MATH 4820: 2ND PROBLEM SET
IDENTIFYING COMPLEX NUMBERS //FRACTIONAL POWER SERIES

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1. Motivation for the F(undamental)T(heoem)ofA(lgebra)

On pgs. 283-284 the text discussed motivation from integrating rational functions as an important piece of considering the **FTA** as significant. Mark off the points of a regular n-gon on the unit circle at $A_k = (\cos(2\pi k/n), \sin(2\pi k/n))$, $k = 0, \dots, n-1$. Consider the distance from the point $P = (x, 0)$ to A_k. Denote it $\mathrm{D(P, A_k)}$.

The particular problem used this discovery of Cotes (1716):

$$\prod_{k=0}^{n-1} \mathrm{D(P, A_k)} = 1 - \mathrm{x^n}. \tag{1.1}$$

In class we used the law of cosines from the following two principles.

(1.2a) Consider these two points

$$\boldsymbol{v}_1 = t_1(1, 0) \text{ and } \boldsymbol{v}_2 = t_2(\cos(\theta), \sin(\theta)) \text{ with } t_1, t_2 \in \mathbb{R} \setminus \{\mathbf{0}\}.$$

Then, $\frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{|\boldsymbol{v}_1||\boldsymbol{v}_2|}$ is equal to $\cos(\theta)$, independent of (t_1, t_2).

(1.2b) Write $\mathrm{D(P, A_k)}^2 = \mathrm{(P - A_k) \cdot (P - A_k)}$ as $x^2 + 1 - 2\cos(2\pi k/n)x$.

1. Show Cotes' formula.:

(1.3a) **5 pts**: Factor $x^2 + 1 - 2\cos(2\pi k/n)x$ into two terms with complex zeros.
Ans: $(x - (\cos(2\pi k/n) + i\sin(2\pi k/n)))(x - (\cos(2\pi k/n) - i\sin(2\pi k/n)))$.

(1.3b) **5 pts**: Why is $e^{2\pi i k/n}$ a zero of $1 - x^n$ for each k?
Ans: $(e^{2\pi i k/n})^n = e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$; DeMoivre's formula.

(1.3c) **10 pts**: Use (1.3a) and (1.3b) to show Cotes's formula.
Ans: **10 pts**: From above, $e^{2\pi i k/n}$ is a zero of $x^n - 1$, $0 \le k \le n-1$. From (1.3a) it is also a zero of

$$\prod_{k=0}^{\frac{n-1}{2}} x^2 + 1 - 2\cos(2\pi k/n)x \ n \text{ odd}, \prod_{k=0}^{\frac{n}{2}} x^2 + 1 - 2\cos(2\pi k/n)x \ n \text{ even}.$$

Two monic polynomials with the same zeros must be equal.

2. Fractional power series for $P(y) - x$, P a monic polynomial:

(1.4a) **5 pts**: For each zero y' of $\frac{dP(y)}{dy}$, of multiplicity $e_{y'} - 1$, show there is a value x' such that $P(y) - x'$ has $(y - y')$ as a zero of multiplicity $e_{y'}$.
Ans: If $P(y') = x'$, then $P(y) - x'$ has a zero at y' of multiplicity one more than its derivative.

(1.4b) **10 pts**: Imitating what we did in class for $y' = \infty$, for y' in (1.4a), then $P(y) - x$ has a fractional power series zero of form $y' + a_1(x - x')^{1/e_{y'}} + a_2(x - x')^{2/e_{y'}} + \cdots$. Express a_1 in terms of P and y'.[1]
Ans: Write $P(y) - x' + (x' - x) = (y - y')^{e_{y'}} M(y) + (x' - x)$ where $M(y)$ has no zero at y'. Calculate the lowest term in

$$M(y' + a_1(x - x')^{1/e_{y'}} + \text{ higher terms})(a_1(x - x')^{1/e_{y'}})^{e_{y'}} + \text{ higher terms})$$

as $M(y')a_1(x - x')$. To continue inductively requires you be able to find a_2 from the next term in the series, which means that $M(y')a_1(x - x') + (x' - x) = 0$: $a_1 = 1/M(y')$.

(1.4c) **5 pts**: Show, summing over all the y' in (1.4b), including ∞,

$$(\sum_{y'} e_{y'} - 1)/2 - (n-1) = 0. \tag{1.5}$$

Ans: The sum of the $e_{y'} - 1$ is the same as the count of the zeros of $\frac{dP(y)}{dy}$, so that is $n - 1$. In class we found $e_\infty - 1$ is also $n-1$. [2]

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[1]Hint: From a Taylor series expansion use that $P(y' + a_1(x - x')^{1/e_{y'}} + \text{ higher terms })$ as $P(y') + b_1((x - x')^{1/e_{y'}}) + $ higher terms.

[2]Extra: The right side of (1.5) is the genus of the curve $P(y) - x = 0$. We will get the general formula for the genus for a general $P(x, y) = 0$ in class. It will look be similar to the left side.