CONFIGURATION SPACES FOR WILDLY RAMIFIED COVERS

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Abstract. The classification for wildly ramified covers is more mysterious than for tame covers. A natural configuration space — the space of unordered branch points of a cover — for tame covers allows the construction of Hurwitz spaces, and an often very effective theory for families of tame covers. We define invariants generalizing Nielsen classes for Hurwitz families. The result is a configuration space for classifying families of wildly ramified covers. The center of this construction is what we call local ramification data. This notion does not assume the covers are Galois. The potential of the method comes out with problems posed by the construction of the Galois closure of a family of covers with r distinct branch points. In the tamely ramified case this provides one monodromy invariant — evidence that global construction of the Galois closure gives serious data. Families of wildly ramified covers, by contrast, show influences of local data on global Galois closure constructions. These ideas are forced even on problems that superficially appear to deal with one curve cover (field extension) at a time.

Contents

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This paper expands on an unpublished manuscript of the first author from 1973, shown to many over the intervening years. Its goal was to contrast tamely and wildly ramified covers to benefit from [Gr] insights. Both authors received support from the MSRI semester on Arithmetic Fundamental Groups. Partial support for Fried came from NSF #DMS-9970676 and a senior research Alexander von Humboldt award.

1. Introduction and motivation

Let $\phi : X \to \mathbb{P}^1$ be a connected tame cover of the Riemann sphere defined over an algebraically closed field k of characteristic $p > 0$. Typically, that field will be $\bar{\mathbb{F}}_q$ with $\mathbb{F}_q$ the finite field of order $q = p^t$ for some t. All covers and field extensions are assumed separable. Deformations of branch points of ϕ in characteristic p parametrize deformations (locally in the étale topology; not globally) of the tame cover ϕ. This is Grothendieck's theorem [Gr]. Hurwitz spaces of simple branched covers go back to [Hu] and [Kl]. The modern study of fields of definition of components of Hurwitz spaces defined by conjugacy classes in a group (Nielsen classes) is in [Fr77], [FV], [MM], [Vö]. Such families give an effective parametrization of families of tame covers over k. We form something similar, retaining some Hurwitz space virtues, for wildly ramified covers.

Our formulations notationally simplify (with no loss to our applications) by focusing on covers of the projective line, $\mathbb{P}^1$. Assume now $\phi : X \to \mathbb{P}^1$ to be a connected wildly ramified cover of projective nonsingular curves. Let $x_1, \dots, x_t$ run over all wildly ramified points of X. Abusing notation, consider the same symbol x_i as also denoting a uniforming parameter for the corresponding point. We attach invariants (§1) — *ramification data* $\mathcal{R}_i$ — to each of the local field extensions $k((x_i))/k((\phi(x_i)))$, $i = 1, \dots, t$. Briefly: Suppose $\phi(x_i) = \sum_{j=0}^{u} (f_j(x_i))^{p^{v_j}}$ with $f_j(x_i) = \sum_{\substack{\ell \ge d_j \\ (\ell,p)=1}} a_{j,\ell} x_i^\ell \in k[[x_i]]$ where $d_j > 0$, $(d_j, p) = 1$, $a_{j,d_j} \ne 0$ and v_j is strictly decreasing, $j = 0, \dots, u$. Then ramification data is a subset of $\{d_j p^{v_j}\}_{0 \le j \le u}$ formed by the aid of a sequence of differential operators (Def. 2.1). It is an invariant of the field extension $k((y))/k((x))$ with $x = \phi(y)$ (Lem. 2.5).

Unlike the tame case, wildly ramified extensions $k((x_i))/k((\phi(x_i)))$ often (usually) are not Galois! The paper notes that no cheap trick will allow reverting to the Galois case with wildly ramified covers unless you consider only one cover at a time. Even for families of tamely ramified families of covers there are subtleties to going to the Galois closure. (The distinction between absolute and inner Hurwitz spaces: [BF, §3.5] captures this.)

We define an *explicit* open subset $\mathcal{P}(\mathcal{R})$ of affine space parametrizing local field extensions with ramification data $\mathcal{R}$ (§2, §3). In fact, it is just a product of copies of 1-dimensional affine space and copies affine 1-space minus the origin. Finitely many points in $\mathcal{R}$ correspond to each isomorphism class of extensions $k((y))/k((x))$ with ramification data $\mathcal{R}$. A subsequence of the ramification data is called the *regular ramification data* $\mathcal{R\!R}(k((y))/k((x))$, defining a convex hull for the pairs $\{(p^{v_j}, d_j p^{v_j}), j = 0, \dots, u\}$ appearing in $\mathcal{R}$. Prop. 3.5 computes the number of points of $\mathcal{P}(\mathcal{R})$ corresponding to a given extension as the cardinality of a ratio $I_1(\mathcal{R})/\mathrm{Aut}(k((y))/k((x)))$ with $I_1(\mathcal{R})$ computed from the regular ramification data. Then, $\mathcal{P}(\mathcal{R})$ is versal for the finite topology for algebraic families of (local) field extensions with ramification data $\mathcal{R}$ (Thm. 4.4). As $\mathrm{Aut}(k((y))/k((x)))$ varies with the field extension, this forces use of the finite, rather than the étale, topology.

Section §5 gives properties of the ramification data related to the Galois closure of the extension $k((y))/k((x))$. A reader should find this material enlightening about the definition of ramification data. It starts with a simple rubric for an extension $k((y))/k((x))$ showing how ramification data computes the number of embeddings of $k((y))$ (fixed on $k((x))$) in the tame closure of $k((y))$ (Lem. 5.1). In particular, from the *slopes* in the regular ramification data one computes the composite ramification index of all the tame embeddings. If $k((y))/k((x))$ is a tame extension, then it is a cyclic Galois extension, so its degree alone determines the lattice of field extensions between $k((y))/k((x))$. Some applications benefit from finding the actual decomposition into primitive subextensions of local extensions with ramification data $\mathcal{R}$. §5.2 has beginning observations in that direction.

In section §6 we consider families of (not necessarily Galois) covers and the canonical formation of their (global) Galois closure. We first remind of what to expect of the configuration space in characteristic 0, especially considering the Galois closure of a family (§6.1). We briefly review results in this direction ([BF], [FV]) in the tame case: Describing the configuration space for tamely ramified families and the Galois closure construction (§6.2-6.4). There has been little development of global families of covers mixing tame and wild ramification. We restrict to three beginning aspects, with §6.5 having the structure of a research program.

- How a family of covers with a given type of wild ramification produces a map to the configuration space for that type (Prop. 6.7).
- Why, if this map is constant, the family is isotrivial (Prop. 6.8).
- The effect of the Galois closure of the local field extensions on the global Galois closure construction.

Section §6.6 concludes with some Galois closure questions for families of wildly ramified covers.

2. Ramification data

Let p be a prime number and let k be an algebraically closed field of characteristic p. We study deformations of finite separable totally ramified extensions $k((y))$ of $k((x))$ defined by the equation $x = f(y)$ with $f \in k[y]$. All field extensions, unless noted otherwise, are separable. Suppose $\xi = \sum_{i=0}^{\infty} a_i y^i \in k[[y]]$ with $a_0 \neq 0$. Then, the standard order ord_y on $k((y))$ assigns $y^n\xi$ the value $n = \mathrm{ord}_y(y^n\xi)$. Also, $\mathrm{ord}_y(0) = \infty$. For $\ell, \ell' \in \mathbb{Z}$, let (ℓ, ℓ') be the greatest common divisor of ℓ and ℓ'. For $n \in \mathbb{N}$, denote by $\pi(n)$ the greatest power of p dividing n: for $n = mp^v$ with $(m, p) = 1$, $\pi(n) = p^v$.

Assume $k((y))/k((x))$ is finite.

We first define the ramification data of the extension $k((y))/k((x))$ (1.1). The ramification data is a practical invariant of the extension $k((y))/k((x))$. We present its effective computation in (1.2).

