ALGEBRAIC FUNCTIONS WITH EVEN MONODROMY

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Abstract. Let X be a compact Riemann surface of genus g and $d \geq 12g+4$ be an integer. We show that X admits meromorphic functions with monodromy group equal to the alternating group A_d.

1. Introduction

Let X be a compact Riemann surface of genus g and $f : X \to \mathbb{P}^1$ be a meromorphic function of degree d. The function field $C(X)$ is a finite algebraic extension of degree d of $C(\mathbb{P}^1) = \mathbb{C}(x)$. The *monodromy group* $M(f)$ is the Galois group associated to the Galois closure of the extension $C(X)/\mathbb{C}(x)$. The group $M(f)$ has a natural transitive representation in the symmetric group S_d. In this paper, we prove the existence of meromorphic functions on X with *even monodromy*. This means that the monodromy group is contained in the *alternating group* A_d on d elements. In fact, we study the case:

$$M(f) = A_d.$$

The problem of finding Riemann surfaces with given monodromy group is a classical one (see for example [5]). In particular, a difficult question is that of determine all possible monodromy groups for the generic Riemann surface of genus g. Several aspects of this problem were considered by Zariski in [22]. Notice that the definition of monodromy group can be extended to any holomorphic map between compact Riemann surfaces. In fact, Zariski observed that there are no holomorphic maps from the generic Riemann surface of genus $g > 1$ to one of positive genus. Thus the critical case to study is that of meromorphic functions. Besides, he proved that the monodromy groups of a generic Riemann surface of genus $g > 6$ are not solvable. Zariski also introduced the reduction to the *primitive* case. This means that we can consider only minimal extensions of $\mathbb{C}(x)$ or, equivalently, *indecomposable* meromorphic functions.

It was first shown by Guralnick and Thompson in [13] that there are groups that cannot occur as the monodromy group of a meromorphic function on a Riemann surface of genus g.

Later, it was proved that the monodromy group of an indecomposable meromorphic map of degree d on a generic Riemann surface of genus $g > 3$ is either S_d, $d \geq (g+2)/2$ or A_d, $d \geq 2g+1$. This result is contained in a series of three papers: [11], [10] and [12].

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It is well known that the symmetric group S_d is the monodromy group of every curve of genus g for $d \geq g+1$ (see [8]) and of the generic one for $d \geq (g+2)/2$ (see [16]). As noticed by Guralnick in [9], this makes the case of the alternating group very interesting.

In a recent article K. Magaard and H. Völklein prove that the general curve of genus $g \geq 3$ admits meromorphic functions with monodromy group A_d and 3-cycles (or double transpositions) as branch cycles if and only if $d \geq 2g+1$ (see [17]).

In this paper we prove that, for d sufficiently large compared to g, every compact Riemann surface of genus g realizes the monodromy group A_d.

Theorem 1. *Let X be a compact Riemann surface of genus $g > 0$ and $d \geq 12g+4$ be an integer. Then there exist indecomposable meromorphic functions $f \in \mathbb{C}(X)$ of degree d with $M(f) = A_d$. Moreover, there exists on X a family $\mathcal{F}(X,d)$ of such maps with dimension:*

$$\dim \mathcal{F}(X,d) = [\frac{d+3}{2}] - 2g + 2.$$

We give a brief summary of the proof, which is contained in section 4.

We fix a *spin bundle* S on a compact Riemann surface X. This is a line bundle on X such that $S^2 = K_X$, where K_X is the canonical bundle. We then take a divisor D on X having large degree and small support (i.e. $D = n_1P_1 + n_2P_2 + n_3P_3$ where P_1,P_2 and P_3 are points of X). Let $[D]$ be the divisor given by the formal sum of points in the support of D. The square of a global section s of $S(D)$ is a meromorphic form $\omega = s^2$ on X with at most poles at $[D]$. It has even order at each point. If ω is "exact" i.e. $\omega = df$, then f is a meromorphic function on X with local monodromy in A_d where $d = \deg(f)$. We call such meromorphic functions *odd ramification coverings* of the projective line. [1] These coverings were studied in [7], [20] and [21] in connection with the spin lifting.

