MATH 4820/5820-3: 4TH–8TH WEEK SYLLABUS
2:00 – 2:50 MWF ECCR 139
CHAPS. 7–10: GEOMETRY AND ALGEBRA OF EQUATIONS

PROF. MICHAEL D. FRIED, OFFICE MATH. DEPT. #223

Comments on the 1st Problem set: Notice on our first problem set, the use of the parametric form of line is only a slight generalization of Diophantus' method of finding rational solutions on the points on a circle $\{(x,y) \mid x^2+y^2=1\}$. We parametrized lines L_m through the point $(-1,0)$ as $\{(-1,0)+t(1,m) \mid t \in \mathbb{R}\}$ with m giving the slope of the line, t the parameter giving the points on the line L_m.

Note, too, the material I did on Monday, Sept. 10, consisted – with slight change in notation – the Pell equation exercises on p. 80 of the text.

Finally, the 2nd problem on continued fractions was like the material I did in class on repeating continued fractions giving quadratic irrationalities. I did it using some algebra rather than Stillwell's historical approach with geometry that goes back to Euclid. One reason for that is that it wasn't until Lagrange that it was possible to finish the work of Baskara.

1. Chap. 7: Analytic Geometry

p. 112: Rotations put quadrics in standard form:

Consider rotations

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \text{ operating on } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives the two vectors for rotating the x-axis and y-axis by θ.

If we apply this matrix to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ to get $\begin{pmatrix} x \\ y \end{pmatrix}$, then the result in (x',y') gives a rotation of $\{(x,y)| \mid ax^2+bxy+cy^2\}$ to

$$\{(x',y') \mid a'(\theta)(x')^2+b'(\theta)x'y'+c'(\theta)(y')^2=0\}.$$

Write it out explicitly. For $\theta=0$, get $bx'y'$, as the middle term. For $\theta=\frac{\pi}{2}$ get $-bx'y'$. From the *mean value theorem*, somewhere in between a value of θ gives 0.

p. 113: "curves": Unlike the Greeks, Descartes restricted to "curves" defined by a polynomial equation. We call these algebraic versus transcendental, using the degree – in each variable separately, or the total degree – as a classifier. 3rd degree curves have inflections, double points and cusps.

Example 1.1. $\{(x,y) \mid x^3+y^3-axy=0\}$ has a *double point* at the origin. Again use m as a parameter for the slopes of different lines through the origin. Then, find the value of t, as a function of m, that hits the curve. This only works for special degree 3 equations. That leads to the major themes of Chaps. 11, 12, and 16.

So much work for moving up just one degree?

That material eventually relates to the excitement over Fermat's *Last Theorem*.

§*7.5 p. 118: Observations on solving equations in one variable:* Descartes found a particular cubic whose intersections with a suitable circle yield solutions for any given for any given 5th or 6th degree equation. *Bezout's Theorem* suggests that solutions of any $r(x)$ of degree $m \cdot n$ might be the intersections of a suitable degree m with a suitable degree n, an example of an *inverse problem*.

Suppose $p(x, y) = 0$ and $q(x, y)$ have no nonconstant common factor in y. Can we apply the Euclidean algorithm to conclude there exists $a(x, y)$ and $b(x, y)$ with

$$ap + bq = r(x) \text{ a \textit{resultant} polynomial in } x \text{ alone?}$$

(1.1a) How does this *resultant* come about from $p(x, y)$ and $q(x, y)$?
(1.1b) How do you know $r(x)$ has $\deg(r)$ solutions, and where are they?

2. Chap. 8: Projective Geometry

The idea was to create a geometry in which

(2.1a) two points determine a line (like in Euclidean geometry); and
(2.1b) *any* two lines determine a point (unlike Euclidean geometry) where parallel lines are an exception.

p. 130: Creating points at ∞*:* The practical goal was – in art – to give an image of tiled lines on a floor. Alberti solved how to do this when one set of floor lines is horizontal (parallel to the horizon).

He started with an arbitrary choice of a horizontal line supposedly behind which someone is standing. Point at ∞ chosen above it. Then, put equal spacing on the tiles marked on the horizontal line, and choose another horizontal line above it. Then, the diagonal in 1st quadrilateral intersects the other nonhorizontal lines at the correct horizontals. The principles (p. 129):

(2.2a) Lines in perspective remain lines.
(2.2b) parallel lines remain parallel or converge to a single (vanishing) point.
(2.2c) Using parametric lines we can actually find those horizontal lines.

So, Kepler and Desargues tried to set up a geometry in which each line has one point at ∞, and parallels share the same point at ∞. Nonparallel lines also have exactly one (finite) point in common; the horizon consists of the points at ∞.

p. 131: Alberti's viewpoint: Only from the viewpoint of the painter using Alberti's veil does the viewpoint look correct. Distortion occurs mostly at the extreme edges. Some Italian artists took to making one of those extreme views work as in Holbein's "The Two Ambassadors."

Theorem 2.1. *Points of intersection of corresponding sides of line intersections of triangles in perspective drawing from a point lie on a line. If done in space that line is the intersection of the planes of the two triangles. For triangles in the plane this requires a different proof (p. 134).*

p. 134: Invariance of cross-ratios: What is preserved of 4 points on a line since any 3 can be projected to any other 3. If the points are A, B, C, D (in that order) on the line and O is the point of projection, then any line with four points A', B', C', D' that are in perspective with the original four points will have the same cross ratio:

Definition 2.2. Such ratios *projectively relate* if given by *composing* perspectives.

