CHAPTER 1

SCOPE OF THE EXISTENCE THEOREM

This chapter is an overview of this book's three main topics.

- How Riemann's Existence Theorem describes moduli spaces of Riemann surface covers of the Riemann sphere.
- How finite group theory puts practical — for applications — structures into collections of such covers.
- How each finite group generates its own nilpotent theory of fundamental groups, forming systems of moduli spaces with $G_{\mathbb{Q}}$ actions.

1. Context for the book

We note the beautiful 1st year graduate text of L. Ahlfors [**Ahl79**] on the algebra and geometry of complex variables. This author could do no better than use it as his underpinning. Still, that book leaves the full scope of monodromy a mystery, it prepares little on coordinates describing Riemann surfaces, and none on families of Riemann surfaces.

E. Hille wrote a function theoretic encyclopedia [**Hil62**]. As a graduate student, I enjoyed how relevant were its historical comments to sophisticed mathematics in the 1800s. For example, mathematicians seeking immortality (in private, of course) might ponder its many serious references to H. Schwarz's work [**Sch1890**]. Few present day complex variable enthusiasts know the coherence or context of that work. Two authors, G. Springer [**Spr57**] and R.C. Gunning ([**Gun66**] and [**Gun67**]), did great service bringing Riemann surfaces to graduate students by the 1960s. For the former, that was H. Weyl's uniformization approach (as in his projection lemma). For the latter it was the Cartan-Serre vector bundle view of the algebro-differential geometry that works on Riemann surfaces.

E. Neuenschwanden's perspective answers many questions on what took so long for Riemann surfaces to make their mark [**Ne81**]. He documents contention between Weierstrass's algebraic and Riemann's harmonic function approaches. This is relevant to the relation between Riemann and Abel and Galois. For Weierstrass admits the influence of Abel on his work. Still, one can't see it directly on Riemann despite serious documentation of his intellectual activities, including the direct influence on him of Gauss. Further, [**Ne81**] leaves unanswered other questions about the assimilation of mathematics.

These modern works have little group theory; not even including the original approaches of Abel, Galois and Riemann. Few presented group theory so dramatically as did H. Weyl. Yet, even Weyl (on quantum mechanics) met resilient resistance to group theory. My convictions are here; I advocate using the *power* of group theory. Showing how finite and profinite group theory can handle intricate monodromy and moduli, and apply practically to algebra and complex variables, is my goal. Still, there's a fence to walk. We can't afford to let group theory overwhelm us. Galois was first to note group theory's power. Also, he wrote on its potential to dominate the subject techically.

The introductions of two books, [**MM95**] and [**Vö96**], show they closely connect through group theory with this book. [**Fri94**] and [**Fri95c**] specifically discuss connnections of our topics to [**Se92**]. These three books concentrate on how Riemann's Existence Theorem applies to the Inverse Galois Problem. By contrast, classical topics appear here more often than in the first two. Also, this author uses standard formulations of the Inverse Galois Problemmuch less. Yet, the reader can find here a leisurely track through Riemann surface theory guided by problems requiring little preparation for their statements, a virtue of the Inverse Galois Problem. My choices often have a long literature *before* the connection to Riemann surfaces appeared. By occasional referring to topics from these three books, starting in Chap. 4, I have added efficiency to this liesurely pace.

By being leisurely, we (I and the reader) may also consider the struggle of many generations with whether punctured Riemann surfaces and their moduli variation belong to function theory or to algebra. If we aim to please and appeal to Abel, Galois and Riemann on this score, we realize — in ultra-rational moments — that is an impossibility. Further, since that is a triumvirate of geniuses, such an appeal detracts from showing why even *they* struggled, and despite the time that has passed we too, with the whole topic. There is a serious question for mathematics. When does *mathematics* (versus Riemann) have a firm grasp on a significant subject?

Is it when an elite institution husbands a handful of caretakers of an industry of supporting research? Is it when myriad papers allude to consequent deep theories, even if they don't directly involve the roiling concepts? Is it when some text has nailed the subject completely to a prestigious group's satisfaction? Is it when a blithely confident prestigious group claim the subject's foundations are firm and available to any sincere seeker? Is it when the subject successfully supports several independent and competing schools derived from its basic problems?

We don't know what would convince most research mathematicians of the security of a subject. The author has a point in writing this book; though he cannot easily pick one affirmative viewpoint for the maturity of this book's subject. Its techniques quickly worked to reveal the nature of long standing problems in his hands. On that basis a fair observer might support that the techniques work. Still, there are geniuses beyond Abel, Galois and Riemann who have their viewpoints. Examplars of thinking with great scope and imagination certainly include [**An02**], [**De89**] [**Moc96**]. We end the book at the wealth of analytic questions and applications raised by Modular Towers, a little before the influence of these writers on the author. So, only a shadow of their influence is here.

All, however, support connecting profinite groups to function theory. That leads to final, painful consideration. Will we, and the world outside mathematics, ever be able to tolerate the inundation that often overwhelms us from the connections bridged by mathematical language?

2. A quick summary

A fuller overview follows this brief summary.

Compact Riemann surfaces as branched covers of a sphere appear in 1st year graduate courses as *elementary* discussions of *multi-valued functions*. We expand this brief treatment in Chap. 2, to carefully treats analytic continuation motivate the geometry behind it. It introduces the Existence Theorem sufficiently to get lessons from the theory of *abelian* algebraic covers of the punctured Riemann sphere (§3.2). It starts with two different definitions of algebraic functions, one from algebraic equations another phrasing from analytic continuation. An imprecise version of Riemann's Existence Theorem is that these describe the same functions. This is an elementary investigation, requiring only graduate complex variables.

In this book Riemann's Existence Theorem means the precise statement from Chap. 4. That *really* organizes all algebraic functions (of z). Chap. 4 fully develops Riemann's Existence Theorem. It emphasizes data determining a branched cover of the sphere up to equivalence. Abel and Galois started a tradition. Our version: Translate complex analytic and arithmetic geometry problems into group theory through application of forms of Riemann's Existence Theorem.

Advanced texts often append another statement. It is that *any* compact Riemann surface (Chap. 3) has an analytic (nonconstant) map to $\mathbb{P}^1_z \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$, the Riemann sphere. Springer's book [**Spr57**] dedicates much space to proving this last statement. We rarely use it; our basic data already includes such a function and (given the Riemann-Roch Theorem) includes Springer's goals (see below).

Suppose, however, $\varphi : X \to \mathbb{P}^1_z$ is such an analytic map. Let z_0 be a particular z value, and consider X_{z_0}, the fiber of φ over z_0. Then, finding algebraic equations for X, necessary for most applications, depends on producing another function $\varphi' : X \to \mathbb{P}^1_w$ that *separates* points of X_{z_0}. The explicit production of such a φ' is a consequence of *uniformization* of X by the appropriate simply-connected domain (disk, plane or sphere). As uniformization plays an important role in advanced applications, say, related to θ functions, we often raise elementary aspects of it.

The globally defined functions, φ and φ' have an algebraic relation $F(\varphi, \varphi') \equiv 0$ between them: $F \in \mathbb{C}[z, w]$. Let $L_{\varphi'} \subset \mathbb{C}$ be the field generated by the ratios of all coefficients of F. Let K be a field containing $L_{\varphi'}$. A frequent application of this relation F is to give meaning to the expression *a K point on X*. From F and $z_0 \in K$, there is an equivalence class of permutation representations of the absolute Galois G_K of K. This comes from its action on points of X over z_0. Refined applications of covers analyze the dependence of this statement on the choice of φ'.

Chap. 4 shows the following. Let $L_{\boldsymbol{z}}$ be the field generated by the symmetric functions in $\boldsymbol{z}$ (with ∞ removed if it appears).

(2.1a) There is a choice of φ' giving K algebraic over $L_{\boldsymbol{z}}$.

(2.1b) The complete set of minimal fields $\mathcal{L}_\varphi$ appearing as $L_{\varphi'}$ in the algebraic closure of $L_{\boldsymbol{z}}$ is an intrinsic (moduli) invariant of $\varphi : X \to \mathbb{P}^1_z$.

(2.1c) Sometimes (the Existence Theorem shows) $\mathcal{L}_\varphi$ consists of a unique field.

When $\boldsymbol{z}$ consists of algebraically independent values, the analysis of $\mathcal{L}_\varphi$ includes the moduli (deformation) theory of a cover. That is Part II of the book. Comparing this case with the case $L_{\boldsymbol{z}} = \mathbb{Q}$ (or some other explicit algebraic number field) is tantamount to approaches to the Inverse Galois Problem.

We assume students with one semester each of a graduate algebra course and a graduate complex variables course. Few students master Galois theory from their algebra courses. Thus, we give an analytic continuation approach to showing the field of convergent Puiseux expansions around a point is algebraically closed. This supports many elementary subtopics that could otherwise be baffling. For example, Riemann's Existence Theorem uses an infinite number of incompatible algebraically closed fields containing the field $\mathbb{C}(z)$. Let $\boldsymbol{z} = (z_1, \dots, z_r)$ be a fixed set of points on the sphere. Denote the complement of $\boldsymbol{z}$ on the sphere by $U_{\boldsymbol{z}}$.

Riemann's Existence Theorem is about algebraic functions *extensible* on $U_{\boldsymbol{z}}$. These are functions with analytic continuations along any path (from an explicit base point) avoiding $\boldsymbol{z}$. At each point z_0, not in $\boldsymbol{z}$, these algebraic extensible functions embed in the algebraically closed field of Puiseux expansions in z_0. Isomorphisms between their different embeddings is coded in the *fundamental groupoid*.

Chap. 2 describes *abelian* functions of z through analytic continuing branches of the log function. It demonstrates many basic definitions and some advanced concepts. Among these is that of a *group* attached to monodromy action. For books motivated by θ functions and their applications, this book is unusually persistent in emphasizing finite group theory.

Chap. 3 has basics on fundamental groups and permutation representations. Though our definitions and first examples of *manifolds* are traditional, our aim is to illustrate practical use of deformations of Riemann surfaces. We concentrate on very explicit manifolds. Chap. 5 produces highly structured *moduli* spaces parametrizing equivalence classes of Riemann surfaces.

Consider the notation around (2.1). For $\boldsymbol{z}$ fixed, and $K = L_{\boldsymbol{z}}$, if $z_0 \in K$, there is an action of G_K on the profinite completion of the fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$ (Chap. 4). Moduli parameters appear with the following question.

PROBLEM 2.1. What happens with covers of $U_{\boldsymbol{z}}$ as $\boldsymbol{z}$ varies?

First appearances give the following impression.

(2.2a) The fundamental group of $U_{\boldsymbol{z}}$ doesn't change with $\boldsymbol{z}$.
(2.2b) $G_{\mathbb{Q}}$ action changes drastically if you can even consider it varying with $\boldsymbol{z}$.

Both (2.2a) and (2.2b) are wrong.

Suppose we try to write equations (with coefficients in $\boldsymbol{z}$) for the deformations of an algebraic function $f = f_{\boldsymbol{z}}$ extensible on $U_{\boldsymbol{z}}$ (Chap. 2). Locally in $\boldsymbol{z}$ this is possible. Going, however, around various closed paths in the space for $\boldsymbol{z}$, $f_{\boldsymbol{z}}$ might return to a different extension field of $\mathbb{C}(z)$. Riemann's Existence Theorem tells precisely how to calculate which paths return to the original function field (§5.4.1). *Hurwitz monodromy action* is the phrase for our most important calculations. This produces coordinates for coefficients relating $f_{\boldsymbol{z}}$ algebraically to $(z, \boldsymbol{z})$ (Chap. 5).

Choosing generators and a base point are what allow covering applications of the fundamental group. A response to (2.2a) is that this extra data produces a refined moduli space setup. This motivates a Lie algebra approach to (2.2b) putting the two parts of (2.2) under a common framework. We use ideas from renown papers of Y. Ihara and J.P-Serre and moduli space that give the proper context for the Inverse Galois Problem.

Abelian covers of $U_{\boldsymbol{z}}$ for any $\boldsymbol{z}$ comes from branches of log (Chap. 2). Ihara studied (parts of the) arithmetic of nilpotent covers of $U_{\boldsymbol{z}}$ when $r = 3$ [**Ihar86**]. Nilpotent theory appears in applications to the Inverse Galois Problem. Here it starts from nonsplit nilpotent extensions extending data about covers with any given finite (often simple) group G. For p a prime dividing the order of G, a univeral totally nonsplit extension ${}_p\tilde{G}$ of G produces sequences of refined moduli spaces (§8.3).

[**Fri78**] and [**Ihar86**] had common elements: use of the theory of *complex multiplication*, and an arithmetic philosophy using the *braid group*. The former used analytic geometry and finite group theory. There is now a natural way to join this to the profinite and function theory approach of the latter. This means joining *Modular Towers* to the *Grothendieck-Teichmüller* technology. The tools include extension of Deligne's tangential base points [**De89**] with insight from Riemann's θ functions.

2.1. Meaning of the word, elementary in the title. The first four chapters are truly elementary. Still, understanding later chapters on moduli requires mastery of the first four. Few readers will easily skip those first four chapters. The approach is elementary because it allows a newcomer into the area through examples and techniques using finite group theory. Traditionally, for example, with modular curves, one must have serious training in complex analysis. The action happens with automorphic functions on the upper half plane.

Here we often use *uniformization from below*, replacing the upper half plane and representations of $\mathrm{SL}_2(\mathbb{R})$ with the Riemann sphere $\mathbb{P}^1_z$ and finite group theory. Then, modular curves and their associated towers are an example of the moduli of dihedral group covers. The same technique works by replacing the dihedral group by any finite group. This opens up applications beyond the traditional modular curve approach.

This modular curve generalization uses a construction attached to each prime p dividing the order of a finite group G: The *universal p-Frattini cover* ${}_p\tilde{G}$ of G. This especially considers those primes p for which G is *p-perfect* (it has no cyclic quotient of order p).

Add to this a collection $\mathbf{C}$ of conjugacy classes from G whose elements have order prime to p. Then, $(G, p, \mathbf{C})$ produces a sequence of *moduli spaces of curves*. Example: G is the dihedral D_p of order $2p$ (p an odd prime) and $\mathbf{C}$ consists of four repetitions of the conjugacy class of involutions. Then, the sequence of moduli spaces is the classical modular curve series $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$: Quotients of the upper half-plane by well-known subgroups denoted $\Gamma_1(p^{k+1})$ of $\mathrm{PSL}_2(\mathbb{Z})$. The kth *level* of the sequence in this case is $Y_1(p^{k+1}$. Introducing the generalizing sequences of spaces, *Modular Towers*, is the book's main advanced topic.

When G is an alternating group A_n ($n \geq 4$) and $p = 2$, the Modular Tower sequences generalize applications of θ functions in two ways. First: In the dihedral case the levels are connected. Here, sometimes there are several — mysterious — components. Ihe famous mod 2 (*half-canonical class*) invariant from θ functions often explains these components. Modular representation theory of characteristic quotients of ${}_p\tilde{G}$ give these invariants (§10.2). This extends Schur's theory of *universal central extensions*.

Second: Function theory, as in *cusp forms* and *Eisenstein series* from modular curves appears here. Since the levels are moduli spaces of curves, we know most about those functions by relating them to θ functions of curves representing points in the moduli spaces. Such varying θ functions produces θ-null automorphic forms. Our main examples illustrate this when the moduli spaces are quotients of the upper half plane, giving covers of the classical j-line. This exactly corresponds (for any (G, p)) to the case $\mathbf{C}$ consists of four conjugacy classes in G.

Modular curves, though a guide, are a small portion of the noncongruence quotients of the upper half plane with a tower structure related to a prime p. New applications reveal the value of a Riemann's Existence Theorem approach. Function wise it generalizes both the *braid group* approach to the Inverse Galois Problem and the *Tate module*.

Early chapters develop detailed motivation for using classical functions. The deeper function theory, however, appears in outline (with exposition on applications related to the literature). Developing this completely is a topic for a later book.

3. Early historical motivation

A renown problem from the early 19th century was to express *in radicals* solutions x of the general nth degree polynomial equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{3.1}$$

with f of degree n in x. The goal specifically asks for solutions x using *known* functions of the coefficients $a_1, \dots, a_n$. The explicitly known functions of the time were what we call radicals.

Traditional books tackle this using Galois theory with pure algebra. They reproduce Galois' Theorem characterizing when a field extension L/K is a subfield of a chain of radical field extensions of K. This happens if the Galois closure of L/K has solvable group.

It is a pretty story. Still, Galois' Theorem is not a common object of mathematical pilgrimage (even if Galois is). This treatment hides ingredients that still seize the imagination of modern mathematicians, as it possessed Abel, Galois and Riemann. Abel and Galois recognized *group theory* for showing, with a_i s and $n \ge 4$, the field of radical sequences in the a_i s do not contain the solutions. Still, these books lack problems motivating present research. Further, the subject's character falls outside the neatly compartmental introduction of rings, groups, modules and elementary classification results of the rest of 1st year graduate algebra. These historically come long after it, leaving the impression Galois theory is both mildly exotic and slightly moribund.

3.1. Consider functions of one variable. To be more explicit turn to complex variables, as did Abel. Instead of $a_1, \dots, a_n$ being general, specialize to functions $a_1(z), \dots, a_n(z)$ of one complex variable z. Assume $a_1(z), \dots, a_n(z)$ are in the field $\mathbb{C}(z)$: rational functions of z with complex coefficients. It is convenient to replace x by a variable w taking complex values. Refer in this specialized form to the equation $f(a_1(z), \dots, a_n(z), w) = m(z, w) = 0$.

The left side of (3.1) does not factor into lower degree polynomials over the field $a_1, \dots, a_n$ generate. The specialized expression $m(z, w) = 0$ may factor over $\mathbb{C}(z)$. To simplify, assume m is an irreducible polynomial in w over $\mathbb{C}(z)$. *Analytic continuation* displays the n solutions in w as n manifestations of one solution. The manifestations cohere through a group. Here is how it arises.

3.2. Motivating integrals. Critical values $\boldsymbol{z} = z_1, \dots, z_r$ of m are places z' where $m(z', w)$ has repeated roots. Fix $z_0 = z$ not equal to a critical value of m. Then the zeros w of $m(z, w)$ have expressions $w_1(z; z_0), \dots, w_n(z; z_0)$, meromorphic functions in z around z_0. This holds for any z_0 outside $\boldsymbol{z}$. So these algebraic functions are *extensible* on $\mathbb{C} \cup \{\infty\} \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}$ (Chap. 2). The *group* of $m(z, w)$

(relative to z) is all permutations of the w_i s from continuation around closed paths in $U_{\boldsymbol{z}}$ based at z_0. Call the w_i s *abelian* if this group is abelian.