Given an inferior bound on the degree of D, the existence of not trivial sections s of $S(D)$ with exact square follows easily for dimensional reasons. In section 4, we compute the dimension of the families of these sections when $d \geq 12g+4$. As in the theory of special divisors (see [1]), it is possible to compute the tangent space to our families in the cohomological setting. For this, we use a rational variation of De Rham's algebraic theory and the adjustment of a technical result contained in [19] (we recall its proof in section 6). The generic smoothness of our families follows from a cohomology vanishing. Thus, comparing dimensions, we find indecomposable functions with the expected degree: their poles have maximum order at $[D]$. An algebraic lemma and a count of parameters show that the monodromy group of the general element in the family is exactly A_d.

In section 5, using the theory of admissible coverings, we show that odd ramification coverings are limits of *simple odd ramification coverings* i.e. with ramification points all of order 3 with distinct images in $\mathbb{P}^1$. Then, Theorem 1 implies that the generic Riemann surface of genus g admits simple odd ramification coverings of degree $d \geq 12g+4$ with monodromy group A_d (see Theorem 2). This proves in a completely different way a weaker version of the result contained in [17].

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[1] We avoid the tempting term "spin covering" from a suggestion of M. Fried who explained us a different use of this word.

In [3] S. Brivio and the second author apply Theorem 1 to give an analogous result for rational functions on complex surfaces.

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2. Preliminaries and Notation

We start recalling some known results about the monodromy group of a meromorphic function (see [18]). Let X be a compact Riemann surface of genus g, $f : X \to \mathbb{P}^1$ be a meromorphic function of degree d on X with branch locus B and ramification locus R. Let $p : X \setminus R \to \mathbb{P}^1 \setminus B = Y$ be the associated topological covering. Fixed a base point $y \in Y$, the fundamental group $\Pi_1(Y, y)$ acts on the fibre $f^{-1}(y)$ by path lifting giving a transitive subgroup of the symmetric group S_d. This is the *monodromy group* of f (it is determined by f up to conjugacy). In fact, the monodromy group of f is isomorphic to the Galois group associated to the Galois closure of the extension $\mathbb{C}(X)/\mathbb{C}(x)$ (see [14]).

Conversely, let us consider $B = \{b_1, \dots, b_r\} \subset \mathbb{P}^1$, $Y = \mathbb{P}^1 \setminus B$ and fix standard geometric generators of $\Pi_1(Y, y)$. Riemann's existence theorem gives a bijection between meromorphic functions of degree d with branch locus B (up to isomorphism) and ordered r-uples of permutations $\{\sigma_1, \dots, \sigma_r\}$ with $\sigma_1 \cdots \sigma_r = id$ and generating a transitive subgroup of S_d (up to conjugacy).

It can be easily seen that f is indecomposable if and only if its monodromy group is primitive.

We need the following group theoretic lemmas (the first one is a classical result, the second one is just an exercise):

Lemma 1. *Let G be a transitive and primitive subgroup of the alternating group A_d containing a 3-cycle. Then $G = A_d$.*

Lemma 2. *Let G be a transitive subgroup of S_d with $g \in G$ having exactly two cycles of relatively prime lenght or three cycles of odd and relatively prime lenght. Then G is a primitive subgroup.*

Notation 1. Let D be a divisor on X, we denote by $\mathrm{supp}(D)$ its support and by $[D]$ the divisor given by the formal sum of points in $\mathrm{supp}(D)$. Let $X \backslash [D]$ be the open set $X \backslash \mathrm{supp}(D)$. We write $[D_1] \cap [D_2] = \emptyset$ if D_1 and D_2 have disjoint supports. If $D_1 - D_2$ is effective we use the notation $D_1 - D_2 \geq 0$.

