SPHERICAL, TOROIDAL AND HYPERBOLIC GEOMETRIES

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These notes use groups (of rigid motions) to make the simplest possible analogies between Euclidean, Spherical,Toroidal and hyperbolic geometry. We start with 3-space figures that relate to the unit sphere. With spherical geometry, as we did with Euclidean geometry, we use a group that preserves distances. The approach of these notes uses the geometry of groups to show the relation between various geometries, especially spherical and hyperbolic. It has been in the background of mathematics since Klein's *Erlangen* program in the late 1800s. I imagine Poincaré responded to Klein by producing hyperbolic geometry in the style I've done here. Though the group approach of these notes differs philosophically from that of [**?**], it owes much to his discussion of spherical geometry. I borrowed the picture of a spherical triangle's area directly from there. I especially liked how Henle has stereographic projection allow the magic multiplication of complex numbers to work for him. While one doesn't need the rotation group in 3-space to understand spherical geometry, I used it gives a direct analogy between spherical and hyperbolic geometry.

It is the comparison of the four types of geometry that is ultimately most interesting. A problem from my Problem Sheet has the name *World Wallpaper*. Map making is a subject that has attracted many. In our day of Landsat photos that cover the whole world in its great spherical presence, the still mysterious relation between maps and the real thing should be an easy subject for the classroom. The topic, why you can't make World Wallpaper, says that Toroidal geometry and Euclidean geometry are close, while Spherical geometry and Euclidean geometry remain tantalizingly far apart. Projects related to and explainations of World Wallpaper appear at the URL [Fr92].

The young students who attended this course had heard about Fermat's Last Theorem, though they didn't remember its statement. Many had seen the PBS broadcast. I took the occasion in my last lecture — surely more bewildering than the others — to say a few words on the Shimura-Taniyama-Weil conjecture fitting the topic of these lectures. They interpret as saying $\mathbb{Q}$-toroidal geometries are all *given* by those special hyperbolic geometries we call modular curves. So doing, I gave the word *given* new meaning. Using the analogy as I did between spherical and hyperbolic geometry had one goal. I wanted to introduce these young people to the word *group*, through geometry; then turning through algebra, to show it as the master creative tool it is. Combining rotations and translations in the plane, through composition of each as functions on the points of the plane, contains extraordinary lessons about combining algebra and geometry. Geometry pictures and then overwhelms us, until algebra computes and gives proofs. One must wonder that students never see examples of noncommutative groups until upper division undergraduate courses. Wait! One needn't wonder too much. Almost no courses seriously combine algebra and geometry, though Lagrange in his famous Celestial Mechanics book in the late 1700s tried to teach us to do just that.

Many know it is impossible for me to speak with young people and not convey the significance of Galois [Rig96]. He, modular curves, Goro Shimura and Gerhard Frey all got into that last lecture (of eight). I should apologize to everyone and everything of those mentioned, though at least one of them will forgive me; he too tries hard to make mathematics include everything rightfully in its domain.

1. Making convex polyhedrons

First we make a sphere S^2 (for simplicity of radius 1), and then we put down n points $P_1, \dots, P_n$ on the sphere with no three lie on a line. So any three $\{P_{i_1}, P_{i_2}, P_{i_3}\}$ have a unique plane P_{i_1,i_2,i_3} through them. Next step: Select collections of distinct triples $\{P_{i_1}, P_{i_2}, P_{i_3}\}$ from $\{P_1, \dots, P_n\}$ with these properties.

(a) The plane P_{i_1,i_2,i_3} has a side without points from $\{P_1, \dots, P_n\}$.
(b) If for all $\{P_{i_1}, P_{i_2}, P_{i_3}\}$ you throw away the part of the sphere on the side of P_{i_1,i_2,i_3} with no points, the surface of the result is a polyhedron S.
(c) Each P_j appears in the final figure.