2.1. **Definition of the ramification data.** We first introduce some operators. Regard the binomial coefficient $\binom{t}{n} = \frac{t(t-1)\cdots(t-n+1)}{n!}$ as an integer reduced modulo p. For integers $n \geq 0, t \geq 1$ and z a variable, consider $D_y^{(n)}$: $y^t \mapsto \binom{t}{n} y^{t-n}$. Extend it linearly to $k((y))$. We understand from M. Jarden that it was F. K. Schmidt who first introduced this replacement for the derivative, and who noticed its iterative properties [Sch]:

- Leibniz's rule: $D_y^{(n)}(\xi_1\xi_2) = \sum_{u=0}^n D_y^{(u)}(\xi_1)D_y^{(n-u)}(\xi_2)$ when all terms are defined;

and from $(a^{p^r}+b^{p^r})^\ell = (a+b)^{p^r\ell}$, $a,b\in k$, $\ell,r\in\mathbb{N}$,

- for $n,\ell,r\in\mathbb{N}^*$, $D_y^{(n)}(y^{\ell p^r}) = \begin{cases} 0 & \text{for } p^r \not| n \\ (D_y^{(\frac{n}{p^r})}(y^\ell))^{p^r} & \text{for } p^r|n. \end{cases}$

Given y and $\ell\in\mathbb{N}$, consider the operators $L_y^{(\ell)} = \sum_{v=0}^\ell y^{p^v}D_y^{(p^v)}$. Apply $L_y^{(\ell)}$ to $\xi\in k[[f(y)]]$ with positive order. Let $h(-1,\xi)=\infty$. Inductively:

$$h(\ell,\xi) = \min\big(\mathrm{ord}_y(L_y^{(\ell)}(\xi)), h(\ell-1,\xi)\big),\ \ell\in\mathbb{N}. \tag{2.1}$$

Similarly, let $h(-1)=\infty$ and

$$h(\ell) = \min_{\substack{\xi\in k[[x]]\\ \mathrm{ord}_y(\xi)>0}} \min\big(\mathrm{ord}_y(L_y^{(\ell)}(\xi)), h(\ell-1)\big),\ \ell\in\mathbb{N}. \tag{2.2}$$

Definition 2.1. The *ramification data* of the extension $k((y))/k((x))$ is

$$\mathcal{R}(k((y))/k((x))) = \{h(\ell) \mid h(\ell)-h(\ell-1)\neq 0,\ \ell\in\mathbb{N}\}.$$

From Leibniz's rule this doesn't depend on the choice of y.

Complete the notation above. For $\xi\in k[[y]]$ of positive order,

$$\mathcal{R}(\xi) = \{h(\ell,\xi) \mid h(\ell,\xi)-h(\ell-1,\xi)\neq 0,\ \ell\in\mathbb{N}\}.$$

Lemma 2.2 allows computing $\mathcal{R}(\xi)$ explicitly. Example 2.3 illustrates this. With the ord of the term of minimal order written as $p^{v_0}d_0$, $(d_0,p)=1$, there is no loss for computing $\mathcal{R}(k((y))/k((x)))$ in replacing $f(y)$ by its truncation past the lowest nonzero term with exponent prime to p. In this local context we regard $k((y))$ as fixed and consider $f(y)=x$ as the subject of study. So, Lem. 2.4 shows that, while several possible choices of $f(y)=x$ have $\mathcal{R}(k((y))/k((x)))$ equal $\mathcal{R}(x)$, any one computes the ramification data.

Lemma 2.2. *Let $\sum_{\substack{0\le i\le r\\ 0\le a_i<p}} a_ip^i$ (resp. $\sum_{\substack{0\le i\le r\\ 0\le b_i<p}} b_ip^i$) be the p-adic expansion of the positive integer n (resp. m). Then, $\binom{n}{m} = \prod_{k=0}^r \binom{a_k}{b_k} \bmod p$.*

Proof. Do an induction on r by writing $n=a_0+pn_1$ and $n=b_0+pm_1$. Compute the coefficient of $x^{b_0+pm_1}$ in both sides of

$$(1+x)^{a_0+pn_1} = (1+x)^{a_0}(1+x^p)^{n_1} \bmod p.$$

The formula is $\binom{n}{m}\equiv\binom{a_0}{b_0}\binom{n_1}{m_1} \bmod p$. □

Example 2.3. Take $p=3$. Let $f(y)=x$ be

$$\begin{aligned} f(y) &= y^{729}+(y^4+y^5)^{243}+(y^{110})^9+(y^{13})^{81}+(y^{370})^3+y^{10000} \\ &= y^{3^6}+(y^4+y^5)^{3^5}+(y^{110})^{3^2}+(y^{13})^{3^4}+(y^{370})^3+y^{10000}. \end{aligned}$$

The terms are in order as to their ords. The 3-adic expansion for 10000 is

$$10000 = 1+9+27+2\cdot 243+729+2187+6561 = 1+3^2+3^3+2\cdot 3^5+3^6+3^7+3^8.$$

Apply Lemma 2.2 for the following congruences:

$$\binom{10000}{3^6}\equiv 1 \bmod 3,\ \binom{10000}{3^5}\equiv 2 \bmod 3,\ \binom{10000}{3^4}\equiv 0 \bmod 3,$$
$$\binom{10000}{3^3}\equiv 1 \bmod 3,\ \binom{10000}{3^2}\equiv 1 \bmod 3 \text{ and } \binom{10000}{3}\equiv 0 \bmod 3.$$

Similarly, $\binom{370}{3^3} \equiv 1 \bmod 3$ and $\binom{370}{3^2} \equiv 2 \bmod 3$ and $\binom{370}{3} \equiv 0 \bmod 3$. Then:

$$\begin{aligned}
L_y^{(0)}(f) &= y^{10000};\\
L_y^{(1)}(f) &= y^{3\cdot 370} + y^{10000};\\
L_y^{(2)}(f) &= 2y^{3^2\cdot 110} + y^{3\cdot 370} + 2y^{10000};\\
L_y^{(3)}(f) &= 2y^{3^2\cdot 110};\\
L_y^{(4)}(f) &= y^{3^4\cdot 13} + 2y^{3^2\cdot 110} + y^{3\cdot 370};\\
L_y^{(5)}(f) &= y^{3^5\cdot 4} + 2y^{3^4\cdot 13} + 2y^{3\cdot 370} + 2y^{3^5\cdot 5} + 2y^{10000};\\
L_y^{(6)}(f) &= y^{3^6} + 2y^{3^5\cdot 4} + y^{3^2\cdot 110};
\end{aligned}$$

Thus, $\mathcal{R}(x) = \{729 = 3^6, 972 = 3^5 \cdot 4, 990 = 3^2 \cdot 110, 1110 = 3 \cdot 370, 10000\}$.

Now we show in generality that $\psi = x$ computes $\mathcal{R}(k((y))/k((x)))$.

Lemma 2.4. *In the notation above, $\mathcal{R}(k((y))/k((x))) = \mathcal{R}(x)$.*

Proof. Let $n \in \mathbb{N}, t = p^r m \in \mathbb{N}$ with $(p, m) = 1$. By definition

$$\binom{t}{n} = \frac{p^r m(p^r m - 1)\cdots(p^r m - n + 1)}{n!}$$

If $n < p^r$, the powers of p dividing i and $p^r m - i$ for $i = 1, \ldots, n-1$ are the same, then $\binom{t}{n} \equiv 0 \bmod p$.
If $n = p^r$ then $\binom{t}{n} = m \not\equiv 0 \bmod p$.
Since $y^n D_y^{(n)}(y^t) = \binom{t}{n} y^t$, for $\ell \in \mathbb{N}$,

$$\min_{v\le\ell} \operatorname{ord}_y\big(y^{p^v} D_y^{(p^v)}(y^t)\big) = \min_{n\le p^\ell} \operatorname{ord}_y\big(y^n D_y^{(n)}(y^t)\big) = \begin{cases} \infty & \text{if } \ell \ge \mathrm{r} \\ t & \text{if } \ell < \mathrm{r} \end{cases}$$

From additive properties of ord_y one obtains for all $\ell \in \mathbb{N}$

$$h(\ell, x) = \min_{v\le\ell} \operatorname{ord}_y\big(y^{p^v} D_y^{(p^v)}(x)\big) = \min =_{n\le p^\ell} \operatorname{ord}_y\big(y^n D_y^{(n)}(x)\big)$$

From Leibnitz's rule, for all $v \in \mathbb{N}$, $r \in \mathbb{N}$

$$\operatorname{ord}_y\big(y^{p^v} D_y^{(p^v)}(x^r)\big) = \operatorname{ord}_y\Big(\sum_{s=0}^{p^v} y^s D_y^{(s)}(x) y^{p^v - s} D_y^{(p^v - s)}(x^{r-1})\Big)$$

The right side is at least as large as $\min_{s\le p^v} \operatorname{ord}_y\big(y^s D_y^{(s)}(x)\big)$. By taking $\min_{v\le\ell}$, one obtains

$$h(\ell, x^r) \ge h(\ell, x), \quad r \in \mathbb{N}^*, \ell \in \mathbb{N}$$

So, for any $\xi \in k[[x]]$ with $\operatorname{ord}_x \xi > 0$, $h(\ell, \xi) \ge h(\ell, x)$, $\ell \in \mathbb{N}$. The result follows. □

2.2. **Refined computing of $\mathcal{R}$.** Write x as

$$x = f(y) = \sum_{i=0}^{u} (f_i(y))^{p^{v_i}} \quad \text{with} \quad f_i(y) = \sum_{\substack{j\ge d_i\\ (j,p)=1}} a_{i,j} y^j \in k[[y]]$$

where $d_i > 0$, $(d_i, p) = 1$, $a_{i,d_i} \neq 0$ and v_i is strictly decreasing, $i = 0, \ldots, u$. Serious ramification occurs with lower order terms of the separable power series f having exponents divisible by high powers of p. Still, there will be terms of higher order with exponents divisible by no powers of p. So from now on, assume $\operatorname{ord}_y(f(y))$ is a power of p. Change of y allows assuming f is a polynomial of degree the only term of exponent prime to p (see Prop. 3.5). Thus $k((y))/k((x))$ is a wildly ramified extension. Denote $e_i = d_i p^{v_i}$ and $\pi(e_i) = p^{v_i}$, $i = 0, \dots, u$.

