References

1. M. P. F. du Sautoy, D. Segal and A. Shalev (eds), *New horizons in pro-p groups*, Progress in Math. 184 (Birkhäuser, Boston, 2000).
2. John S. Wilson, *Profinite groups*, London Math. Soc. Monogr. (NS) 19 (Clarendon Press, Oxford, 1998).

University of Oxford — Dan Segal

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E GALOIS THEORY

(Springer Monographs in Mathematics)

By Gunter Malle and B. Heinrich Matzat: 436 pp., £37.50, ISBN 3-540-62890-8 (Springer, Berlin, 1999).

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In this review, 'simple group' always means a non-abelian finite simple group. For K a field, G_K is the absolute Galois group of K. We denote the field generated by all roots of 1 over the rationals $\mathbb{Q}$ by $\mathbb{Q}_{\text{ab}}$.

In the late 1700s and early 1800s, unsolved problems about equations motivated the device of attaching a (Galois) group to equations. These applications produced keywords such as: Abelian, solvable and simple groups; composition factors of a finite group; and transitive and primitive permutation representations. This *group classification domain* had a parallel *equation classification domain*, with its own keywords: elliptic functions, moduli of equations and uniformization. Galois applied *solvability* in algebraic equations to measure how, by varying equation coefficients, one changes their solutions. Today, a great divide still separates abelian equations from those with simple groups. While equations with nilpotent groups are a natural (though difficult) extension of abelian equations, general solvable equations are another matter. Solvable equations still contribute to graduate algebra, through Galois' famous equation: solvable group = solvable algebraic relation.

Galois considered modular curve covers of the j-line. The curves are upper half-plane quotients by the subgroups $\Gamma_0(p^{k+1})$ of $\mathrm{SL}_2(\mathbb{Z})$. Modular functions give coordinates for such curves. It is their relation to the variable $j = j(\tau)$ from complex variables that he was testing for solvability. He found that the groups of these covers were rarely solvable, and noted the exceptions. These groups have $\mathrm{PSL}_2(p)$, where p is a prime, as a quotient. This is usually simple. Yet, as k increases, the $\mathrm{PSL}_2(p^{k+1})$ quotients accrue more p-group behavior.

Extensions of simple groups by nonsplit p-group tails were a big theme in Galois' short life. Galois and modular curves: could that be? Yes! Documented on the last pages of **[10]** is a story corroborating Galois' problems with Cauchy and what was essentially his 'suicide' on the morning of May 30, 1832. This is a story far sadder, yet more significant for mathematics, than any legend preceding it. There are examples where modern mathematics honours this tradition. Galois recognized the significance of the two special embeddings of $\mathrm{PSL}_2(5) = A_5$ in $\mathrm{PSL}_2(11)$. In their relation to the *buckyball*, **[8]** suggests the way Nature picks one embedding over the other. That is the simple group representation theory part of Galois' work. The book **[11]** was a long investigation into that p-Frattini tail that Galois discovered.

Its topic is the dynamics of $G_{\mathbb{Q}}$ action on projective systems of points on modular curve towers: Serre's renowned *open image theorem.*

You will not, however, find the relation between the inverse Galois problem and the works of Abel, Galois and Riemann in the book under review. Finding which groups are Galois groups of regular extensions of arithmetic fields dominates any secondary themes. Regular extensions over $\mathbb{Q}$ are synonymous with geometric curve covers whose automorphisms have definition field $\mathbb{Q}$. This book assumes without question that the inverse Galois problem is significant. Few of the special problems presented inform us beyond their direct computational consequences.

The book under review has the feel of group theory emphasizing computation over inspiration. Yet, with all that it tackles, it cannot avoid resonant problems that resist manifold techniques. For example, on page 245 the authors apply their version of **[3]**. This is a (braid group) criterion for checking, in a family of genus 0 curves, whether some member has a $\mathbb{Q}$ rational point. Their example starts with a regular realization of the Mathieu group M_{24}. It comes from the Galois closure of members of a family of genus 0 covers. The goal is a regular realization of M_{23}, such as would arise if one of those genus 0 curves in the family had a rational point. The style is reminiscent of examples of Hilbert. Mestre (**[9]** or **[12**, Section 9.3]) recently applied it to go from realizations of spin cover representations of A_n with n odd to n even. It brings an echo of a Thompson phrase: 'In a lecture of an hour, I'd have more success explaining the *Monster* than the Mathieu groups.'

