LINEAR ALGEBRA MATH 2130
SECTION 6,
3RD PROBLEM SET
COMPARING BASES OF FUNCTIONS; AND
THE SIMPLEST CASE OF NON-DIAGONALIZABILITY
DUE FRIDAY, APRIL 5TH

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1. Bases of cosine functions **20 pts**

Consider these two sets of rational functions

$$M_1 = \{(x + 1/x)^k, k = 0, 1, 2, 3, 4\} \text{ and } M_2 = \{x^k + 1/x^k, k = 0, 1, 2, 3, 4\}.$$

Let V_1 (resp. V_2) be the vector space that M_1 (resp. M_2) spans.

(1.a) **10 pts**: Show that $V_1 = V_2$.[1]
Ans: For $k = 0$ or 1, the sets M_1 and M_2 are the same. For $k = 2, 3, 4$, the binomial expansion gives

$$(x + 1/x)^2 = x^2 + 1/x^2 + 2; (x + 1/x)^3 = x^3 + 1/x^3 + 3(x + 1/x);$$
$$\text{and } (x + 1/x)^4 = x^4 + 1/x^4 + 4(x^2 + 1/x^2) + 6.$$

The right hand sides are clearly in the span of M_2, and since V_1 is contained in V_2, both having the same dimension, they are equal.

(1.b) **10 pts**: Now substitute $e^{i\theta}$ for x, and using the two identities in a recent e-mail, show that the functions in the two sets

$$\Theta_1 = \{(\cos(\theta))^k, k = 0, 1, 2, 3, 4\} \text{ and } \Theta_2 = \{\cos(k\theta), k = 0, 1, 2, 3, 4\}$$

span the same vector space of functions. I will use that $\cos(0) = 1$.
Ans: As the hint says: Plug in $e^{i\theta}$ to and use that $e^{ik\theta} + e^{-ik\theta} = 2\cos(\theta)$ (as done in class):

$$(2\cos(\theta))^2 = 2\cos(2\theta) + 2; (2\cos(\theta))^3 = 2\cos(3\theta) + 6\cos(\theta);$$
$$\text{and } (2\cos(\theta))^4 = 2\cos(4\theta) + 8\cos(2\theta) + 6.$$

Again, the space Θ_1 is contained in the space Θ_2.

Extra Credit **10 pts**: Why are the functions in Θ_2 linearly independent. There are many ways to see this. Here is the one I hinted at in my e-mail. Suppose:

$$c_0 + c_1\cos(\theta) + c_2\cos(2\theta) + c_3\cos(3\theta) + c_4\cos(4\theta) \equiv 0,$$
$$\text{with some } c_k \neq 0.$$

[1] Hint: Use the binomial theorem to write $(x+1/x)^k$ as a sum of scalars times $x^j + 1/x^j$, $j \leq k$.

The expression $\equiv 0$ means for all values of θ. Take the second derivative with respect to θ to get

$$-c_1\cos(\theta) - 4c_2\cos(2\theta) - 9c_3\cos(3\theta) - 16c_4\cos(4\theta) \equiv 0.$$

Do it again to get

$$c_1\cos(\theta) + 4^2c_2\cos(2\theta) + 9^2c_3\cos(3\theta) + (16)^2c_4\cos(4\theta) \equiv 0.$$

And then again to get four different linear equations in the $\cos(k\theta)$ s with coefficients in the c_k s. Now plug in $\theta = 0$, to get linear equations in just the c_k s. Row reduce their matrix by hand to get the identity matrix. Therefore the only solution in the c_k s is that they are 0.

2. A special normal form **20 pts**

Suppose $T : V \to V$ is a linear transformation on a vector space V of dimension 6. Construct the following series of vectors from $\vec{\boldsymbol{v}} \in V$:

$$B_{\boldsymbol{v}} \overset{\text{def}}{=} \{\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_1, T(\vec{\boldsymbol{v}}_1) = \boldsymbol{v}_2, \dots, T(\vec{\boldsymbol{v}}_5) = \vec{\boldsymbol{v}}_6\}. \tag{2.1}$$

Assume $\vec{\boldsymbol{v}}_6$ is not zero, but $T(\vec{\boldsymbol{v}}_6) = \vec{\boldsymbol{0}}$. You may assume that 6 linearly independent vectors in V form a basis of V.

(2.a) **8 pts**: Show that the vectors of (2.1) are linearly independent, and therefore a basis of V. [2]

Ans: Take k as given in the hint, and write that $c_6\vec{\boldsymbol{v}}_6 + c_5\vec{\boldsymbol{v}}_5 \dots c_{k+1}\vec{\boldsymbol{v}}_{k+1} = \vec{\boldsymbol{v}}_k$. Apply T to both sides to get

$$c_6\vec{\boldsymbol{v}}_6 \dots c_{k+2}\vec{\boldsymbol{v}}_{k+2} = \vec{\boldsymbol{v}}_{k+1},$$

which is a nontrivial linear dependence relation with k replaced by $k+1$, contrary to assumption.

(2.b) **8 pts**: What is the matrix, $A_{B_{\vec{\boldsymbol{v}}}}$, of T with respect to the basis $B_{\vec{\boldsymbol{v}}}$?

Ans: Apply T to each element of $B_{\boldsymbol{v}}$ in order, but use the order on $V_{\boldsymbol{v}}$ starting with $\vec{\boldsymbol{v}}_6$ and going down.

$$\begin{aligned}
T(\vec{\boldsymbol{v}}_6) &= 0\cdot\vec{\boldsymbol{v}}_1 + 0\cdot\vec{\boldsymbol{v}}_2 + 0\cdot\vec{\boldsymbol{v}}_3 + \cdots + 0\cdot\vec{\boldsymbol{v}}_6\\
T(\vec{\boldsymbol{v}}_5) &= 0\cdot\vec{\boldsymbol{v}}_1 + 0\cdot\vec{\boldsymbol{v}}_2 + 0\cdot\vec{\boldsymbol{v}}_3 + \cdots + 1\cdot\vec{\boldsymbol{v}}_6\\
&\dots\\
T(\vec{\boldsymbol{v}}_2) &= 0\cdot\vec{\boldsymbol{v}}_1 + 0\cdot\vec{\boldsymbol{v}}_2 + 1\cdot\vec{\boldsymbol{v}}_3 + \cdots + 0\cdot\vec{\boldsymbol{v}}_6\\
T(\vec{\boldsymbol{v}}_1) &= 0\cdot\vec{\boldsymbol{v}}_1 + 1\cdot\vec{\boldsymbol{v}}_2 + 0\cdot\vec{\boldsymbol{v}}_3 + \cdots + 0\cdot\vec{\boldsymbol{v}}_6
\end{aligned}$$

By definition the resulting matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note: If you took the basis vectors from 2.a in the opposite order, then you would have gotten a different matrix, with the 1's below the diagonal and the rest 0's. That's not wrong, (the result though is numbers – 0's

[2] Hint: Do proof by contradiction. Start by considering the largest integer k such that $\vec{\boldsymbol{v}}_k, \dots, \vec{\boldsymbol{v}}_6$ are linearly dependent.

and 1's – in a simple looking matrix), but you can get them in the order I have them, and that gives a form, so that everyone can get the same thing. In class we will finish what this gives us.

(2.c) **4 pts**: With I_V the linear transformation that takes $\vec{\boldsymbol{w}} \in V$ to $\vec{\boldsymbol{w}}$, what is the matrix attached to $3I_V + T$ relative to the basis $B_{\boldsymbol{v}}$?

Ans: To the k row you just add $3\vec{\boldsymbol{v}}_k$ and the result is

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$