3. Spin sections with exact square

Definition 1. Let X be a compact Riemann surface, we call *spin bundle* a line bundle S on X with: $S^2 = K_X$, where K_X is the canonical bundle.

It is known (see [1]) that there are exactly 2^{2g} non equivalent spin bundles on any compact Riemann surface X of genus g.

Fix X a compact Riemann surface of genus g, S a spin bundle on X and $D \in Div(X)$ a divisor of the form:

$$D = n_1 P_1 + n_2 P_2 + n_3 P_3 \quad (\star)$$

where n_1, n_2, n_3 are integers with $n_1 \geq n_2 \geq n_3 \geq 0$ and P_1, P_2, P_3 are distinct points in X. Let k be the degree of $[D]$. Notice that $1 \leq k \leq 3$ and $D = \sum_{i=1}^{k} n_i P_i$. We set:

$$D' = 2D - [D].$$

Consider now the line bundle $S(D)$. The squares of sections $s \in H^0(X, S(D))$ give meromorphic forms $\omega = s^2$ on X with at most poles in $[D]$. More precisely, their divisor is:

$$(\omega)_0 - (\omega)_\infty = 2div(s) - 2D = 2E,$$

with $E = div(s) - D$. If ω is an exact form with $\omega = df$ then $f \in H^0(X, \mathcal{O}_X(D'))$ and

$$\begin{aligned} e_f(x) &= ord_\omega(x) + 1 = 2ord_E(x) + 1 \quad \text{if } x \in (\omega)_0, \\ e_f(x) &= -ord_\omega(x) - 1 = -2ord_E(x) - 1 \quad \text{if } x \in (\omega)_\infty, \end{aligned}$$

where $e_f(x)$ is the ramification index of f in x. The map f has odd ramification index in each point, hence *even monodromy*, i.e. $M(f)$ is contained in the alternating group. In fact each permutation generating the monodromy group $M(f)$ can be decomposed in cycles of odd length.

Definition 2. We call a meromorphic map $f : X \to \mathbb{P}^1$ an *odd ramification covering* if all ramification points have odd index.

We define the analytic map:

$$\Upsilon : H^0(X, S(D)) \to H^1(X \backslash [D], \mathbb{C}) \quad s \mapsto [s^2].$$

Firstly, we want to show the existence of sections $s \in H^0(X, S(D)), s \neq 0$ with $\Upsilon(s) = 0$.

Notation 2. $h^i(X, L) = \dim H^i(X, L)$ (i=1, 2); $\mathcal{H}(X, D) = \Upsilon^{-1}(0)$; $\mathcal{F}(X, D) = \{f \in \mathbb{C}(X) : df = s^2,\ s \in \mathcal{H}(X, D)\}$.

By the Riemann-Roch theorem and De Rham theory we have :

$$h^0(X, S(D)) = \deg(D) \text{ if } \deg(D) > g - 1;$$

$$h^1(X \backslash [D], \mathbb{C}) = 2g + k - 1$$

where k is the degree of $[D]$.

Suppose $\Upsilon(s) \neq 0$ for every $s \in H^0(X, S(D)) \setminus \{0\}$. This gives a contradiction if $\deg(D) > 2g + k - 1$. Therefore we have:

Proposition 1. *Let X be a compact Riemann surface of genus g and D a divisor as in $(\star)$. If $\deg(D) > 2g + k - 1$ then $\dim_{\mathbb{C}} \mathcal{H}(X, D) > 0$. In particular, there exists a not constant odd ramification covering $f \in \mathcal{F}(X, D)$.*

