FINAL EXAM, SATURDAY, MAY 4, 2019, 4:30–7PM
LINEAR ALGEBRA 180 TOTAL POINTS

Use the following notion:

- $\mathbb{M}_{m,n}$ for the matrices with m rows and n columns.
- $0_{m,n}$ is $m \times n$ zero matrix; I_n, the $n \times n$ identity matrix.
- $\langle \vec{\boldsymbol{v}}_1, \dots, \vec{\boldsymbol{v}}_k \rangle$ for the span of the vectors $\vec{\boldsymbol{v}}_1, \dots, \vec{\boldsymbol{v}}_k$.
- For $\mathcal{B} = \vec{\boldsymbol{v}}_1, \dots, \vec{\boldsymbol{v}}_n$ a basis of $\mathbb{R}^n$, $A_{\mathcal{B}}$ for the matrix of $A : \mathbb{R}^n \to \mathbb{R}^n$ relative to that basis.
- Remember, an eigenvector $\vec{\boldsymbol{x}}$ for a linear transformation refers to the space spanned by $\vec{\boldsymbol{x}}$. So, one eigenvector means a dimension 1 space.

If matrix appears in a question on its eigenvalues, assume its eigenvalues are real. Use the following results from class. Refer to them by the labels used below.

C_1 Any linear transformation $T : V \to V$ has at least one eigenvector.
C_2 If $A \in \mathbb{M}_{m,n}$ then its row reduced echelon form can be written as $D \cdot A$ with $D \in \mathbb{M}_{m,m}$ invertible.

Question 1: *Finding an invariant subspace:* **Pts 35:** Suppose A_1 is a 3×3 matrix, and A_2 is a 2×2 matrix and consider $A = \begin{pmatrix} A_1 & 0_{3,2} \\ 0_{2,3} & A_2 \end{pmatrix}$.

(1.a) **Pts 10:** Explicitly give spanning vectors for a dimension 3 subspace V_1 and a dimension 2 subspace V_2 of $\mathbb{R}^5$, both invariant under A. [1]

Answer: For V_1 take $\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3$. Since A applied to each of these gives the first 3 columns of A. each of which is in the space $\langle \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3 \rangle$, the range of A is in V_1. Similarly, for $V_2 = \langle \vec{\boldsymbol{e}}_4 . \vec{\boldsymbol{e}}_5 \rangle$ since A applied to these gives the 4th and 5th columns of A, both of which are in V_2.

(1.b) **Pts 15:** Suppose P is the invertible matrix $(\vec{\boldsymbol{e}}_4 | \vec{\boldsymbol{e}}_2 | \vec{\boldsymbol{e}}_5 | \vec{\boldsymbol{e}}_3 | \vec{\boldsymbol{e}}_1)$. Now answer question 1.a) for PAP^{-1}. Show why this is correct.

Answer: Take $\vec{\boldsymbol{p}}_i$, $i = 1, 2, 3$, to be the first three columns of P, and let $V_1' = \langle P(\vec{\boldsymbol{e}}_i) = \vec{\boldsymbol{p}}_i, i = 1, 2, 3 \rangle$. Then, $PAP^{-1}(P(\vec{\boldsymbol{e}}_i)) = P(A(\vec{\boldsymbol{e}}_i))$. Now each $A(\vec{\boldsymbol{e}}_i)$ is a linear combination of $\vec{\boldsymbol{e}}_1, \dots, \vec{\boldsymbol{e}}_3$, and so P applied to each is a linear combination of $\vec{\boldsymbol{p}}_1, \dots, \vec{\boldsymbol{p}}_3$ and so is in V_1'. Similarly, take V_2' to be the space spanned by $\vec{\boldsymbol{p}}_i$, $i = 4, 5$, the 4th and 5th columns of P.

(1.c) **Pts 10:** Why must A (no matter what are A_1 and A_2) have at least two eigenvectors? [2]

Answer: The matrix A acting on V_i, $i = 1, 2$ has at least one eigenvector from C_1. So A has at least two eigenvectors.

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[1] Hint: Your answer should not depend specifically on either A_1 or A_2.
[2] Hint: Use C_1.

