COSMOS: EUCLIDEAN GEOMETRY TRANSFORMATIONS

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Elements of Euclidean geometry include distances, angles, areas and some basic shapes that we understand through using these three measurements.

1. Relation between distance and angles

In the (x, y)-plane consider three points: $(0, 0)$, (4,5) and $(2, 7)$. With the origin as a vertex these three points generate a parallelogram that includes the angle at the origin as an interior angle. It's fourth vertex is $(4, 5) + (2, 7) = (6, 12)$. The geometry of parallelograms justifies *vector* addition.

This works in 3-space. The three points $(0, 0, 0)$, (4,5,1) and $(2, 7, 3)$ generate a parallelogram in three space. It's fourth vertex is $(4, 5, 1) + (2, 7, 3) = (6, 12, 4)$.

In the plane we have the *law of cosines*. Draw a triangle with sides of length a, b, c and opposite angles are α, β, γ (measured counterclockwise) and corresponding vertices A, B, C. Draw the perpendicular from A to side a. Let the meeting point be D. Then

$$c^2 = (AD)^2 + (DB)^2 = (b\sin(\gamma))^2 + (c - b\cos(\gamma))^2,$$

or $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$.

The *area* of ΔABC, expressed as $\frac{1}{2}ab\sin(\gamma)$ relates angles, distance and area.

2. Translating parallelograms

Suppose P is the parallogram with vertices $(0, 0)$, (4,5), $(2, 7)$ and $(6, 12)$ above. What is the parallelogram *parallel* to this one, with the side corresponding to the origin at $(-1, -2)$? We call it $P_{(-1,-2)}$. The operation of translating its points by $(-1, -2)$ is $T_{(-1,-2)}$.

If $T_{(x_0,y_0)}$ is the operation of translating any point in the (x, y) plane by (x_0, y_0), then there is an *inverse* to this operation: $T_{(-x_0,-y_0)}$. The collection of all such operations $\mathcal{T}_2 = \{T_{(x_0,y_0)} | (x_0, y_0) \in \mathbb{R}^2\}$ form a group with the addition

$$T_{(x_0,y_0)} \circ T_{(x_1,y_1)} = T_{(x_0,y_0)} + T_{(x_1,y_1)} = T_{(x_0+x_1,y_0+y_1)}.$$

It is part of Euclidean geometry that if you translate any line segment, the new line segment has the same length: The group $\mathcal{T}_2$ preserves distances.

3. Rotations and the Galilean group

We can rotate any point, so any line segment, around the origin through an angle. Hidden in Euclidean geometry is that if you rotate a line segment around the origin, the new line segment has the same length. The operation of rotation around the origin through the angle θ we denote by R_θ. The inverse to R_θ is $R_{-\theta}$ and $R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1+\theta_2}$ so the rotations form a group $\mathcal{O}_2^+$.

When you write elements of a group abstractly, the inverse of an element g is usually denoted g^{-1}.

The observation I need is that if we only had *one* line segment of length a we could get all others by applying elements of $\mathcal{T}$ and $\mathcal{O}_2^+$. We can even get a line segment to map into itself, switching the end points. This shows it is a good idea to combine the operations $\mathcal{T}_2$ and $\mathcal{O}_2^+$ into one group, which we call $\mathcal{G}_2^+$ the *special Galilean group* in the plane.

4. Computing with $\mathcal{G}^+$ and $\mathcal{G}$

We need a concrete way to combine operations (multiply; compose the functions).

Theorem 1. *Given $R_\theta \in \mathcal{O}_2$ and $T_{(x_0,y_0)} \in \mathcal{T}_2$, $R_\theta \circ T_{(x_0,y_0)} = T_{R_\theta(x_0,y_0)} \circ R_\theta$. The groups $\mathcal{T}_2$ and $\mathcal{O}_2$ are commutative (abelian), while $\mathcal{G}^+$ is not.*

Here is an exercise in the investigation of these groups.

(a) Write the element $T_{(-x_0,-y_0)} \circ R_\theta \circ T_{(x_0,y_0)}$ of $\mathcal{G}_2^+$ as $T \circ R_\theta$ with T a translation.

(b) Suppose $\cos(\theta) \neq 1$. Given T, find (x_0, y_0) so that

$$T_{(-x_0,-y_0)} \circ R_\theta \circ T_{(x_0,y_0)} = T \circ R_\theta.$$

(c) Show an element of $\mathcal{G}_2^+$ has no fixed point if and only if it is a translation. Hint: $T_{(-x_0,-y_0)} \circ R_\theta \circ T_{(x_0,y_0)}$ has a fixed point (what is it?).

In any group with two elements h and g, when you have an expression hgh^{-1} refer to this as h *conjugating* g.

Lemma 2. *Elements of $\mathcal{G}_2^+$ have either no fixed points (translations) or one fixed point. Reflection in the x-axis given by $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not in $\mathcal{G}_2^+$.*

Proof. As reflection in the x-axis has the x-axis as fixed points, it is not a conjugation of a rotation or a translation. From the last exercise these are the only possibilities for elements of $\mathcal{G}_2^+$. □

Another way to write R_θ is as a matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Check it with the effect on $(1, 0)$ and with the effect on $(0, 1)$. Now use matrix multiplication to compute $R_\theta(x_0, y_0)$. Let $\mathcal{G}$ be the group by combining $\mathcal{G}_2^+$ and M_x.

Crucial properties of $\mathcal{G}_2$:

- It preserves distances (and so by the law of cosines, it preserves angles);
- Given a measure $d(x)$ from the origin to the point $(x, 0)$, invariance under $\mathcal{G}_2$ would give a measure of any line segment in the plane.

5. The subgroup of $\mathcal{G}_2$ fixing a regular polygon

Let C_n be a regular n-gon. The set of elements in $\mathcal{G}_2$ that map C_n into itself form a *subgroup* of $\mathcal{G}_2$. It has a name: The *dihedral group* D_n of degree n.

Lemma 3. *If C_n is centered at the origin, and one vertex is at $(1, 0)$, the elements of D_n composed from $\{R_{\frac{2\pi}{n}}, M_x\}$. What is the group D_n' corresponding to C_n', which is C_n rotated by θ', then translated by (x_0, y_0)? Hint: It is D_n except, conjugate each element by $T_{(x_0,y_0)} \circ R_{\theta'}$.*

The groups attached to different regular n-gons are different (we say conjugate) subgroups of $\mathcal{G}_2$. Still, they are so so similar (*isomorphic*) we use for each only the symbol D_n.

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