The result S is a *convex* figure with n *vertices*, and for each *face* three *edges* and for each edge two vertices. Further, each edge is on two faces.

Given two P_1 and P_2 on the sphere, if they are not antipodal (opposite ends of a diameter), you can define the *great circle* through them by taking the unique plane T through the origin, P_1 and P_2. (What would be wrong if P_1 and P_2 were antipodal?). The great circle consists of the points of intersection of the sphere with T. A straight line joining any two vertices lies under an arc along a great circle on the sphere. So, any triangle in our polyhedron corresponds to a triangle on the sphere made of arcs from great circles. Such a polyhedron gives a *triangulation* of the sphere. This is a polyhedron with triangles as faces.

Question 1. If we choose the n points so no three lie on a line, is there a triangulation using just those points?

Example: In the plane draw the unit circle. Put down vertices at 0^0, 120^0 and 240^0. Let R_0, R_{120}, R_{240} be the rays at each of the corresponding $\measuredangle$ s: $E = 9$, $F = 6$ and $V = 4$, but we are missing something that would turn this into a polyhedron on the sphere: The point at ∞. Adding it we get $V = 5$.

The *Euler Characteristic* of a polyhedron is $F - E + V$.

Question 2. Why is $F - E + V$ always 2 for polyhedron?

Consider a *filled connected graph* in the plane: made from vertices and edges (no two of which cross). *Connected* means all pairs of vertices are connected by a path of edges. The edges partition the plane into connected regions, one of which is unbounded. Think of the bounded regions as faces, and call the result *filled*. It too has an Euler Characteristic made from its vertices, edges and faces.

Question 3. Why is $F - E + V$ always 1 for a *triangulated connected graph* in the plane? Hint: See the relation between graphs of k edges and graphs of $k + 1$ edges by removing one edge, but not its vertices.

What is the relation between Question 2 and Question 3? Suppose S is a polyhedron whose vertices are on a sphere. Then, we make something we call a *spherical polyhedron*, S^{sph}, from S by the following process: For each edge of S take the corresponding arc of the great circle on the sphere. Use arcs of great circles in place of straight line segements; and use faces that these arcs bound. That gives vertices, edges and faces for an Euler Characteristic for $S^{\rm sph}$. Why is it also 2?

2. Platonic Solids

Platonic solids are convex polyhedron satisfying these two conditions:

(a) All faces are identical regular polygons; and
(b) the same number F_0 of faces meet at each vertex.

Suppose F_0 is the number of faces at a vertex. If the faces are triangles, then the $\measuredangle$ is 60^0, so you can only have $F_0 = 3, 4$ or 5 or else it won't be convex. This gives the *tetrahedron, octohedron and icosahedron.*

Question 4. How can you count the vertices and edges from the Euler Characteristic in each case? Hint: $3F/2 = E$ (use that the faces are triangles) and $3F/F_0 = V$ and $F - E + V = 2$. Example: 20 faces for an icosahedron.

If the faces are squares, then $F_0 = 3$, and the result is a cube. If the faces are pentagons, there are $\frac{3}{5}\cdot\pi$ radians (for an n-gon that is $\frac{n-2}{n}\cdot\pi$) at each interior $\measuredangle$. So there are exactly three $F_0 = 3$ faces at each vertex. This gives the dodecahedron. How many faces does it have?

Definition 5 (Semiregular polyhera)**.** A semiregular polyhedron is also convex, its faces are regular polygons, and each vertex looks identical. We don't, however, assume all faces are the same regular polygon.

The hypo-truncated icosahedron (*buckyball* is the icosahedron with a shallow area around each vertex cut off, so the former vertices are replaced by pentagons, and each former triangle face is replaced by a hexagon.

You can describe the semiregular polyhedra by using the notation $(n_1, n_2, \dots, n_k)$ with $n_1 \le n_2 \le \cdots \le n_k$ indicates k faces are at each vertex, with the given number of sides on each face.