This study of zeros jibed nicely with another problem of Abel's day: Analyze elementary antiderivatives, like the watershed example $\int \frac{dx}{\sqrt{x^3+ax+b}}$. Specifically, what is the dependence of these antiderivatives on the parameters a and b?

Here $m(z,w) = w^2 - (z^3 + az + b)$. Write $G(z) = \frac{1}{\sqrt{z^3+az+b}}$ acknowledging (Chap. 2) that plugging in values of z near z_0 requires choosing one of two functions $G(z)$ analytic in a disc about z_0 with $G(z)^{-2} = z^3 + az + b$. Consider $F(z)$, an antiderivative of $G(z)$, locally. An integral gives $F(z)$. So, it has analytic continuations around $U_{\boldsymbol{z}}$. These continuations produce an *abelian group* of periods (Chap. 2). Chap. 4 shows the group is $\mathbb{Z} \times \mathbb{Z}$. Further, its fit with the analytic continuations of $G(z)$ appears in the semidirect product $\mathbb{Z} \times \mathbb{Z} \times^s \{\pm 1\}$ (§8). Let D_n be the dihedral group of order $2n$.

Classical modular curves parametrize four branch point D_n extensions of $U_{\boldsymbol{z}}$. Galois checked with his theorem for which n these modular curve parameters were solvable functions of the classical j parameter [**Rig96**, p. 133]. Properties of $F(z)$ entwine integration and the appearance of abelian extensions:

(3.2) $F(z)$ is a *versal abelian* extensible function on $U_{\boldsymbol{z}}$ with monodromy around $\boldsymbol{z}$ bounded by $G(z)$ (Chap. 4).

Restricting to $U_{\boldsymbol{z}}$ still shows the full scope of Riemann's version of (3.2). The next three sections base a story of his program on analytic continuation.

4. Algebraic functions among extensible functions

Denote Laurent series expansions about z_0 by $\mathcal{L}_{z_0}$. Let $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ be extensible (meromorphic) elements of $\mathcal{L}_{z_0}$ on $U_{\boldsymbol{z}}$. Call $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ algebraic if it satisfies $m(z, f(z)) \equiv 0$ with $m \in \mathbb{C}[z,w]$ a nonzero polynomial. Characterizing such f through analytic continuation, the main topic of Chap. 2, is the first step to classifying algebraic functions. Any analytic continuation of f around a closed path in $U_{\boldsymbol{z}}$ also gives a zero of m. So, there are only finitely many analytic continuations of f. Analytic continuations of f along paths whose end points have limits in $\boldsymbol{z}$ take values nowhere dense (a finite set) in the Riemann sphere. This qualitative statement characterizes algebraic f. The full force of Riemann's Existence Theorem is in phrasing this through fundamental group representations (Chap. 4). Denote the algebraic elements of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ by $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$.

4.1. One element of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ is versal for $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$. There are so many algebraic functions in $\mathcal{E}(U_{\boldsymbol{z}}, z_0) = \mathcal{E}(U_{\boldsymbol{z}})$ (if the cardinality, $r = |\boldsymbol{z}|$ exceeds two). We can explain little about them by listing their polynomial equations. Yet, there is much structure in this collection.

4.1.1. *Setup for uniformizaton.* Riemann provided such by finding one function $\tilde{f}_{\boldsymbol{z}}$ giving all of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$ through a type of Galois correspondence. An outline for this appears in Chap. 3.

(4.1a) Recognize each algebraic function $f \in \mathcal{E}(U_{\boldsymbol{z}})$ has an attached topological cover $\varphi_f : X_f \to U_{\boldsymbol{z}}$.

(4.1b) Produce a (uni)versal cover $\varphi_{\boldsymbol{z}} : \tilde{U}_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ with a discrete group $\pi_1(U_{\boldsymbol{z}}, z_0)$ acting on $\tilde{U}_{\boldsymbol{z}}$.

(4.1c) Show X_f is a topological quotient of $\tilde{U}_{\boldsymbol{z}}$ by a subgroup of $\pi_1(U_{\boldsymbol{z}})$.

(4.1d) Show $\tilde{U}_{\boldsymbol{z}}$ has a complex analytic embedding in $\mathbb{C}$: $h : \tilde{U}_{\boldsymbol{z}} \to \mathbb{C}$.

As in Chap. 3, (4.1d) produces $\tilde{f}_{\boldsymbol{z}}$ as follows. Let U_{z_0} be *any* disk around z_0 (on $U_{\boldsymbol{z}}$). *Cauchy's Theorem* (we return soon to that) shows this:

(4.2) Each $g \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ extends to a unique meromorphic function on U_{z_0}.

4.1.2. *$\tilde{U}_{\boldsymbol{z}}$ and Hurwitz equivalence.* Riemann's Existence Theorem shows why $\tilde{U}_{\boldsymbol{z}}$ identifies with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. Apply the Existence Theorem (see Chap. 4 or §5.1) with a branch cycle description of form

$$g_1 = (1 \,\dots\, s_1) \cdots (s_{t-1}{+}1 \,\dots\, s_t),$$

$s_1 + s_2 + \cdots + s_t = n$; $g_2 = (s_1\, s_2\, \dots\, s_t)$ and $g_3 = (g_1 g_2)^{-1}$. Count points over branch points: $t + (n - t + 1) + 1 = n + 2$.

Uniformize $U_{\{0,1,\infty\}}$ with the classical λ function (§7.1.1). Choose $n = r - 2$. This produces a genus 0 cover of $\varphi_{\boldsymbol{g}} : X_{\boldsymbol{g}} \to \mathbb{P}^1_z$ unramified over $U_{\{0,1,\infty\}}$ with exactly r points over $\{0, 1, \infty\}$. Further, λ factors through this cover:

$$\mathbb{H} \to X_{\boldsymbol{g}} \setminus \varphi_{\boldsymbol{g}}^{-1}(0, 1, \infty) \to U_{\{0,1,\infty\}}.$$

This uniformizes one copy of $\mathbb{P}^1_z$ minus r points. Deform (differentiably) $X_{\boldsymbol{g}} \setminus \varphi_{\boldsymbol{g}}^{-1}(0, 1, \infty)$ to *any* other copy of $\mathbb{P}^1_z$ minus r points (Chap. 5).

Regard algebraic functions $f = y$ (of z) as giving a relation between two variables x and y. Classical literature often chooses the isomorphism class of the *function field* $\mathbb{C}(z, y)$ as the unique goal of an algebraic relation. If $\mathbb{C}(z, y)$ is isomorphic to $\mathbb{C}(z^*, y^*)$, this views the algebraic relation between (z^*, y^*) (take the minimal polynomial of y^* over $\mathbb{C}(z^*)$) as elementary equivalent to the relation between z and y. The history of considering algebraic relations had its motivation in integrals. There the most telling invariant of a function field $\mathbb{C}(z, y)$ is the *genus* g (maximal number of linearly independent holomorphic differentials §6.2) on the function field.

A connected algebraic space parametrizes all algebraic relations of genus g (Chap. 5). Investigating this and subtler problems about algebraic relations suggest a more delicate equivalence between function fields. In addition to the isomorphism of $\mathbb{C}(z^*, y^*)$ with $\mathbb{C}(z, y)$, this isomorphism includes that $\mathbb{C}(z^*) = \mathbb{C}(z)$. Call this *Hurwitz equivalence*. Even in restricting to genus g function fields there are many components to the parameter spaces of Hurwitz equivalences of algebraic relations. Hurwitz (equivalence) spaces all derive from the elementary notion of deforming points as in the construction above for $\tilde{U}_{\boldsymbol{z}}$.

4.1.3. *The value of $\tilde{f}_{\boldsymbol{z}}$.* Since $\varphi_{\boldsymbol{z}} : \tilde{U}_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ is a covering space, $\varphi_{\boldsymbol{z}}^{-1}(U_{z_0})$ has countably many connected components $\{U_i\}_{i=1}^{\infty}$, each homeomorphic to U_{z_0} by restriction of $\varphi_{\boldsymbol{z}}$. Let $\varphi_1 : U_1 \to U_0$ be this one-one restriction. Then, (4.1d) produces the function

$$\tilde{f}_{\boldsymbol{z}} = h \circ \varphi_1^{-1} : U_0 \to \mathbb{C}. \tag{4.3}$$

This *one* function distinguishes *homotopy classes* of paths on $U_{\boldsymbol{z}}$ by analytic continuation. It separates homotopy classes of paths (based at z_0) by its values at end points of analytic continuations. Since $\tilde{U}_{\boldsymbol{z}}$ is simply connected and in $\mathbb{C}$, Riemann's mapping theorem says it is analytically isomorphic to a disk (or to $\mathbb{C}$, if $r = 1$ or 2) for each $\boldsymbol{z}$.

4.2. Uniformizing from above versus below. Thus, $\tilde{U}_{\boldsymbol{z}}$ is a domain for parametrizing X_f for all $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$, as $\boldsymbol{z}$ varies. This complements how we use Riemann's Existence Theorem.

4.2.1. *Shortcomings of $h \circ \varphi^{-1}$.* The universal covering space helps organize functions and differential forms. Still, algebraists find it hides phenomena close to their interests. For example, $h \circ \varphi_1^{-1}$ is neither algebraic nor known: Its values at algebraic points of $U_{\boldsymbol{z}}$ are rarely algebraic. Though based on $\lambda(\tau)$ in §4.1.2, it changes with $\boldsymbol{z}$. Yet, it provides no explicit equations for algebraic functions.

Even proving a cover from Riemann's Existence Theorem is algebraic still goes through a hard proof that we now separate from other, more algebraic, observations. Suppose $\varphi : X \to \mathbb{P}^1_z$ is a cover. Let $\varphi_w : X \to \mathbb{P}^1_w$ be any function separating all points on the fiber X_{z_0} over z_0. Then, $X \to \mathbb{P}^1_z \times \mathbb{P}^1_w$ by $x \mapsto (\varphi(x), \varphi_w(x))$ has closed image birational to X in the algebraic variety $\mathbb{P}^1_z \times \mathbb{P}^1_w$. Apply Chow's Lemma (Chap. 4) to get that X is algebraic.

Classical construction of φ_w relies on a uniformization $\mathbb{H} \to U_{\boldsymbol{z}}$ presenting $U_{\boldsymbol{z}}$ as a quotient $\mathbb{H}/H$, H a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. One must find nontrivial H invariant functions on $\mathbb{H}$ [**K72**, Chap. III]. Variants are in [**Spr57**, Chap. 6-10] and [**Vö96**, Chap. 5]. We rely on the treatment from the last of these references —especially well adapted to Riemann's Existence Theorem. How to find φ_w (or some related differential form) algebraically appears in many of our examples.

4.2.2. *Virtues of $h \circ \varphi^{-1}$.* The phrase "abelian theory" means here Riemann's unified generalization of Abel's results. This includes describing functions, abelian covers and the results of integration of differentials on a Riemann surface. It includes Riemann's extension of Cauchy's integral theorem to open Riemann surfaces. We discuss it, and our reason for including a *nilpotent theory* below. There is no denying the value of $h \circ \varphi^{-1}$.

(4.4a) It organizes tool the abelian and nilpotent theory.
(4.4b) It coordinates analyzing real points on moduli spaces of curves.
(4.4c) It is suspiciously close to being algebraic, producing an algebraic object (a flat $\mathbb{P}^1$-bundle) capturing its uniformizing properties.

4.2.3. *The Existence Theorem and classical uniformization meet.* Each item in (4.4) has Existence Theorem and $\tilde{U}_{\boldsymbol{z}}$ aspects: *Uniformization from below versus above*. The literature neglects the former, though it is constructive and practical. The latter has had elegant developments.

Both work best as tools for analyzing properties of families (moduli spaces) of curves. They give enhancements when the moduli spaces themselves fit in natural sequences. The abelian theory gave the first such natural sequences. This shows in modular curve sequences (§8.3, Chap. 5).

4.2.4. *Illustrating with modular curves.* When the parameter r (cardinality of $\boldsymbol{z}$) is 4, the comparison between Modular Towers and modular curves is direct. For example, these properties hold for Modular Towers when $r = 4$.

- Their levels are curves.
- They include modular curve towers and come with an essential prime p: Its powers correspond to Modular Tower levels.
- They lie over the classical j-line and have useful cusps over $j = \infty$.
- All levels are moduli spaces, with variants corresponding to structures going with modular curve notation $X_0(p^{k+1})$, $X_1(p^{k+1})$ and $X(p^{k+1})$.

Any finite group G and prime p dividing $|G|$ produces many Modular Towers; many more than there are modular curve towers. The name Modular Tower comes from this comparison and the group (*modular representation*) theory that appears in their analysis.

An elementary comparison occurs in analyzing *real* points on a Modular Tower. Through Riemann's Existence Theorem this gives the essential data about *cusps*. From that come their geometric properties (Chap. 5), including genuses of their components. This is especially interesting when the finite group G producing the Modular Tower is simple and the prime p is 2. We now discuss the Existence Theorem, then the abelian theory.

5. $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\rm alg}$ and data from groups

Riemann's Existence Theorem (Chap. 4) compactifies $\varphi_f : X_f \to U_{\boldsymbol{z}}$ to a ramified cover of Riemann surfaces $\bar\varphi_f : \bar X_f \to \mathbb{P}^1_z$. It then turns the process around by using special generators of the fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$ of $U_{\boldsymbol{z}}$. From these it produces all elements of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\rm alg}$.

5.1. Identifying a fundamental group requires generators. Suppose G is a finite transitive subgroup of S_n. A surjective homomorphism $\psi : \pi_1(U_{\boldsymbol{z}}, z_0) \to G$ canonically produces a cover $X_\psi \to U_{\boldsymbol{z}}$ from homotopy classes of paths. We don't need generators of $\pi_1(U_{\boldsymbol{z}})$ to define these covers Chap. 3. They, however, handily *list* all such homomorphisms and therefore all such covers. Convenient listing of covers allows explicitly computing properties of Hurwitz spaces (Chap. 5).

The collections of r paths (based at z_0) we call *classical generators* of $\pi_1(U_{\boldsymbol{z}}, z_0)$ appear in Chap. 3. Points in $\boldsymbol{z}$ produce conjugacy classes $\mathbf{C}_{\boldsymbol{z}}$ in $\pi_1(U_{\boldsymbol{z}}, z_0)$. Classical generators are homotopy classes of paths respectively representing these conjugacy classes. Choose representing paths that pair wise meet only at their beginning and end point z_0. Label one as $\bar g_1$. Label the others from their having a clockwise order in leaving the point z_0. These r paths $\bar g_1, \dots, \bar g_r$ now satisfy

(5.1) $\bar g_1 \bar g_2 \cdots \bar g_r = 1$: The *product-one condition*.

Invariants of Hurwitz space components appear from (5.1) (§10.1 illustrates). Classical generators — satisfying these conditions — automatically generate the fundamental group (Chap. 3). Solving for $\bar g_r$ presents the fundamental group as a free group on $r-1$ generators. Yet, that violates the product-one symmetry. So, that free group presentation appears only in stray computations.

This part of Riemann's theory works *very* well. It successfully applies to many problems. These require some finite group theory. It is the center of the first third of the book. Polynomial equations describe algebraic curves. This is what gives structure allowing fields of definitions and interpreting rational points. The Riemann's Existence Theorem approach, however, emphasizes effective group theory over manipulating explicit equations. Exercises and examples illustrate this (Chap. 3, Chap. 4, Chap. 9).

5.2. Changing classical generators. There is no canonical set of classical generators for $\pi_1(U_{\boldsymbol{z}}, z_0)$. The necessary variation of this choice produces the *braid* and *mapping class* groups (Chap. 5). This complication enriches mathematics. Still, it requires explanation.

The second third of the book organizes collections of elements from $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\rm alg}$. This allows $\boldsymbol{z}$ to vary. Sets of r (unordered) distinct points on $\mathbb{P}^1_z$ have a topology and analytic structure extending that of $\mathbb{P}^1_z$. This set is $\mathbb{P}^r \setminus D_r = U_r$: Projective r-space minus the discriminant locus (Chap. 5). Think of U_r as monic polynomials of degree either r or $r-1$ with distinct roots. Or, consider it the quotient of $(\mathbb{P}^1_z)^r \setminus \Delta_r$

by permutation action of S_r, the symmetric group of degree r, on ordered r-tuples of points. Here Δ_r is r-tuples with distinct coordinates.

The fundamental group of U_r is the degree r *Hurwitz monodromy group* H_r (Chap. 5), an *Artin braid group* quotient. A permutation representation of H_r produces the space of deformations of X_f. These are *unreduced Hurwitz spaces*.

A given function $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$ deforms in many ways as $\boldsymbol{z}$ varies. Local deformation, however, of $\bar{\varphi}_f : \bar{X}_f \to \mathbb{P}^1_z$ is unique along any path. This allows analyzing parameters for these moduli spaces. Yet, it leads further from explicit equations. To paraphrase Joni Mitchell's "Both Sides Now" (from the 60's): *Something's lost and something's gained in putting equations away*. Explicit functions, however, return with the *abelian* and *nilpotent* theory.

5.3. Moving $\boldsymbol{z}$, even with z_0 fixed, forces changing generators. Picture: z_1 and z_2 follow semicircles, producing

$$Q_1 : (\bar{g}_1, \dots, \bar{g}_r) \mapsto (\bar{g}_1 \bar{g}_2 \bar{g}_1^{-1}, \bar{g}_1, \dots, \bar{g}_r). \tag{5.2}$$

Replacing 1 by $i \le r-1$ gives the full generating collection $Q_1, \dots, Q_{r-1}$ of the *Hurwitz monodromy group* H_r (Chap. 5). The H_r action from (5.2) on classical generators is the technical tool for describing families of covers.

Let G be a fixed finite group. Assume these further ingredients.

(5.3a) $\boldsymbol{z}'$ is a specific point of U_r.
(5.3b) $\psi_{\boldsymbol{z}'} : \pi_1(U_{\boldsymbol{z}'}, z_0) \to G$ is a specific surjective homomorphism to G using classical generators $\bar{g}_1, \dots, \bar{g}_r$ (§5.1).
(5.3c) $T : G \to S_n$, n an integer, is a faithful permutation representation.