The proof is by taking the height h from O to the line. Then computing the area of the triangle of base, say, $\bar{AC}$ as $\frac{h}{2}\bar{AC}$ in terms of $\sin(\angle COA)$. Result: The crossratios come out as a pure formula in sins of these angles.

p. 138: Effect on conics: What does it mean to *tilt* a parabola or hyperbola into perspective view (Alberti's) to get an ellipse (1 point at ∞) versus 2 points at ∞.

§8.5: The real projective plane $\mathbb{RP}^2$: Gives the model of it as the plane at $z = -1$ viewed from the origin $(0, 0, 0) = \mathbf{0}$.

Points on the plane $\leftrightarrow$ lines through $\mathbf{0}$. Lines are given by planes through $\mathbf{0}$.

Some other interpretations:

(2.3a) A horizontal line through $\mathbf{0}$ does not correspond to an actual point of the plane. It is the line of sight to a point at infty on the plane.
(2.3b) The horizontal plane through $\mathbf{0}$ models the horizon line.

Another way to look at it is as the sphere around $\mathbf{0}$ with *antipodal points* identified, wherein the equator becomes a circle with its antipodal points identified.

Then, any two distinct projective lines meet at exactly one point.

Problem 2.3. Make sense of Prob. 8.5.3 that a strip of the projective plane that surrounds a projective line is a Möbius band.

Skipped §8.6 (an introduction to linear fractional transformations).

§8.7: Homogenous coordinates: A *point* is the multiplies $\{t(x, y, z) \mid t \in \mathbb{R}\}$. We give an algebraic curve in $\mathbb{RP}^2$ using a *homogenous* polynomial

$$\{(x, y, z) \neq \mathbf{0} \mid p(x, y, z) = 0 \text{ with } (x, y, x) \sim t(x, y, z), t \in \mathbb{R}\}.$$

Homogeneous for p means that $p(tx, ty, tz) = t^{\deg(p)}p(x, y, z)$.

Here is a degree 1 homogeneous polynomial: $ax + by + cz$ with not all $a, b, c = 0$.

Example 2.4. The points $\{(y^2 - xz = 0 \mid (x, y, z) \equiv t(x, y, z), t \neq 0\}$ is just the parabola $\{(x, y) \mid y^2 - x = 0\}$ with a point at ∞ (represented by $(0, 0, 1)$) added.

Bezout's Theorem: p. 148-150 are a section which revisits Bezouts theorem. It says that if p and q are homogeneous polynomials, with $m = \deg(p)$ and $n = \deg(q)$, then they intersect in $m \cdot n$ points if the following.

(2.4a) You properly count those points with multiplicity.
(2.4b) You include the points with coordinates in the complex numbers, and consider their intersections in the complex plane $\mathbb{CP}^2$ (§15.1).

This joins several different topics:

(2.5a) Apply the Euclidean algorithm to $P(y)$ and $Q(y)$ (of degree m and n) first with coefficients in $\mathbb{Q}$, $A(y)P(y) + B(y)Q(y) = r$, with r a constant.

(2.5b) Do the same when P and Q have coefficients in $\mathbb{Q}[x]$ to get

$$A(x,y)P(x,y) + B(x,y)Q(x,y) = r(x).$$

(2.5c) Recognize that if x_0 satisfies $r(x_0) = 0$, that means $P(x_0, y)$ and $Q(x_0, y)$ have a common zero in y.

(2.5d) Understand this in terms of the homoginization $p(x,y,z)$ and $q(x,y,z)$ of P and Q as necessary according to the conditions of (2.4).

(2.5e) Consider the equation on p. 150 Prob. 8.7.3 for $r(x,z)$ that is homomogenous in x and z of degree $\deg(p)\deg(q) = m \cdot n$.

In (2.5a), we assume P and Q have no common factors. Then, you can find A and B by taking A of degree n and B of degree m, with unknown coefficients.

Solve for them by setting the left side coefficients of all powers of $y^i, i \geq 1$ to 0. In (2.5b) use the coefficient of $y^i, i \leq n$ in A as a polynomial with unknown coefficient in x of degree $n-i$ (similarly in B). Then, write the equations for solving for these unknown coefficients.

In (2.5e) Stillwell has taken r to be homogeneous in (x,y) rather than in (x,z). The determinant that arises is the condition from Cramer's rule that there is a nontrivial solution.

Assume the highest y term is $a_0 y^m$ in P, $b_0 y^n$ in Q with $a_0 \cdot b_0 \neq 0$.
Then, neither $p(x,y,z) = 0$ nor $q(x,y,z) = 0$ contains $(0,1,0)$.

Two very serious mathematical concerns arise from this.

First: We have to know something about $\mathbb{C}$ to know $r(x/z, 1)$ has $m \cdot n$ solutions.

For that discussion we return to solving the cubic equation $y^3 + ay^2 + by + c = 0$, as done by Cardono (p. 98). Immediately there are complications about complex numbers, $\mathbb{C}$, even for polynomials with only real zeros.

Second: It turns out we can't always display zeros of polynomials as *radicals*.

In Chap. 10, §10.5 we will take up the topic of *fractional power series* solutions of $P(x,y) = 0$, starting with the special case $M(y) - x$ where

$$M(y) = y^m + a_1 y^{m-1} + \cdots + a_m$$

with the a_i s in $\mathbb{Q}$. This is a different kind of solution, than given by radicals. Still, we can always do it, and it is very much related to Bezout's Theorem.

E-mail address: `michael.fried@Colorado.edu`

E-mail address: `michaeldavidfried@gmail.com`