Question 6. How many square faces are on a semiregular polyhedron represented by $(3, 4, 4, 4)$?

3. Triangles and $\measuredangle$s on the sphere

We already have the notion of a triangle on the sphere. Call it a *spherical triangle*. Let a and b be the sides meeting at a vertex C. To get a notion of the $\measuredangle$ γ at a vertex of the triangle, consider the *tangents* $\mathbb{T}_a$ and $\mathbb{T}_b$ to the great circles formed by the sides going through C. The two lines $\mathbb{T}_a$ and $\mathbb{T}_b$ meet in 3-space. So, some plane contains them both. Call this $P_{a,b}$. The $\measuredangle$ of their meeting is the $\measuredangle$ at C. Notice we need the direction of the lines to define the $\measuredangle$ between them. This is a definition from using $\measuredangle$ in 3-space!

Question 7. In the problem about World Wallpaper and Johnny B. Badde, if the decorator could make world wallpaper, why would he be giving a function from a plane to the sphere that is onto and *continuous*.

Question 8. Continue using $\mathbb{T}_a$, $\mathbb{T}_b$ and $P_{a,b}$. Map the point that is a distance of θ from C along $\mathbb{T}_a$ to the point that is a distance of θ radians along the great circle G_a along a. Denote this function $d_a : \mathbb{T}_a \to G_a$. Similarly form the function $d_b : \mathbb{T}_b \to G_b$. How can you fill this in with a function f from the whole plane $P_{a,b}$

mapping to the sphere? Where does f map the circle on $P_{a,b}$ around C that has radius 2π? What does this have to do with World Wallpaper?

There is a different map f^* from a plane P the sphere S^2 called *sterographic projection*. Let P be the plane passing through the equator of S^2 (the locus where $z = 0$ in (x, y, z)-space). Let C be the North Pole on S^2. For any point $(x, y, 0) \in P$ draw the line $L_{x,y}$ through C and $(x, y, 0)$. Let $f^*(x, y, 0)$ be the intersection of $L_{x,y}$ and S^2.

Question 9. Why isn't stereographic projection a solution to the problem of World Wallpaper?

4. Distances on the sphere and the orthogonal group $\mathcal{O}_3$

Motions of 3-space (maps of 3-space to 3-space) that preserve distances form a group: If you take two of them and compose them you get another one that also preserves distances. This is the Galilean group in 3-space $\mathcal{G}_3$. This has a subgroup of elements that fix the origin, called $\mathcal{O}_3$. In turn, this has a subgroup of motions consisting of rotations around an axis through the origin. Call this set $\mathcal{O}_3^+$ (or SO(3)).

It is not obvious that $\mathcal{O}_3^+$ is a group: If R_{L_i,θ_i} is rotation at an $\measuredangle$ θ_i around the (directed) line L_i through the origin, $i = 1, 2$, what is $R_{L_1,\theta_1} \circ R_{L_2,\theta_2}$? The group $\mathcal{O}_3$ also includes $M_{x,y}$, reflection in the (x, y) plane (where $z = 0$). The group made from compositions of M_x with elements of $\mathcal{O}_3^+$ is $\mathcal{O}_3$.

Notice: Suppose you have an arc along a great circle. Then some element of $\mathcal{O}_3^+$ will take it to an arc along the equator. So, to measure distances along arcs on the equator, we could declare that $\mathcal{O}_3^+$ is distance preserving. Then, this would measure distances along any arc on any great circle. Does this sound silly? Yet, that is exactly what happens in Hyperbolic geometry where there is no obvious way to give a measure of distances along *hyperbolic lines*.