Question 2: *Row reduced form:* **Pts 25:** Suppose $A \in \mathbb{M}_{3,n}$ is in reduced form with $\vec{\boldsymbol{e}}_1$ in the 1st column, $\vec{\boldsymbol{e}}_2$ in the 3rd column and $\vec{\boldsymbol{e}}_3$ in the 5th column.

(2.a) **Pts 10:** Give a matrix $D \in \mathbb{M}_{n,3}$ that is independent of any other information about A, such that $A \cdot D = I_3$.

Answer: Take $D = (\vec{\boldsymbol{e}}_1|\vec{\boldsymbol{e}}_3|\vec{\boldsymbol{e}}_5)$.

(2.b) **Pts 15:** Suppose A' is any matrix whose row reduced echelon form equals A. Show there is a matrix $D' \in \mathbb{M}_{n,3}$ such that $A' \cdot D' = I_3$. [3]

Answer: From C_2, $A' = D^*A$ with invertible 3×3 matrix D^*. Multiply both sides by D from 2.a) on the right. Get $D^*AD = D^*I_3 = D^*$. Multiply both sides of this by $(D^*)^{-1}$ on the right. The result is

$$(D^*A)(D \cdot (D^*)^{-1}) = A'(D \cdot (D^*)^{-1}) = I_3.$$

Question 3: *The matrix condition $B^2 = B$:* **Pts 30:** In this problem B is a square, $n \times n$, *symmetric* matrix satisfying $B^2 = B$. Denote its range by R_B.

(3.a) **Pts 10:** If $\boldsymbol{x}$ is an eigenvector for B, what is the attached eigenvalue if it is not 0?

Answer: $B(\boldsymbol{x}) = \lambda\boldsymbol{x}$ implies $B^2(\boldsymbol{x}) = \lambda^2\boldsymbol{x} = B(\boldsymbol{x}) = \lambda\boldsymbol{x}$. So, $\lambda = 1$.

(3.b) **Pts 10:** If $\vec{\boldsymbol{y}} \in \mathbb{R}^n$, why is $B(\vec{\boldsymbol{y}})$ perpendicular to $\vec{\boldsymbol{y}}^* \stackrel{\text{def}}{=} \vec{\boldsymbol{y}} - B(\vec{\boldsymbol{y}})$?

Answer: Notice that $B(\vec{\boldsymbol{y}} - B(\vec{\boldsymbol{y}})) = B(\vec{\boldsymbol{y}}) - B^2(\vec{\boldsymbol{y}})) = \vec{\boldsymbol{0}}$. Therefore $\vec{\boldsymbol{y}}'$ is in the null space of B. While $B(\vec{\boldsymbol{y}})$ is in the range, R_B, of B. Since $B = B^T$, the null space of B is perpendicular to the columns of B, which give the range of B. Therefore, $\vec{\boldsymbol{y}}^*$ is perpendicular to $B(\vec{\boldsymbol{y}})$.

(3.c) **Pts 10:** For any subspace $V \subset \mathbb{R}^n$ define

$$V^\perp \stackrel{\text{def}}{=} \{\vec{\boldsymbol{w}} \in \mathbb{R}^n \mid \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}} = 0 \mid \text{ for each } \vec{\boldsymbol{v}} \in V\}.$$

Why is any $\vec{\boldsymbol{y}} \in \mathbb{R}^n$ the sum of a vector in R_B and a vector in $R_B^\perp$?

Answer: From 3.b), write $\vec{\boldsymbol{y}} = \vec{\boldsymbol{y}}^* + B(\vec{\boldsymbol{y}})$. The 1st term is in $R_B^\perp$ from 3.b) and the second is in R_B.

Question 4: *Null space of a matrix:* **Pts 35:** Consider

$$A_1 = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_{4,6}.$$

(4.a) **Pts 10:** Give three vectors in $\mathbb{R}_6$ that span the null space of A_1.

[3]Hint: use result C_2.

