{\pm}_j$ (not Hausdorff) from $\mathbb{P}_j$ by replacing each point z along any one of the γ_is by two points: z^+ and z^-. Let $D_{i,z}$ be a disk around z. Write this as $D^+_{i,z}$ (resp. $D^-_{i,z}$) , all points on and to the left (resp. right) of γ_i.

The construction

Proposition 6. *Form a manifold from an equivalence on* $\cup_{i=1}^{n}\mathbb{P}_{j}^{\pm}$ *based on using the* r*-tuple* $\boldsymbol{g}$*. Running over all* n *and product-one* r*-tuples* $\boldsymbol{g}$ *(even with the cuts fixed), forming the compactification gives all possible compact Riemann surfaces mapping to* $\mathbb{P}_{z}^{1}$ *ramified over* $\boldsymbol{z}$*.*

Proof. If g_i maps k to l, then identify $z^- \in \mathbb{P}_{k}^{\pm}$ in the g_i cut with $z^+ \in \mathbb{P}_{l}^{\pm}$. In the resulting set, put on a topology where the neighborhood of such a z^- is $D_{l,z}^{+} \cup D_{k,z}^{-}$ identified along the part of γ_i running through z. $\square$

Where did the cuts come from?: [# 6]

Group $G(\boldsymbol{g})$ generated by product-one $\boldsymbol{g}$ is the (monodromy) group: Galois closure of the cut map $\varphi : X \to \mathbb{P}^1_z$. Trivial to form covers with just about any group. New issues: No clue present about any other meromorphic function on X other than φ. Clues universal covering space is analytically a disk.

Holomorphic differential ω on X from a cut construction $\varphi : X \to \mathbb{P}^1_z$ has description around $z_0 \in U_{\boldsymbol{z}}$ as $h(z)\,dz$. Integrate along any $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$ or in $\pi_1(X, x_0)$. Generalize Abel by taking a basis $\mathcal{B} = (\omega_1, \dots, \omega_g)$ of holomorphic differentials and integrals along paths on X. Define L_X to be $\{\int_\gamma \mathcal{B} \mid \gamma \in H_1(X, \mathbb{Z})\}$.

Riemann's generalization: [# 6]

Proposition 7. *If* $D^0 = \varphi^{-1}(0)$ *and* $D^\infty = \varphi^{-1}(\infty)$*, for an* n*-tuple of lifts* $\gamma_1, \dots, \gamma_n$ *of* $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$*,*

$$(\sum_{i=1}^{n} \int_{\gamma_i} \omega_1, \dots, \sum_{i=1}^{n} \int_{\gamma_i} \omega_g) = \int_{\boldsymbol{\gamma}} \mathcal{B} = \mathbf{0}. \qquad (4)$$

First part: $\gamma_1, \dots, \gamma_n$ *starting at* D^0*/ending at* D^∞ *is sufficient for existence of* φ*.*

We want an odd function θ on $\mathbb{C}^g$ whose translates allow us to craft any $\varphi : X \to \mathbb{P}^1_w$:

$$\varphi(x) = \prod_{i=1}^{n} \theta(\int_{a_i}^{x} \mathcal{B}) / \prod_{i=1}^{n} \theta(\int_{b_i}^{x} \mathcal{B}). \qquad (5)$$

In θ you see g coordinates; the ith entry is $\int_{a_i}^{x} \omega_i$. Each holds an integral over one basis element from $\mathcal{B}$. It is totally well-defined if the integrations are on a space that is to X as $\mathbb{C}$ was to $\mathbb{C}/L_{c,d}$ in Abel's problem.

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