5. Geometry on a sphere

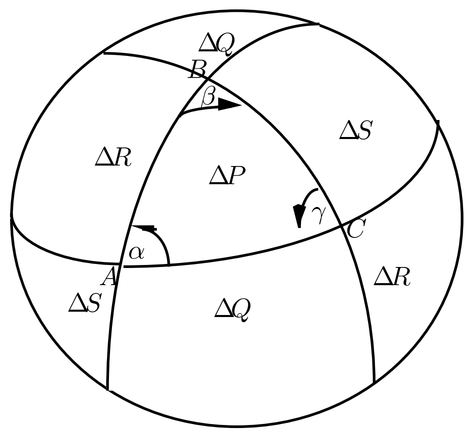
Suppose we declare great circles on the sphere to be the *lines* in our geometry. Then, one of the first axioms of Euclidean geometry fails. There is not a unique line containing any two distinct points.

5.1. **Points in spherical geometry.** To get rid of this, we change the definition of points: A point in *elliptic* or *spherical* geometry is a pair of *antipodal* points on the sphere. There is a unique line containing two distinct points if we *lines* are the pairs of antipodal points on a great circle. Stereographic projection gives a picture of elliptic points as being in a disc around the origin in the (x, y) plane.

Lemma 10. *The 3×3 matrix $M_{x,y,z}$ mapping each point on the sphere to its antipodal point gives an element in $\mathcal{O}_3$. Further, $M_{x,y,z}^2$ acts like the identity: $M_{x,y,z}$ generates a group of order 2.*

5.2. **Use of complex numbers.** Complex numbers appear in the next lemma. By regarding (x, y) as a complex number $x + iy$, we retain that there is a first and second coordinate. The *conjugate* of $x + iy$ is $x - iy$. That is like the phrase: The conjugate of $1 + 2\sqrt{2}$ is $1 - 2\sqrt{2}$. The conjugate of $x + iy$ is the point that corresponds to flipping (x, y) in the x axis.

Figure 1. Cutting the elliptic plane into triangles



Suppose two lines L_1 and L_2 in the (x, y) plane have slope m_1 and m_2. Then, you can test if they meet at a right angle by the formula $m_1 m_2 = -1$. So, if you know m_1, then you also know $m_2 = -1/m_1$. The following formula works the same way: Knowing $x + iy$), solve for $x' + iy'$ as the conjugate of $-1/(x + iy)$.

Lemma 11. *Suppose a point (x, y) is a distance of at most 1 from the origin. Let (x', y') be the point coming from the antipodal point on the sphere to (x, y). Its distance from the origin is at least 1. Relate these two points by the following formula: $(x + iy)(x' - iy') = -1$. Hint: Try it first with a point on the circle bounding the unit disk.*

Suppose a triangle ΔABC has respective $\measuredangle$ s α, β, γ as in Figure 1. A sphere of radius 1 has area 4π. So the elliptic plane has area 2π (half of this, right!). Notice the areas we can compute easily.

A sector on a sphere given by an $\measuredangle$ α has area 4α on the sphere. So it has area 2α on the elliptic plane. The two areas in Figure 1 with the label ΔQ together give one triangle on the elliptic plane with vertices A, B, C (the same vertices as has ΔABC). The triangle $\Delta Q + \Delta P$ is called a *2-gon* because it has two sides and only one vertex, whose $\measuredangle$ is α.

Lemma 12. *The whole elliptic plane is $\Delta Q + \Delta R + \Delta S + \Delta P$:*

$$(\Delta Q + \Delta P) + (\Delta R + \Delta P) + (\Delta S + \Delta P) - 2\Delta P.$$

So, the area of ΔP is $\alpha + \beta + \gamma - \pi$.

6. Geometry on a torus

Recall the lattice Γ_2 of pairs (m, n) with m and n integers. As we made Spherical Geometry, we now make Torioidal (Torus) Geometry. Instead of taking points in 2-space to be points, as in Euclidean geometry, for each point (x, y) in 2-space, take $P_{x,y}$ to be the collection of points $\{(x, y) + (m, n) \mid (m, n) \in \Gamma_2\}$. Example: If L is a line in Euclidean space, then a corresponding line in toroidal space would be $P_L = \{P_{x,y} | (x, y) \in L\}$.

