LINEAR ALGEBRA MATH 2130
SECTION 6,
2ND PROBLEM SET
LINEAR TRANSFORMATIONS
DUE MONDAY, FEBRUARY 25TH

PROF. MICHAEL D. FRIED, OFFICE MATH. DEPT. #223

1. Use a matrix in row reduced form. **20 pts**

(1.a) **5 pts**: Suppose $A_1 : \mathbb{R}^5 \to \mathbb{R}^2$ is a linear transformation. Show it *cannot* have null space

$$N = \{\vec{\boldsymbol{u}} \in \mathbb{R}^5 \mid u_1 = 3u_2 \text{ and } u_3 = u_4 = u_5\}.$$

Ans: If A_1 exists then its range is spanned by ≤ 2 linearly independent vectors, so its null space is spanned by $\ge 5 - 2 = 3$ linearly independent vectors. Here though the nullspace is

$$\{u_2\vec{\boldsymbol{v}}_1 + u_5\vec{\boldsymbol{v}}_2 \mid (u_2, u_5) \in \mathbb{R}^2\},$$

with $\vec{\boldsymbol{v}}_1 = \begin{pmatrix}3 & 1 & 0 & 0 & 0\end{pmatrix}^T$ and $\vec{\boldsymbol{v}}_2 = \begin{pmatrix}0 & 0 & 1 & 1 & 1\end{pmatrix}^T$.

(1.b) **7 pts**: Give a linear transformation $A_2 : \mathbb{R}^5 \to \mathbb{R}^3$ that has the null space N of (1.a).
Ans: Use an A_2 in row reduced form.

$$A_2 = \left(\vec{e}_1|\vec{\boldsymbol{a}}_2|\vec{e}_2|\vec{e}_3|\vec{\boldsymbol{a}}_5\right), \text{ with } \vec{\boldsymbol{a}}_2 = \begin{pmatrix}-3\\0\\0\end{pmatrix} \text{ and } \vec{\boldsymbol{a}}_5 = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}.$$

(1.c) **8 pts**: Use your answer in (1.b) to give a description of all matrices $A : \mathbb{R}^5 \to \mathbb{R}^3$ that have null space N.
Ans: Row reduction of a 3×5 matrix A is affected by multiplying on the left by 3×3 invertible matrices. That doesn't change the null space, just the range. The collection of matrices A with N as null space consists of

$$\{D_L A_2 \mid D_L \text{ is a } 3 \times 3 \text{ invertible matrix}\}.$$

2. Relating two bases of a vector space. **20 pts**

Denote the vector space of polynomials of degree at most 3 by

$$\mathcal{P}_4 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Consider the function $T : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$f \in \mathcal{P}_4 \mapsto x\frac{df}{dx} - f.$$

(2.a) **5 pts**: Find the matrix A_1 of T with respect to the basis

$$\{1, x, x^2, x^3\} \text{ of } \mathcal{P}_4.$$

Ans: Compute: $T(x^0) = -x^0, T(x^1) = x - x = 0, T(x^2) = 2x^2 - x^2 = x^2, T(x^3) = 3x^3 - x^3 = 2x^3$. So, if we take $\vec{\boldsymbol{v}}_i = x^{i-1}$, then the matrix that expresses the affect of T on these vectors is

$$A_1 \stackrel{\text{def}}{=} A_{\vec{\boldsymbol{v}}_1,\vec{\boldsymbol{v}}_2,\vec{\boldsymbol{v}}_3,\vec{\boldsymbol{v}}_4} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(2.b) **8 pts**: What is the matrix A_2 of T with respect to the basis

$$\{1, x-1, (x-1)^2, (x-1)^3\} \text{ of } \mathcal{P}_4?$$

Ans: Compute:

$$T((x-1)^0) = -(x-1)^0, T((x-1)^1) = x - (x-1) = (x-1)^0,$$
$$T((x-1)^2) = 2x(x-1) - (x-1)^2 = (x-1)^2 + 2(x-1),$$
$$T((x-1)^3) = 3x(x-1)^2 - (x-1)^3 = 2(x-1)^3 + 3(x-1)^2.$$

So, if we take $\vec{\boldsymbol{w}}_i = (x-1)^{i-1}$, then the matrix that expresses the affect of T on these vectors is

$$A_2 \stackrel{\text{def}}{=} A_{\vec{\boldsymbol{w}}_1,\vec{\boldsymbol{w}}_2,\vec{\boldsymbol{w}}_3,\vec{\boldsymbol{w}}_4} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(2.c) **7 pts**: Describe the range and null space of A_1.
Ans: I was trying to get you to see that A_1 is a diagonal matrix. Also, it is easy to describe the range and null space of a diagonal matrix.

The range of A_1 is the span of $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4$, correspond to the 1st, 3rd and 4th columns:

$$\text{Range}(A) = \{a_0 + a_2x^2 + a_3x^3 \mid a_0, a_2, a_3 \in \mathbb{R}\}.$$

The null space of A_1 consists of the span of $\vec{\boldsymbol{v}}_2$:

$$\text{Null}(A) = \{a_1x \mid a_1 \in \mathbb{R}\}.$$