Then, $\psi_{\boldsymbol{z}'}$ gives a finite (ramified) cover $\varphi_{G,T,\boldsymbol{z}'} = \varphi_{\boldsymbol{z}'} : X_{\boldsymbol{z}'} \to \mathbb{P}^1_z$ of Riemann surfaces of degree n. The images of $\bar{g}_1, \dots, \bar{g}_r$ give generators $g_1, \dots, g_r$ of $G \le S_n$, with an associated set of r conjugacy classes $\mathbf{C}$ in G. Riemann's Existence Theorem labels covers by $g_1, \dots, g_r$ (branch cycles). It gives $\varphi_{G,T,\boldsymbol{z}'}$ as an equivalence relation on homotopy classes of paths based at z_0. Suppose $\boldsymbol{z}'$ moves to nearby $\boldsymbol{z}''$, with $z_0 \in \mathbb{P}^1_z$ and paths representing $\bar{g}_1, \dots, \bar{g}_r$ fixed. Then, there is a unique isomorphism of $\pi_1(U_{\boldsymbol{z}'}, z_0)$ and $\pi_1(U_{\boldsymbol{z}''}, z_0)$ commuting with their maps to G.

An automorphism α of $\pi_1(U_{\boldsymbol{z}}, z_0)$ sends generators to new generators, changing $\psi_{\boldsymbol{z}}$ to $\psi_{\boldsymbol{z}} \circ \alpha$. Inner automorphisms of $\pi_1(U_{\boldsymbol{z}}, z_0)$, however, produce covers equivalent to the old cover. It is moduli of covers we use; equivalence two homomorphisms if they differ by an inner automorphism. Further, only automorphisms from the Hurwitz monodromy group H_r send classical generators to classical generators (possibly changing the intrinsic order of the paths). Such automorphisms arise from deforming the pair $(\boldsymbol{z}, z_0)$ along closed paths in U_r. They preserve the conjugacy classes of classical generators. So, $\mathbf{C}$, the conjugacy class set in G, is an H_r invariant of any given homomorphism ψ.

5.4. The moduli spaces appear. The *Nielsen class* of $(G, \mathbf{C})$ (Chap. 5) consists of r-tuples $(g_1, \dots, g_r)$ satisfying the product-one condition attached to $(G, \mathbf{C})$. The Existence Theorem uses classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$ to produce equivalence class of covers.

5.4.1. *Writing equations in $\boldsymbol{z}$.* The Nielsen class $\text{Ni}(G, \mathbf{C})$ has entries in a set of conjugacy classes $\mathbf{C}$ in G, independent of the braid action. Thus, H_r acts on elements of $\text{Ni}(G, \mathbf{C})$ (similar to (5.2)). An aside: We need to quotient by conjugation from G. Here is how to think of this action.

Let $\varphi_0 : X_0 \to \mathbb{P}^1_z$ be a cover from the Existence Theorem using $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$. Take the branch points to be $\boldsymbol{z}_0$. What if someone asks for *explicit* equations for this cover? That could mean either:

(5.4a) equations just for φ_0; or
(5.4b) equations for $\varphi_{\boldsymbol{z}} : X_{\boldsymbol{z}} \to \mathbb{P}^1_z$, with branch points $\boldsymbol{z}$, valid for $\boldsymbol{z}$ near $\boldsymbol{z}_0$ (where it specializes to φ_0).

Don't those seem like asking too little? Why concentrate on one set of branch points $\boldsymbol{z}_0$, or even on a neighborhood of $\boldsymbol{z}_0$? You'd want $\varphi_{\boldsymbol{z}}$ valid for *all* $\boldsymbol{z} \in U_r$. If, however, this were possible, then analytically continuing $\varphi_{\boldsymbol{z}}$ around any closed path $\mathcal{P}$ in U_r would return you to φ_0.

The homotopy class of $\mathcal{P}$ is an element $Q_{\mathcal{P}}$ of H_r. Further, Chap. 5 shows the cover at the end of $\mathcal{P}$ has a branch cycle description $(\boldsymbol{g})Q_{\mathcal{P}}$. (Compute that with the starting classical generators of $\pi_1(U_{\boldsymbol{z}_0})$.) So, finding equations for $\varphi_{\boldsymbol{z}}$ valid for all $\boldsymbol{z}$ requires $(\boldsymbol{g})Q_{\mathcal{P}}$ be $\boldsymbol{g}$ (modulo conjugation by G or closely related). This you can check: Is $(\boldsymbol{g})Q$ essentially $\boldsymbol{g}$ for all $Q \in H_r$. Example: Consider

$$\boldsymbol{g} = ((1\,2\,3), (3\,2\,1), (1\,4\,5), (1\,5\,4)) \in \mathrm{Ni}(A_5, \mathbf{C}_{3^4})$$

(§10.1). Then $(\boldsymbol{g})Q_2 = ((1, 2\,3), (2\,4\,5), (3\,2\,1), (1\,5\,4))$. This is not conjugate to $\boldsymbol{g}$ even under S_5. So, as typical when $r \geq 4$, there are no such equations for $\varphi_{\boldsymbol{z}}$.

5.4.2. *Analytic continuations of* $\varphi_{\boldsymbol{z}_0}$. Nontrivial H_4 action means coefficients of equations for $\varphi_{\boldsymbol{z}}$ act as coordinates for a nontrivial cover of U_r. What cover?

It comes from the action of H_r, the fundamental group of U_r, on $\mathrm{Ni}(G, \mathbf{C})$ produced by covering space theory. Notation for this cover depends on the equivalence used for elements of the Nielsen class (as in (5.5)). Typical notation is $\mathcal{H}(G, \mathbf{C}, T)$. Each point of $\mathcal{H}(G, \mathbf{C}, T)$ corresponds to an equivalence class of covers: A point over $\boldsymbol{z} \in U_r$ is an element from $\mathrm{Ni}(G, \mathbf{C})$ attached to $\boldsymbol{z}$. Then, $\mathcal{H}(G, \mathbf{C}, T)$ itself covers the space U_r of distinct unordered r-tuples of points from $\mathbb{P}^1_z$ (Chap. 5).

Various equivalences among covers produce different versions of this space. Two predominate in early applications. Denote the subgroup of S_n normalizing G and permuting the conjugacy classes in $\mathbf{C}$ by $N_{S_n}(G, \mathbf{C})$.

(5.5a) $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$: T is the *regular representation* and the Galois cover comes with a fixed isomorphism between its Galois group and G (inner spaces).
(5.5b) $\mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$: T any faithful representation, with r-tuples equivalenced by $N_{S_n}(G, \mathbf{C})$ conjugation (absolute spaces).

See the main example of this chapter at §10.1.

The combinatorial groups in Chap. 5 have long histories: the *Artin braid group*, the *Hurwitz monodromy group* and the *mapping class group*. As in Chap. 4, we give formal proofs. Pictures appear only to convey conceptual symbolic data. Absolute spaces are the work horses in applications (Chap. 9). Inner spaces, however, directly connect the Inverse Galois Problem to generalizations of *modular curves* (§7.4).

5.4.3. *The statics and dynamics of a cover.* In the game of mentally *writing equations* for a cover, why would one cover be more significant than another? Many historical applications, such as the Inverse Galois Problem, consider a cover with equations over $\mathbb{Q}$ as most significant. For example, many arithmetic problems gain solutions if one can produce a cover with a particular monodromy group over $\mathbb{Q}(z)$ or over $\mathbb{Q}$. Such a cover provides solutions to related problems over another field by extending its equations to that field.

We picture such a cover $\varphi_0 : X_0 \to \mathbb{P}^1_z$ as being at the crossroads of a network of roads. The real points on $\mathcal{H}(G, \mathbf{C})^{\text{in}}$ would go through the point corresponding to φ_0, as would all p-adic points for every prime p. Concentrate on a real point, $\boldsymbol{p}_0 \in \mathcal{H}(G, \mathbf{C})^{\text{in}}$ corresponding to a cover φ_0 over $\mathbb{R}$. To get a measure of the potential energy of this point we measure its distance from boundary points on $\mathcal{H}(G, \mathbf{C})^{\text{in}}$. Developing such a measure, depends on measuring something that goes to 0 as we deform φ_0 along a real component going to a boundary point, and the measuring coordinates must be canonical functions of the coordinates of the point $\boldsymbol{p}$ as it moves from $\boldsymbol{p}_0$ to the chosen boundary point.

The theory of abelian covers on $\bar{X}_0$ gives classical functions that we can use for making such measurements. As easily this could be on $\bar{X}_0$ minus a finite number of points, as with $U_{\boldsymbol{z}}$. Still, in the compact case, functions in $\mathcal{E}(\bar{X}_0)$ with finitely many analytic continuations are algebraic.

6. Abelian theory on $\bar{X}_f$ and integration

Let $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. Suppose analytic continuations f_γ of $f(z)$ have this property.
(6.1) $f_\gamma(z) = f(z)$ for each closed path γ based at z_0.
Rather than extensible, Chap. 2 calls f *extendible*. Denote extendible elements of $\mathcal{E}(U_{\boldsymbol{z}})$ by $\mathcal{E}(U_{\boldsymbol{z}})^{\text{ext}}$.

Consider $f \in \mathcal{E}(U_{\boldsymbol{z}})^{\text{ext}}$. Cauchy's Theorem in $U_{\boldsymbol{z}}$ shows precisely the nature of integrals $f(z)\,dz$ around certain closed paths. Since these are integrals, assume without further mention the paths miss any poles of $f\,dz$. Let $\boldsymbol{z}_{f,\infty}$ be the set of these poles. Assume for simplicity it is a finite set (appropriate for algebraic functions) which may include ∞: $z^n\,dz$ has a pole of order $n+2$ at ∞.

The definition of integral makes sense. Let $F(z)$ be an antiderivative of f in a neighborhood of z_0. For any (simplicial) path $\gamma : [0,1] \to U_{\boldsymbol{z}}$, take the indefinite integral to the end point of γ to be F_γ (Chap. 2).

Cauchy's Residue Theorem: Let γ be a closed path *homologous* to 0 in $U_{\boldsymbol{z}}$. Compute $\int_\gamma f(z)\,dz$ from the *winding number* of γ and residue of f at each $z' \in \boldsymbol{z}_{f,\infty}$ (Chap. 2). Winding numbers are values of integrals $\int_\gamma \omega$ where ω is a differential form —*logarithmic*, or of *3rd kind*—taking the shape $\frac{1}{2\pi i}\frac{dz}{z-z'} = \omega_{z'}$ with $z' \in \boldsymbol{z}_{f,\infty}$. Also, winding numbers appear in the definition of being homologous to 0: The path has winding number 0 about each point in $\boldsymbol{z}$.

6.1. Changes of significance for algebraic f. Here is a paraphrase of Cauchy. Suppose (6.1) holds. Then, poles of f and the map $\gamma \to (\int_\gamma \omega_{z_1}, \dots, \int_\gamma \omega_{z_r})$ determine $\int_\gamma f(z)\,dz$ when γ is closed and homologous to 0.

Suppose, however, f is both algebraic and extendible. That means it is a rational function on $\mathbb{P}^1_z$. Then, there is no significant difference between the points in $\boldsymbol{z}$ and those in $\boldsymbol{z}_{f,\infty}$. By combining them both in the set $\boldsymbol{z}_{f,\infty}$ this allows dropping the homologous to 0 condition. We may consider integrals around any closed path.

Riemann made a more abstract change. Antiderivatives of ω_{z_i} are (up to an additive constant) *branches of* $\log(z - z_i)$, $i = 1, \dots, r$. Recognizing Cauchy's Theorem as a statement entirely about integrals of meromorphic differentials (not of functions) immediately allowed generalizations. Here is what the abelian theory does for $U_{\boldsymbol{z}}$ (Chap. 2).

(6.2a) It gives explicit differentials providing details on integrals of any meromorphic differentials around any closed paths.

(6.2b) It describes elements of $\mathcal{E}(U_{\boldsymbol{z}})^{\mathrm{alg}}$ with associated group abelian.

Chap. 2 does (6.2b) by corresponding such functions to an r-tuple in $(\mathbb{Q}/\mathbb{Z})^r$ with entries summing to 0.

6.2. Extending Cauchy's Theorem to $\bar{X}_f$. Riemann extended Cauchy's Theorem to $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ not satisfying (6.1). Compatible with (6.2), he extended it to meromorphic differentials on $\bar{X}_f$. This emphasis on differentials over functions didn't throw functions out. They were still there through the definition of df, the *differential* and df/f, the *logarithmic* differential of f (Chap. 3).

The serious step was analyzing the space of *holomorphic* (or *first kind*) differentials on $\bar{X}_f$ (Chap. 3, Chap. 4): differentials with no poles anywhere. Standard notation for this $g = g(\bar{X}_f)$ dimensional space over $\mathbb{C}$ is $\Gamma(\bar{X}_f, \Omega^1)$: Global sections of the sheaf of holomorphic differentials on $\bar{X}_f$. The genus g of $\bar{X}_f$ now attaches to $f = f(z)$, toward pinning its place among algebraic functions of z.

Guidance came from the Abel-Jacobi-Legendre differentials like $\frac{dz}{\sqrt{z^3+az+b}}$ from §3.2. Just giving the dimension of $\Gamma(\bar{X}_f, \Omega^1)$ called for a more abstract approach. Riemann needed a full basis to solve the *Jacobi-Inversion problem*. Relying on coordinates from $\mathbb{P}^1_z$ was a confining kludge.

With points removed from $\bar{X}_f$, add further *logarithmic* (or *3rd kind*) differentials. In $U_{\boldsymbol{z}}$, the (vector-)space of logarithmic differentials has a preferred basis by reference to classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$ (§5.1).

Extending this to $\bar{X}_f$ still leaves an infinite set of choices for a $\Gamma(\bar{X}_f, \Omega^1)$ basis, with all choices related by the action of a group: The *symplectic group* $\mathrm{Sp}_{2g}(\mathbb{Z})$. Different basis choices correspond to different choices of $2g$ closed paths whose homology classes determine integration of any meromorphic differential around closed paths. This is an imprecise statement of Cauchy's Theorem on $\bar{X}_f$.

As with $U_{\boldsymbol{z}}$, there is a notion of classical generators. With $U_{\boldsymbol{z}}$ the paths were nonintersecting, except at the base point. On $\bar{X}_f$ classical generators signifies normalizing information about the intersection of these $2g$ paths. Given classical generators for $U_{\boldsymbol{z}}$ there is a process for producing classical generators on X_f. This provides explicit actions of appropriate subgroups of H_r on the homology of $\bar{X}_f$. Suppose $\bar{X}_f$ appears in the moduli space of curves of genus g. Then, the whole action may well give $\mathrm{Sp}_{2g}(\mathbb{Z})$. On Hurwitz spaces, however, the data is more refined. The significant group action may be much smaller.

6.3. The Jacobian and generalizing Abel's Theorem. Suppose $\omega_1, \dots, \omega_g$ is a specific $\Gamma(\bar{X}_f, \Omega^1)$ basis. As with $U_{\boldsymbol{z}}$, Cauchy's Theorem on $\bar{X}_f$ builds from this data an abelian group. In this case it is a compact complex torus $J(\bar{X}_f)$, the *Jacobian* of $\bar{X}_f$. Follow Mumford's view [**Mum76**, p. 58-67]. Consider the space of locally defined holomorphic tangent vectors for $\bar{X}_f$ as dual to locally defined holomorphic differential forms (Chap. 3). Then, paths are dual to holomorphic differentials (by integration). The problem is to interpret a dual to *global* holomorphic differentials. This generalizes the Abel-Jacobi approach to Cauchy's Theorem and it produces an abelian covering theory.

Let h be any meromorphic function on $\bar{X}_f$ (Chap. 3) of degree u. Then, $h : \bar{X}_f \to \mathbb{P}^1_z$ has zeros $x^0_1, \dots, x^0_u$ and poles $x^\infty_1, \dots, x^\infty_u$. A mysterious identification then occurs: $\bar{X}_f$ appears in $J(\bar{X}_f)$. So, each zero x^0_i and pole of h produces a point in $J(\bar{X}_f)$. List these as $\boldsymbol{p}_{x^0_i}, \boldsymbol{p}_{x^\infty_i}$, $i = 1, \dots, u$.

6.3.1. *Logarithmic differentials.* Yet, finding the $\boldsymbol{p}$s doesn't require giving h. It only needs points $x_1^0, \dots, x_u^0$ and $x_1^\infty, \dots, x_u^\infty$ on $\bar{X}_f$ viewed as inside $J(\bar{X}_f)$. Define $[D_{\boldsymbol{x}}] = [D(\boldsymbol{p}_{x_i^0}, \boldsymbol{p}_{x_i^\infty}, i = 1, \dots, u)]$ as the sum of all the $\boldsymbol{p}_{x_i^0}$s minus all the $\boldsymbol{p}_{x_i^\infty}$s on $J(\bar{X}_f)$. To say $[D]$ is zero means it is the origin of $J(\bar{X}_f)$. Abel's Theorem (generalized) says existence of h with these zeros and poles characterizes exactly when $[D]$ is zero.

If h exists, consider the logarithmic derivative dh/h. This is a meromorphic differential of 3rd kind with pure imaginary periods. Even if h doesn't exist, given the divisor $D_{\boldsymbol{x}}$ above, the following holds.

(6.3) There is a unique differential $\omega_{\boldsymbol{x}}$ with residue divisor $D_{\boldsymbol{x}}$ having pure imaginary periods (Chap. 4).

6.3.2. *Coordinates from holomorphic differentials.* Suppose $[D]$ is not zero, but $m[D]$ is zero on $J(\bar{X}_f)$ for some integer m. Then, repeating all the zeros and poles m times produces a function h on $\bar{X}_f$. The mth root of h defines an abelian unramified cover $Y \to \bar{X}_f$. So, the abelian theory of $\bar{X}_f$ appears from this version of Cauchy's Theorem. Riemann produced $\theta = \theta_{\bar{X}_f}$ functions to provide global coordinates (uniformization) for this construction. They are functions on $\mathbb{C}^g$ (§6.5).

Many mathematical items on $\bar{X}_f$ appear constructively from this. This includes functions and meromorphic differentials (with particular zeros and poles). This was a central goal in generalizing Abel's Theorem: To provide Abel(-Jacobian) constructions for a general Riemann surface. For the function h it has this look:

$$h(x) = \prod_{i=1}^{u} \theta\Big(\int_{x_i^0}^{x} \boldsymbol{\omega}\Big) \Big/ \prod_{i=1}^{u} \theta\Big(\int_{x_i^\infty}^{x} \boldsymbol{\omega}\Big). \tag{6.4}$$

In θ you see g coordinates; the *ith* entry is $\int_{x_i^0}^{x} \omega_i$. Each holds an integral over one basis element from $\boldsymbol{\omega}$. Integration paths join respective points on $\bar{X}_f$'s universal covering space. The integrals make sense up to integration around closed paths. So, they define a point in $J(\bar{X}_f)$.

