MIDTERM EXAM, LINEAR ALGEBRA
FEBRUARY 27, 2019

Question 1: *Range of a matrix:* **Pts 20:** Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(1.a) **Pts 10:** Find a vector $\vec{\boldsymbol{b}}$ for which $A(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{b}}$ has no solution.

Answer: $\vec{\boldsymbol{e}}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is *not* in the span of the columns of A.

(1.b) **Pts 10:** Why are the columns of A linearly independent?

Answer: If we row reduce A we get $(\vec{\boldsymbol{e}}_1|\vec{\boldsymbol{e}}_2|\vec{\boldsymbol{e}}_3)$. Those are linearly independent, and if D is the invertible matrix whose left multiplication gives these row reductions, then as we have seen several times, the columns of D^{-1}, $D^{-1}(\vec{\boldsymbol{e}}_1)$, $D^{-1}(\vec{\boldsymbol{e}}_2)$ and $D^{-1}(\vec{\boldsymbol{e}}_3)$ must also be linearly independent. You can also directly use the definition of linear independence.

Question 2: *Transpose of a matrix:* **Pts 20:**

(2.a) **Pts 10:** Find the null space of the transpose, A^T, of A in Prob. 1.

Answer: The null space of a matrix B is the same as that of its reduced matrix, B_{red}. The reduced matrix of A^T is $(\vec{\boldsymbol{e}}_1|\vec{\boldsymbol{e}}_2|\vec{\boldsymbol{e}}_3|\vec{\boldsymbol{0}})$. So, the null space is the span of $\vec{\boldsymbol{e}}_4$.

(2.b) **Pts 10:** A matrix B is symmetric if $B = B^T$. For any matrix A, why does multiplying A times A^T make sense, and why is AA^T symmetric?

Answer: If A is $m \times n$, then A^T is an $n \times m$ matrix. We have the formula $(AA^T)^T = (A^T)^T A^T$. But if you apply transpose twice to a matrix, you get the original matrix back. So, this is AA^T.

Question 3: A linear transformation: **Pts 20:** Suppose $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ is a linear transformation for which

$$T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } T \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \text{ What must } T \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ be?}$$ [1]

[1] Hint: Write $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$.

Answer: Applying T to both sides of $2\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}-1\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}$ gives

$$2T\begin{pmatrix}1\\1\\1\end{pmatrix}+T\begin{pmatrix}-1\\0\\-1\end{pmatrix}=2\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}7\\8\end{pmatrix}.$$

Question 4: Commuting and noncommuting matrices **Pts 30:** Assume A and B are 4×4 matrices. Use these definitions: the diagonal entries of A are those in the (i,i) positions, $i=1,2,3,4$. The trace of A is the sum of the diagonal positions, and A is a diagonal matrix if all the nondiagonal positions are 0.

(4.a) **Pts 20:** If A is a diagonal matrix, with all of the diagonal positions different and nonzero, what are the matrices B that commute with A. [2]

Answer: Consider the (i,j), for $i\neq j$, term in both products. The (i,j) term in AB is $\sum_k a_{i.k}b_{k,j}$ where $a_{i,k}=0$, unless $k=i$. So the term is $a_{i,i}b_{i,j}$. Similarly, the (i,j) term in BA is $b_{i,j}a_{j,j}$. But these should be equal. Since $a_{i,i}\neq a_{j,j}$, then $b_{i,j}=0$: B is also a diagonal matrix.

(4.b) **Pts 10:** Use without proof that the trace of AB equals the trace of BA. Consider the matrix $AB-BA=C$. Show that no matter what matrix A is, you cannot get all 4×4 matrices C by varying B.[3]

Answer: The trace of $AB-BA$ is [trace of AB] - [trace of BA]. Since they are equal applying trace to $AB-BA$ is 0. If there is a B for which $C=I_4$, then the trace of C would be 4. This is a contradiction.

Question 5: *The geometry of the space of solutions:* **Pts 30:** Suppose, for a 5×4 matrix A, the null space of A is spanned by a single (nonzero) vector $\boldsymbol{v}$.

(5.a) **Pts 15:** Assume $A(\vec{\boldsymbol{x}})=\vec{\boldsymbol{b}}$ has a solution. Why is the set of solutions of this equation a line in $\mathbb{R}^4$?

Answer: Since $\vec{\boldsymbol{b}}$ is in the range of A, there is $\vec{\boldsymbol{b}}'\in\mathbb{R}^4$ with $A(\vec{\boldsymbol{b}}')=\vec{\boldsymbol{b}}$.

Then, $L=\{u\boldsymbol{v}+\vec{\boldsymbol{b}}' \mid u\in\mathbb{R}\}$ is the set of solutions of $A(\vec{\boldsymbol{x}})=\vec{\boldsymbol{b}}$.

. The set L is the parametric form of a line.

(5.b) **Pts 15:** What are if and only if conditions on A that guarantee (5.a) holds for any $\vec{\boldsymbol{b}}$ in the range of A?

Answer: The null space of A has dimension 1 if and only if the range of A is of dimension $4-1=3$ since the dimensions of the null space and the range add to 4. The dimension of the null space is the number of nonpivotal columns. Therefore A has exactly one nonpivotal column.

[2] Hint: Look at the term in the (i,j) position in AB and in BA. They should be equal if A and B commute.

[3] Hint: Consider whether there is B for which $C=I_4$.