Question 13. In toroidal geometry we can ask about the Euler characteristic of a toroidal polyhedron. Why must it be 0?

The figure of a donut in 3-space is like a torus — so people say. Do you agree?

Question 14. Look again at the problem World Wallpaper. Why is Johnny B. Badde asking for a geometry on the sphere that comes from Euclidean geometry like the geometry we just got on a torus?

7. Lessons from spherical geometry

In spherical geometry, *points* were pairs of antipodal points $\{P, P^d\}$ on the sphere, and *lines* were the pairs of antipodal points moving along great circles on a sphere. Distances along these great circles came from the distance in 3-space. Given any arc along any great circle, some rotation would take the arc to any other arc that went through exactly the same angle along that great circle.

We say this another way: $\mathcal{O}_3$ preserves distances between points on the sphere. We switch this around now, and give a group first, acting on a space, then distances by making them invariant under this group. This group is very famous, though it has a funny notation, $\mathrm{PSL}_2(\mathbb{R})$. The set of the geometry is not a plane or a sphere. Rather, it is $\mathbb{H} = \{x, y \mid y > 0\}$, called the *upper half plane*. We start by describing distances along a special subset of $\mathbb{H}$.

We draw pictures of spherical triangles as in Figure 1: These figures are in the unit disk in the plane. This disk, which we call D, is the image by stereographic projection of the southern hemisphere. Each spherical point $\{P, P^d\}$ creates a unique point in this disk. Here are further thoughts motivating what is the group $\mathrm{PSL}_2(\mathbb{R})$. Denote the complex plane with the point at ∞ by $\mathbb{C}_\infty$.

(a) Stereographic projection maps points on the sphere to $\mathbb{C}_\infty$, and the map is one-one and onto. Call this map f.
(b) A rotation R on the sphere should correspond to a map R^* of $\mathbb{C}_\infty$. The rule: If P is on the sphere, then $R^*(f(P))$ should be $f(R(P))$.
(c) Stereograph projection maps a great circle on the sphere to a circle (or straight line segment) in $\mathbb{C}_\infty$.
(d) If $\{P, P^d\}$ is a pair of antipodal points on the sphere, and R is a rotation of the sphere, then

$$\{R^*(f(P)), R^*(P^d)\} = \{f(R(P))), f(R(P^d))\}.$$

Expression (d) says R^* *preserves* antipodal pairs. Write $x + iy$ as w (the typical complex number). Use $\bar{w}$ to mean its conjugate, and if w_1 and w_2 come from antipodal points, then we know $w_1\bar{w}_2 = -1$. Call w_2 the *antipodal* to w_1. Suppose w is the point on the unit circle at a counterclockwise angle of θ. The notation for this is $e^{i\theta}$. Let P_{SP} be the south pole on the sphere.

Lemma 15. *If R_θ is rotation around the north-south (z) axis on the sphere, then R_θ^* consists of multiplication of w by $e^{i\theta}$, and it preserves antipodal points. Suppose U_θ is rotation around the y-axis on the sphere. Then, U_θ^* maps the real line into the real line. Suppose $m_1, m_2 \in \mathbb{R}$ and $m_1 m_2 = -1$. Use that $U_\theta^*(m_1)U_\theta^*(m_2) = -1$ to see $U_\theta^*(w) = \frac{w - w_0}{w_0 w + 1}$ with for some $w_0 \in \mathbb{R}$. How can you figure what is w_0 and its relation to the south pole?*

Let $\mathcal{E}$ be the group composed from $\{R^*_\theta|\theta \in \mathbb{R}\}$ and $\{U^*_\theta|\theta \in \mathbb{R}\}$: the *Elliptic group*. Then, $\mathcal{E}$ is the group of rigid motions of the figures in elliptic geometry whose points are pairs of antipodal points in $\mathbb{C}_\infty$.