Even if h doesn't exist, the logarithmic differential of (6.4) does. It gives the third kind differential from (6.3). Here you see the differential equation defining θ functions. In the expression for h, replace $\int_{x_i^\infty}^{x} \boldsymbol{\omega}$ by a vector $\boldsymbol{w}$ in the universal covering space of the Jacobian. Form the logarithmic differential of it: $d\theta(\boldsymbol{w})/\theta(\boldsymbol{w})$. Translations by periods will change it by addition of a constant. With ∇ the gradient in $\boldsymbol{w}$, $\nabla(\nabla\theta(\boldsymbol{w})/\theta(\boldsymbol{w}))$ is invariant under the lattice of periods.

Thus, $J(\bar{X}_f)$ provides transparent coordinates for differentials, and their periods, through a mysterious embedding of $\bar{X}_f$ in it. Then, objects from the abelian structure on $J(\bar{X}_f)$ restrict to $\bar{X}_f$ (§10.6). To use, however, Riemann's theory an algebraist faces two major complications.

6.4. Complication 1: The role of f. Suppose $\bar{X}_f$ varies in the Hurwitz space $\mathcal{H}(G, \mathbf{C})$ attached to $(G, \mathbf{C})$. It moves along a path in U_r with the coordinates for $\boldsymbol{z}$. Is Riemann's theory sufficiently algebraic to express the changes using equations with coefficients in the the point of $\mathcal{H}(G, \mathbf{C})$ corresponding to $\bar{X}_f$. Answer: It is algebraic in many ways, though rarely will coordinates from $\mathcal{H}(G, \mathbf{C})$ support all the identifications. Here is why.

6.4.1. *The Picard components.* There are three geometric ingredients in Riemann's theory: $J(\bar{X}_f)$, $\bar{X}_f$ and the zero (Θ) divisor of the function $\theta = \theta_{\bar{X}_f}$ (§6.5). The first identifies with divisor classes $\mathrm{Pic}^0(\bar{X}_f) = \mathrm{Pic}^0_f$ of degree 0 on $\bar{X}_f$ (Chap. 4).

The second embeds naturally (algebraically) in Pic^1_f, divisor classes of degree 1 on $\bar{X}_f$. Then, Θ_f is the dimension $g-1$ variety of positive divisor classes in Pic^{g-1}_f.

Further, Pic^g_f interprets the Riemann-Roch Theorem and the *Jacobi Inversion Problem* geometrically (Chap. 4). It takes its group structure from adding two positive divisors of degree g together modulo linear equivalence. Weil used this for an algebraic construction of Pic^0_f years after his thesis. His principle: The nearly well defined addition on positive divisors produced a unique complete algebraic group on the homogeneous space of divisor classes. Therefore Pic^0_f is almost the symmetric product of $\bar{X}_f$ taken g times. Riemann's theory was an inspiration to Weil's 1928 thesis (§10.6). Still, Weil was not certain until later that Pic^0_f and $\bar{X}_f$ have the same field of definition. This reminds that what now looks obvious is the result of many mathematical stories.

6.4.2. *Half-canonical classes.* All Picard components Pic^k_f are pair wise analytically isomorphic. Yet, finding an isomorphism analytic in the Hurwitz space coordinates may require moving to a cover of the Hurwitz space (§10.6).

Applying Riemann's theory directly requires having $\bar{X}_f$ and the Θ_f divisor on Pic^0_f. For example, suppose there is an analytic assignment of a divisor class of degree $g-1$ on each curve $\bar{X}_f$ in the Hurwitz family. Then, translation of Θ_f by this divisor class puts it in Pic^0_f. Here it would be available to construct the θ function. Convenient for this might be a *half-canonical* class: two times gives divisors for meromorphic differentials (Chap. 4).

Places marked by $\oplus$ in the Constellation Table of §10.1 signify *inner* Hurwitz spaces components that support such an assignment of half-canonical classes. This example shows how the *Schur multiplier* of a finite group appears in describing connected components of Hurwitz spaces (§10.2). It is a taste of the nilpotent theory arising in Modular Towers (§8.3). One last subtlety, however, occurs. Only some half-canonical translates work to give a formula like (6.4). They must be *odd*; the linear system has odd dimension (Chap. 4). This includes that $\theta(\mathbf{0}) = 0$: When you plug in $x = x_i^0$ you expect $h(x_i^0) = 0$. For the correct multiplicity of a zero on the right of (6.4), the gradient of the θ at $\mathbf{0}$ also must be nonzero. Such half-canonical classes always exist (Chap. 4).

Half-canonical classes, however, attached to $\oplus$ components in §10.1 are *even*. Sometimes they provide nontrivial *θ-nulls* along the moduli space.

Riemann was even less algebraic in relating $\bar{X}_f$ and its Jacobian. He used coordinates from $\tilde{X}_f$, its universal covering space, to uniformize $\bar{X}_f$.

6.5. Complication 2: $\tilde{X}_f$ and nilpotent covers. The analytic isomorphism class of $\tilde{X}_f$ depends on the genus g of $\bar{X}_f$. If $g = 0$ it is the sphere, if $g = 1$ it is $\mathbb{C}$ and it is *the upper half plane* $\mathbb{H}$ (or disk) if $g \geq 2$. As with $U_{\boldsymbol{z}}$ (§4.1), suppose we accept that $\tilde{X}_f$ is an analytic subspace of the Riemann sphere. Then, this comes from the Riemann mapping theorem. Still, it is not the uniformizing space we would expect. That would be $\tilde{X}^{\mathrm{ab}}_f$, the quotient of $\tilde{X}_f$ by the subgroup of $\pi_1(\bar{X}_f)$ generated by commutators. This is the maximal quotient of $\tilde{X}_f$ that is an abelian cover of $\bar{X}_f$.

6.5.1. *Abelian Frattini covers.* Mathematics rarely looks directly at $\tilde{X}^{\mathrm{ab}}_f$. It embeds in the universal covering space $\mathbb{C}^g$ of $J(\bar{X}_f)$. It is on $\mathbb{C}_g$ that $\theta_{\bar{X}_f}$ lives with its zeros, the Θ divisor, meeting $\tilde{X}^{\mathrm{ab}}_f$ transversally. Periods of differentials on $\bar{X}_f$

translate $\tilde{X}_f^{\rm ab}$ into itself. Yet, it is sufficiently complicated there seems to be no device for picturing it.

There are two models for picturing this. A standard picture shows the complex structure on a complex torus (like the Jacobian). It is of a fundamental domain (parallelpiped) in $\mathbb{C}^g$. Then, $2g$ vectors representing generators of the lattice defining the complex torus (Chap. 3) give the sides of the parallelpiped. Inside this sits the pullback of $\bar{X}_f$. The geometry for this picture uses geodesics (straight lines) from the flat (Euclidean) metric defining distances on the complex torus.

Assume the genus of $\bar{X}_f$ is at least 2. Then, the universal covering $\tilde{X}_f$ of $\bar{X}_f$ is the upper half plane $\tilde{X}_f$. A standard picture for $\bar{X}_f$ appears by grace of this having the structure of a negatively curved space. Geodesics here provide a polygonal outline of a set representing points of $\bar{X}_f$ (Chap. 4). Since $\tilde{X}_f \to \tilde{X}_f^{\rm ab}$ is unramified, $\tilde{X}_f^{\rm ab}$ inherits a metric tensor with constant negative curvature. Yet, it sits snuggly in a flat space. Every finite abelian (unramified) cover Y of $\bar{X}_f$ is a quotient of $\tilde{X}_f^{\rm ab}$; it is a minimal cover of $\bar{X}_f$ with that property. Recall: We started with $\varphi_f : \bar{X}_f \to \mathbb{P}^1_z$. Assume it is a Galois cover, with group G.

Let $\mathcal{G}_f$ denote the abelian covers $\psi : Y \to \bar{X}_f$ with $\psi_f = \psi \circ \varphi_f : Y \to \mathbb{P}^1_z$ also Galois. Call ψ_f a (relatively abelian) *Frattini cover* if the following holds. For any sequence $Y \to W \to \mathbb{P}^1_z$, of covers with $W \neq \mathbb{P}^1_z$, there is always a proper cover of $\mathbb{P}^1_z$ that $W \to \mathbb{P}^1_z$ and $\bar{X}_f \to \mathbb{P}^1_z$ factor through. A Frattini cover has no differentials and functions that pull back from covers disjoint from φ_f, so its function theory isn't accessible by knowing smaller degree covers. The most mysterious quotients of $\tilde{X}_f^{\rm ab}$ are these *relatively abelian* Frattini covers.

This Frattini cover notion does not require an abelian cover ψ. Still, a Frattini cover arises always from ψ being a Galois cover with *nilpotent* (a product of its p-Sylows) group.

6.5.2. *No universal nilpotent cover.* Relatively nilpotent Frattini covers produce natural sequences of moduli spaces generalizing sequences of modular curve covers (§8.3). Further, these moduli space sequences interpret many expectations about the regular version of the Inverse Galois Problem (§8). Relatively nilpotent covers and especially relatively Frattini covers bring up a combination of group theory and function theory. This includes many problems around new aspects of the abelian theory using the Frattini property. This book explores aspects of it through these sequences of moduli spaces.

A complete understanding of all nilpotent (versus abelian) covers of $\bar{X}_f$ requires new, recent, ideas. An immediate difficulty is that there is no $\tilde{X}_f^{\rm nil}$ similar to $\tilde{X}_f^{\rm ab}$. Equivalently, no nontrivial subgroup of $\Gamma_0 = \pi_1(\bar{X}_f, x_0)$ is in the intersection of all iterates of commutators in this group.

That is, let $\Gamma_k < \pi_1(\bar{X}_f, x_0)$ be elements of form $(g_1(g_2(\dots g_{k-1}, g_k)\dots)$ with $g_1, \dots, g_k \in \Gamma_0$. Only $\{1\}$ is in all the Γ_ks. So, putting structure on the complete collection of algebraic nilpotent covers of $\bar{X}_f$ requires profinite limits. First consider how profinite limits appear in $G_{\mathbb{Q}}$ acting on points of the moduli spaces.

7. Acting with $G_{\mathbb{Q}}$

What changes in replacing $\boldsymbol{z}$ by $\boldsymbol{z}'$, another r-tuple of elements? You might expect the fundamental group of $U_{\boldsymbol{z}}$ to tell nothing about changes. As a group it remains the same. We don't, however, use it as an abstract group. Its generators appear directly in applications. Changing $\boldsymbol{z}$ forces changing generators. Yet, we understand the braiding changes from H_r (§5.2). From elementary principles they give a profinite guide for action of $G_{\mathbb{Q}}$.

7.1. Acting on Laurent series. Suppose $\sigma \in G_{\mathbb{Q}}$ and $z_0 \in \mathbb{Q}$. Assume $f(z) = \sum_{n=N}^{\infty} a_n(z-z_0)^n$ has coefficients in $\bar{\mathbb{Q}}$. Then, σ acts on the a_n s, producing f_σ. The hypothesis, however, of algebraic coefficients won't hold for $\tilde{f}_{\boldsymbol{z}}$ from (4.3).

7.1.1. *Setup for a test Case:* $r = 3$. Suppose z_1, z_2, z_3 are in $\mathbb{Q}$. Change the variable z by an element of $\mathrm{SL}_2(\mathbb{Z})$ to map $\{z_1, z_2, z_3\}$ in some order to $\{0, 1, \infty\}$. Six different permutations $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ do this, depending on the order we choose. Composing $\tilde{U}_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ with one of these produces $\lambda : \tilde{U}_{\boldsymbol{z}} \to U_{0,1,\infty} = \mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$. Riemann's uniformization appears from a classical function, $\lambda : \mathbb{H} \to U_{0,1,\infty}$ (Chap. 4).

7.1.2. *Uses for* $\lambda(\tau)$. Periods of an antiderivative of $F(z)$ form an additive subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (§3.2). In that notation, consider

$$m(z, w) = w^2 - z(z-1)(z-\lambda)$$

with $\lambda \in \mathbb{P}^1_z \setminus \{0, 1, \infty\}$. Choose $\tau \in \mathbb{H}$ so the function λ takes τ to the value λ (appearing in $m(z, w)$). Identify $\mathbb{Z} \times \mathbb{Z}$ with the subgroup H_τ of $\mathbb{C}$ that 1 and τ generate. Other choices of τ give the same lattice H_τ. It only depends on λ. Let $\Gamma(2)$ be the group of integral matrices congruent to the identity matrix modulo 2. Suppose $\lambda(\tau_0) = \lambda_0$. Then, $\tau \mapsto \lambda(\tau)$ has as preimage of λ_0 the set $\Gamma(2)(\tau_0) = \{\alpha(\tau_0) \mid \alpha \in \Gamma(2)\}$: λ uniformizes $\mathbb{H}/\Gamma(2)$.

Picard used λ to show any nonconstant function $f(z)$ meromorphic on $\mathbb{C}$ excludes at most three values. Assume otherwise, and $f(\mathbb{C})$ excludes $0, 1, \infty$. Then the *monodromy theorem* (Chap. 3) analytically continues $\lambda^{-1} \circ f$ to a function $\mathbb{C} \to \mathbb{H}$. The maximum modulus principle prevents existence of nonconstant holomorphic function maps $\mathbb{C}$ into the upper half plane. This contradiction shows f must be constant [**Ahl79**, p. 307].

7.1.3. *Another valuable function.* Ordering the coordinates of $\boldsymbol{z}$ violates some of our goals. The origins of the subject kept that in mind. Use the notation $U_{\lambda:0,1,\infty}$ when the variable for $U_{0,1,\infty}$ is λ. Six elements of $\mathrm{PSL}_2(\mathbb{Z})$, forming a subgroup S, leave stable the set $\{0, 1, \infty\}$. Then, S acts on $U_{\lambda:0,1,\infty}$. The quotient is $\mathbb{P}^1_j \setminus \{\infty\} = U_{j:\infty}$. The composite from $\mathbb{H} \to U_{j:\infty}$ is a Galois cover with group $\mathrm{PSL}_2(\mathbb{Z})$ (Chap. 4). It is ramified (not a topological cover) over fixed points of elements in $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalues 4th or 6th roots of 1. We use $j(\tau)$ to display how Modular Towers of reduced Hurwitz spaces when $r = 4$ (four elements in $\mathbf{C}$) generalize classical modular curves.

7.1.4. $G_{\mathbb{Q}}$ *won't directly act on* λ *and* j. A theorem of Schneider-Siegel says $\tau(z_0)$ and z_0 are simultaneously algebraic only if τ is the ration of periods for an elliptic curve with complex multiplication. Therefore, even the constant term in the expansion of $\lambda^{-1}(z)$ around z_0 won't often by algebraic. That illustrates the extent previous generations sought to prove properties of $\lambda(\tau)$. Here, however, it shows using $\tilde{f}_{\boldsymbol{z}}$ directly for the action of $G_{\mathbb{Q}}$ won't work.

7.2. Profinite fundamental groups. Suppose $X \to U_{\boldsymbol{z}}$ is a finite (unramified) cover, and $\boldsymbol{z}$ consists of algebraic points. Then, $X = X_f$ where f has the following properties (Chap. 4).

(7.1a) It is defined by a nontrivial polynomial equation $m(z, f(z)) \equiv 0$.
(7.1b) $m = m(z, w)$ has algebraic coefficients.

(7.1c) $\frac{\partial m}{\partial w}(z_0)$ and $m(z_0, w)$ have no simultaneous zeros.

Apply the implicit function theorem (Chap. 2). It says $m(z, w)$ has $\deg_w(m)$ distinct zeros in $\mathcal{L}_{z_0}$. Conclude: Coefficients of $f(z)$ around z_0 are algebraic.

7.2.1. *Grothendieck's Alternative.* Define $\sigma \in G_{\mathbb{Q}}$ acting on a path γ through what the result does to algebraic functions f:

$$f \mapsto f_{\sigma^{-1}\circ\gamma\circ\sigma} = f_{\gamma^\sigma}.$$

In words: Apply σ^{-1} to the coefficients of f, analytically continue f around γ and then apply σ to the coefficients of the result. The effect of γ on algebraic functions determines it. So this determines γ^σ.

PROBLEM 7.1. What does γ^σ look like?

Only if σ is complex conjugation $\mathfrak{c}$ will there be a *path* γ' (independent of f) so that represent $f_{\gamma^\sigma} = f_{\gamma'}$. To see this, apply the theorem of Artin-Schreier: σ, if not complex conjugation $\mathfrak{c}$, either has infinite order or it is $\mu\mathfrak{c}\,\mu^{-1}$ where all powers of μ give distinct conjugates of $\mathfrak{c}$. Further, σ and μ generate an *uncountable* subgroup of $G_{\mathbb{Q}}$. If all the γ^σ s were paths, $\{\gamma^{\sigma'}\}_{\sigma'\in\langle\sigma\rangle}$ would have to be a countable, therefore finite, set. Simple considerations show this is impossible.

7.2.2. *Where can we put γ^σ?* Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The collection $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ is in the Laurent series about z_0. With no loss we're allowed to assume the coefficients are in $\bar{\mathbb{Q}}$.

This gives an ordering: $f \le g$ if $\bar{\mathbb{Q}}(z, g) \supset \bar{\mathbb{Q}}(z, f)$. Action of a path on $\bar{\mathbb{Q}}(z, g)$ determines its action on $\bar{\mathbb{Q}}(z, f)$. So, paths act on the equivalence classes and respect this ordering. Each equivalence class defines a specific function field inside $\mathcal{L}_{z_0}$. It is the exact data you get from a cover and a point on the cover over z_0. The ordering allows considering $\mathcal{P}_{z_0}$, projective systems of (algebraic) points over z_0. Thus, paths act on $\mathcal{P}_{z_0}$ (Chap. 4 or [**Ihar91**, p. 104]).

