LINEAR ALGEBRA MATH 2130
SECTION 6,
1ST–7TH WEEK SYLLABUS
2:00 – 2:50 MWF ROOM MUEN E131
FUNDAMENTALS ON LINEAR TRANSFORMATIONS; THE ALGEBRA AND GEOMETRY OF SOLVING THEM

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Book: Linear Algebra and its Applications (5th edition) by David Lay, Steven Lay and Judi McDonald.

1. An overview to the start

Linear equations are defined by matrices, which give a certain special class of functions, called linear functions. Much of the time we take the domain and range of the function defined from a matrix $A, B, \dots$ or some other capital letter at the beginning of the alphabet using the notation $A : \mathbb{R}^n \to \mathbb{R}^m$. The notation for the matrix is then that it has m rows and n columns.

For an $m \times n$ equation with $m = 2$ and $n = 3$, we have the notation:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

1.1. Weeks 1-4: The mechanics of solving equations.

The equations have the form $A(\boldsymbol{x}) = \boldsymbol{b}$, with $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$ (except – following the book, p. 26 – we tip both *vectors* on their sides using their *transposes*). Transpose of any matrix is defined on p. 101. As explained in class:

(1.1a) All vectors are $m \times 1$ (one column) matrices. That allows us to use the notation of matrix multiplication: $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, with A above, the expression $A(\boldsymbol{x}) = \boldsymbol{b}$ looks like

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{1.2}$$

(1.2b) Because it takes up too much space writing column vectors on the board or in e-mails, we still use the notation $(x_1, \dots, x_n) \in \mathbb{R}^n$ for vectors. The symbol $\vec{\boldsymbol{x}}$ will also be used in class for indicating $\boldsymbol{x}$ is a vector.

(1.2c) The special vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ in $\mathbb{R}^n$ make up the *standard basis* in $\mathbb{R}^n$. So, $\boldsymbol{e}_2$ in $\mathbb{R}^4$ (resp. $\mathbb{R}^3$) is $(0,1,0,0)$ (resp. $(0,1,0)$) in the calculus vector notation, but $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ (resp. $\begin{pmatrix}0\\1\\0\end{pmatrix}$) in the column matrix form.

It defines these vectors formally on p. 150, but it already used them from the beginning of the book (say on p. 13) to define reduced echelon form.

The mechanics of solving (1.2) uses elementary matrix operations as here:

(1.3a) Replace a row by itself added to a scalar multiple of another row.
(1.3b) Interchange two rows.
(1.3c) Multipy a row by a nonzero constant (scalar).

Eventually (p. 108-109) we learn that all matrices can be written as a product of *elementary matrices*. That becomes very useful in understanding the nature of solutions of linear equations.

Given A, whether there are solutions depends on $\boldsymbol{b}$. Then, if there is one solution, we want to know the nature of the whole set of solutions.

1.2. **Introducing the geometry of solutions.** I will do a presentation, that will appear in the course web site, called *Lines in the Curriculum*, early in the course. This will emphasize that, say, lines in 3-space, $\mathbb{R}^3$, appear in two different ways:

(1.4a) *Equation form*: As the intersection of two planes.
(1.4b) *Solution form*: Parametrized by a point and a vector direction.

This takes us through sections §1.1–§1.8. We also do some work in §1.9 on particular matrices that give us linear transformations that we can describe geometrically.

Many courses use these – especially engineering, the physical sciences, but economics and social sciences, too – when dealing with objects in special configurations.

2. Vectors describing the range and domain

Given A, the set of $\boldsymbol{b}$ for which $A(\boldsymbol{x}) = \boldsymbol{b}$ has a solution, is the *Range* of A. There is a one case that is especially useful: where $\boldsymbol{b} = (0, \dots, 0) \stackrel{\text{def}}{=} \mathbf{0}$, the 0-vector. Then, we call the solutions the *kernel* or *null-space* of A.

2.1. **The special case $m = n$.** When $m = n$, an especially interesting case is when there is a *unique* solution to $A(\boldsymbol{x}) = \boldsymbol{b}$ for every value $\boldsymbol{b}$. For that we have the notion of when a linear transformation is invertible.

That corresponds to the matrix A having an *inverse*. For that, there are several characterizations. Beginning material on that phrases this as saying we may write such a matrix as a product of elementary matrices. That material is in §2.1–§2.3.

A more geometric approach is given by Cramer's rule, and the use of the *determinant*. This has an extremely important application interpreting the determinant as the volume of a geometric (parallelpiped) figure. This material is in §3.1–§3.3.

2.2. **Vector spaces.** Both the range (in $\mathbb{R}^m$) and kernel (in $\mathbb{R}^n$) of A are *vector subspaces*. In this part of the material we learn to describe them in efficient ways using the concepts of *linear independence* and *basis* of a vector subspace.

Then, we turn studying the *equation* given by A around, to learnto what extent we can create an equation that has a particular null-space and a particular range.

While this is basically in §2.8 and §2.9, I will also use some of the vocabulary of Chap. 4, which considers one of the most practical of vectors spaces, given by polynomials of a bounded degree. We may not finish this before the midterm.

3. Extra Comments about the Course

3.1. **Office Hours.** I will use two of them regularly, after the course on Wednesday and Friday. We'll check if that suffices. Also, I expect to communicate by e-mail with you. You can arrange another time to meet with me, and you can also ask then any question you like.

3.2. **Grading.**

- **Problem sets 160 points**: In each half of the course, there will be two problem sets, each worth 40 points.
- **Midterm 120 points**: I expect the midterm to be what you might expect from the problem sets. Emphasis will be on definitions which I will repeat often in class.
- **Final 240 points**: Based on the material for the whole course. The idea is that I hope you get a picture of the whole course by the time we are done.
- **Class participation 80 points**: I consider your class participation and contributions, to the best of your ability, as part of the course. It is best if I can see what you are getting, and I can adjust through answering questions in class, or by adding material that is helpful.
- **Total 600 points**

3.3. *Syllabus statements.* from a document university hands out.

- Accommodation for Disabilities: If necessary, submit your accommodation letter from Disability Services to me.
- Classroom Behavior: in particular refers to my use of your name.
- Honor Code: All incidents of academic misconduct will be reported to the Honor Code (honor@colorado.edu); 303-492-5550).
- Sexual Misconduct, Discrimination, Harassment and/or Related Retaliation, refers to the Office of Institutional Equity and Compliance.
- Religious Holidays: I will adjust to the best of my ability to your religious and family related needs.

3.4. **Website.** This document, Syll1Spring19-LineAlg.pdf, and Syllabus Statements are at

http://www.math.uci.edu/˜mfried/linalglist-spring19.html.

Up-dated versions, problem sets, etc. will go there.

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