Chapter I explains and applies the rigidity method, a special case of the so-called *braid-rigidity* method from **[4]** (see below). While this applies only to groups with very special conjugacy classes, it opened up a territory of Galois group realizations in the late 1980s.

In Chapter II, on applications of rigidity, the authors apply to Chevalley simple groups the rigidity technique alone. Satisfaction of simple linear algebra conditions allows the realization of such groups over $\mathbb{Q}_{ab}$. It starts with generators g_1, g_2 of the classical groups that satisfy Belyi's criterion **[1]**: $g_1 - 1$ has rank 1. This chapter is the book's attempt to prove Shafarevich's conjecture: that $G_{\mathbb{Q}_{ab}}$ is pro-free. Since $G_{\mathbb{Q}_{ab}}$ is projective (in the category of profinite groups), a technical result reduces this conjecture to proving that every single finite simple group has a special regular realization over $\mathbb{Q}_{ab}$. You cannot leave out even one simple group. Thus, the chapter runs parallel to aspects of the classification of finite simple groups. The authors manage to obtain all the sporadic simple groups, although (as expected) exceptional Lie-type groups are a big problem. The simple groups that they have obtained, and those they did not, appear in a list in Section 10.

Dedicated experts might already know much of the material given in Chapter I. That also holds for Chapter II, except that the relevant experts change. Exceptional groups of Lie type do not have appropriate matrix representations; rather, one uses pure character theory to apply basic rigidity. The prestige of the Deligne–Lusztig results will recommend Section 5 especially; even near-experts may not have encountered this compendium of facts.

A Matzat idea, *GAR realizations* **[13**, Chapter 8], appears here, and should attract many; it recurs in the continuation to Section 7 for the sporadic groups.

Serre's book **[12]** constructs a story around the 'Monster', a simple group that is still fascinating to mathematicians. Serre and others worked it into questions such as: 'Where does the classification stand?' and 'Can the general mathematician use it efficiently?' By contrast, the end of Chapter II feels like the end of the trail; applying rigidity to the sporadic simple groups is a finite task if one believes that there are no unknown sporadic simple groups.

Chapter III concerns the action of braids. These are necessary to understand moduli of algebraic equations. The inverse Galois technique started with monodromy action through braids (what [13] calls 'braid-rigidity'), in [4]. Though technically more difficult than its special rigidity technique, it also allows the inclusion of connections to classical spaces like modular curves. The group for such moduli questions is the *Hurwitz* (*monodromy* or braid) group H_r, corresponding to the monodromy from a deformation of Riemann surface covers of the sphere branched at r points. The Artin braid group on r strings has H_r as a quotient. The chapter starts with pleasant presentations of group results: Theorem 1.13 shows that H_r is residually finite. (It misses an opportunity, though, when it alludes, without precise quotation, to a result of Lyndon and Schupp.) The consequence is that H_r has a solvable word problem. Compatible with practice, H_r is a group that you can understand, even though it has many relations, so you can often compute nicely with the moduli spaces produced from it.

The authors use early results (including [2] and [5]) that turned the inverse Galois problem into an existential diophantine problem about rational points on Hurwitz spaces. This led in two directions. One assumed conditions that ensure that the Hurwitz space cover of the projective r-space is somewhat trivial. Völklein has been single-minded and successful in this direction. The book under review documents this work from six years ago, with [14] giving an extensive update. In their quest for a solution to the inverse Galois problem, the authors add to the potential of their method in Section 4.2, on the braid orbit genera. So far, this has worked only for covers with four branch points. Still, it is now included in their book, for the next generation. The other direction has been to prove versions of the inverse Galois problem over large fields. Here, too, one can go for either classical connections or entertaining results. For the former, [7] showed that the field of totally real numbers, from the theory of complex multiplication, has absolute Galois group profreely generated by involutions. For the latter, [6] showed that there is an exact sequence $1 \to \tilde{F}_\omega \to G_{\mathbb{Q}} \to \prod_{n=2}^{\infty} S_n \to 1$. The group on the left is the profree group on a countable number of generators. The group on the right is the direct product of the symmetric groups, one copy for each integer; $G_{\mathbb{Q}}$ is caught between two known groups.