PROPOSITION 7.2. *This action on $\mathcal{P}_{z_0}$ determines paths in $\pi_1(U_{\boldsymbol{z}}, z_0)$. The collection $\{\gamma^\sigma\}_{\gamma\in\pi_1(U_{\boldsymbol{z}},z_0),\sigma\in G_{\mathbb{Q}}}$ also acts on $\mathcal{P}_{z_0}$. Define π_1^{alg} to be the projective completion of this action. Then, π_1^{alg} is the completion of π_1 by all normal subgroups of finite index. Further, $G_{\mathbb{Q}}$ acts on this.*

7.3. Extending $G_{\mathbb{Q}}$ action. Extend the homomorphism $\pi_1(U_{\boldsymbol{z}'}, z_0) \to G$ to $\psi_{\boldsymbol{z}',z_0} : \pi_1(U_{\boldsymbol{z}'}, z_0)^{\mathrm{alg}} \to G$. As a profinite group, $\pi_1(U_{\boldsymbol{z}'}, z_0)^{\mathrm{alg}}$ is also a free group on r (topological) generators modulo one relation. Here, however, there are many more sets of *classical generators*.

For $G_{\mathbb{Q}}$ to act requires $\boldsymbol{z}$ is stable under $G_{\mathbb{Q}}$. Then, $G_{\mathbb{Q}}$ acts on $\pi_1(U_{\boldsymbol{z}'}, z_0)^{\mathrm{alg}}$ through Ihara's *pro-braid group* if $z_0 \in \mathbb{Q}$. Again, recognize this action through its effect on classical generators of $\pi_1(U'_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$. Dependence on z_0 is so subtle, that any two distinct choices of z_0 give different actions. One remedy is to consider only the induced action of $G_{\mathbb{Q}}$ modulo inner automorphisms by $\pi_1(U_{\boldsymbol{z}',z_0})^{\mathrm{alg}}$. Two further points guide investigations.

(7.2a) Unless the cover $X_{\boldsymbol{z}',z_0} \to \mathbb{P}^1_z$ coming from $\psi_{\boldsymbol{z}',z_0}$ is Galois and defined (with its automorphisms) over $\mathbb{Q}$, the action of $G_{\mathbb{Q}}$ won't respect $\psi_{\boldsymbol{z}',z_0}$.

(7.2b) The action is so big, interesting properties of $G_{\mathbb{Q}}$ are hard to detect at the level of finite covers.

7.4. Motivation from the Inverse Galois Problem**.** Consider a finite group G and the *regular* version of the Inverse Problem. It says for some $\boldsymbol{z}$, G should be the group of a cover of $U_{\boldsymbol{z}}$ with it and its automorphisms over $\mathbb{Q}$. That is, G should be an r-branch point realization over $\mathbb{Q}$. To find r and this cover needs structure.

You won't want to do one group at a time. So, we look at various quotients of $\pi_1(U_{\boldsymbol{z}',z_0})^{\text{alg}}$ with classical generators up to an action by H_r. Then, use $G_{\mathbb{Q}}$ action to investigate when there might be a value of r and a corresponding $\boldsymbol{z}'$ to realize such a quotient over $\mathbb{Q}$. Rather, however, than taking finite group quotients of $\pi_1(U_{\boldsymbol{z}',z_0})^{\text{alg}}$, take them maximally *Frattini*. Then dependence of $G_{\mathbb{Q}}$ action on $\boldsymbol{z}'$ has some uniformity. This gives the application generalizing modular curves Chap. 5 calls *Modular Towers*.

Start with a finite group G. Call a surjective homomorphism $\mu : H \to G$ *Frattini* if for any subgroup $H^* \le H$, $\mu(H^*) = G$ implies $H^* = H$. This is the exact group translation of the cover property from §6.5.1. Suppose μ corresponds to a sequence of covers $\mu^* : X \to X/\ker(\mu) \to X/H$. Then, any proper cover W appearing in the factorization $X \to X/H$ must factor properly through the cover $X/\ker(\mu) \to X/H$. A profinite group $\tilde{G}$ gives the maximal Frattini cover of G. All other group covers of $\mu : H \to G$ are targets for the map $\tilde{G} \to G$. Given $\psi : \pi_1(U_{\boldsymbol{z}})^{\text{alg}} \twoheadrightarrow G$, a significant geometric invariant of ψ is the set of maximal Frattini quotients of $\pi_1(U_{\boldsymbol{z}})^{\text{alg}}$ (quotients of $\tilde{G}$) appearing as factors of ψ. These *Frattini invariants* interpret properties of the levels of Modular Towers. Their simplest instances refine Riemann's theory of θ characteristics (§10.1). They give many implications for the Inverse Galois Problem.

Conjugacy classes $\mathbf{C}$ hit by classical generators separate these homomorphisms discretely. This data gives structure to the problem. A preliminary investigation with $(G, \mathbf{C})$ from the Branch Cycle Lemma (Chap. 9, see §8.2) produces a necessary condition for a $(G, \mathbf{C})$ realization (over $\mathbb{Q}$). It is that $\mathbf{C}$ be a *rational union* of conjugacy classes.

8. Extensible nilpotent functions and the group $\tilde{G}$

We explain the *universal Frattini cover* $\tilde{G}$ of G following the guide of Abel. He solved an inverse problem to part of the expression by radicals problem. This produced dihedral group extensions, labeled by parameters still appearing in treatments of modular curves. For a prime p, $\mathbb{Z}_p$ denotes the p-adic numbers. Suppose A and B are two abelian groups. Assume elements of A act as automorphisms of B: $a \in A$ acts on $b \in B$ giving $a(b)$. Then, form a group on $A \times B$ (called $A \times^s B$) using multiplication of 2×2 matrices:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+a' & ab'+b \\ 0 & 1 \end{pmatrix}.$$

8.1. A guide from dihedral groups. Case: $G = D_p = \mathbb{Z}/p \times^s \{\pm 1\}$ has $\mathbb{Z}_p \times^s \{\pm 1\}$ and $\mathbb{Z}_p \times^s \mathbb{Z}_2$ as the pieces of its universal Frattini cover. Patch these together as a fiber product over D_p. This generalizes: For each prime p dividing $|G|$, there is a universal p-Frattini cover ${}_p\tilde{G}$ (Chap. 5). You can deal with one prime at a time. So, for investigating the arithmetic properties of quotients of $\pi_1(U_{\boldsymbol{z}})^{\text{alg}}$, consider the biggest quotients compatible with r and $\mathbf{C}$ satisfying the Branch Cycle Lemma. Let p be a prime. Recall: A conjugacy class in a finite group is called p' if its elements have order prime to p.

Certain properties of ${}_p\tilde{G}$ suggest levels of a tower of moduli spaces.

(8.1a) ${}_p\tilde{G} \to G$ has a pro-free pro-p group $\ker_0$ as kernel.

(8.1b) It has a characteristic sequence of quotients G_k, $k = 0, 1, \dots$.
(8.1c) Each p'-conjugacy class of G lifts uniquely to a p'-conjugacy class of ${}_p\tilde{G}$.
(8.1d) Elements of G_k whose images in G generate, already generate G_k.

Form $\ker_1$ as the closed subgroup of $\ker_0$ generated by $\ker_0^p$ and the commutators $(\ker_0, \ker_0)$. This gives G_1 in (8.1b) as the quotient ${}_p\tilde{G}/\ker_1$. Continue inductively to form the other G_k s.

8.2. Applying the Branch Cycle Lemma. When there is profinite data, or over $\mathbb{R}$ or $\mathbb{Q}_p$, the explicit formula from the Branch Cycle Lemma is valuable.

Suppose $\sigma \in G_{\mathbb{Q}}$ maps to $n_\sigma \in \hat{\mathbb{Z}}^* = G(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$. Find $\pi \in S_r$ to satisfy $z_i^\sigma = z_{(i)\pi}$. Then, a $(G, \mathbf{C})$ realization (over $\mathbb{Q}$ at $\boldsymbol{z}$) implies

$$C_{(i)\pi}^{n_\sigma} = C_i,\ i = 1, \dots, r. \tag{8.2}$$

Suppose the following:

(8.3) $\mathbf{C}$ consists of r conjugacy classes whose elements have orders prime to p.

Note: Classes $\mathbf{C}$ from G uniquely extend to p' classes in all G_k s. Also, suppose $(G, \mathbf{C})$ passes Branch Cycle test (8.2). Then, so does $(G_k, \mathbf{C})$ for *all* values of k. This illustrates a phenomenon: The groups G_k are similar. So, they produce a guiding question.

QUESTION 8.1. Are the G_k s so similar their realizations fall to the Inverse Galois Problem with a k-free bound on the number of branch points?

The answer is conjecturally "No!" If you bound the number of branch points, there should be a bound on the values of k for which G_k has a K regular realization where K is a number field. Making this bound explicit, however, is another matter. The Mazur-Merel Theorem is well-known. It says, for any number field K, there is an explicit bound C_K on p^{k+1} so that for $p^{k+1} > C_K$, there are no non-cusp rational points on the modular curve $X_1(p^{k+1})$. Below we see this interprets as the easiest special case of this conjecture: There are but finitely many four branch point, dihedral group involution realizations. The first step in the process forces us into investigating the structure of some Modular Tower. An H-M (*Harbater-Mumford*) representative of $(G, \mathbf{C})$ is an r-tuple $\boldsymbol{g} \in \mathbf{C}$ with this property:

(8.4) $\langle \boldsymbol{g} \rangle = G$ and $g_{2i-1} = g_{2i}^{-1}$, $i = 1, \dots, s$ with $r = 2s$.

Approach the following statement by considering r' to be very large (say, two trillion). Then, consider if you can see a difference between the following cases.

(8.5a) G is the *monster* (or use your favorite simple group) and $p = 2$.
(8.5b) G is D_5 and $p = 5$.

THEOREM 8.2. *Fix r'. Suppose there are $(G_k, \mathbf{C}_k)$ realizations over $\mathbb{Q}$ with $r_k \le r'$ conjugacy classes in $\mathbf{C}_k$, for each $k \ge 0$. Then, there exists $r \le r'$ and p'-conjugacy classes $\mathbf{C}$ with $(G_k, \mathbf{C})$ realizations over $\mathbb{Q}$ for all k.*

If $p = 2$, each $(G_k, \mathbf{C})$ realization falls on a Hurwitz space component corresponding to an H_r orbit containing H-M representatives.

8.3. Thm. 8.2 and Modular Towers. Thm. 8.2 (Chap. 5) produces p' conjugacy classes $\mathbf{C}$ in ${}_p\tilde{G}$ and a sequence $\{\boldsymbol{z}_k\}_{k=0}^\infty$ of $\mathbb{Q}$-stable unordered r-tuples of distinct points from $\mathbb{P}_z^1$. This sequence has the property that $\boldsymbol{z}_k$ lies under a $(G_k, \mathbf{C})$ realization. Further, suppose $p = 2$. Then, the attached homomorphisms $\pi_1(U_{\boldsymbol{z}_k})^{\mathrm{alg}} \to {}_p\tilde{G}$ send classical generators of $\pi_1(U_{\boldsymbol{z}_k})^{\mathrm{alg}}$ to H-M representatives in ${}_p\tilde{G}$ so the induced quotient to G_k has $G_{\mathbb{Q}}$-stable kernel.

Chap. 5 shows how this system of realizations fits into a system of moduli spaces generalizing classical modular curves. Consider all maps $\pi_1(U_{\boldsymbol{z}}, z_0) \to {}_p\tilde{G}$ with generators $\boldsymbol{g}$ mapping to $\mathbf{C}$ as $\boldsymbol{z}$ runs over U_r ($z_0 \notin \boldsymbol{z}$). For each k this produces an affine algebraic variety $\mathcal{H}_k$. Its $\mathbb{C}$ points correspond to equivalence classes of maps $\pi_1(U_{\boldsymbol{z}}, z_0) \to G_k$ (with $\boldsymbol{z}$ variable). The group $\mathrm{GL}_2(\mathbb{C})$ acts on these spaces. The quotient is another affine variety $\mathcal{H}_k^{\mathrm{rd}}$, level k of the Modular Tower for $(G, \mathbf{C}, p)$.

A significant case: $G = D_p$ (p odd), p the prime and $\mathbf{C}$ is $r = 4$ repetitions of the conjugacy class of involutions (elements of order 2) in D_p. Then, $\mathcal{H}_k^{\mathrm{rd}}$ is the modular curve $X_1(p^{k+1})$ minus its cusps. Each case with $r = 4$ produces a tower of curves, respective quotients of the upper half plane by finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Usually the Modular Tower levels are noncongruence covers. They always have a useful moduli space structure.

8.4. A diophantine view of a nilpotent theory. Generalizations of theorems of Mazur and Serre now have formulations through the action of $G_{\mathbb{Q}}$ on projective systems of points on the spaces

$$\cdots \to \mathcal{H}_{k+1}^{\mathrm{rd}} \to \mathcal{H}_k^{\mathrm{rd}} \to \cdots \to \mathcal{H}_0^{\mathrm{rd}} \to U_r^{\mathrm{rd}} = J_r. \tag{8.6}$$

CONJECTURE 8.3 (Main Conjecture). Suppose $(G, \mathbf{C}, p)$ is data for a Modular Tower. Assume G is centerless and does not have $\mathbb{Z}/p$ as a quotient. For k large, $\mathcal{H}_k^{\mathrm{rd}}$ has no $\mathbb{Q}$ points.

8.4.1. *Interpreting the Main Conjecture.* Thus, $\mathbb{Q}$ realizations of G_k require increasing large sets of conjugacy classes for k large. This is more refined information than from any known versions of the Branch Cycle Lemma. If $p = 2$, Thm. 8.2 says rational points will appear only on H-M components of the sequence, and this refines the problem immensely. Changing $\mathbb{Q}$ to another number field K requires significant generalization (Chap. 5).

Here is a response to the setup of cases from (8.5). Both require information on the geometry of Modular Tower levels we don't know yet. The dihedral group case (with r equal two trillion) looks easier because it translates to statements about classical moduli spaces: The moduli of cyclic 5^{k+1} degree covers of hyperelliptic curves (of genus 1,000,000,000,000-1). No one knows if this space is without $\mathbb{Q}$ points for large k. Suppose the curves in the family have genus 1. Then we know much since the Modular Tower levels are modular curves.

Yet, with the monster, there could be surprises. For example, for $(A_6, \mathbf{C}_{3^5})$ with $p = 2$, there are no $\mathbb{Q}$ points at level 1 of the Modular Tower. Reason: There are no points at level 1 at all, the result of the $\otimes$ symbol at (6,5) in the Constellation Table of §10.1. The case $r = 4$ gets much attention for problems that immediately generalize those for modular curves (§10.5).

8.4.2. *Nilpotency from projective systems of points.* Let X be a compact Riemann surface. Denote the pro-p quotient of the fundamental group of X by $\pi_1(X)^{(p)}$. When this group appears only up to inner automorphism, we drop the notation for the base point. Thm. 8.2 includes a nilpotent theory. Consider one of the homomorphisms $\psi_{\boldsymbol{z}} : \pi_1(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}} \to {}_p\tilde{G}$ mapping a fixed set of classical generators of into the p'-conjugacy classes $\mathbf{C}$.

Let $X_0 \to \mathbb{P}_z^1$ be the G quotient cover from this homomorphism. For investigating all possible such maps $\psi_{\boldsymbol{z}}$, note it factors through a smaller quotient group of $\pi_1(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$. This is an extension $M_{\boldsymbol{z}}$ (independent of $\psi_{\boldsymbol{z}}$ as a group extension) of $G = G_0$ by $\pi_1(X_0)^{(p)}$.

Call two such homomorphisms $M_{\boldsymbol{z}} \to {}_p\tilde{G} \to G_0$ *inner equivalent* if they differ by inner automorphisms from $\ker_0$ in (8.1a). Suppose $X_0 = X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ corresponds to $\boldsymbol{p} \in \mathcal{H}_0^{\text{rd}}$. Projective systems of points on the Modular Tower over $\boldsymbol{p}$ correspond to inner homomorphism classes of $M_{\boldsymbol{z}} \to {}_p\tilde{G} \to G_0$. Shorten this phrase to *a point* on the Modular Tower. In this case refer to $M_{\boldsymbol{z}}$ as $M_{\boldsymbol{p}}$. Let the set of inner homomorphism classes be $\mathcal{T}_{\boldsymbol{p}}$.

Homomorphisms factoring through ${}_p\tilde{G}$, surjective to G_0, map surjectively to ${}_p\tilde{G}$ (from (8.1d)). Let $g = g(X_{\boldsymbol{p}})$ be the genus of $X_{\boldsymbol{p}}$ — transparent from $\mathbf{C}$ by the Riemann-Hurwitz formula. So, $\pi_1(X_{\boldsymbol{p}})^{(p)}$ is a free pro-p group on $2g$ generators modulo one *commutator* relation.

8.4.3. *$G_{\mathbb{Q}}$ action on* $\text{Ni}({}_p\tilde{G}, \mathbf{C})^{\text{in}}$. The notion of Nielsen class (§5.4) applies uniformly to $({}_p\tilde{G}, \mathbf{C})$. Its absolute and inner versions inherit an H_r action. Orbits for this action correspond to projective systems of components at the levels of the Modular Tower. Reducing this action modulo $\ker_0$ maps each orbit to an H_r orbit at level 0. Components of $\mathcal{H}_k^{\text{rd}}$ (over $\bar{\mathbb{Q}}$) map among each other by $G_{\mathbb{Q}}$ acting on the coefficients of their equations.

We don't often see equations for these moduli spaces. So, figuring this action from the data is one of our main problems. From this, regard $G_{\mathbb{Q}}$ as acting on the H_r orbits in $\text{Ni}({}_p\tilde{G}, \mathbf{C})^{\text{in}}$. In the A_n examples of §10.1, there are components at a finite level k that have no projective system of components above them. This could happen with any $(G, \mathbf{C})$. The invariant in §9.1 catches these *obstructed components* precisely, when you can compute it (Chap. 5).

PROBLEM 8.4. Compute the $G_{\mathbb{Q}}$ action on H_r orbits of $\text{Ni}({}_p\tilde{G}, \mathbf{C})^{\text{in}}$. Also, compute the pattern of chains of obstructed components.

8.4.4. *A nilpotent Tate Grassmanian.* For G *any* finite group this theory has a large pro-nilpotent part. Thus, it generalizes the abelian theory setup.