To compute $\mathcal{R}(x)$, compute $L_y^{(\ell)}(x) = L_y^{(\ell)}(f(y))$. For ℓ small, the operators $L_y^{(\ell)}$ kill terms with orders divisible by powers of p higher than ℓ, exposing to view those of higher order, divisible by low powers of p. As ℓ increases, ramification data tracks values of ℓ with this property:

$$\text{A term } e_i \text{ contributes to } \mathcal{R}(x) \text{ if } \pi(e_i) = p^\ell \text{ and } e_i \text{ is less than } e_k \text{ with } \pi(e_k) < p^\ell. \quad \text{That is } h(\ell, x) = \min_{i,\ v_i \le \ell} e_i,\ \ell \ge 1. \tag{2.3}$$

Here is a summary of how this produces $\mathcal{R}(x)$.

Lemma 2.5. *Plot all pairs $\{(\pi(e_i), e_i)\}_{i=0}^u$. (Recall: $\pi(e_u) = 1$ though e_u is large; while e_0 is small though $\pi(e_0)$ will be the largest value.) Join points between $(\pi(e_u), e_u)$ and $(\pi(e_0), e_0)$ by a single broken line $\tilde{L}$ with negative slope segments so all plotted points are on or above $\tilde{L}$. Vertices of $\tilde{L}$ correspond to elements of $\mathcal{R}(x)$.*

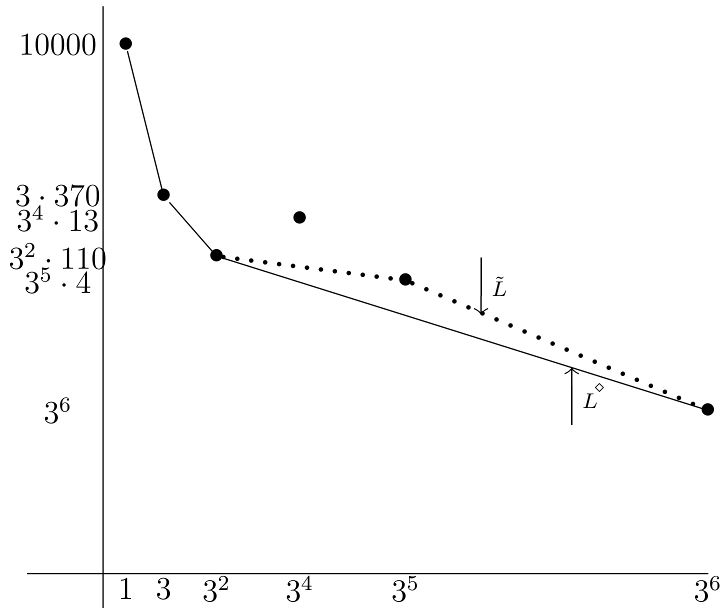
Remark 2.6. The definitions of ramification data and regular ramification data appear in [Fr74, p. 231–233]. This is an extended treatment of that, with the development of the configuration space properties.

2.3. **Regular ramification data $\mathcal{RR}$.** Refer to either $\mathcal{R}(x)$, or its graph as indicated in Lem. 2.5, as the *ramification data* for x. The convex hull of this graph is also significant.

Definition 2.7. *Regular ramification data* $\mathcal{RR}(k((y))/k((x)))$ of $k((y))/k((x))$ is the subset $\{E_0, \dots, E_t\}$ of $\mathcal{R}(k((y))/k((x)))$ with $e_0 = E_0$ formed inductively as follows. For $m > 0$: E_m is the vertex on $\tilde{L}$ with second coordinate e_j, $j > m-1$ chosen so the line segment joining E_{m-1} to $(\pi(e_j), e_j)$ has (negative) slope of minimal absolute value.

Assume $|\mathcal{RR}(k((y))/k((x)))| = t$. Joining $(\pi(E_0), E_0), \dots, (\pi(E_t), E_t)$ produces a new broken line $\overset{\diamond}{L}$. Denote by $\overset{\diamond}{L}_1, \dots, \overset{\diamond}{L}_{\overset{\diamond}{t}}$ the broken line segments of $\overset{\diamond}{L}$ with slopes strictly increasing in absolute value. Note!: Several points of $(\pi(E_j), E_j)$ may lie on a single line segment $\overset{\diamond}{L}_m$.

Definition 2.8. Let n be the least positive integer for which n times each slope of $\overset{\diamond}{L}_1, \dots, \overset{\diamond}{L}_{\overset{\diamond}{t}}$ is integral. Denote this $\text{ind}^{\text{tm}}(y/x)$: the *tame index* of $k((y))/k((x))$.



$\mathcal{R}(x)$ and $\tilde{\mathcal{R}}(x)$ for $f(y)$ in Ex. 2.3

$f(y) = y^{3^6} + (y^4 + y^5)^{3^5} + (y^{110})^{3^2} + (y^{13})^{3^4} + (y^{370})^3 + y^{10000}$

Remark 2.9 (Using ramification data in applications). Applications systematically change the value of x at which ramification occurs. The point $x = \infty$ is typical when applying information about ramification of covers by curves from global equations. For example, suppose

$$X_f = \{(x, y) \in \bar{\mathbb{F}}_q \times \bar{\mathbb{F}}_q \mid f(y) = x\} \text{ with } f \in k[y].$$

To inspect ramification over $x = \infty$, substitute $1/x$ for x. In many applications, $y = \infty$ might be the ramified value of y. To revert to where ramification corresponds to the place $y = 0$ over the place $x = 0$, replace X_f by $X'_f = \{(x, y) \mid f(1/y) = 1/x\}$.

Example 2.10 (Degree p ramification over $x = \infty$)**.** Let

$$x = f(y) = y^p + a_{p-1}y^{p-1} + a_{p-2}y^{p-2} + \cdots + a_0 \in k[y]$$

in the above discussion, with $a_{p-1} \neq 0$. Inspect how $y = \infty$ ramifies over $x = \infty$ by writing $x = g(y) = y^p/(1+a_{p-1}y+\cdots+a_0y^p) = y^p - a_{p-1}y^{p+1} +$ higher terms. In our usual notation, with $y = 0$ ramified over $x = 0$ from this new expression, $\mathcal{R}(x)$ consists of $\{p+1, p\}$. There is only one segment on the ramification data graph. Its slope is $\frac{p-(p+1)}{p-1} = \frac{-1}{p-1}$. So, $\text{ind}^{\text{tm}}(y/x) = p - 1$. The case where $a_{p-k} \neq 0$, but $a_{p-j} = 0$, $j = 1, \ldots, k-1$, is similar. Then $\text{ind}^{\text{tm}}(y/x) = p - k$.

3. Extensions $k((y))/k((x))$ with ramification data $\mathcal{R}$

This section introduces an affine variety $\mathcal{P}(\mathcal{R})$ at the heart of our construction. Its points correspond to valuated field extensions of $k((y))/k((x))$ with ramification data $\mathcal{R}$ where the expression for x from y has a special normalized form. Every isomorphism class of extensions $L/k((x))$ with ramification data $\mathcal{R}$ corresponds to some point of $\mathcal{P}(\mathcal{R})$. Further, the cardinal number of points on $\mathcal{P}(\mathcal{R})$ corresponding to a given isomorphism class of extensions is independent of the extension.

3.1. **Definition of $\mathcal{P}(\mathcal{R})$.** Let $k((y))/k((x))$ be an extension with $x = f(y)$ having ramification data $\mathcal{R} = \mathcal{R}(k((y))/k((x)))$. Use Lemma 2.5 to list points of $\mathcal{R}$ as $\{e_0, \cdots, e_s\}$: $e_0 < e_1 < \cdots < e_s$, $e_i = d_i p^{v_i}$, with $\pi(e_i) = p^{v_i}$ a strictly decreasing function of i.

For $d_0 \neq 1$, let $g(y) = f(y)^{\frac{1}{d_0}} = ay^{p^{v_0}} +$ higher order terms. So, $k((f(y))) \subset k((g(y))) \subset k((y))$ is a sequence of extensions with $k((g(y)))/k((f(y)))$ ramifying tamely and having degree d_0. This often allows assuming $d_0 = 1$ in exposition and examples and the smallest ord term has coefficient 1. With this, $k((y))/k((x))$ is totally wildly ramfied (as in 2.2).