Question 16. You can compose the maps R^*_α and U^*_β in either order. Rewrite $U^*_\beta \circ R^*_\alpha$ as $R^*_\alpha \circ U$ with $U(w) = \frac{w-w'}{w''w+1}$ with $w', w'' \in \mathbb{C}$. What are w' and w''?

8. The group $\mathrm{PGL}_2(\mathbb{R})$ and hyperbolic geometry in $\mathbb{H}$

Let $\mathbb{R}^+$ be the positive real numbers. We start by taking a distance along one *line* in our new geometry, the line consisting of the complex numbers $\{iy|y \in \mathbb{R}^+\}$.

8.1. **The ray from 0 to ∞ and distance along the iy axis.** The usual distance along the y axis between two values y_1, y_2 is $|y_1 - y_2|$. Suppose both values are positive. What if we use $|\log(y_1/y_2)|$? What is the distance from $y = 1$ to $y = 0$ along the y-axis? Now regard the y-axis as the set $\{iy|y \in \mathbb{R}^+\}$.

The group we take is an analog of that for elliptic geometry: $\mathrm{PGL}_2(\mathbb{R})^+ \stackrel{\text{def}}{=} \{w \mapsto \frac{aw+b}{cw+d}|a, b, c, d \in \mathbb{R}, ad - bc > 0$. Here are some properties of $\mathrm{PGL}_2(\mathbb{R})^+$. This includes the special affine group $\mathcal{A}^+ = \{w \mapsto aw + b|a > 0, b \in \mathbb{R}\}$. By our definition, for $a > 0$, iay_1, iay_2 are the same distance apart as are iy_1, iy_2. By declaring $\mathcal{A}^+$ distance preserving, we know how to compute distances along any line perpendicular to the iy-axis. Just translate it by some $b \in \mathbb{R}$ to the iy-axis and compute the distances of the image points.

Let sC be the collection of half circles, whose points (except the endpoints) are in $\mathbb{H}$ and are perpendicular to the x-axis. Include also translates of the iy-axis.

(a) $ad - bc \neq 0$ means exactly that $w \mapsto \frac{aw+b}{cw+d}$ is one-one. What does $ad - bc > 0$ guarantee addition for $\mathbb{H}$ under the map?
(b) Every pair of points in $\mathbb{H}$ is on a half circle in the upper half plane with endpoints on the x-axis.
(c) Compositions of $\mathcal{A}^+$ with the map $w \mapsto \frac{-1}{w}$ include everything in $\mathrm{PGL}_2(\mathbb{R})$. Call this last map $\tau(w)$. Why the minus sign?
(d) Any $g \in \mathrm{PGL}_2(\mathbb{R})$ takes the iy axis to another element of $\mathcal{C}$. Hint: $\mathcal{A}^+$ maps elements of $\mathcal{C}$ to other elements of $\mathcal{C}$. So, it only requires the same is true of $\tau(w)$. This is easy to see for any half circle around the origin; leaving only to do the same for a translate of the unit half circle.
(e) For $c \in \mathcal{C}$, there is $g_c \in \mathrm{PGL}_2(\mathbb{R})$ that takes c to the iy-axis.

Question 17. Why is $\mathrm{PGL}_2(\mathbb{R})^+$ for the upper half plane like $\mathcal{E}$ for the elliptic plane? Hint: The elements of $\mathcal{E}$ preserve the relation between antipodal points, in this form. For $w_1, w_2 \in \mathbb{C}_\infty$, $w_1\bar{w}_2 = -1$, then $g(w_1)g(\bar{w}_1) = -1$ (Lem. 15). Replace $w_1\bar{w}_2 = -1$ with the relation $w_1 - \bar{w}_2 = 0$.

