LINEAR ALGEBRA MATH 2130
SECTION 6,
1ST PROBLEM SET
LINES AND PROJECTIONS
DUE MONDAY, FEBRUARY 4TH

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1. Using reduced row echelon form. 20 pts

Suppose A is a 3×4 matrix, it is in row echelon reduced form and its range consists of a plane in $\mathbb{R}^3$.

(1.a) **10 pts**: What are all the possibilities for A?[1]
Ans: Since the pivot columns span the range, A contains just two pivot columns. Therefore the possibilities are

$$\big(\vec{e}_1|\vec{e}_2|\vec{a}_3|\vec{a}_4\big)\ \big(\vec{e}_1|\vec{a}_2|\vec{e}_2|\vec{a}_4\big)\ \big(\vec{e}_1|\vec{a}_2|\vec{a}_3'|\vec{e}_2\big)\ \big(\vec{0}|\vec{e}_1|\vec{e}_2|\vec{a}4\big)$$
$$\big(\vec{0}|\vec{e}_1|\vec{e}_2|\vec{a}_4\big)\ \big(\vec{0}|\vec{e}_1|\vec{a}_3'|\vec{e}_2\big)\ \big(\vec{0}|\vec{0}|\vec{e}_1|\vec{e}_2\big)$$

where $\vec{a}_3$ and $\vec{a}_4$ are $\begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix}$, and $\vec{a}_2$ and $\vec{a}_3'$ and $\begin{pmatrix} \bullet \\ 0 \\ 0 \end{pmatrix}$

(1.b) **10 pts**: What is the null space of A in each case of (1.a) for which $A(\vec{e}_1) = \vec{e}_1$ and $A(\vec{e}_3) = \vec{e}_2$?

In (1.b), write your answer by giving two vectors spanning the null space whose entries are functions of the entries of A.[2]

Ans: The conditions force $A = \big(\vec{e}_1|\vec{a}_2|\vec{e}_2|\vec{a}_4\big)$ with $\vec{a}_2 = \begin{pmatrix} a_{1,2} \\ 0 \\ 0 \end{pmatrix}$ and $\vec{a}_4 = \begin{pmatrix} a_{1,4} \\ a_{2,4} \\ 0 \end{pmatrix}$.

This is the case from class. We wrote the solutions for the nullspace as $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ and dotted this vector into A starting from the 3rd row to the 1st row to get the condition forced on the us. Dotting into the first row forced nothing. Dotting into the second row forced $u_3 + a_{2,4}u_4$ to be 0. Finally, dotting into the first row forced $u_1 + a_{1,2}u_2 + a_{1,4}u_4 = 0$. That gave $\vec{u} = \begin{pmatrix} -a_{1,2}u_2 - a_{1,4}u_4 \\ u_2 \\ -a_{2,4}u_4 \\ u_4 \end{pmatrix}$. As in class: This

[1] Notation Hint: In this part I suggest using a notation like $\bullet$ to indicate entries that can be any real numbers.

[2] Notation Hint: In this part, you will need notation like $a_{2,3}$, say, to indicate an entree in A in the 2nd row, 3rd column.

gave the set of nullspace solutions as

$$\left\{ u_2 \begin{pmatrix} -a_{1,2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + u_4 \begin{pmatrix} -a_{1,4} \\ 0 \\ -a_{2,4} \\ 1 \end{pmatrix} \mid (u_2, u_4) \in \mathbb{R}^2 \right\}.$$

2. Lines contained in a given plane. **20 pts**

Here is a statement from classical geometry: If two distinct points lie on a plane, then all points on the line determined by these two points lie on the plane. This problem considers parametric lines and a plane defined by an equation.

Here is a parametric line line through $(1,3,-1)$ in the direction (v_1,v_2,v_3):

$$L_{\boldsymbol{v}} = \{(1,3,-1) + t(v_1,v_2,v_3) \mid t \in \mathbb{R}\},$$

Consider the function $f(x_1,x_2,x_3) = -x_1 + 2x_2 - 3x_3 - 8$ and the plane

$$P = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid f(x_1,x_2,x_3) = 0\}.$$

(2.a) **4 pts**: Find nonzero vectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ for which $\vec{\boldsymbol{v}}_1$ is not a scalar multiple of $\vec{\boldsymbol{v}}_2$ and P contains both lines $L_{\vec{\boldsymbol{v}}_i}$, $i = 1, 2$.
Ans: Check that $(1,3,-1)$ lies on P by plugging its coordinates into $f(x_1,x_2,x_3) = 0$.

To find $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$, find (v_1,v_2,v_3) solutions of $A(\vec{\boldsymbol{v}}) = -v_1 + 2v_2 - 3v_3 = 0$.

Take $\vec{\boldsymbol{v}}_1 = (2,1,0)$ and $\vec{\boldsymbol{v}}_2 = (-3,0,1)$. From linearity of A, since $A(1,3,-1) = 8$, the set of solutions of $A(\vec{\boldsymbol{x}}) = 8$ is

$$\{(1,3,-1) + t_1\vec{\boldsymbol{v}}_1 + t_2\vec{\boldsymbol{v}}_2 \mid (t_1,t_2) \in \mathbb{R}^2\}.$$

(2.b) **12 pts**: Use the $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ you found in (2.a). The points

$$\boldsymbol{u}_1 = (1,3,-1) + \vec{\boldsymbol{v}}_1 \text{ and } \boldsymbol{u}_2 = (1,3,-1) + \vec{\boldsymbol{v}}_2$$

both lie on P. Show the parametric line L containing $\boldsymbol{u}_1$ and in the direction from $\boldsymbol{u}_1$ to $\boldsymbol{u}_2$ has all of its points on P. Give a parametric description for the points on L.
Ans: The direction from $\boldsymbol{u}_1$ to $\boldsymbol{u}_2$ is given by the subtraction $\boldsymbol{u}_2 - \boldsymbol{u}_1 = \vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2$. The line L therefore consists of the points $\{\boldsymbol{u}_1 + t(\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2) \mid t \in \mathbb{R}\}$.

(2.c) **4 pts**: Why does $(1,3,-1) + t_1\boldsymbol{v}_1 + t_2\boldsymbol{v}_2$ lie on P for any $t_1, t_2 \in \mathbb{R}$?
Ans: The answer for Part 2.a already shows this.