Suppose $\boldsymbol{z} \in U_r$ lies below a $\mathbb{Q}$ point $\boldsymbol{p} \in \mathcal{H}_0^{\text{rd}}$. Then, $G_{\mathbb{Q}}$ acts on $\pi_1(X_{\boldsymbol{p}})^{(p)}$ (modulo inner automorphisms) as a quotient of the action on $\pi_1(X_{\boldsymbol{p}})^{\text{alg}}$. Act by $G_{\mathbb{Q}}$ on the quotient of $\pi_1(X_{\boldsymbol{p}})^{(p)}$ by the closed subgroup of commutators. Denote this quotient by $T_{\boldsymbol{p}}$, the Tate module for p. This gives the theory of abelian covers of $X_{\boldsymbol{p}}$ with group order a power of p. Its relation to the Jacobian of $X_{\boldsymbol{p}}$ is clear. It is the projective system of points of p-power order on the Jacobian.

Continue the actions of $G_{\mathbb{Q}}$. Suppose α is in $\mathcal{T}_{\boldsymbol{p}}$. Then, $\sigma \in G_{\mathbb{Q}}$ acts on α (on the right) through the composition $\alpha \circ \sigma$ (Chap. 9). There is a *Lie algebra structure* on $\pi_1(X_{\boldsymbol{p}})^{(p)}$. Using it and the *Weil pairing* allows dualizing these maps. The result is $\mathcal{T}_{\boldsymbol{p}}^*$, a nilpotent version of $G_{\mathbb{Q}}$ acting on a *Grassmannian* of a *Tate module* of the Jacobian for $X_{\boldsymbol{p}}$ (Chap. 5).

One goal of Modular Towers is to provide *small* actions for $G_{\mathbb{Q}}$. Modular Towers retains the feel of finite groups. Though a generalization of modular curves, the group theory reminds of situations yielding groups as Galois groups. Chap. 9 reviews achievements of that program, appearing in detail in [**Se92**], [**MM95**] and [**Vö96**] (see [**Fri94**]). In particular, the Dettweiler-Völklein generalization of Katz's *rigid tuples* [**DVo98**] pushes realization of Chevalley simple groups to a new place. It produces many cases with G_0 simple where $\mathbb{Q}$ points are dense in $\mathcal{H}_0^{\text{rd}}$.

These give a setting for $\widehat{\mathcal{G}T}$ relations close to the Inverse Galois Problem territory. Yet, the pro-finite elements of Modular Towers are like those of modular curve towers, suitable for checking the effect of these constraints. One goal is to see if $\widehat{\mathcal{G}T}$ relations force significant quotients of ${}_p\tilde{G}$ to have $\mathbb{Q}$ realizations.

9. The Grothendieck-Teichmüller group

When $G_{\mathbb{Q}}$ acts on fundamental groups related to moduli spaces, that action preserves underlying geometry. Often that geometry is not obvious to us. So, asking what to expect from a $G_{\mathbb{Q}}$ action has us delving more deeply to where the geometry appears. The principle everyone uses occurs in divining components of a moduli space. The expectation is $G_{\mathbb{Q}}$ should map these components among each other, unless a geometric reason prevents it.

9.1. Moduli spaces with several components. The Constellation Table of §10.1 illustrates this. Superficially the two components appearing at the locus (n, r) $(r \geq n)$ have much in common. Action of $G_{\mathbb{Q}}$, however, on their equations leaves them fixed. Setup: The only alternative is it maps one of them to the other, because their union is a moduli space. Finish: The Schur multiplier invariant gives a geometric condition separating the components (§10.2.2).

Does $G_{\mathbb{Q}}$ have relations appearing everywhere in moduli space actions? These would induce relations for $G_{\mathbb{Q}}$ acting on all related moduli spaces (Chap. 5). The Grothendieck-Teichmüller group offers such relations. We discuss now the implication of these for the Inverse Galois Problem. Recall the space $J_r = U_r/\mathrm{PGL}_2(\mathbb{C})$ and its relative $\Lambda_r = (\mathbb{P}^1_z)^r \setminus \Delta_r$ when $r = 4$: $\Lambda_4 = U_{\lambda:0,1,\infty}$ (§7.1.1).

§4.1 has a description of the extensible algebraic functions $\mathcal{E}(\Lambda_4, \lambda_0)^{\mathrm{alg}}$. Each starts from a Laurent series in λ_0 that analytically continues along any path in Λ_4.

9.2. Deligne's tangential base points. Deligne suggested an extra structure to $\mathcal{E}(\Lambda_4, \lambda_0)^{\mathrm{alg}}$ by expanding the choices of base point [**De89**]. The elements of $\mathcal{L}_{\lambda_0}$ sit inside an algebraically closed field $\mathcal{P}_{z_0}$, convergent Puiseux expansions around λ_0 (Chap. 2). They look like Laurent series in $(\lambda - \lambda_0)^{1/e}$ for some integer e. They don't, however, work as functions in a neighborhood of λ_0 (Chap. 2).

Give the special case $\lambda^{1/e}$ meaning by making it take positive values along the real axis pointing from 0 to 1. This produces an analytic expression convergent in a neighborhood of any point on the *positive* real axis between 0 and 1. An alternative would ask $\lambda^{1/e}$ to take positive values along the real axis in the negative direction from 0 to $-\infty$.

Distinguish between those two choices. Extend the meaning of the first to all Puiseux expansions about 0 using the notation $\mathcal{P}_{\overline{0\,1}}$. Each produces a meromorphic function defined near 0 to the right of 0. Similarly, for the second choice use the notation $\mathcal{P}_{\overline{0\,\infty}}$. Each element in this defines a meromorphic function near 0 to the left of 0. To be explicit, choose an open disk (on $\mathbb{P}^1_\lambda$). It should be symmetric about the real axis, tangent to the imaginary axis and contain part of the real axis from 0 to 1 (Chap. 2). Denote this disk $D_{\overline{0\,1}}$.

For any i and j, distinct elements from $\{0, 1, \infty\}$ form the similar set of *functions* $\mathcal{P}_{\overline{i\,j}}$. The ordering from §7.2.2 on algebraic functions in $\mathcal{L}_0$ extends to algebraic elements of $\mathcal{P}_{\overline{0\,1}}$. So does the action of $G_{\mathbb{Q}}$ extend (Chap. 4).

Denote the set of ordered arrows by $\mathbb{B}$. Label the linear fractional transformations that permute $\{0,1,\infty\}$: $t_{\overline{i\,j}}$ takes i to 0, j to 1 and k to ∞. Apply $t_{\overline{i\,j}}{}^{-1}$ to $D_{\overline{0\,1}}$ to get similar disks $D_{\overline{i\,j}}$ attached to $\mathcal{P}_{\overline{i\,j}}$.

PRINCIPLE 9.1 (Branch Extensibility). *Consider $f \in \mathcal{E}(\Lambda_4,\lambda_0)^{\rm alg}$ and i,j distinct elements from $\{0,1,\infty\}$. Suppose $\gamma:[0,1]\to\Lambda_4$ is a path with $\gamma(0)=\lambda_0$ and $\gamma(1)$ in $D_{\overline{i\,j}}$. Then, there exists a unique $F_{f_\gamma}\in\mathcal{P}_{\overline{i\,j}}$ restricting to f_γ. The collection of order preserving maps on the equivalence classes of fields $\mathbb{C}(\lambda,F_{f_\gamma})$ is $\pi_{\overline{0\,1}}=\pi_1(\Lambda_4,\overline{0\,1})^{\rm alg}$. It has a natural $G_{\mathbb{Q}}$ action (*Chap. 4*).*

Let x be a clockwise circle ([**Ihar91**] takes counterclockwise; see comments of §11) around 0 meeting $D_{\overline{0\,1}}$. It represents an element of $\pi_{\overline{0\,1}}$ from Princ. 9.1. For example, suppose in the definition of F_{f_γ} that $\gamma(1)$ is on x. Take $F=F_{f_\gamma}$ equal to $h(\lambda^{1/e})$ with h meromorphic around 0. Let $\zeta_e=e^{\frac{2\pi i}{e}}$.

The effect of x on F is the substitution $\lambda^{1/e}\mapsto\zeta_e^{-1}\lambda^{1/e}$. So, $\sigma^{-1}\circ x\circ\sigma$ (following §7.2.1) gives this sequence of operations on a power series. Act on coefficients with σ^{-1}, then substitute $\zeta_e^{-1}\lambda^{1/e}$, then act by σ on the resulting coefficients. Use the notation of §8.2: n_σ is restriction of σ to cyclotomic numbers. The total effect is the substitution $\lambda^{1/e}\mapsto\zeta_e^{-n_\sigma}z^{1/e}$. So, $x^\sigma=x^{n_\sigma}$.

9.3. The first two relations. Following [**AnIh88**], the $t_{\overline{i\,j}}$s act on Puiseux expansions. So, they give maps among the fundamental groups $\pi_{\overline{i\,j}}$.

9.3.1. *Continuations from $\overline{0\,1}$ to $\overline{1\,0}$.* Extend this to the *fundamental groupoid* (Chap. 3), to give $\pi_{\overline{0\,1}\,\overline{1\,0}}=\pi_1(\Lambda_4;\overline{0\,1},\overline{1\,0})$. Let $\gamma_p:[0,1]\to\Lambda_4$ be a path running along $\mathbb{R}\cup\{\infty\}$ from 0 toward 1, with $\gamma_p(0)\in D_{\overline{0\,1}}$ and $\gamma(1)\in D_{\overline{1\,0}}$. As with x it defines an element of $\pi_{\overline{0\,1}\,\overline{1\,0}}$.

Let x' be the transform of x by $t_{\overline{1\,0}}$ ($\lambda\mapsto 1-\lambda$). Take $y=\gamma_p\circ x'\circ\gamma_p^{-1}$. Then, y represents an element of $\pi_{\overline{0\,1}}$. Even easier than x, $\sigma^{-1}\circ y\circ\sigma$ has the effect of $\gamma_p^\sigma(x')^{n_\sigma}(\gamma^{-1})^\sigma$. Let $m_\sigma=\gamma_p^\sigma\gamma_p^{-1}$. Then y^σ equals $m_\sigma y^{n_\sigma}m_\sigma^{-1}$.

Since x and y are topological generators of $\pi_{\overline{0\,1}}$, the effect of σ on them determines the action of σ. It makes sense to write $m_\sigma(x,y)$. If P_1 and P_2 are two both homotopy classes of paths with the same end points, then they are conjugate. Even though this is a profinite group, apply this to γ_p and γ_p^{-1}. Therefore, m_σ is a commutator in the pro-free group x and y generate.

9.3.2. *The product-one relation.* Most significant is what σ does to xy. Equivalently: What is z^σ, with $z=(xy)^{-1}$ the 3rd element in a product-one relation (as in (5.1)). The formula for this comes from the first two Drinfeld-Ihara relations:

(9.1a) $m_\sigma(x,y)m_\sigma(y,x)=1$; and with $u_\sigma=\frac{n_\sigma-1}{2}$,
(9.1b) $m_\sigma(z,x)z^{u_\sigma}m_\sigma(y,z)y^{u_\sigma}m_\sigma(x,y)x^{u_\sigma}=1$.

Apply $t_{\overline{1\,0}}$ to $m_\sigma(x,y)$ to see (9.1a). Let r be the half-circle from the center of $D_{\overline{1\,0}}$ to the center of $D_{\overline{1\,\infty}}$ going clockwise. Then, r defines an element of $\pi_{\overline{1\,0},\overline{1\,\infty}}$. Expression (9.1b) comes from applying σ to the geometric relation

$$t_{\overline{1\,\infty}}{}^2(r\circ\gamma_p)\circ t_{\overline{1\,\infty}}(r\circ\gamma_p)\circ(r\circ\gamma_p)=1.$$

We left out the famous *5-cycle relation* [**Ihar91**, p. 107]. It forcefully appears soon.

9.3.3. *Return of the j-line.* There is a conspicuous quotient of the fundamental group of $\pi_1(\mathbb{P}^1_{j:0,1,\infty})$ (§7.1.3). It has generators $\boldsymbol{\gamma}$:

$$\gamma_0=q_1q_2,\ \gamma_1=q_1q_2q_1 \text{ and } \gamma_\infty=q_2$$

from a quotient of H_4 (Chap. 5; see §5.3). These satisfy

(9.2) $\gamma_0^3 = 1$, $\gamma_1^2 = 1$, $\gamma_0\gamma_1\gamma_\infty = 1$; the group $\langle\gamma_0, \gamma_1\rangle$ is $\mathrm{PSL}_2(\mathbb{Z})$.

When $r = 4$ a reduced Hurwitz space has a Riemann's Existence Theorem description coming from these generators acting on a *reduced Nielsen class* (Chap. 5). The geometry of the reduced Hurwitz spaces $\{\mathcal{H}_k^{\mathrm{rd}}\}_{k=0}^{\infty}$ shows from analyzing $\boldsymbol{\gamma}$. Most crucial are disjoint cycles of $\gamma_{\infty,k}$, the result of γ_∞ in its action on $\mathrm{Ni}_k^{\mathrm{rd}}$.

PRINCIPLE 9.2 (Cusp Principle). *Each disjoint cycle of $\gamma_{\infty,k}$ corresponds to a cusp point for $\bar{\mathcal{H}}_k^{\mathrm{rd}}$ over $j = \infty$. Further, each cusp has its own geometry.*

9.4. Detecting $\widehat{\mathcal{G}T}$ at the level of a Modular Tower. Relations (9.1) have versions for action of $G_{\mathbb{Q}}$ on $\boldsymbol{\gamma}$. Yet, we must generalize them beyond their present shape to have them suit the geometry of a Modular Tower. Here is why.

9.4.1. *Viewing tangential base points from $\mathbb{P}^4$.* Deligne's tangential base points come from components of real points on $(\mathbb{P}_z^1)^4 \setminus \Delta_4 = U^4$. An example is R_{z_1,z_2,z_3,z_4}: 4-tuples of distinct points on $\mathbb{R} \cup \{\infty\} = \mathbb{R}_\infty$ where the four points are in the same order as $(0, 1, \infty, -1)$ around the circle. Rearrangements from permuting the elements $\{z_1, z_2, z_3\}$ produce new connected components. To get to $\mathbb{B}$, mod out by the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ stabilizing each component.

Formulas similar to (9.1) allow working directly with H_4. Replace elements of $\mathbb{B}$ with the image of R_{z_1,z_2,z_3,z_4} in $\mathbb{P}^4 \setminus D_4 = U_4$. This is in [**IhMa95**] which also treats higher values of r.

9.4.2. *Other real component configurations.* The sets R_{z_1,z_2,z_3,z_4} often fail to capture the cusp geometry on a Modular Tower. Here is an example. Real points on level 1 of the $(A_5, \mathbf{C}_{3^4})$ Modular Tower (§10.1, §10.3) lie on the genus 12 component of $\bar{\mathcal{H}}_1^{\mathrm{rd}}$. Denote that $\bar{\mathcal{H}}_1^+$.

Real points on $\bar{\mathcal{H}}_1^+$ collect in eight disjoint components, each associated to a cusp (of width 20). Four attach to H-M representatives in this Nielsen class (§8.2). Let $CP_{z_1,z_2,z_3,z_4} = \{z_1, z_2 \in \mathbb{H} \mid z_3 = \bar{z}_1,\ z_4 = \bar{z}_2\}$: Two sets of complex conjugate pairs of points, with the first two in the upper half plane. The preimage in the inner Hurwitz space of these eight components is 32 real components. Each lies over a locus of real points in U_4 with preimage in U^4 of type CP_{z_1,z_2,z_3,z_4}.

There are $\binom{4}{2} = 6$ choices for which two coordinates to put in the upper half plane. Then, counting possible lower half plane pairings with z_1 gives a total of 12 such real components of U_4. Action of $\mathrm{PSL}_2(\mathbb{R})$ on CP_{z_1,z_2,z_3,z_4} has a hyperbolic description (Chap. 2). Put z_1 at i under this action, so orbits of z_2 are points at a fixed (hyperbolic) distance from i.

To choose an explicit representative from each orbit, take a (hyperbolic) circle from i to $i + 1$: A half circle perpendicular to the real axis through i and $i + 1$. Then, the interval description for Deligne's tangential base points has as analog the portion of the circle from i to the right of i going through $i + 1$.

9.5. Variants of the Drinfeld-Ihara relations Chap. 9**.** There was a first definition of $\widehat{\mathcal{G}T}$. It was a subgroup of the automorphisms $\mathrm{Aut}(\hat{F}_2)$ of the profree group on generators x and y.

9.5.1. *$\widehat{\mathcal{G}T}$: A moving target.* $\widehat{\mathcal{G}T}$'s elements are automorphisms of the form $(x, y) \mapsto (x^n, my^n m^{-1})$: $n \in \hat{\mathbb{Z}}^*$ and $m \in (\hat{F}_2, \hat{F}_2)$ satisfy relations 9.1 (with the 5-cycle relation). The composition of two is another automorphism. That this composition also satisfies the relations is more serious. This gives a group structure to such pairs (n, m).

After the first definition, there was speculation $\widehat{\mathcal{G}T}$ might contain $G_{\mathbb{Q}}$ as an open subgroup. These days, however, $\widehat{\mathcal{G}T}$ presents a moving target. Recent joint work of Nakamura and Schneps reveals new relations satisfied by the image of $G_{\mathbb{Q}}$ in $\widehat{\mathcal{G}T}$. Its unclear whether to relabel $\widehat{\mathcal{G}T}$ appropriately for these relations or to start indexing a sequence of $\widehat{\mathcal{G}T}$-like groups. Yet, there are still only few of them and each is precious.

9.5.2. *Cusps producing other base points.* Consider $G_{\mathbb{Q}}$ acting as permutations on H_r orbits of a reduced Nielsen class (§8.4.2).

Several steps are necessary to include $\widehat{\mathcal{G}T}$ type relations (Chap. 9). First: Develop corresponding relations from tangential base points using components like CP_{z_1,z_2,z_3,z_4} (as suggested in [**Fri95a**, App. C-D]).

Second: Complete comparing with $\widehat{\mathcal{G}T}$ by extending this action to $\mathrm{Ni}({}_p\tilde{G},\mathbf{C})^{\mathrm{rd}}$. This works because H_4 acting on generators of the 4-punctured sphere identifies with a subgroup H_5' of H_5. We explain.

As in §5.2, consider $(\mathbb{P}^1_z)^5 \setminus \Delta_5 = U^5$. There is a *fibration*, $U^5 \to U^4$ by projection on the first four coordinates. Embed S_4 in S_5 as the permutations leaving 5 fixed. Then, S_4 acts to give a new fibration, $U_4 \times \mathbb{P}^1_{z_5} \setminus D_5' \to U_4$ with D_5' the image of Δ_5 in $\mathbb{P}^4 \times \mathbb{P}^1$ (Chap. 5, [**BF82**], [**DFr99**] for the $\mathbb{R}$ analysis). Even without this quotient, analogs of all $\widehat{\mathcal{G}T}$ relations would appear. The fiber is a copy of $\mathbb{P}^1$ minus four points, with classical generators identified with words in $Q_1,\dots,Q_5$. So, even analogs of the 5-cycle relation (Chap. 9) show in identifying the $G_{\mathbb{Q}}$ action on $\mathrm{Ni}({}_p\tilde{G},\mathbf{C})^{\mathrm{rd}}$ when $r=4$.