To get more precise information about the dimension of $\mathcal{H}(X, D)$, we are interested in the surjectivity of the differential of Υ in $s \in \mathcal{H}(X, D)$:

$$d\Upsilon(s) : H^0(X, S(D)) \to H^1(X \backslash [D], \mathbb{C}).$$

Take the holomorphic curve $s(t) = s + tv$ in $H^0(X, S(D))$ then:

$$[s(t)^2] = [s^2 + 2tsv + t^2 v^2] = 2t[sv] + o(t^2).$$

Therefore:

$$d\Upsilon(s)(v) = [2sv].$$

We now recall a technical lemma contained in [19], its proof is contained in section 6. Let $A, B \in Div(X)$ be effective divisors with $[A] \cap [B] = \emptyset$, $a = \deg[A]$ and $b = \deg[B]$. Consider the map:

$$\mathcal{D} : H^0(X, K_X(A-B)) \to H^1(X \setminus [A], \mathbb{C}) \quad \omega \mapsto [\omega].$$

Lemma 3. *The map $\mathcal{D}$ is onto if* $\deg(A-B) - a - b > 2g-2$.

We consider a divisor D as in $(\star)$ and a section $s \in \mathcal{H}(X,D)\backslash\{0\}$ with zero divisor $div(s) = E$. The multiplication by s gives an isomorphism:

$$m(s) : H^0(X, S(D)) \to H^0(X, K_X(2D-E)).$$

Composing with $\mathcal{D}$ we obtain:

$$\mathcal{D} \circ m(s) : H^0(X, S(D)) \to H^1(X \setminus [D]), \mathbb{C}) \quad v \mapsto [sv],$$

which is $d\Upsilon(s)$ (up to a costant).

We would like to apply Lemma 3. We first decompose the divisor of s in the sum of two effective divisors as follows:

$$E = E_1 + E_2 \text{ with } [D] - [E_1] \geq 0,\ [E_2] \cap [D] = \emptyset.$$

Let us define:

$$A = 2D - E_1,\ B = E_2.$$

With this choice, Lemma 3 gives the surjectivity of $d\Upsilon(s)$ if the couple (D,E) satisfies :

a) $2D - E_1 \geq 0$,
b) $\deg(2D-E) - k - \deg[E_2] > 2g-2$

Notation 3. Let $T \in X \setminus [D]$ be a generic point and $r \geq 1$ be an integer. Set

$$\mathbb{P}(\mathcal{H}(X,D)) := \{(s) \in \mathbb{P}H^0(X, S(D)) \ : s \in \mathcal{H}(X,D)\},$$
$$\mathbb{P}H^0(X, S(D-rT)) := \{(s) \in \mathbb{P}H^0(X, S(D)) \ : div(s) \geq rT\}.$$

Notice that:

$$\dim \mathbb{P}(\mathcal{H}(X,D)) \geq \deg(D) - 2g - k,$$
$$\dim \mathbb{P}H^0(X, S(D-rT)) = \deg(D) - 1 - r.$$

Therefore, if $r \leq \deg(D) - 2g - k$:

$$\mathbb{P}(\mathcal{H}(X,D)) \cap \mathbb{P}H^0(X, S(D-rT)) \neq \emptyset.$$

Let $s \in \mathcal{H}(X,D)$ be a not trivial section with $div(s) \geq (\deg(D) - 2g - k)T$. Then we have:

$$\deg(E_1) \leq (g - 1 + \deg(D)) - (\deg(D) - 2g - k) = 3g + k - 1.$$

In particular a) holds for $E = div(s)$ when:

$$2n_i > 3g + k - 1,\ i \in \{1, \dots, k\}.$$

Then *b*) becomes: $\deg(D) - \deg[E_2] > 3g - 3 + k$. Since

$$\deg[E_2] \leq \deg[E] \leq 3g + k - 1,$$

condition b) holds when

$$\deg(D) > 6g + 2k - 4.$$

Under these hypotheses, Lemma 3 implies that all components of $\mathcal{H}(X,D)$ contain a point s such that $d\Upsilon(s)$ is surjective. The implicit function theorem finally gives:

Proposition 2. *Let X be a compact Riemann surface of genus g and D be a divisor as in $(\star)$ with support of degree k. Suppose that:*

1) $2n_i > 3g + k - 1,\ i \in \{1, \dots, k\}$
2) $\deg(D) > 6g + 2k - 4$.