Let $M_1(\mathcal{R}) = \{j \in \mathbb{N} \mid j < e_0, \text{ or } e_{i-1} < j < e_i \text{ and } \pi(e_{i-1})p \not| j\}$.

Remark 3.1. Let $S(\mathcal{R})$ be powers series $\{g \in k[[y]] \mid g(y) = \sum_{i=0}^{\infty} b_i y^i\}$ with

(3.1a) $b_i = 0$ if $i \in M_1(\mathcal{R})$ and
(3.1b) $b_{e_0} \cdots b_{e_s} \neq 0$.

Then, for $g \in k[[y]]$, $\mathcal{R}(g) = \mathcal{R}$ if and only if $g \in S(\mathcal{R})$.

Let $M_2(\mathcal{R}) = \{j \in \mathbb{N} \mid \exists \mu \in \mathbb{N}^*, j = j(\mu) = \min_{0 \leq \ell \leq t}(E_\ell + \mu\pi(E_\ell))\}$. Let $N(\mathcal{R}) = \mathbb{N} \setminus M_1(\mathcal{R}) \cup M_2(\mathcal{R})$.

Remark 3.2. For $\mu \in \mathbb{N}^*$ suitably large, $j(\mu) = E_t + \mu\pi(E_t) = E_t + \mu$. So, large values of j large are not in $\mathbb{N} \setminus M_2(\mathcal{R})$. Thus, $N(\mathcal{R})$ is finite.

Let $\mathcal{P}(\mathcal{R})$ be

$$\{g \in k[y] \mid g(y) = \sum_{j \in N(\mathcal{R})} b_j y^j,\ b_{e_0} = 1,\ b_{e_1} \cdots b_{e_s} \neq 0\}.$$

By definition, $\mathcal{P}(\mathcal{R}) \equiv (\mathbb{A}^*)^s \times (\mathbb{A})^{|N(\mathcal{R})|-1-s}$ and $g \in \mathcal{P}(\mathcal{R})$ implies $\mathcal{R}(g) = \mathcal{R}$.

3.2. **First properties of $\mathcal{P}(\mathcal{R})$.** We need further notation to study moduli properties of the affine variety $\mathcal{P}(\mathcal{R})$.

Definition 3.3. Call a polynomial $h \in k[z]$ *p-developed* if $h(z) - h(0) = \sum_{i=0}^{t} b_i z^{p^{v_i}}$. Suppose p-developed h has a nonzero term of degree 1. Then $\frac{dh}{dz}$ is identically a constant and h has no repeated roots.

Recall $k((y))/k((x))$ is an extension with $x = f(y)$ having ramification data $\mathcal{R} = \mathcal{R}(k((y))/k((x)))$. As in Def. 2.7, let $\overset{\diamond}{L}_1, \ldots, \overset{\diamond}{L}_{\overset{\diamond}{t}}$ be the broken line segments of $\overset{\diamond}{L}$ corresponding to regular ramification $\mathcal{RR} = \mathcal{RR}(k((y))/k((x)))$ with respective slopes $-m_j$, $j = 1, \ldots, \overset{\diamond}{t}$, strictly increasing in absolute value. Denote by E_{j_+} (resp. E_{j_-}) the element of $\mathcal{RR}$ corresponding to the right (resp. left) vertex of $\overset{\diamond}{L}_j$.

For $n \in \mathbb{N}^*$, let $\mathrm{Ind}_n(\mathcal{R}) = \{j \mid nm_j \in \mathbb{Z}, 1 \leq j \leq \overset{\diamond}{t}\}$. Also,

$$I_n(\mathcal{R}) = \prod_{j \in \mathrm{Ind}_n(\mathcal{R})} \frac{\pi(E_{j_+})}{\pi(E_{j_-})}.$$

Then, $I_1(\mathcal{R})$ is the product indexed by those m_j s in $\mathbb{Z}$.

Lemma 3.4. *There are exactly $I_1(\mathcal{R})$ power series expressions $\alpha \in \mathrm{Aut}(k[[y]])$ with $f(\alpha(y)) \in \mathcal{P}(\mathcal{R})$.*

Proof. With $f(y)=\sum_i c_i y^i$, we seek $\alpha(y)=a_0y+\sum_{i=1}^{\infty} a_i y^{1+i}\in k[[y]]$ so

$$f(\alpha(y))=g(y)\in \mathcal{P}(\mathcal{R}).$$

Write $g(y)=\sum_{j\in N(\mathcal{R})} b_j y^j$. We must find α so the b_j 's with $j\in M_1(\mathcal{R})\cup M_2(\mathcal{R})$ are all 0. By hypothesis, $a_0=1$. Assume we know $a_1,\dots,a_{i-1}$. The term of least ord from $f(\alpha(y))$ in which α_i is one from those of least order in which a_i appears in $c_u(\alpha(y))^{e_u}$, $u=0,\dots,s$. Here, the lowest order appearance of a_i is to the power $\pi(e_u)$ in the coefficient of $y^{e_u+i\pi(e_u)}$. This coefficient is $d_u c_u a_i^{\pi(e_u)}$.

So, a_i appears in the term of $f(\alpha(y))$ having order

$$j(i)=\min_{0\le u\le s}(e_u+i\pi(e_u)). \tag{3.2}$$

By hypothesis $b_{j(i)}=0$. If only one value of u achieves the minimum in (3.2), then $a_1,\dots,a_{i-1}$ uniquely determine a_i.

Suppose more then one value achieves the minimum in (3.2). Then there is a $j=j(i)$ order term of $f(\alpha(y))$ (as in (3.2))with coefficient a p-developed polynomial $h_i(a_i)$. Also, h_i has a nonzero degree $\pi(e_u)$ term for each u achieving the minimum in (3.2). So, $h_i(a_i)=\bar{h}_i(a_i^{\pi(E_{j\_})})$ with $\bar{h}_i$ a p-developed polynomial of degree $\frac{\pi(E_{j_+})}{\pi(E_{j\_})}$ with a nonzero term of degree 1. Therefore $\frac{d\bar{h}_i}{dz}$ is identically a constant and $\bar{h}_i$ has no repeated roots. Conclude: There are exactly $\frac{\pi(E_{j_+})}{\pi(E_{j\_})}$ solutions for a_i given any previous values of $a_1,\dots,a_{i-1}$.

Conclude: Exactly $I_1(\mathcal{R})$ power series expressions $\alpha\in \mathrm{Aut}(k[[y]])$ satisfy

$$f(\alpha(y))=\sum_{i\not\in M_2(\mathcal{R})} b_i y^i = g(y) \text{ with } b_{e_0}=1.$$

We show $f(\alpha(y))=g(y)\in\mathcal{P}(\mathcal{R})$. With β the inverse of α, $\alpha(\beta(y))=y$ and $g(\beta(y))=f(y)$. As $\mathcal{R}$ is an invariant of $k((y))/k((x))$, $\mathcal{R}(k((z))/k((x)))=\mathcal{R}$ for $g(z)=x$ and $g\in\mathcal{P}(\mathcal{R})$. □

Proposition 3.5. *There is a natural surjective map from $\mathcal{P}(\mathcal{R})$ to isomorphism classes of valued field extensions of $k((x))$ with ramification data $\mathcal{R}$. The number of points of $\mathcal{P}(\mathcal{R})$ mapping to a given isomorphism class $k((y))/k((x))$ of extensions of $k((x))$ is $I_1(\mathcal{R})/|\mathrm{Aut}(k((y))/k((x)))|$.*

Proof. Consider $g\in\mathcal{P}(\mathcal{R})$. As above, assume $d_0=1$. If $x=g(y)$ has ord p^{v_0}, then $g(y)=x$ has p^{v_0} solutions y_g in y with ord 1. Choose a y_g to $k((y_g))/k((x))$. The isomorphism class of the field extension is independent of y_g.

Suppose a field extension for some $f(y)=\sum_i c_i y^i$ has ramification data $\mathcal{R}$. After Lemma 3.4, there are exactly $I_1(\mathcal{R})$ elements $\alpha\in k[[y]]$ of order 1 such that $g=f(\alpha(y))\in\mathcal{P}(\mathcal{R})$.

Two series α,α' define the same element $g\in\mathcal{P}(\mathcal{R})$ if and only if

$$f(\alpha(y))=f(\alpha'(y)):\ \alpha\circ(\alpha')^{-1}\in \mathrm{Aut}(k((y))/k((x))).$$

□

4. Families of covers with ramification data $\mathcal{R}$

This section displays $\mathcal{P}(\mathcal{R})$ as a parameter space for a convenient family of curve covers of the x-line so that $x=0$ is a branch point for each member of the family, lying below a point having ramification data $\mathcal{R}$. So we can use $\mathcal{P}(\mathcal{R})$ as local coordinates (in the finite topology) for all such families of curve covers.