Answer: The null space of A_1 is

$$\left\{\begin{pmatrix} -3u_2-2u_4-4u_6 \\ u_2 \\ -u_4-2u_6 \\ u_4 \\ -3u_6 \\ u_6 \end{pmatrix} \mid u_2,u_4,u_6\in\mathbb{R}\right\} \text{ or the span of}$$

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{\boldsymbol{v}}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{\boldsymbol{v}}_3 = \begin{pmatrix} -4 \\ 0 \\ -2 \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

(4.b) **Pts 10:** Suppose the null space of a matrix $A_2 \in \mathbb{M}_{4,6}$ equals the nullspace of A_1. Consider

$$\mathcal{N}_{A_1} = \{S_L \cdot A_1 \mid S_L \in \mathbb{M}_{4,4} \text{ and invertible}\}.$$

Show A_2 is in $\mathcal{N}_{A_1}$ by finding S_L with $S_L \cdot A_1 \in \mathcal{N}_{A_1}$ equal to A_2. [4]

Answer: The nullspace of A_2 is the same as the nullspace of $A_{2,\text{red}} = S_1 \cdot A_2$, with S_1 an invertible matrix in $\mathbb{M}_{4,4}$. We have only to choose S_L invertible, so that

$$S_L(\vec{\boldsymbol{e}}_1) = \vec{\boldsymbol{a}}_1, S_L(\vec{\boldsymbol{e}}_2) = \vec{\boldsymbol{a}}_2, S_L(\vec{\boldsymbol{e}}_3) = \vec{\boldsymbol{a}}_3$$

are the first 3 columns of A_2. For the 4th column of S_L take *any* vector $\vec{\boldsymbol{a}}_4$ that is not in the span of $\vec{\boldsymbol{a}}_1, \vec{\boldsymbol{a}}_2, \vec{\boldsymbol{a}}_3$ so that S_L is invertible.

(4.c) **Pts 10:** Answer these two questions about other choices of S_L.

- **Pts 5:** Why does the choice of the 4th column of S_L not affect the result of $S_L \cdot A_1$?

  Answer: The result of dotting column vectors of A_1 into each of row of S_L doesn't depend on what is in the 4th column of S_L because there is a 0 in every entry of the 4th row of A_1.
- **Pts 10:** How would you change your answer to (4.b) about columns of S_L if you were to consider matrices A_2 whose null space *contains* the null space of A_1 using $\mathcal{N}^*_{A_1} = \{S^*_L \cdot A \mid S_L \in \mathbb{M}_{4,4}\}$.

  Answer: No matter what you choose for S^*_L, the null space of $S^*_L \cdot A_1 = A_2$ contains the null space of A_1. But if the first three columns of S^*_L are not linearly independent, then the nullspace of A_2 would be bigger than the null space of A_1. The change would be: $\mathcal{N}^*_{A_1}$ consists of the matrices in $\mathbb{M}_{4,6}$ whose nullspace contains the nullspace of A_1.

Question 5: *Upper triangular matrices:* **Pts 35:** Suppose for $A \in \mathbb{M}_{4,4}$ there is a basis $\mathcal{B} = \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_4$ for which $A_{\mathcal{B}}$ is upper triangular.

[4] Hint: Use spanning vectors of the range of A_2 to give columns of S_L.

(5.a) **Pts 20:** Show for each k, $1 \leq k \leq 4$, the range of A restricted to $V_k = \langle \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k \rangle$ is contained in V_k. [5]

Answer:

$$\text{Write } A_{\mathcal{B}} \text{ as } \begin{pmatrix} a_1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_2 & a_{2,3} & a_{2,4} \\ 0 & 0 & a_3 & a_{3,4} \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

For each k the range of $A_{\mathcal{B}}$ is the range of the first k columns of $A_{\mathcal{B}}$. The 1st column is $a_1\vec{\boldsymbol{e}}_1$, the 2nd column is $a_{1,2}\vec{\boldsymbol{e}}_1 + a_2\vec{\boldsymbol{e}}_2$, and the two of them are in V_2'; the 3rd column is $a_{1,3}\vec{\boldsymbol{e}}_1 + a_{2,3}\vec{\boldsymbol{e}}_2 + a_3\vec{\boldsymbol{e}}_3$ which is in V_3'.
Now consider A where $A = PA_{\mathcal{B}}P^{-1}$ with $P(\vec{\boldsymbol{x}}_i) = \vec{\boldsymbol{e}}_i$. Then,

$$A(\vec{\boldsymbol{x}}_i) = PA_{\mathcal{B}}(\vec{\boldsymbol{e}}_i) = \text{ the linear combination of the } \vec{\boldsymbol{x}}_i\text{ s}$$
$$\text{obtained by substituting } \vec{\boldsymbol{x}} \text{ for } \vec{\boldsymbol{e}} \text{ as above.}$$

The final statement then is that A applied to V_k is in V_k by substituting $\vec{\boldsymbol{x}}$ for $\vec{\boldsymbol{e}}$ in the statement that $A_{\mathcal{B}}$ maps V_k' into V_k'.