Then the general point of any component of $\mathcal{H}(X, D)$ is smooth and

$$\dim \mathcal{H}(X, D) = \deg(D) - 2g - k + 1.$$

4. Constructing maps with even monodromy

In the previous section we have seen that sections in $\mathcal{H}(X, D)$ give in a natural way odd ramification coverings. However Proposition 2 only gives a bound on the ramification at infinity of such maps. In fact "a priori" these sections could have zeros at D, hence the degree of the corresponding maps could drop. Moreover, we need to provide indecomposable maps.

Let $D = n_1P_1 + n_2P_2 + n_3P_3$ as in the previous sections with k the cardinality of its support. Set:

$$d_1 = 2n_1 - 1,\ d_2 = 2n_2 - 1,\ d_3 = 2n_3 - 1.$$

Definition 3. We will say that (d_1, d_2, d_3) is an *indecomposable triple* in these cases:

- if $k = 1$: d_1 is prime number;
- if $k = 2$: d_1, d_2 are relatively prime,
- if $k = 3$: d_1, d_2, d_3 are relatively prime.

The result is the following:

Proposition 3. *Let X be a compact Riemann surface of genus $g > 0$ and D be a divisor as in $(\star)$ with support of degree $1 \le k \le 3$. Let d_1, d_2 and d_3 as before. Suppose:*

a) $d_i > 3g + k$ *for all* $i \in \{1, \dots, k\}$
b) $\deg(D) > 6g + 2k - 3$,
c) (d_1, d_2, d_3) *is an indecomposable triple.*

Then there exists a family $\mathcal{F}(X, D)$ of meromorphic functions f on X with these properties:

1) $deg(f) = 2\deg(D) - k$,
2) $M(f) \subset A_d$ *and* $M(f)$ *is primitive i.e.* f *is indecomposable.*
3) f *has maximum ramification in* $[D]$.

Moreover: $\dim \mathcal{F}(X, D) = \deg(D) - 2g - k + 2$.

Proof. By construction the divisors

$$D_i = D - P_i,\ i \in \{1, \dots, k\}$$

satisfy the hypothesis of Proposition 2. Then we get:

$$\dim \mathcal{H}(X, D_i) = \deg(D) - 2g - k = \dim \mathcal{H}(X, D) - 1, \quad i \in \{1, \dots, k\}.$$

Comparing dimensions, we get the existence of sections s in:

$$\mathcal{H}(X, D) \setminus \bigcup_{i=1}^{k} \mathcal{H}(X, D_i).$$

They have the property:

$$ord_{P_i}(s^2) = -2n_i,\ i \in \{1, \dots, k\}.$$

This proves the existence of a family of odd ramification coverings $\mathcal{F}(X, D)$ with maximum ramification in the points of $[D]$. Notice that maps in $\mathcal{F}(X, D)$ have degree:

$$d = 2\deg(D) - k.$$

Moreover:

$$\dim \mathcal{F}(X, D) = \deg(D) - 2g - k + 2.$$

Finally, condition c) and Lemma 2 give indecomposability of maps in $\mathcal{F}(X, D)$. □

Proposition 3 specializes to three cases as follows:

- $k = 1$: we get families of indecomposable odd ramification coverings with a pole P_1 of total ramification and prime degree d.
- $k = 2$: we get families of indecomposable odd ramification coverings of (even) degree $d = 2n_1 + 2n_2 - 2$ and ramification $d_1 = 2n_1 - 1$ at P_1, $d_2 = 2n_2 - 1$ at P_2 with d_1 and d_2 relatively prime.
- $k = 3$: we get families of indecomposable odd ramification coverings of (odd) degree $d = 2n_1 + 2n_2 + 2n_3 - 3$ and ramification $d_1 = 2n_1 - 1$ at P_1, $d_2 = 2n_2 - 1$ at P_2, $d_3 = 2n_3 - 1$ at P_3 with d_1, d_2, d_3 relatively prime.