4.1. **A family over $\mathcal{P}(\mathcal{R})$.** Let $\mathbb{A}^1_y$ (resp. $\mathbb{P}^1_y$) denote a copy of affine 1-space (resp. projective 1-space) uniformized by the variable y. Consider the map

$$\Psi : \mathbb{A}^1_y \times \mathcal{P}(\mathcal{R}) \to \mathbb{A}^1_x \times \mathcal{P}(\mathcal{R}) \text{ by } (y, g) \mapsto (g(y), g).$$

Then Ψ finite of degree E_0. It is flat because its fibers are locally complete intersections. Denote projection of $\mathbb{A}^1_x \times \mathcal{P}(\mathcal{R})$ onto $\mathbb{A}^1_x$ by pr_x.

For each $g \in \mathcal{P}(\mathcal{R})$, restriction of $\mathrm{pr}_x \circ \Psi$ to the fiber $\mathbb{A}^1_y \times g$ over $g \in \mathcal{P}(\mathcal{R})$ gives a cover $\mathbb{A}^1_y \to \mathbb{A}^1_x$ by $y \mapsto g(y)$. In this cover $y = 0$ is ramified over $x = 0$ with ramification data $\mathcal{R}$. Of course, other points of $\mathbb{A}^1_y$ ramify over (usually) other values of x. So, regard this as a family of covers with a section picking out a choice of ramified fiber. Replacing $\mathbb{A}^1_x$ and $\mathbb{A}^1_y$, respectively, by $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$ works the same, giving a family of covers of the projective line.

4.2. **Locally versal families of covers with given ramification data.**

Definition 4.1. Let V be an affine variety equipped with morphisms

$$\phi^V_{i,j} : V \to \mathbb{A}^1,\ \ 0 \le i \le e_0 - 1,\ \ j \in \mathbb{N},\ \phi^V_{0,0} = 0.$$

For each closed point $P \in V$, consider the polynomial

$$f_P(x, y) = \sum_{i=0}^{e_0-1} \Big(\sum_{j=0}^{\infty} \phi^V_{i,j}(P) x^j \Big) y^i + y^{e_0}$$

The collection $V, \{\phi^V_{i,j}\}$ defines an *algebraic family of field extensions with ramification data* $\mathcal{R}$ if the following hold.

(4.1a) $f_P(x, y)$ is nonsingular at $(0, 0)$.
(4.1b) $f_P(x, y)$ defines an irreducible polynomial (in y) over $k((x))$ whose corresponding extension is $k((y))$ with $\mathcal{R}(k((y))/k((x))) = \mathcal{R}$.

Remark 4.2. Suppose $f(x, y) = 0$ defines the field extension $k((y))/k((x))$. Apply the implicit function theorem to find $g(y) \in k[[y]]$ such that $f(x, y) = 0 \iff x = g(y)$. Indeed, solve inductively for the coefficients of $g(y)$ from the the expression $f(g(y), y) = 0$. The reciprocal is also true after Lemma 4.3: Given $g(y) = x$, we may rediscover $f(x, y)$.

Lemma 4.3. *The affine variety $\mathcal{P}(\mathcal{R})$ defines an algebraic family of field extensions with ramification data $\mathcal{R}$.*

Proof. Let $P \in \mathcal{P}(\mathcal{R})$. By definition P gives $g_P \in k[y]$ such that $g_P(y) = x$ defines a field extension $k((y))/k((x))$ with ramification data $\mathcal{R}$. □

Denote the infinite number of morphisms in Lem. 4.3 from $\mathcal{P}(\mathcal{R})$ by $\{\phi^{\mathcal{P}(\mathcal{R})}_{i,j}\}$. It is the algebraic family of field extensions with ramification data $\mathcal{P}(\mathcal{R})$.

Theorem 4.4. *The affine variety $\mathcal{P}(\mathcal{R})$ is versal for the finite topology for algebraic families of field extensions with ramification data $\mathcal{R}$. That is, if $V, \{\phi^V_{i,j}\}$ is an algebraic family of fields extensions with ramification data $\mathcal{R}$, then there exists*

- *$\phi : Y \to \mathcal{P}(\mathcal{R})$ a finite surjective cover,*
- *$\psi : Y \to V$ a morphism such that for all $P \in Y$, $\psi(P)$ corresponds to the same isomorphism class of extension of $k((x))$ as $\phi(P)$,*

- *and a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\psi_{i,j}} & \mathbb{A}^1 \\ \downarrow^{\phi} & \nearrow^{\phi^V_{i,j}} & \\ V & & \end{array}$$

where $\psi_{i,j}$ is the composite in the sequence $Y \xrightarrow{\psi} \mathcal{P}(\mathcal{R}) \xrightarrow{\phi^{\mathcal{P}(\mathcal{R})}_{i,j}} \mathbb{A}^1$.

The affine variety $\mathcal{P}(\mathcal{R})$ is the configuration space for $\mathcal{R}$ ramification data.

Proof. Let $P \in V$ and consider the associated polynomial $f_P(x, y)$. After Remark 4.2, $f_P(x, y)$ corresponds to $g_P(y) \in k[y]$ for which $g_P(y) = x$ defines an extension $k((y))/k((x))$ associated to P.

As in the proof of Lemma 3.4, form a cover $\phi : Y \to V$ where the points in the fiber over P are in one-to-one correspondence with the power series $\bar{g}_P(y) \in k[[y]]$ with $\bar{g}_P(\beta(y)) = g_P(y)$ with $\beta \in \text{Aut}\ k[[y]]$. The map $\psi : Y \to \mathcal{P}(\mathcal{R})$ is the unique map identifying the polynomial $\bar{g}_P$ to a polynomial in the set $\mathcal{P}(\mathcal{R})$.

From Chevalley's theorem [Mu, Lemma 3.5.1], ϕ is finite since it is proper and has finite fibers. □

Remark 4.5. In Prop. 4.4, the morphism $\phi : Y \to \mathcal{P}(\mathcal{R})$ ramifies if the order of the automorphism group of the field extension corresponding to points of V is not locally constant.

4.3. **Generalization appropriate for considering families of covers.** Applying the previous ideas to collections (families) of covers requires some simple general notation to apply to the configuration space of §6. For that use the variable z to parametrize a copy $\mathbb{P}^1_z$ of the projective line. The families there will be of covers of $\mathbb{P}^1_z$. To decorate the ramification data appropriately for this, denote by $\mathcal{P}_{z'}(\mathcal{R})$ the same space where we replace the variable x by $z - z'$. So, each point $g \in \mathcal{P}_{z'}(\mathcal{R})$ represents the expression $g(y) = z - z'$.

Further, we will be considering families of covers $\phi : X \to \mathbb{P}^1_z$ whose individual members have a given number, say u, points that ramify over a particular branch point z', and that will associate points from different spaces to each of these ramified points. In that case, attached to z' we must consider the disjoint union of spaces $\cup_{i=1}^u \mathcal{P}_{z'}(\mathcal{R}_i)$, though we don't a priori need an order on these.

Finally, the cover ϕ representing a fiber in the family might have several branch points (though these may have no natural ordering) with varying numbers of ramified points above each. Suppose there are u_i ramified points attached to the ith branch point. Denote the collection $\{(j, i) \mid 1 \le i \le, 1 \le j \le u_i\}$ by $I_{\boldsymbol{u}}$, with $\boldsymbol{u} = \{u_1, \dots, u_r\}$. Running over each point z_i in the collection of branch points $\{z_1, \dots, z_r\} = \boldsymbol{z}$, consider the disjoint union of the collections $\cup_{(j,i)\in I_{\boldsymbol{u}}} \mathcal{P}_{z_i}(\mathcal{R}_{j,i})$. The space of r distinct unordered points on $\mathbb{P}^1_z$ has the usual notation $\mathbb{P}^r \setminus D_r$ (as in §6 and [BF, §2.1.3]). Fix all of these $\mathcal{R}_{j,i}$, $(j, i) \in I_{\boldsymbol{u}}$. Then, consider the *configuration* space attached to these $\mathcal{R}_{j,i}$ s:

$$S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i}) \stackrel{\text{def}}{=} \cup_{\boldsymbol{z}\in\mathbb{P}^r\setminus D_r} \{\cup_{i=1}^r \cup_{j=1}^{u_i} \mathcal{P}_{z_i}(\mathcal{R}_{j,i})\}.$$

This already has an affine space structure from the coordinates we have used previously, and a natural map $S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i}) \to \mathbb{P}^r \setminus D_r$.

Remark 4.6. At first sight the configuration space for wild ramification appears to workless well than that for tame ramification, especially as one must substitute the finite topology in statements for the étale topology. Still, close inspection of the role of regular ramification and Galois closure shows precisely properties of such finite covers. So we prefer to express that the wild configuration space captures more demanding phenomena. It deserves precise labeling of concepts, like p-developed polynomials, figuring in the necessary finite covers, and also the relation of all families to those explicitly used in §4. This will be the focal point for a later paper.