(5.b) **Pts 15:** Suppose there is another basis $\mathcal{B}' = \vec{\boldsymbol{v}}_1', \ldots, \vec{\boldsymbol{v}}_4'$ of $\mathbb{R}^4$ for which A restricted to $V_k' = \langle \vec{\boldsymbol{v}}_1', \ldots, \vec{\boldsymbol{v}}_k' \rangle$ is contained in V_k'. Show that $A_{\mathcal{B}'}$ is upper triangular.

Answer: If $A(\vec{\boldsymbol{v}}_k') \in V_k'$, then

$$A(\vec{\boldsymbol{v}}_k') = a_{k,1}\vec{\boldsymbol{v}}_1' + a_{k,2}\vec{\boldsymbol{v}}_2' + \cdots + a_{k,k}\vec{\boldsymbol{v}}_k'.$$

So, the kth column of $A_{\mathcal{B}}$ has only 0s below the diagonal (k,k) entry. Since this is true for every column of $A_{\mathcal{B}}$, the matrix is upper triangular.

Question 6: *Non-conjugate matrices:* **Pts 30:** Consider the matrix

$$A' = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

(6.a) **Pts 10:** Why does A' have exactly one eigenvector?

Answer: First note that $\vec{\boldsymbol{e}}_1$ is an eigenvector of A' with eigenvalue λ since it is an eigenvector of $N_\lambda = A - \lambda I_6$ with eigenvalue 0: $A'(\vec{\boldsymbol{e}}_1) = \lambda\vec{\boldsymbol{e}}_1$.

[5] Hint: First show why the statement is true for $A_{\mathcal{B}}$ with $V_k' = \langle \vec{\boldsymbol{e}}_1, \ldots, \vec{\boldsymbol{e}}_k \rangle$ in place of V_k.

Now try a general $\vec{\boldsymbol{x}}$ to see if it could be another eigenvector of N_λ with eigenvalue 0.

$$N_\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ 0 \end{pmatrix} \text{; but this is supposed to be } \vec{\mathbf{0}}.$$

So, $x_2 = x_3 = x_4 = x_5 = x_6 = 0$. The only nonzero vector possible $\vec{\boldsymbol{x}} = x_1 \vec{\boldsymbol{e}}_1$ with $x_1 \neq 0$, the same eigenvector as in 6.a).

(6.b) **Pts 20:** Consider a matrix $A \in \mathbb{M}_{6,6}$ and $\mathcal{B} = \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_6$ is a basis of $\mathbb{R}^6$. Now assume, $A_1 \in \mathbb{M}_{k_1,k_1}$, $1 \leq k_1 \leq 6$, and $A_2 \in \mathbb{M}_{6-k_1,6-k_1}$ and $A_{\mathcal{B}}$ is block diagonal with A_1 and A_2 its blocks along the diagonal. Why is there *no* invertible matrix P such that $A' = PA_{\mathcal{B}}P^{-1}$? [6]

Answer: Each block, A_i, $i = 1, 2$, gives $A_{\mathcal{B}}$ an eigenvector with eigenvalue λ_i, according to Prob. 1.c), $\vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{x}}_2$. Since P is invertible, P maps $\vec{\boldsymbol{x}}_i$ to $\vec{\boldsymbol{y}}_i$, $i = 1, 2$. The result is two distinct eigenvectors. Since

$$PA_{\mathcal{B}}P^{-1}(\vec{\boldsymbol{y}}_i) = PA_{\mathcal{B}}(\vec{\boldsymbol{x}}_i) = P(\lambda_i \vec{\boldsymbol{x}}_i) = \lambda_i \vec{\boldsymbol{y}}_i, i = 1, 2.$$

Therefore, $PA_{\mathcal{B}}P^{-1}$ has two eigenvectors. But from 6.a), A' has only one eigenvector: a contradiction.

[6] Hint: Do proof by contradiction by assuming they are conjugate, and apply the conclusion of Prob. 1.c) whether you answered it or not.