Notice that condition b) in Proposition 3 implies:

$$d > 12g + 3k - 6.$$

It can be easily proved that, for every integer $d \geq 12g + 4$, there exists a divisor D in the hypotheses of Proposition 3 with $d = 2\deg(D) - k$. Fixed d a positive integer, we use the following notation:

$$\mathcal{F}(X, d) = \bigcup_{D \in \mathcal{D}(d)} \mathcal{F}(X, D),$$

where $\mathcal{D}(d) = \{D \in Div(X) : D \text{ as in Proposition 3},\ k > 1,\ 2\deg(D) - k = d\}$. Notice that, for every $D \in \mathcal{D}(d)$:

$$\dim \mathcal{F}(X, d) = \deg(D) - 2g + 2 = [\frac{d+3}{2}] - 2g + 2.$$

Proof of Theorem 1. Notice that Proposition 3 gives a weak version of Theorem 1. In fact, we need to show that the generic map in $\mathcal{F}(X, d)$ has monodromy group equal to A_d. By Lemma 1, it is enough to prove that its monodromy group contains at least a 3-cycle.

Let d be an even integer. Suppose, by contradiction, that the monodromy group of a generic map $f \in \mathcal{F}(X, d)$ doesn't contain any 3-cycle. It follows that any branch point has ramification order ≥ 4. Then Hurwitz's formula gives:

$$2g - 2 = -2d + (d-2) + \sum_{P \in X} (e_f(P) - 1) \geq -d - 6 + 4b$$

where b is the number of branch points. Thus: $b \leq (2g + 4 + d)/4$. Then, from the theory of Hurwitz schemes:

$$\dim \mathcal{F}(X, d) \leq (2g - 4 + d)/4.$$

On the other hand:

$$\dim \mathcal{F}(X,d) = d/2 - 2g + 2 > (2g-4+d)/4 \text{ if } d > 10g-12.$$

Similar considerations hold for odd d. □

Remark 1. Let $\mathcal{F}_1(d)$ be the union of all families $\mathcal{F}(X,D)$ with $k=1$, $2\deg(D)-1=d$ and X moving in the moduli space of smooth projective curves $\mathcal{M}_g$. A parameters' count shows that the generic element in $\mathcal{F}_1(d)$ is a couple (X,f) where f has total ramification at infinity and all other ramification points of index 3 with distinct images in $\mathbb{P}^1$.

5. Simple odd ramifications

Let X be a compact Riemann surface of genus g and $f : X \to \mathbb{P}^1$ a meromorphic function with branch locus B and ramification locus R. The simplest odd ramification type is a $3:1$ ramification covering (see [7]).

Definition 4. We say that an odd ramification covering is *simple* if all points in R have ramification index equal to 3 and have distinct images in $\mathbb{P}^1$

Using admissible coverings we prove:

Proposition 4. *1) Any odd ramification covering $f : X \to \mathbb{P}^1$ is the specialization of a family of simple odd ramification coverings $f_s : X_s \to \mathbb{P}^1$.*
2) The monodromy group of f is a subgroup of the monodromy group of f_s for general s.

Notation 4. Set $Y_0 = \mathbb{P}^1$ and let $T \in Y_0$ be a branch point of f. Let Y_1 be another copy of $\mathbb{P}^1$ and P a point in Y_1. We call $\tilde{Y}$ the singular rational curve obtained by gluing Y_1 and Y_0 in T and P: $\tilde{Y} = Y_0 \cup Y_1$. Let $D = f^{-1}(T) = \sum_{i=1}^{r}(2n_i-1)P_i$. For any $i \in \{1,\dots,r\}$ we take a copy X_i of $\mathbb{P}^1$ with a selected point $Q_i \in X_i$. Let

$$\tilde{X} = (\cup_{i=1}^{r} X_i) \cup X$$

be the curve obtained identifying P_i and Q_i for every $i \in \{1,\dots,r\}$.