Definition 4.7. Suppose $\phi : X \to \mathbb{P}^1_z$ is a cover branched over $\boldsymbol{z} = (z_1, \dots, z_r)$ so that for some ordering of $\boldsymbol{z}$ and some ordering of the ramified points $x_{j,i}$ over each z_i, the local ramification extensions have ramification data $\mathcal{R}_{j,i}$. We say ϕ is a cover with ramification type $\{\mathcal{R}_{j,i}\}_{(j,i)\in I_{\boldsymbol{u}}}$.

5. Tame embedding extensions and Galois closure

As previously, assume k is an algebraically closed field of characteristic $p > 0$. The equation $x = f(y)$ defines a finite separable totally ramified extensions $k((y))$ of $k((x))$ for $f \in k[[y]]$. List $\mathcal{R} = \mathcal{R}(k((y))/k((x))) = \{e_0 \dots, e_s\}$ with $e_0 < \cdots < e_s$, $e_i = d_i p^{v_i}$ with $\pi(e_i) = p^{v_i}$ strictly decreasing in i. Let $\widehat{k((y))}$ be the Galois closure of the extension $k((y))/k((x))$ with its Galois group $G = \mathrm{Gal}(\widehat{k((y))}/k((x)))$. This section expresses properties of the Galois group G using ramification data. We begin with a description of tamely ramified embeddings of $k((y))$ in $k((x))$ (§5.1). Then a decomposition $f(y) = f_1(f_2(y))$ produces the field $k((f_2(y)))$ between $k((y))$ and $k((f(y)))$. We study this decomposition using the group of the Galois closure (§5.2) and ramification data (§5.3).

5.1. **Tame embedding extensions.**

Our next calculations develop from finding all field embeddings

$$\sigma : k((y)) \to \bigcup_{\substack{n\geq 1\\(n,p)=1}} k((y^{1/n})), \text{ fixed on } k((x)). \tag{5.1}$$

The least integer n for which $k((y^{1/n}))$ contains the image of σ in (5.1) is the *ramification order* of σ. By definition, the field $\widehat{k((y))}$ is tamely ramified over $k((y))$ if and only if there are exactly $n_{y/x} \stackrel{\text{def}}{=} [k((y)) : k((x))]$ embeddings in (5.1). The notation of Lemma 5.1 comes from §3.2.

Lemma 5.1. *Let n be a positive integer. Each embedding σ in* (5.1) *with ramification order n, comes from a power series in $y^{1/n}$ having the form $\sigma(y) = a_0 y + \sum_{i=1}^{\infty} a_i y^{1+i/n}$ satisfying*

$$f(\sigma(y)) \equiv f(y), \text{ an identity as power series in } y^{1/n}. \tag{5.2}$$

If $n_{y/x} = mp^v$ with $(m,p) = 1$, then a_0 is some mth root of 1. Moreover n divides $\mathrm{ind}^{tm}(y/x)$ *and there are at most $mI_n(\mathcal{R})$ such embeddings.*

Proof. We proceed as in proof of Lemma 3.4. Construct σ by induction on the index of the coefficient a_i. By definition, a_0 is an mth root of 1. Assume we know $a_0, \dots, a_{i-1}$. The automorphism σ satisfies the equation $f(y) = f(\sigma(y))$. The lowest order of the terms in $f(\sigma(y))$ in which a_i appears is given by

$$M(i) = \min_{0\leq u\leq s} (e_u + \frac{i}{n}\pi(e_u))$$

If only one value of s achieve the minimum then the equation for a_i uniquely determines a_i from $a_0, \dots, a_{i-1}$. If more than one value achieves the minimum, then $\frac{i}{n}$ is the value of a slope of $L^{\diamond}_j$ for some j. Then a_i is a solution of a p-developed polynomial of degree $\frac{\pi(E_{j_+})}{\pi(E_{j_-})}$. Then there are at most $m\prod_{j\in \mathrm{Ind}_n(\mathcal{R})}\frac{\pi(E_{j_+})}{\pi(E_{j_-})} = mI_n(\mathcal{R})$ power series.

Moreover if more than one value achieves $M(i)$ then there exists u, u' such that $E_u + \frac{i}{n}\pi(E_u) = E_{u'} + \frac{i}{n}\pi(E_{u'})$. Hence the denominator of $\frac{i}{n}$ divides $\frac{E_{u'}-E_u}{\pi(E_u)-\pi(E_{u'})}$. In particular, n divides $\mathrm{ind}^{tm}(y/x)$. ☐

When $\widehat{k((y))} = k((y))$ ($k((y))/k((x))$ is Galois) we may interpret Lemma 5.1 from higher ramification groups. The ith higher ramification group is

$$G_i = \{\sigma \in \mathrm{Gal}(k((y))/k((x))),\ \ \mathrm{ord}_y(\sigma(y)-y) \geq i+1\} \text{ [Se, IV]}.$$

Hence

$$\frac{|G_i|}{|G_{i+1}|} = \begin{cases} \frac{\pi(E_{j_+})}{\pi(E_{j_-})} & \text{if } i = m(j), \text{ the slope of } L^{\diamond}_j \\ 1 & \text{otherwise} \end{cases}$$

Therefore computation of the ramification data includes the computation of the higher ramification groups.

5.2. **Decomposing** $k((y))/k((x))$**.** This subsection studies the decomposition of $f(y)$ in the form $f_1(f_2(y))$ using the group of the Galois closure $\widehat{k((y))}$.

Lemma 5.2. *Suppose $n_{y/x} = mp^v$, with $(m,p) = 1$. It is always possible to write $f(y) = g(y)^m$ with $g(y)$ a power series where $\mathrm{ord}_y(g) = p^v$. It is also possible to write g as $h_1(g_1(y))$ with g_1 a polynomial and $h_1(y)$ a power series in y with $\mathrm{ord}_y(h_1) = m$.*

Proof. Let I be the (unique) p-Sylow of $G = \mathrm{Gal}(\widehat{k((y))}/k((x)))$. The fixed field T of I is the maximal tamely ramified extension of $k((x))$ in $\widehat{k((y))}$. Then, $k((y)) \cap T = T'$ is the unique extension of $k((x))$ in $k((y))$ of degree m. From the classification of tamely ramified extensions over an algebraically closed field, $T' = k((x^{1/m}))$. Write $x^{1/m}$ as a power series $g(y)$ to rewrite $x = f(y)$ as $g(y)^m$. As an alternative, write $k((g(y)))$ as $k((g_1(y)))$ for $g_1 \in k[y]$. For this, choose $g_1(y)$ a polynomial generating the maximal ideal of the integral closure of $k[[x]]$ in $k((g(y)))$. Then, some power series $h_1(y)$ satisfies $h_1(g_1(y)) = f(y)$. ☐

Let G_1 be the (Galois) group of $\widehat{k((y))}/k((y))$. List cosets of G_1 in G as $\sigma_1 G_1, \ldots, \sigma_r G_1$, for σ_i in (5.1). Let I_1 be the minimal normal subgroup of G_1 whose fixed field, $k((z))$, tamely ramifies over $k((y))$.

Lemma 5.3 (Tame embeddings)**.** *Embeddings of $k((y))$ into $k((z))$, fixed on $k((x))$, correspond to integers i with $\sigma_i^{-1} I_1 \sigma_i$ a subgroup of G_1. The ramification order of the embedding from such a σ_i is*

$$[k((z)) : k((y))]/[G_1 \cap \sigma_i G_1 \sigma_i^{-1} : I_1].$$

Decompositions of $f \in k((y))$ as $f_1(f_2(y))$ with $f_1, f_2 \in k((y))$ (up to equivalencing f_2 and $\beta(f_2(y))$ with β an automorphism of $k((y))$) correspond one-one with the groups between G and G_1.

Proof. Let $\sigma(y)$ be the image of y under an embedding of $k((y))$ into $k((z))$. Then, σ extends to an element of G. So, σ is in the coset $\sigma_i G_1$ where $\sigma(y) = \sigma_i(y)$. Acting on the left of y, $\sigma_i G_1$ is the subset of G mapping y to $\sigma(y)$. The Galois correspondence says $\sigma(y)$ is in $k((z))$ if and only if each $\alpha \in I_1$ fixes $\sigma_i(y)$. That is, if $\sigma_i(y) = \alpha(\sigma_i(y))$, or $\sigma_i^{-1} I_1 \sigma_i \le G_1$ ($\sigma_i^{-1} I_1 \sigma_i$ subgroup of G_1).