Comparison between $\widehat{\mathcal{G}T}$ and Modular Towers then has these practical goals. Use all cusps on a Modular Tower to define the $\widehat{\mathcal{G}T}$ attached to that Modular Tower.

PROBLEM 9.3. What do $\widehat{\mathcal{G}T}$ relations applied to Modular Towers detect about $\mathbb{Q}$ orbits on $\mathrm{Ni}({}_p\tilde{G},\mathbf{C})^{\mathrm{rd}}$. Compare with the Branch Cycle Lemma and ω (§10.2.2) invariant combination?

We conclude by tying together four advanced goals of the research motivating this book. It is convenient to do this by joining classical θ-functions to Modular Towers. Each diophantine element of this section gives specific detailed results on the Modular Towers of this example (Chap. 5).

10. Combining the Existence Theorem and θ functions

The first Hurwitz spaces were moduli spaces of simple branched covers. In this case the Hurwitz spaces are connected. An easy application of the Riemann-Roch theorem then shows connectedness of the moduli space of curves of genus g.

10.1. Theta functions and Hurwitz spaces. An example with many applications comes from covers with alternating groups A_n as monodromy groups. Take A_n, $n \ge 4$, with the prime $p=2$ and 3-cycles (r of them) as data for a Modular Tower. The usual representation T_n gives an absolute space of degree n covers with group A_n. There is a corresponding inner space of Galois covers (as in (5.5)). The following diagram displays the complete set of inner Hurwitz space components at level 0 of their Modular Tower.

Locations in this diagram have an attached integer pair (n,r). The location shows components of the inner Hurwitz space for $(A_n,\mathbf{C}_{3^r})$. Abbreviate this to $\mathcal{H}^{\mathrm{in}}_{n,r}$. The corresponding absolute spaces would be for the data $(A_n,\mathbf{C}_{3^r},T_n)$, or

$\mathcal{H}^{\rm abs}_{n,r}$. The group is the alternating group A_n. Conjugacy classes are r repetitions of 3-cycles. There is a famous *spin* group cover of A_n, $\tilde{A}_n$ where $\tilde{A}_n \to A_n$ is a central nonsplit extension with kernel $\mathbb{Z}/2$. The universal 2-Frattini cover of A_n (as in (8.1) automatically factors through $\tilde{A}_n$. This is a special case of a general phenomenon. The universal p-Frattini cover ${}_p\tilde{G}$ of any perfect group G factors through the universal central extension of G.

Labels for rows are by the genuses of the degree n covers. The relation between the spaces $\mathcal{H}^{\rm abs}_{n,r}$ and $\mathcal{H}^{\rm in}_{n,r}$ comes from a corollary in [**Fri96**].

PROPOSITION 10.1 (Absolute-Inner). *The natural map $\mathcal{H}^{\rm in}_{n,r} \to \mathcal{H}^{\rm abs}_{n,r}$ has degree 2. Each component of the former maps to a corresponding component of the latter.*

10.1.1. *Explanation of the symbols.* Two primitive icons appear. The symbol $\otimes$ corresponds to a(n irreducible) component whose points represent covers $\hat{X} \to \mathbb{P}^1$ with this property. A special degree two unramified cover $\hat{Y} \to \hat{X}$ satisfies

$$\hat{Y} \to \mathbb{P}^1 \text{ composed from } \hat{Y} \to \hat{X} \text{ and } \hat{X} \to \mathbb{P}^1 \text{ is Galois with group } \tilde{A}_n. \quad (10.1)$$

Then, $\oplus$ denotes a component of covers in $\mathcal{H}^{\rm in}_{n,r}$ having no such $\hat{Y}$ cover. Excluding the genus 0 row, all rows have exactly two components. One is of $\otimes$ type, the other of $\oplus$ type. The spin group cover of alternating groups reveals its presence in components of Hurwitz spaces.

10.1.2. *Subtleties about Schur multipliers.* This phenomenon holds in general. Schur multipliers of finite groups produce distinct components of the Hurwitz space. For each conjugacy class C in $\mathbf{C}$, let $b_{\rm C}$ be its multiplicity of appearance in $\mathbf{C}$. A generalization of a Conway-Parker result has as hypothesis that $b_{\rm C}$ is suitably large for all C in $\mathbf{C}$. Conclusion: Distinct components in level k of a Modular Tower correspond exactly to elements in a subgroup of the Schur multiplier.

Yet, whether $b_{\rm C}$ is suitably large depends on G_k (or on k) with $G = G_0$ fixed. This is the issue of §10.2. The Constellation Table shows level 0 of Modular Towers for all alternating groups with $p = 2$ and 3-cycle conjugacy classes.

Further, covers in one component differ from those in another in a simple striking way. Suppose $\hat{\varphi}_{\boldsymbol{p}} : \hat{X}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ is a cover attached to $\boldsymbol{p} \in \mathcal{H}^{\rm in}_{n,r}$. Then the differential $d\hat{\varphi}_{\boldsymbol{p}}$ has a divisor of form $2\hat{D}_{\boldsymbol{p}}$. (This happens whenever all elements in the conjugacy classes $\mathbf{C}$ have odd order.) The divisor $\hat{D}_{\boldsymbol{p}}$ is canonically defined over $\boldsymbol{p}$. Let $\dim(\hat{D}_{\boldsymbol{p}})$ be the dimension of the space of meromorphic functions h on $\hat{X}_{\boldsymbol{p}}$ for which $(h) + D_{\boldsymbol{p}} \geq 0$ (Chap. 3, Chap. 4).

TABLE 1. **Constellation of spaces $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\rm in}$**
Components correspond to lifting invariant values.
Genus at (n,r) of degree n cover: $g = r - n + 1$
of the Galois cover: $g^* = \frac{(r-3)n!}{6}$

$\xrightarrow{g\geq 1}$	$\otimes\oplus$	$\otimes\oplus$	...	$\otimes\oplus$	$\otimes\oplus$	$\xleftarrow{1\leq g}$
$\xrightarrow{g=0}$	$\otimes$	$\oplus$	...	$\otimes$	$\oplus$	$\xleftarrow{0=g}$
$n \geq 4$	$n = 4$	$n = 5$	...	n even	n odd	$4 \leq n$

So, $\hat{D}_{\boldsymbol{p}}$ defines a half-canonical divisor at each point on $\mathcal{H}^{\rm in}_{n,r}$, and a half-canonical divisor class on $\mathcal{H}^{\rm in,rd}_{n,r}$. A formula of Fried-Serre ([**Fri96**], [**Ser90b**]) says the components of $\mathcal{H}^{\rm in,rd}_{n,r}$ separate according to $\dim(\hat{D}_{\boldsymbol{p}})$ modulo 2. For $r \ge n$, there is an $\oplus$ component of *even* half-canonical classes, the other of *odd*. For the components of $\mathcal{H}^{\rm abs}_{n,r}$, define a similar divisor $D_{\boldsymbol{p}}$. Then, the formula for even or odd half-canonical classes is $\dim(D_{\boldsymbol{p}})+r$ [**Ser90b**, Thm. 3]. Note: When $X_{\boldsymbol{p}}$ has genus 0, $\dim(D_{\boldsymbol{p}})$ is 0. Alternating $\otimes$ and $\oplus$ signs in the first row of the Constellation Table correspond exactly to r. §10.2 shows this is a small piece of an invariant applying to every Modular Tower.

10.2. Conjugacy class products. Examples show the Branch Cycle Lemma and ω invariant (§10.2.2) combination work well in this profinite context. Still, computing ω is not yet easy.

10.2.1. *How modular representations appear.* Computing the ω invariant for a Modular Tower relies on modular representation theory. The ω invariant is trivial for the usual modular curve tower. Here the kernel of ${}_p\tilde{G} \to G$ is one dimensional (${}_p\tilde{G} = \mathbb{Z}_p \times^s \{\pm 1\}$ and $\ker_0 = p\mathbb{Z}_p$). It is, however, more interesting for Modular Towers in the Constellation Table of §10.1.

Consider the location $(5,4)$. Four repetitions of the conjugacy class C_3 of 3-cycles appear there. Here consider it a conjugacy class in ${}_2\tilde{A}_5$ (${}_p\tilde{G}$ for A_5 and $p=2$). As above, let C_3^4 be all products of four elements from C_3. Let M_k be G_k/G_{k+1}, the G_k module associated to level k. For any G_k submodule M_k' of M_k there is a quotient $G_{k+1}/M_k' = G_k'$. A special case is when $M_k/M_k' = W_k$ is maximal for G_k acting trivially on it.

Suppose G_0 is a perfect group (includes any simple group). Then, W_k is the maximal exponent p Schur multiplier of G_k and $G_k' = R_k$, the *representation* cover of G_k (Chap. 5). This A_5 case has $p=2$ and $R_k \to G_k$ has kernel $\mathbb{Z}/2$ for each k.

Let O be an H_r orbit of $\mathrm{Ni}(G_k, \mathbf{C})$ with $\boldsymbol{g}$ a representative. Since $R_k \to G_k$ is a central extension, $\boldsymbol{g}$ has a unique lift to $\tilde{\boldsymbol{g}} \in R_k^r \cap \mathbf{C}$. If the product-one condition holds for $\tilde{\boldsymbol{g}}$, then it is in $\mathrm{Ni}(R_k, \mathbf{C})$. Otherwise let $s(\boldsymbol{g}) \in W_k$ be the product of the $\tilde{\boldsymbol{g}}$ entries. Running over all such orbits O creates a subset $\mathbf{Obs}_{1,k} = \mathbf{Obs}_1(G_k, \mathbf{C})$ of W_k not containing the identity.

Suppose O is an H_r orbit with $s(\boldsymbol{g}) = 1$. Consider $M_{k+1} \lneq M_k' \lneq W_k$, with M_k' a G_k submodule. Call O *obstructed* at M_k' if these two properties hold.

(10.2a) $\boldsymbol{g}$ lifts to $\mathrm{Ni}(G_k', \mathbf{C})$, but not to $\mathrm{Ni}(G_{k+1}, \mathbf{C})$.
(10.2b) M_k' is minimal with this property.

From [**FrK97**, §2] (or Chap. 5), G_{k+1}/M_k' has a nontrivial p part to its Schur multiplier. Also, M_k' contains a proper submodule distinct from $\mathbf{1}_{G_k}$. Under these assumptions (running over allowable orbits O) put M_k' in the set $\mathbf{Obs}_{2,k}$. We state a problem only $\mathbf{C} = \mathbf{C}_{3^r}$ (general formulation in Chap. 5). The answer is not known even if $r = 4$.

Problem 10.2 (Commutator Problem). Fix $r \ge 4$ even. What are the elements of $\mathbf{Obs}_{1,k} = \mathrm{C}_3^r \cap W_k \setminus \{1\}$? Suppose k is large. Is this set just the identity? Then, the same question for $\mathbf{Obs}_{2,k}$ where we ask if it is empty for k large.

10.2.2. *Interpreting Problem 10.2.* Notice the problem is about commutators. Suppose r is even and C is any conjugacy class with $\mathrm{C} = \mathrm{C}^{-1}$. Then, elements of C^r are products of $r/2$ commutators of form (g, g') with $g, g' \in \mathrm{C}$. Now assume G_0 is a perfect group. Then, so are the G_k s for all k. Therefore, for r large, all elements of G_k are in C^r. The crucial elements are in W_k? For example, make a graph on the group G_k. Elements of G_k are the vertices, and edges are pairs $g_1, g_2 \in R_k$ with $g_1 g_2^{-1} \in \mathrm{C}$. As a function of k, what is the minimal distance between 1 and $W_k \setminus \{1\}$?

The sets $\mathbf{Obs}_{1,k}$ and $\mathbf{Obs}_{2,k}$ give a version of the ω invariant (§10.2.2, Chap. 5, [**Fri95a**, Part III], [**Ser90a**]). This *big* invariant $\omega(O)$ is a collection of conjugacy classes in the kernel of ${}_p\tilde{G} \to G_0$. An H_4 orbit that contributes to the sets **Obs** is obstructed; O has nothing above it at level $k+1$. Suppose we know these sets and they determine the $\bar{\mathbb{Q}}$ components of $\mathcal{H}_k^{\mathrm{rd}}$. Then, it is easy to compute definition fields of obstructed components contributing to $\mathbf{Obs}_{i,k}$ $i = 1, 2$.

10.3. The diophantine effect of few components. Take $r = 4$. Chap. 5 shows the genus of components in the sequence (8.6) goes up with k. That suffices to prove Conj. 8.3 when $r = 4$. For example, level 0 of the $(A_5, \mathbf{C}_{3^4})$ Modular Tower (§10.1) has one genus 0 component. Yet, level 1 has two components of respective genuses 12 and 9. The latter is obstructed [**BFr02**].

This one example illustrates the influence of Schur multipliers (equivalent to distinguishing θ characteristics). Why no obstructed component at level 0, and then such appears at level 1? The Schur multiplier presence at level 1 comes from two same length (1152) H_4 orbits on $\mathrm{Ni}_1^{\mathrm{in}}$. So, the inner Hurwitz space has two absolutely irreducible components of the same degree as covers of U_4. Yet, they aren't conjugate under $G_{\mathbb{Q}}$. The H_4 orbits gave distinct permutation representations that show profoundly in the cusps of the reduced spaces cover $\mathbb{P}_j^1$.

Suppose $r = 4$ and all components at some level of a Modular Tower have genus least 2. This assures only finitely many points (no matter what is the number field K) at some level k. That does come from Falting's Theorem (the former Mordell Conjecture [**Fal83**]), though there are other older techniques that are more explicit about computing the exceptional values of k [**Fr02**, §5].

What, however, will help analyze levels of a Modular Tower when $r \geq 5$; they are no longer curves? We don't know. It would be valuable to show level k components are varieties of *general type* for large k. Then, according to a conjecture of Lang, rational points on that level would lie in a lower dimensional subset. That would be progress, though not the quality of Conjecture 8.3.

More to the point would be a *canonical height* on a Modular Tower. Having in print background for developing this is an important goal of this book.

10.4. Height functions. Let K be a number field. Let $\mathcal{H}_k^\dagger$ be the unobstructed components of $\mathcal{H}_k^{\mathrm{rd}}$ (§10.2.1). The goal is a function $H_{G,\mathbf{C}} : \mathcal{H}_0^{\mathrm{rd}} \to \mathbb{R}$ whose properties prove Main Conjecture 8.3. That's simple enough and too much to expect. So, following [**Fal83**], aim for a finiteness result. Consider finding functions $H_k : \mathcal{H}_0^{\mathrm{rd}} \to \mathbb{R}$, $k = 0, \dots,$ with these properties.

(10.3a) $H_k(\boldsymbol{p})$ is nondecreasing in k for each fixed $\boldsymbol{p}$.
(10.3b) For k large it is positive on a nontrivial Zariski open subset V_k of $\mathcal{H}_0^{\mathrm{rd}}$.
(10.3c) H_k is a sum of local height functions, one for each prime dividing $|G|$.
(10.3d) There are no K points on $\mathcal{H}_k^\dagger$ over V_k.
(10.3e) When $r = 4$, $\mathcal{H}_k^\dagger$ consists of finitely many curves. For k large, H_k should detect that the genus of all components of $\mathcal{H}_k^\dagger$ has gone beyond one.

Should such a function be effective? Bounding k with H_k not positive on an open set is only one critical problem. As important is to describe the *nonordinary* (see §10.5) locus that is the intersection of $\cap_k(\mathcal{H}_0^\dagger \setminus V_k)$. There also must be an overall measure using the branch points. The primes dividing $|G|$ contribute heavily to a measure of how branch points behave.

10.5. Introducing nonordinary points. We prefer to think of Conj. 8.3 as the Main *Operating* Conjecture. It's value is to find failures in nonobvious places. These would provide astounding realizations for Inverse Galois Problem. [**FKVo98**] has an example of a Chevalley group $G_0 = \mathrm{PGL}_n(p)$ (with certain special p and n and conjugacy classes $\mathbf{C}$ (with $r = 5$). The p-adic version $G^\dagger$ is a p-Frattini cover of G_0 (a common phenomenon, attested to in [**Ser86**]). It has characteristic quotients $G_k^\dagger$ formed as in (8.1). Then, there is a projective system of $(G_k^\dagger, \mathbf{C})$ realizations (over some number field K).

Since $G^\dagger$ is a p-Frattini cover of G_0, it is the image of a map ${}_p\tilde{G} \to G^\dagger$. Let $\ker^*$ be the kernel of this map. So, this gives a K point on a significant Modular Tower *quotient*. There is exactly one point in $\mathcal{H}^{\mathrm{in,rd}}(G_0, \mathbf{C})$ under a K point in the tower. It would be proper to call such a point *extraordinary*. The literature, however, uses the name *nonordinary point*. Justifying that name, and locating nonordinary points and there corresponding Modular Tower quotients is a topic motivated by classical problems.

10.5.1. $\mathbb{R}$ *contribution to height.* Cusps of $\mathcal{H}_k^{\mathrm{rd}}$ guide us to the behavior at the real prime. Cusps attached to H-M representatives give a degeneracy that goes with $\mathbb{R}$ contribution to the height function. This is what happens at level 1 of the $(4, 5)$ location. Elementary techniques of [**BFr02**] and [**DFr90**] use uniformization of $\mathbb{R}$ points on Hurwitz spaces.

The less elementary part comes from interpreting them with group theory. Combining this with tangential base points as in §9.4.2 allows analyzing new functions on a Modular Tower. This includes the *even θ-nulls* from §6.4.2, which relate to other functions:

(10.4a) half-canonical differentials on the space $\mathcal{H}^{\mathrm{in,rd}}$; and
(10.4b) Scholl's Eisenstein series associated with cusps [**Sch86**].

The cusp tangential base point geometry allows quantifying the amount of degeneracy as points of the moduli space approach the cusp. Cusps attached to H-M representatives (as in (8.4)) support a total degeneracy. Including contributions for *all* cusps is still an open problem.