Lemma 4. *We can define a map $\tilde{f} : \tilde{X} \to \tilde{Y}$ satisfying the following properties:*

a) $\tilde{f}|_X = f$,
b) $\tilde{f}|_{X_i} = f_i : X_i \to Y_1$ *for every* $i \in \{1,\dots,r\}$,
c) f_i *is an odd ramification covering of degree* $2n_i - 1$ *with total ramification at* Q_i,
d) *ramification points of* f_i *other than* Q_i *have index* 3,
e) *branch points of the* f_i *other than* T *are all distinct.*

Proof. The existence of meromorphic functions $X_i \to \mathbb{P}^1$ satisfying c),d) and e) follows from Riemann's existence theorem. □

Notice that we have constructed an admissible covering. Now we can apply the standard smoothing procedure (see [15] and [1]).

Proof of Proposition 4. We can consider $\tilde{Y}$ as the union of two lines in the projective plane: $\tilde{Y} = \{xy = 0\}$. Choose local coordinates w_i and z_i at $Q_i = P_i$ such that locally $\tilde{X}$ is given by $\{w_i z_i = 0\}$ and $\tilde{f}$ has the form $x = w_i^{2n_i-1}$ on X and

$y = z_i^{2n_i-1}$ on X_i for each $i \in \{1, \ldots, r\}$. Set now: $m_i = 2n_i - 1$ for $i = 1, \ldots, r$. Let Δ be the curve in $\mathbb{C}^{r+1}$ defined by:

$$\Delta = \{(t, t_1, \ldots, t_r) : t = t_i^{m_i}, \mid t \mid < 1, \mid t_i \mid < 1, i = 1, \ldots, r\}.$$

For every $s = (t, t_1, \ldots, t_r) \in \Delta$ consider:

$$I_s = \{(x, 0) : \mid x \mid \leq \mid t(s) \mid\} \cup \{(0, y) : \mid y \mid \leq \mid t(s) \mid\} \subset \tilde{Y},$$

$$J_{i,s} = \{(w_i, 0) : \mid w_i \mid \leq \mid t_i(s) \mid\} \cup \{(0, z_i, s) : \mid z_i \mid \leq \mid t_i(s) \mid\} \subset \tilde{X}.$$

Let $U_s = \tilde{Y} \setminus I_s$ and $U'_s = \tilde{X} \setminus \cup_i J_{i,s}$. Finally, we define:

$$V_s = \{(x, y) \in \mathbb{C}^2 : xy = t(s), \mid x \mid < 1, \mid y \mid < 1\},$$

$$V_{i,s} = \{(w_i, z_i) \in \mathbb{C}^2 : w_i z_i = t_i(s), \mid w_i \mid < 1, \mid z_i \mid < 1\}.$$

We smooth locally $\tilde{Y}$ gluing U_s to V_s in the following way:

$$(x, 0) \mapsto (x, \frac{t(s)}{x});\ (0, y) \mapsto (\frac{t(s)}{y}, y).$$

We can do the same for $\tilde{X}$ gluing U'_s to $V_{i,s}$:

$$(w_i, 0) \mapsto (w_i, \frac{t_i(s)}{w_i});\ (0, z_i) \mapsto (\frac{t_i(s)}{z_i}, z_i).$$

For each $s \in \Delta$ we get a compact smooth surface X_s and a copy Y_s of $\mathbb{P}^1$. We can define a map $f_s : X_s \to Y_s$ with $f_s = \tilde{f}$ on U'_s and $f_s | V_{i,s}$ given by:

$$(w_i, z_i) \mapsto (w_i^{m_i}, z_i^{m_i}).$$

In fact, we get compact Riemann surfaces X_s (the complex structure on X_s is given by the ramified covering) of genus g and odd ramification coverings f_s on X_s with the same degree of f that specialize to $\tilde{f}$ for $s = 0$. Remark that the monodromy group of f is determined by the branches in $Y_0 \setminus \{T\}$. Let $\gamma_1, \ldots, \gamma_b$ be loops in Y_0 corresponding to these branches. Since the coverings f_s and f are topologically the same in U'_s, we can find $\gamma_{1,s}, \ldots, \gamma_{b,s}$ loops in $Y_s \setminus V_s$ giving the same monodromy as $\gamma_1, \ldots, \gamma_b$. Then, the monodromy group $M(f_s)$ contains $M(f)$ as a subgroup for every $s \in \Delta$. Moreover, the coverings f_s have simpler odd ramification. Repeating this procedure for each branch point we finally get a family of simple odd ramification coverings deforming f. □

Remark 2. It is a not trivial problem to find a deformation of f as in Proposition 4 which preserves the conformal structure of X.

Combined with Theorem 1 this implies:

Theorem 2. *The generic Riemann surface of genus g admits simple odd ramification coverings of degree $d \geq 12g + 4$ with monodromy group equal to A_d.*

6. A technical result

Let $A, B \in Div(X)$ be two effective divisors on X with $[A] \cap [B] = \emptyset$, $a = \deg[A]$, $b = \deg[B]$. Let us call $A' = A - [A]$. We define a sheaf $\mathcal{R} = \mathcal{R}(A, B)$ as follows:

$$\mathcal{R}(A, B)(U) := \{f \in \mathcal{O}(A')(U) : \ df \in \Omega(-B)(U)\},$$

$U \subset X$ open set. We have the short exact sequence of sheaves:

$$0 \to \mathcal{O}(A' - B - [B]) \xrightarrow{i} \mathcal{R} \xrightarrow{e_v} \mathbb{C}_B \to 0 \tag{1}$$

where $\mathbb{C}_B$ is the skyscraper sheaf with support in $[B]$ and e_v is the "valutation" morphism. If $f \in \mathcal{R}(A, B)(U)$ then $df \in \Omega(A - B)(U)$, a second exact sequence is then:

$$0 \to \mathbb{C} \xrightarrow{i} \mathcal{R} \xrightarrow{d} \Omega(A - B)^\circ \to 0 \tag{2}$$

where $\Omega(A - B)^\circ$ is the subsheaf of forms with residues zero in $[A]$. Finally, taking residues, we get a third sequence:

$$0 \to \Omega(A - B)^\circ \to \Omega(A - B) \xrightarrow{res} \mathbb{C}_A \to 0. \tag{3}$$

Consider the coboundary operators associated to sequence (2):

$$\Delta_i : H^{i-1}(X, \Omega(A - B)^\circ) \to H^i(X, \mathbb{C}) \quad i = 1, 2.$$

Suppose now: $H^1(X, \mathcal{R}) = 0$. Then Δ_1 is surjective and Δ_2 is an isomorphism. Moreover, the image of the map $res : H^0(X, \Omega(A - B)) \to H^0(X, \mathbb{C}_A) = \mathbb{C}^a$ is contained in $\mathcal{I} = \{\sum_1^a x_i = 0\}$ by the residue theorem. Sequence (3) gives then: $H^1(X, \Omega(A - B)) = 0$. Therefore Δ_1 and res are both surjective. Finally, this implies the surjectivity of the De Rham's map:

$$\mathcal{D} : H^0(X, \Omega(A - B)) \to H^1(X \setminus [A], \mathbb{C}) \quad \omega \mapsto [\omega].$$

Proposition 5. *If $H^1(X, \mathcal{R}) = 0$, then the map $\mathcal{D}$ is surjective.*

In fact, the vanishing of $H^1(X, \mathcal{R})$ and the surjectivity of $\mathcal{D}$ are equivalent conditions (see [19]).

Proof of Lemma 3. We have $H^1(X, \mathbb{C}_B) = 0$ and $H^1(X, \mathcal{O}(A' - B - [B]) = 0$. This implies $H^1(X, \mathcal{R}) = 0$.

□

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