In Chapter IV, the book finally lightens. An interesting example comes from the definition of semi-abelian groups. These are groups generated by a finite set of abelian subgroups, $A_1, \ldots, A_n$, with A_i in the normalizer in G of A_j for $j \geqslant i$. Theorem 2.7 states that G semi-abelian is equivalent to G being a homomorphic image of $A \times^s U$, with A abelian and U a proper semi-abelian subgroup of G. Using wreath products, the book concludes that every semi-abelian group is a Galois group over a Hilbertian field.

Here, attentiveness to useful details leads to wreath products, barely mentioned in [12]. This is one of the few general constructive tools in the area. Further topics include GAR realizations (close in spirit to the way that Hilbert realized A_n). This gives rise to a fine observation: if a group has composition factors with GAR realizations, then the group has a realization over the field. Shafarevich's conjecture would follow from showing that each simple group has a GAR realization over $\mathbb{Q}_{\text{ab}}$. The presentation of a favorite topic of the authors, central Frattini extensions, deserves high marks and a favorable comparison with [12].

This book is much about examples illustrating technique. Most readers would require more applications from the literature, calling for these techniques. These might inspire new researchers to say: 'This work parallels that of Klein for producing Reimann surfaces of significance.' Other writers might mine the book's examples to produce results on the significance of the moduli of algebraic equations.

References

1. G. V. BELYI, 'On Galois extensions of a maximal cyclotonic field', *Izv. Akad. Nauk SSR Ser. Mat.* 43 267–275 (in Russian); English translation, 247–256.
2. R. BIGGERS and M. FRIED, 'Moduli spaces of covers and the Hurwitz monodromy group', *J. Reine Angew. Math.* 335 (1982) 87–121.
3. P. DÈBES and M. FRIED, 'Rigidity and real residue class fields', *Acta Arith.* 56 (1990) 13–45.
4. M. D. FRIED, 'Fields of definition of function fields and Hurwitz families and groups as Galois groups', *Comm. Algebra* 5 (1977) 17–82.
5. M. D. FRIED and H. VÖLKLEIN, 'The inverse Galois problem and rational points on moduli spaces', *Math. Ann.* 290 (1991) 771–800.
6. M. D. FRIED and H. VÖLKLEIN, The embedding problem over an Hilbertian-PAC field', *Ann. of Math.* 135 (1992) 469–481.
7. M. D. FRIED, D. HARAN and H. VÖLKLEIN, 'Absolute Galois group of the totally real numbers', *C. R. Acad. Sci. Paris* 317 (1993) 95–99.
8. B. KOSTANT, 'The graph of the truncated icoshedron and the last letter of Galois', *Notices Amer. Math. Soc.* (1995) 959–968.
9. J.-F. MESTRE, 'Extensions régulières de $\mathbb{Q}(t)$ de groupe de Galois $\tilde{A}_n$', *J. Algebra* 131 (1990) 483–495.
10. L. T. RIGATELLI, *Evariste Galois:* 1811-1832, translated from the Italian by John Denton, Vita Math. 11 (Birkhäuser, Basel, 1996).
11. J.-P. SERRE, *Abelian ℓ-adic representations and elliptic curves*, written in collaboration with Willem Kuyk and John Labute, 1st edn (Benjamin, New York 1968).
12. J.-P. SERRE, *Topics in Galois theory*, Lecture notes prepared by Henri Damon (Bartlett and Jones Publishers, Boston, MA, 1992).
13. H. VÖLKLEIN, *Groups as Galois groups*, Cambridge Stud. Adv. Math. 53 (Cambridge Univ. Press, 1996).
14. H. VÖLKLEIN, 'Review of *Inverse Galois theory* by G. Malle and B. H. Matzat', *Bull. Amer. Math. Soc* 38 (2001) 245–250.

University of California, Irvine MICHAEL D. FRIED

COMPLEX DIFFERENTIAL GEOMETRY
(AMS/IP Studies in Advanced Mathematics 18)

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The central idea of complex differential geometry (at least in the sense of the book under review) is to introduce a Riemannian metric on the differentiable manifold underlying a complex manifold. Then one can study the interaction between the Riemannian differential geometry—curvature, geodesics, and so forth—and the complex analytic properties of the manifold: for example, the existence of holomorphic functions and maps. The subjects abuts on the one hand onto manifold topology, and on the other hand onto algebraic geometry. In the case of complex dimension one (Riemann surfaces), these ideas can be traced back at least as far as, for example, the introduction of the 'Poincaré metric' on the unit disc. The higher-