This is equivalent to $I_1 \le G_1 \cap \sigma_i G_1 \sigma_i^{-1}$ where the group on the right is the group of $\widehat{k((y))}/k((y, \sigma_i(y)))$. The formula in the lemma follows by rewriting indices of fields as indices of subgroups from the Galois correspondence.

A decomposition $f(y) = f_1(f_2(y))$ produces the field $k((f_2(y)))$ between $k((y))$ and $k((f(y))$ and conversely up to composing f_2 with an automorphism of $k((y))$. The Galois correspondence gives a one-one correspondence with such fields and groups contained between G and G_1. □

Remark 5.4 (Primitivity). Use the notation of Lem. 5.3. Call an extension $k((y))/k((x))$ primitive if there is no group properly between G_1 and G. Equivalently, there is no nontrivial decomposition of a normalized polynomial $f(y)$ giving this extension.

Lemma 5.5 (Galois Closure). *Suppose L/K is a finite separable field extension. Denote the Galois closure of this extension by $\widehat{L}$. Then, the group* $\mathrm{Gal}(\widehat{L}/K)$ *of $\widehat{L}/K$ contains no nontrivial normal subgroup C' such that $C' \subseteq \mathrm{Gal}(\widehat{L}/L)$.*

Proof. Suppose there is such a $C' \subseteq \mathrm{Gal}(\widehat{L}/L)$. Consider the fixed field L' of C'. Then, L' is a Galois extension of K contained in $\widehat{L}$ and containing L. From the definition of $\widehat{L}$, $\widehat{L} = L'$ and C' is trivial. □

For $f \in k[y]$ use G_f for the Galois group of the splitting field Ω_{f-x} of $f(y) - x$ over $k((x))$. If $\deg(f) = n$, then $f(y) - x$ has n solutions in the algebraic closure of $k(x)$ over which it is irreducible, and the group G^* of the splitting field over $k(x)$ is transitive. Over, however, $k((x))$, $f(y) - x$ is usually reducible. Then, the Galois group over $k((x))$ has a transitive permutation representation on the solutions of $f(y) - x$ having ord 1. The degree of this representation is the integer we refer to above as $n_{y/x}$.

Proposition 5.6. *Assume $f \in k[y]$ is separable, and $n_{y/x} = \deg(f) = mp^r$, with $(m, p) = 1$. Then, $G_f \le S_{n_{y/x}}$. Let I_f be the (normal) p-Sylow subgroup of G_f. The quotient G_f/I_f is cyclic of order a multiple of m. If $r = 0$, I_f is trivial and $|G_f| = n$. If $r = 1$, then, $I_f \cong (\mathbb{Z}/p\mathbb{Z})^u$ with $1 \le u \le m$ and $|G_f/I_f|$ is $m \cdot v$, v dividing $p - 1$.*

Proof. The one nontrivial part of proposition 5.6 is when we take $r = 1$. Let Ω_{f-x} be the splitting field of $f(y) - x$ over $k((x))$. Let T be the maximal tamely ramified extension of $k((x))$ in Ω_{f-x}. Its Galois group is G_f/I_f.

For $k((y)) \cap T$ use T_y. The extension $k((y))/T_y$ is of degree p. Denote the group of its Galois closure by H. Call C the stabilizer of $k((y))$ in H. The group C is the stabilizer of an extension of degree p. Then, $|C|$ is a subgroup of S_{p-1}, hence $(|C|, p) = 1$. The p-subgroup of H is generated by a p-cycle. So, C is a subgroup of S_p acting by conjugation on the group generated by a p-cycle. This identifies C with a subgroup of $\mathbb{Z}/p^*\mathbb{Z}$, so it has order v dividing $p - 1$.

Take one of the m factors f_1 of the polynomial relation between y and x over T_y. Let L_1 be the splitting field of f_1 over T_y. Then $H = \mathrm{Gal}(L_1/T_y)$ is its Galois group. It has a two step description. It is a degree p extension of a field V. And V is a cyclic degree v extension of T_y; the field V is the maximal tamely ramified extension of T_y in Ω_{f-x}. Then each splitting field of the m factors $f_1, \dots, f_m$ of $f(y) - x$ over T_y contains V. Then, above V, there are at most m cyclic extensions of order p. □

Proposition 5.7. *Suppose $f = f_1(f_2(y)) \in k[y]$ with $f_1, f_2 \in k[y]$. Assume $[k((y)) : k((f_1(y)))] = p$, $[k((y)) : k((f_2(y)))] = m > 1$, and $(m,p) = 1$. Let v be the integer such that $G_{f_1} = \mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/v\mathbb{Z}$ ($v \,|\, p-1$ as in Prop. 5.6). Then, G_f is $\mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/m'\mathbb{Z}$ with m' the least common multiple of m and v.*

Proof. Let $y_1, \dots, y_p$ be the zeros of $f_1(y) - x$. Since the composite of tamely ramified extensions is tamely ramified $\Omega_i = \Omega_{f_2 - y_i}\Omega_{f_1 - x}$, $i = 1, \dots, p$, is a tamely ramified extension of $\Omega_{f_1 - x}$ of degree $\deg(f_2)$. So, the Ω_is are all equal, and equal to Ω_{f-x}. Therefore, $\Omega_{f-x}/k(y_i)((x))$ is a cyclic Galois extension of degree the least common multiple m' of m and v. Since $G_f \le (G_{f_2})^p \times^s G_{f_1}$, G_f is a subgroup of

$$(\mathbb{Z}/m\mathbb{Z})^p \times^s (\mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/v\mathbb{Z}).$$

Hence G_f maps surjectively to the factor $\mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/v\mathbb{Z}$ and onl= y a single power of p divides $|G_f|$.

Also, there exists a field L_1 with $k((x)) \subset L_1 \subset k((y))$ and $[L_1 : k((x))] = m$. Again, G_f is a subgroup of

$$(\mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/v\mathbb{Z})^m \times^s \mathbb{Z}/m\mathbb{Z}.$$

Then, the kernel of the surjection of G_f to the factor $\mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/v\mathbb{Z}$ is surjective onto any factor of $(\mathbb{Z}/m\mathbb{Z})^p$. Since G_f is an extension of a cyclic group of order prime to p by a p-group, this identifies G_f with $\mathbb{Z}/p\mathbb{Z} \times^s \mathbb{Z}/m'\mathbb{Z}$ with $\mathbb{Z}/m'\mathbb{Z}$ acting through $\mathbb{Z}/v\mathbb{Z}$. We are done if we identify the subgroup of G_f fixing $k((y))$. This, however, is a subgroup of index m in $\mathbb{Z}/m'\mathbb{Z}$: the subgroup m generates. □

5.3. **Composition and decomposition of ramification data.** Given two extensions $k((y_1))/k((x))$ and $k((y_2))/k((x))$ we may substitute y_1 for x in the second extension to form a chain $k((y_2))/k((y_1))/k((x))$. Using our normalized polynomial representation of extensions, associate to the chain a composition of polynomials $f_1(f_2(y)) = x$ (with $f_2(y_2) = y_1$ and $f_1(y_1) = x$). This section discusses the ramification data of the composition $f_1(f_2(y))$.

Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are two sets of ramification data. Then, it is natural to form all compositions, running over $(f_1, f_2) \in \mathcal{P}(\mathcal{R}_1) \times \mathcal{P}(\mathcal{R}_2)$. As Ex. 5.8 shows, we cannot expect the extensions corresponding to $f_1(f_2(y))$ to all lie in the parameter space for one set of ramification data. So, there is (usually) no unique set of ramfication data aptly considered the composition of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Further, even if one from these compositions $f_1(f_2(y))$ falls in $\mathcal{P}(\mathcal{R})$ for some ramification data $\mathcal{R}$, we cannot expect the set of $f_1(f_2(y))$ with ramification data $\mathcal{R}$ to fill out the space $\mathcal{P}(\mathcal{R})$. For example, the Galois closure group of extensions with ramification data $\mathcal{R}$ is not constant in the family. So, some Galois closure groups (in their natural permutation representation) might be primitive while others are not. Suppose $f(y)$ corresponds to a point of $\mathcal{P}(\mathcal{R})$, and the Galois closure of the corresponding extension is primitive. Then, as in Rem. 5.4, there is no decomposition of f. So there is no way f could correspond to having polynomials for $\mathcal{R}_1$ and $\mathcal{R}_2$ compose to f. These observations motivate the necessity of stratifying the spaces $\mathcal{P}(\mathcal{R})$ related to the Galois closures of their extensions.

Example 5.8. Take $p = 3$. Let $f_1(y) = f_2(y) = y^3 + y^4$. Then

$$\mathcal{R}(f_1(y)) = \mathcal{R}(f_2(y)) = \{3, 4\} \text{ and } \mathcal{R}(f_1(f_2(y))) = \{9, 12, 13\}.$$

Let $f_1(y) = y^3 - y^4$, $f_2(y) = y^3 + y^4$. Again, $\mathcal{R}(f_1(y)) = \mathcal{R}(f_2(y)) = \{3, 4\}$, yet $\mathcal{R}(f_1(f_2(y))) = \{9, 13\}$.