8.2. **Hyperbolic lines and distances along points on them.** The elements of sC are the *lines* in *hyperbolic geometry*. Suppose $c \in \mathcal{C}$, and $w_0 \in \mathbb{H}$ is not on c. *Many* lines through w_0 don't meet c. So, the *parallel postulate* doesn't hold.

Definition 18. For w_1, w_2 two points in $\mathbb{H}$, let $c_{w_1,w_2} = c$ be the element of $\mathcal{C}$ going through them. The hyperbolic distance between w_1 and w_2 is $|\log(g_c(w_1)/g_c(w_2))|$.

Call $\mathrm{PGL}_2(\mathbb{R})$ the *isometry group* for the hyperbolic geometry. To get toroidal geometry we took any subgroup H of Γ_2 and made the points of the new geometry to be the collection of translates of the points of the plane. The exact analog is to take any subgroup H of the following subgroup of $\mathrm{PGL}_2(\mathbb{R})$:

$$\mathrm{PSL}_2(\mathbb{Z}) \stackrel{\text{def}}{=} \big\{w \mapsto \frac{mw+n}{uw+v} | m,n,u,v \in \mathbb{Z},\ mv-nu=1\big\}.$$

Call the resulting set $\mathbb{H}_H$. As with elliptic geometry, we take collections of points as the points of our new geometry. Each point in $\mathbb{H}_H$ is a collection $P_w \stackrel{\text{def}}{=} \{h(w)|h \in H\}$ for some $w \in \mathbb{H}$. Call this an H point. Line segments in this space are the H points as w moves along an arc of a half circle in $\mathcal{C}$. Compute distances along arcs joining points using the distance of Def. 18. These spaces for various subgroups H are probably the most important spaces in mathematics.

8.3. **What Hs give the spaces in the proof of Fermat's Last Theorem?** Mathematicians call those spaces *modular curves*. Here is how to describe the main examples. Let p be a prime. Take H to be

$$\Gamma_0(p) \stackrel{\text{def}}{=} \big\{w \mapsto \frac{mw+n}{uw+v} | m,n,u,v \in \mathbb{Z},\ mv-nu=1,\ p \text{ divides } u\big\}.$$

Opps! I haven't told you *why* they appear in Fermat's Last Theorem. Here is a phrasing using our discussion on geometries. Gerhard Frey, of the Institut of Experimental Mathetics in Essen, Germany, proposed that if Fermat's Last Theorem were false it would produce a $\mathbb{Q}$-toroidal geometry – called an elliptic curve over the rational numbers $\mathbb{Q}$ – that would not be a possible toroidal geometry type predicted by a famous theorem of Mathematics called the Shimura-Taniyama-Weil Conjecture. This says that every $\mathbb{Q}$-toroidal geometry comes from a modular curve (a very special hyperbolic geometry). When you know where things come from, it usually gives basic information about them. Modular curves have those primes attached to them. A number —called the *conductor* —attached to the toriodal geometry had to reflect the way those primes appear. Frey guessed, and Ken Ribet proved, that the toroidal geometry that violated Fermat's Last Theorem had a conductor that also violated the Shimura-Taniyama-Weil Conjecture.

Ken Ribet at Berkeley showed that Frey's inspired guess was correct. Andrew Wiles proved the Shimura-Taniyama-Weil Conjecture. That, however, is another story about how Number Theorists would interpret the hyperbolic geometries that we call modular curves.

References

[Hen2001] M. Henle, *Modern Geometries*, 2nd Edition, Prentice Hall, 2001.

[Fr92] An explanation of World wallpaper based on rotations and the parallelogram law of vector addition appears at the following URL: *http://www.math.uci.edu/ mfried/wwallfiles/contents.html*.

[Rig96] L.T. Rigatelli, *Evariste Galois: 1811-1832*, Vol. 11, translated from the Italian by John Denton, Vita Mathematica, Birkhäuser, 1996. This is the most plausible view of what was on Galois' mind when he committed suicide, and why exactly he disguised it to be a dual —as the legend goes.

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