10.5.2. *Combining geometry and function theory.* Here is one development with modular curve precedents. Consider a Modular Tower (with $r = 4$) and a degree 0 divisor D supported in cusps of a component at some level of the tower. Sometimes such a D generates a torsion group on the Jacobian of the tower component. Cases include when the support of D consists of cusps associated to H-M representatives (as in $\oplus$ components of §10.1). We give a brief outline.

Under the hypotheses, consider the automorphic function θ_0 on the reduced Hurwitz space coming from the θ-nulls along the fibers of the family. Scholl associates to D a sum E_D of Eisenstein series. Since D is a divisor on the curve giving the Modular Tower component, it corresponds to a logarithmic differential on this curve (§6.3.1). This is E_D.

So, following (6.4), our goal is to evaluate E_D using θ functions. An example place would be the level 0 component $\mathcal{H}^{\rm rd}_{0:(5,4)} = \mathcal{H}^{\rm rd}_0$ of the Modular Tower at locus $(5,4)$ of the §10.1 Constellation Table. This component has genus 0. Its Jacobian is trivial. So we don't mean a θ function on $\mathcal{H}^{\rm rd}_0$ (or on $\mathcal{H}^{\rm odd}_0$ where this computation really happens, see §10.6). Yet, it is much more than a genus 0 curve. It is a moduli space from whose points we gather data.

Evaluate a significant 3rd kind differential such as E_D from θ_0 at each cusp tangential base point (as in §9.4.2) in the support of D. As θ_0 is canonically defined for a family over $\mathbb{Q}$, its expansion at the cusps has algebraic coefficients. A Theorem of Waldschmidt [**Wa79**] interprets this algebraic coefficient statement. It is equivalent to D generating a torsion group in the Jacobian.

Since these components are moduli spaces, this has interpretations for the Inverse Galois Problem. Here is a low-brow corollary of the geometry in this story. There are exactly three regular $\mathbf{C}_{3^4}$ realizations (up to $\mathrm{SL}_2(\mathbb{Q})$ action) of the spin group cover of A_5. These realizations correspond to three points on the genus 1 pullback of $\mathcal{H}^{\rm rd}_0$ to the λ-line. The cusps there generate a group of order 12 over $\mathbb{Q}$. Nine of those points are cusps, but three are not.

A bigger story, however, requires considering a curve $\hat{X}_{\boldsymbol{p}}$ (of genus 21) corresponding to a point $\boldsymbol{p}$ in the real locus of a tangential base point of §9.4.2 type. Calculation of E_D gives a measure of how $\hat{X}_{\boldsymbol{p}}$ degenerates (into unions of copies of $\mathbb{P}^1_z$) as it pushs along toward evaluation at the tangential base point. It is a bigger story because function theory informs about cusps on all projective systems of components in the Modular Tower. Height data involves all levels of a Modular Tower. Chap. 9 tells that story, related to [**R77**], [**CTT98**] and [**GR78**].

This focused example brings together function theory, geometry and arithmetic on a Modular Tower. It illustrates many potential applications of Modular Towers.

10.5.3. *p contribution to the height.* This investigation comes from restricting the action of $G_{\mathbb{Q}}$ to $G_{\mathbb{Q}_p}$, p the prime of the Modular Tower. After Hasse's invariant, the idea of *nonordinary* points for p started with Serre-Tate theory ([**Se72**], [**Se68**]). Ihara used Hasse's invariant in examples that still inform us [**Ihar00**]. Mochizuki's use of canonical Frobenius elements defines the meaning of ordinary (and nonordinary?) directly [**Moc96**]. His theory, however, must descend from the moduli space of curves of genus g to the moduli spaces in a Modular Tower. Defining and identifying *nonordinary* points on a Modular Tower is at the top of the problems this text aims at (Chap. 9).

In Ihara's approach the theory is entirely nilpotent. He has p-adic versions of classical functions. Especially, such have appeared from the action of $G_{\mathbb{Q}}$ on the second commutator quotient of $\pi_{\overline{0\,1}} = \pi_1(\Lambda_4, \overline{0\,1})^{(p)}$ (§9.1). Coordinates arise from going to the induced Lie algebra actions. The rubric comes from *Gassner-Magnus matrices*. These give coordinates for the Lie algebra of an automorphism group acting on the second graded term of the Lie algebra of $\pi^{(p)}_{\overline{0\,1}}$. Abelian covers of Λ_4 are Fermat curves. Similar to the discussion of §8.4.4, this is a p-adic Lie algebra acting on the p-Tate module of Fermat curves. [**Ihar91**] describes the appearance of Jacobi sum grössencharacters.

These use partials (in the Lie algebra) of $m_\sigma(x,y)$ from (9.1). The Ihara-Drinfeld relations are vital here. Nakamura connects Ihara's example and another case: Replace Λ_4 by an elliptic curve minus one point. When it is an elliptic curve with bad degeneration at p, [**Na98**] produces a *Tate Eisenstein series*. This is a logarithmic partial of Ihara's series. For some examples from the Constellation Table, the real Eisenstein series of §10.5.1 have p-adic parallels to Nakamura's examples. This is what we mean by function theory on the nilpotent part of Modular Towers.

The nonlinear part, from G_0 still has a classical function relation as with θ invariants in §10.1. The nilpotent part, in examples, produces global functions on the moduli space. Specifically we expect these functions, at least those from H-M representative cusps, to tell us about nonordinary points.

10.6. Weil's distributions. Look at (6.4) again. Weil's thesis constructed an analog of it: $(h(x)) \equiv \prod_{i=1}^{u} \theta^{w}_{x_i^0}(x)/\prod_{i=1}^{u} \theta^{w}_{x_i^\infty}(x)$. Here is its meaning. Both sides are fractional ideals in the ring of integers $\mathcal{O}_K$ of a number field K. The $\equiv$ sign means the left and right are equal up to a bounded fractional ideal. The left side is the principal fractional ideal that $h(x)$ generates. Most important, of course, are the functions $\theta^w_{x'}\colon x \mapsto \theta_{x'}(x)$ maps K points x into integral ideals. This function is defined only up to $\equiv$. Weil's distribution theorem allowed he (and Siegel [**Si29**]) to perform diophantine magic.

This works to define part of the height data for the commutative quotient of a Modular Tower. We explain. Denote the commutator subgroup of a profinite group H by (H,H). Replace inner homomorphism classes of $M_{\boldsymbol{p}} \to {}_p\tilde{G}$ in §8.4.2 by the sequence $M_{\boldsymbol{p}}/(\pi_1(X_{\boldsymbol{p}}),\pi_1(X_{\boldsymbol{p}})) \to {}_p\tilde{G}/(\ker_0,\ker_0)$. The question is now a refined question about subspaces of the Tate module of $J(X_{\boldsymbol{p}})$.

[**Si29**] starts with a crude set of reductions by going to a finite extension of K. Doing this point-by-point along a Hurwitz space would be a disaster. Canonical heights avoid this. Here is a related allusion to the odd half-canonical classes.

Following a comment from §6.4.2, we should replace $\mathcal{H}_0^{\rm rd}$ by its pullback to a space $\mathcal{H}_0^{\rm odd}$. Points of $\mathcal{H}_0^{\rm odd}$ are pairs, $\boldsymbol{p} \in \mathcal{H}_0^{\rm rd}$ with an *odd* half-canonical class on $X_{\boldsymbol{p}}$. When the general point of $\mathcal{H}_0^{\rm odd}$ carries a non-degenerate half-canonical class (§6.4.2) this starts an effective analysis. We still don't know what to do in the general case.

Here is a final word on even half-canonical classes. The locations in the Constellation Table with $\oplus$ support even half-canonical classes varying analytically with the coordinates of the Hurwitz space. Suppose the attached $\theta_{\boldsymbol{p}}$ is not zero at the origin of $J(\hat{X}_{\boldsymbol{p}})$. Then, taking its value at the origin provides an *automorphic form* (the meaning is precise and conventional when $r=4$) on $\mathcal{H}_0^{\rm in,rd}$ whose value appears in inspecting properties of the cusps.

10.7. Prelude to the general case? Level 1 of Constellation Table Modular Towers has further surprises related to the Schur multipliers of the level 1 groups. These illustrate practical applications of the nilpotent extension theory of covers (Chap. 9). There are lessons for group theory and geometry.

One is that nilpotent extensions (of any given group, simple or solvable) occur in many constructions with underlying geometric meaning. Such events don't naturally extend to solvable extensions much less to general (pro-)finite group theory. Consider lessons from the dihedral group D_p and its association with the modular curve case of Modular Towers. It has a natural series of groups by changing the prime p to any other prime: vary p among primes. That isn't, however, so special.

10.7.1. *Hecke operators.* Consider the notation arising from §8.1 for the dihedral group $D_p = \mathbb{Z}/p \times^s \{\pm 1\}$. Let p' be a prime distinct from 2 or p. The famous

Hecke Operators of modular curve theory come from there being several values of $j(\tau_1),\dots,j(\tau_{p'+1}$ for $\tau \in \mathbb{H}$ where $j(p'\tau_j)$ is a particular value. This produces an algebraic correspondence represented by a curve $T_{p'}$ on $X_0(N)\times X_0(N)$. A natural correspondence automatically induces an action on holomorphic differentials and cohomology, etc. Significantly, this correspondence produces a lift of the Frobenius correspondence from characteristic p': *The Eichler-Shimura congruence formula*.

Here is how to interpret this from a Modular Tower viewpoint. Consider the Nielsen class $\mathrm{Ni}(D_p,\mathbf{C}_{2^4})^{\mathrm{abs}} = \mathrm{Ni}_0$. Suppose $\boldsymbol{g} \in \mathrm{Ni}_0$ is the branch cycle description of a cover $f_{\boldsymbol{g}} : \mathbb{P}^1_w \to \mathbb{P}^1_z$ with D_p as monodromy group and involutions as branch cycles. This description comes from a choice of classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$. Then, the Galois closure of $f_{\boldsymbol{g}}$ is an elliptic curve E which has a canonical degree p isogeny to another elliptic curve E'. Let $A_{p'}$ be any cyclic subgroup of p' order on E and let $A'_{p'}$ be its image in E'. The morphism $E/A_{p'} \to E'/A'_{p'}$ modulo multiplication by -1 produces a new rational function $f_{\boldsymbol{g},p'}$. This is the genesis of the Hecke theory. It won't extend easily to a general Modular Tower. Yet, there are other candidates for constructions like the above.

Let H be any finite group acting irreducibly on a $\mathbb{Z}$ module V of rank m. Consider conjugacy classes $\mathbf{C}$ of H. (Take $H = \{\pm 1\}$ and $V = \mathbb{Z}$ to get the dihedral group situation.) Consider the semi-direct product $V \times^s H$ and then for each prime p, take $V/pV \times^s H = H_p$. Suppose $(p,|H|) = 1$. Extend the conjugacy classes to H_p. Then, apply the Modular Tower construction to $(H_p,\mathbf{C},p)$.

Let p' be a prime distinct from those dividing $|H|$ and p. Add in $V/p'V$ with an H action to get $V/pGV \times V/p'V \times^s H$ with an extension of the branch cycles $\mathbf{C}$ to this. This produces situations analogous to that for Hecke operators. This remains unexplored territory. A few examples will encourage further exploration. Examples of this type should give Modular Towers uniformizing natural collections of varieties defined over $\mathbb{Q}$, when the Branch Cycle Lemma conditions imply $\mathbb{Q}$ structures (§8.2). Given H what varieties have such a natural uniformization? We haven't developed the expertise to consider this in detail. The value of making such a formulation is that all the arithmetic (including rational point statements) will fall under a uniform rubric. This would include using the Main Conjecture 8.3 on Modular Towers.

10.7.2. *Separating the nilpotent tail and the nonnilpotent quotient.* Group extensions of a given G_0 by a solvable group behave no better than general extensions of G_0. Roughly, the only distinction between solvable (excluding nilpotent) and general groups is that only cyclic groups appear as simple composition factors in solvable groups. That is the author's belief. With it goes the feeling that each finite nonnilpotent group G_0 generates its own intrinsic geometry. The discrete invariants of §10.2 capture much of this.

Then, there is a rich function theory appearing in the geometry from the nilpotent tail of a Modular Tower (as in §10.5). Together they separate the nilpotent tail from the nonnilpotent quotient. We believe this separation is natural and inevitable, and will never be breached. Further, our diophantine experience with problems involving solvable groups is that they belong more with the nonnilpotent quotient than with the nilpotent tail. We intend these comments to raise questions about modern understanding of Galois' famous theorem.

11. Aids to the reader and choice of actions

Expression numbers go from the left margin and most running lists use latin letters. For example, item 3 of expression 2 of section 5 of chapter 4 is (5.2c). Reference to it in another chapter would use the variant Chap. 4 (5.2c). Lemmas, corollaries, theorems, remarks, definitions, and examples fall under one collection of numbers: Definition 3 of section 9, written Def. 9.3, might follow Ex. 9.2. Figures have their own numbering system. Exercises appear as the last section in each chapter. References to these follow a special notation: Exercise 3, part c) appearing in section 9 appears as [9.3c]. Again, the chapter is given if it is in another chapter than that being read. Bibliographical items have notational shorthand for the author's(s') name(s), followed by a pinpoint reference, the usual LaTeX scheme. Like [**Ahl79**, p. 31].

There is sufficient material for a year course around two themes: fundamental groups in complex analytic geometry and families of Riemann surfaces. A third semester of complex analysis might cover just Chap. 2, Chap. 3 and Chap. 4. One year of complex analysis and one semester of graduate algebra are sufficient background. We assume undergraduate topology, as in a junior-senior analysis course, for proper background for the treatment of fundamental groups (Chap. 3).

The author spent much time considering on which side permutation groups would act. He chose the *right side* as the primary action side. That is, when $g \in S_n$ is an element of the symmetric group acting on integers $\{1, \dots, n\}$, usually we write g applied to i as $(i)g$. It is not possible to be universally consistent. It is so typical to act with matrices on the *left* that with matrix groups we follow the usual convention. In making this decision there were these problems:

- Eventually, no matter the starting side, situations force simultaneous action on the other side.
- Group products in fundamental groups work with permutation representations only if you act on the right.
- Finite group theorists in the United States act on the right.

Many students trained by such books as [**Lan71**] and [**Jac85**] put group actions on the left. Neither book, however, does enough group theory training to dissuade from the need to spend considerable further time. Of course, there are always notational ways around the difficulties in any one situation.

The exposition on Riemann-Roch and the Picard groups in Chap. 4 quotes such sources as [**FaKr90**], [**Mum76**] and [**Se59**]. In addition, later examples quote finite group theory results outside the scope of this book. This goes with the book's aspiration to teach group theory *interpretation*, rather than detail. It simplifies exposition on examples to use references to [**Vö96**], in place of lengthy computations. The differences between the two books are large, ours geometric, while [**Vö96**] is more group theoretic. We, however, spend as much time on group theory. Our intention is to teach its use through examples to a generation of students interested in using Riemann surfaces who have little training in group theory. Still, the reader will recognize the two authors had more than a passing acquaintance.

12. Poetry and Mathematics

In the solipsistic world of mathematics, there are still many who find the subject matter of moduli of covers — that this book tackles — *beautiful.* The author agrees, with reservations.

Mathematics isn't poetry though Keats gave us hope it might be!

> A thing of beauty is a joy for ever: its loveliness increases; it will never Pass into nothingness; but still will keep a bower quiet for us . . . : *From Endymion*

12.1. The grandest virtues.

The grandest virtue of mathematics is its modularity; That it builds from pieces. Second: That it lasts so long. An ingredient here is its independence of the framing secular language used. One easily sees the appearance of pythagorean triples in the Rind Papyrus. Yet, few would appreciate that the pyramid architect Imhotep was a *god* to the Egyptian Middle Kingdom.

Still, the converse of Keats' rhyme may not hold. The persistence of mathematics does not imply its beauty. The Durants suggest:

> Poetry makes of language and feeling a music that cannot be heard across the frontiers of speech. [**Du54**, p. 77]

Independent of my abilities with written and spoken German, I can thrill to the simplifying structure Riemann brought to algebraic functions. Though I never think to tack a new verse onto *Endymion*, adding consequentially to Abel, Galois and Riemann is an ever present goal.

12.2. The eye of the beholder.

Mathematical colleagues often don't appreciate the goals of other areas. One θ function adherant can't imagine the value of preoccupation with diophantine properties of large fields, and vice-versa. I'm speaking of co-writers I've known for over 30 years. It is one example of many.

If mathematicians sincerely fail to see the beauty of each others' grand enterprises, how could the world at large have the language and intellectual base to agree with what we think beautiful? In practice it is extremely difficult to explain the beauty of mathematics, even on occasion to a Nobel Prize winning Chemist; or to nonmathematical graduates of our elite institutions. Our perceptions can fail from not appreciating the depth of what we already know before we address our papers. Failure to recognize the absorbed contribution of previous generations has much to do with the present hubris of today's mathematical community.

In particular, we (collectively) learned a lot from Abel, Galois and Riemann, though the first two produced very few theorems, and the third influenced mathematics through something strikingly beyond theorems. Abel and Galois used group theoretic interpretation to bring simplicity to an area littered with facts labeled as theorems. Riemann created coordinates for analyzing the details of a world of baffling geometries. All inherited and enhanced the goal of synthesizing algebra and geometry that Lagrange first articulated. In the age of specialization, we still recognize the coherency of mathematics in large part because of these people.

Mathematics is the *only* language supporting rich neologisms that bears its topics unadulterated to other areas and other generations. It overwhelms us locally in our seminars and colloquiums. Our students rail against what they think its incoherence, though its free inundating associations cause far more problems. The world, however, slowly accustoms to it, long forgetting — especially in related sciences — what a miracle of persistence is wrought by the foundation of clear definition. Definition that more than spotlights a resonant example; fluid definition that takes on new shape in each generation. In its fluidity it lasts and lasts and lasts. So we are certain, will the ideas of Abel, Galois and Riemann.

12.3. Two afterthoughts. The following found its motivation from θ functions and diophantine properties of large fields. There is an exact sequence [**FrV92**]:

$$1 \to \tilde{F}_\omega \to G_{\mathbb{Q}} \to \prod_{n=2}^{\infty} S_n \to 1.$$

The group on the left is the profree group on a countable number of generators. The group on the right is the direct product of the symmetric groups, one copy for each integer. The absolute Galois group is caught between two known groups.

Here is a paraphrase from [**Fri99**, Acknowledgements]

> The 20th century of mathematics belongs to group theory applications; I don't mean just Lie groups or classifications.