Lem. 5.9 is clear by construction of ramification data.

Lemma 5.9. *Let f_1, g_1, f_2 be three polynomials such that*

$$\mathcal{R}(f_1(f_2(y))) = \{d_i p^{v_i},\quad 0 \le i \le s\},\ \ \mathcal{R}(g_1(f_2(y))) = \{d'_j p^{v'_j},\quad 0 \le j \le s'\}$$
$$\text{and}\quad \mathcal{R}(f_1(f_2(y))) \cap \mathcal{R}(g_1(f_2(y))) = \emptyset.$$

Let $I = \{p^{v_i}, 0 \le i \le s\} \cup \{p^{v'_j}, 0 \le j \le s'\} \subset \mathbb{N}$. Then,

$$\mathcal{R}((f_1 + g_1)(f_2(y))) = \Big\{\min\Big(x \in \mathcal{R}(f_1(f_2(y))) \cup \mathcal{R}(g_1(f_2(y)))|\pi(x) = n\Big)\Big\}_{n \in I}.$$

Proposition 5.10. *Suppose f_1, f_2 are polynomials with*

$$\mathcal{R}(f_1(y)) = \{d_0 p^{v_0}\}\ \text{and}\ \mathcal{R}(f_2(y)) = \{d'_j p^{v'_j},\ 0 \le j \le s'\}$$

with $d'_j p^{v'_j}$ (resp. $p^{v'_j}$) a strictly increasing (resp. decreasing) function of i, j. Then

$$\mathcal{R}(f_1(f_2(y))) = \Big\{(d_0 - 1)d'_0 p^{v_0 + v'_0} + d'_j p^{v_0 + v'_j}, 0 \le j \le s'\Big\}. \tag{5.3}$$

Proof. First consider the case $f_1(y) = a_0 y^{d_0 p^{v_0}}$, $f_2(y) = \sum_{j=0}^{s'} a'_j y^{d'_j p^{v'_j}}$. The coefficient of $y^{(d_0-1)d'_0 p^{v_0+v'_0} + d'_j p^{v'_j + v_0}}$ in $f_1(f_2(y))$ is non zero: precisely, it is $a_0 d_0 {a'_0}^{d_0 - 1} a'_j$ (resp. $a_0 {a'_0}^{d_0}$) if $j \ne 0$ (resp. $j = 0$). The others terms of $f_1(f_2(y))$ have the form $c \cdot y^n$ with $n = n_0 d'_0 p^{v_0 + v'_0} + \cdots + n_j d'_j p^{v_0 + v'_j}$ for $c \in k$, $n_i \ge 0$, $n_j \ne 0$, $n_0 + \cdots + n_j = d_0$ an $n_1 + \cdots + n_j \ge 2$. By definition, the degree n satisfies $n > (d_0 - 1)d'_0 p^{v_0 + v'_0} + d'_j p^{v'_j + v_0}$ and $\pi(n) \ge p^{v'_j + v_0}$. Then by construction of ramification data, $n \notin \mathcal{R}(f_1(f_2(y))$. Hence

$$\mathcal{R}(f_1(f_2(y))) = \Big\{(d_0 - 1)d'_0 p^{v_0 + v'_0} + d'_j p^{v_0 + v'_j},\ \ 0 \le j \le s'\Big\}$$

Now replace f_2 by $g_2(y) = f_2(y) + a y^{b p^{v'_m}}$ $(0 \le m \le s')$ having the same ramification data. Then the coefficients of $f_1(f_2) - f_1(g_2)$ have the form $c \cdot y^n$ with $n = n_0 d'_0 p^{v_0 + v'_0} + \cdots + n_j d'_j p^{v_0 + v'_j} + \ell b p^{v_0 + v'_m}$, $c \in k$, $n_i \ge 0$, $\ell \ne 0$ and $n_0 + \cdots + n_j + \ell = d_0$. Let $n' = n_0 d'_0 p^{v_0 + v'_0} + \cdots + n_j d'_j p^{v_0 + v'_j} + \ell d'_m p^{v_0 + v'_m}$.

The degree n satisfies $n > n'$ and $\pi(n) \ge \pi(n')$. Hence $n' \notin \mathcal{R}(f_1(g_2(y)))$ and $\mathcal{R}(f_1(f_2(y))) = \mathcal{R}(f_1(g_2(y)))$. By induction, we are then able to prove that

$$\mathcal{R}(f_1(f_2(y))) = \Big\{(d_0 - 1)d'_0 p^{v_0 + v'_0} + d'_j p^{v_0 + v'_j},\ \ 0 \le j \le s'\Big\}$$

for any $f_2 \in k[y]$ with ramification data $\mathcal{R}(f_2(y)) = \{d'_j p^{v'_j},\ 0 \le j \le s'\}$. □

6. Configuration spaces and Galois closure

We usually assume all covers are of the projective line $\mathbb{P}^1_z$, with uniformizing variable z and having $r \ge 2$ branch points. This section considers families of covers having type $\{\mathcal{R}_{j,i}\}_{(j,i) \in I_{\boldsymbol{u}}}$ (for notation see §4.3). We first remind of what to expect of the configuration space in characteristic 0, especially considering the Galois closure of a family (§6.1). We briefly review results in this direction ([BF], [FV]): configuration space for tamely ramified families, Galois closure construction (§6.2-6.4). The paragraph §6.5 initiates a discussion of the new phenomena that require more careful consideration of a Galois closure construction when the covers are wildly ramified. We present an outline of what to expect of the wild case. The paper concludes with a short list of questions about the nature of families of type $\{\mathcal{R}_{j,i}\}_{(j,i)\in I_{\boldsymbol{u}}}$ that parallel properties of Hurwitz families (§6.6).

6.1. **Desirable properties of a configuration space.** The simplicity of the configuration space in characteristic 0 belies the demands made on it.

(6.1a) It should be a natural target for any family of covers of the given type.
(6.1b) Any family of covers of the requisite type should be the pullback of this target map from a family over an étale cover of the configuration space.

In characteristic 0 we expect refined descriptions of certain types of families, that often allows the production of a *unique* cover to serve for that type in (6.1b). This is the condition that the type has *fine moduli*, essential for applications to the Inverse Galois Problem. This gives us a chance to remind that these moduli properties naturally use *going to the Galois closure in families.*

We take as essential the property (6.1a). At this stage, however, we get something like property (6.1b) only by relaxing the situation to produce a finite cover of the range of the target map that gives the family by pullback. For families of tame covers, the monodromy group (geometric Galois closure group) of covers in the family is constant. For families of tame covers with monodromy group having order prime to the characteristic, Riemann's existence Theorem provides all the properties of (6.1). We remind of this in §6.2 and §6.3.

Even, however, for families of tame covers, when the monodromy group has order divisible by the characteristic, the natural configuration space may not be all of $\mathbb{P}^r \setminus D_r$, though it is a constructible subspace of $\mathbb{P}^r \setminus D_r$. Further, we require general finite, not necessarily flat, covers of the configuration space to assure a variant of property (6.1b).

6.2. **The configuration space for tamely ramified families.** Regard $\mathbb{P}^r \setminus D_r \stackrel{\text{def}}{=} U_r$ (projective r space minus the discriminant locus) as the set of unordered distinct points on $\mathbb{P}^1_z$. Suppose $\phi_0 : X_0 \to \mathbb{P}^1_z$ is a cover with unordered branch points $\boldsymbol{z}_0 \in U_r$. Call this an r-branch cover. A family of r-branch covers consists of a smooth cover $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ with the fiber $\phi_{\boldsymbol{p}} = \mathrm{pr}_z \circ \Phi_{\boldsymbol{p}} : \mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ an r-branch cover. Then, there is a (unique) algebraic map $\Psi : \mathcal{P} \to U_r$ mapping $\boldsymbol{p} \in \mathcal{P}$ to the set of branch points of $\phi_{\boldsymbol{p}}$.

Another cover $\phi_1 : X_1 \to \mathbb{P}^1_z$, with unordered branch points $\boldsymbol{z}_1 \in U_r$, is called r-branch connected to ϕ_0 if there is an irreducible family of r-branch covers containing both ϕ_0 and ϕ_1. Let $\mathcal{S}(\phi_0)$ be the collection of r-branch covers connected to ϕ_0, up to the *strong equivalence* given in [BF, §3.5] (§6.3; isomorphism up to commuting with the map to $\mathbb{P}^1_z$).

When $k = \mathbb{C}$, using such families is a natural tool (including for the Inverse Galois problem) though precise natural questions don't have easy answers. There is, however, a combinatorial calculation from computing an orbit of the r-string braid group on Nielsen classes describes the covers r-branch connected to ϕ_0 [BF, §2]. Further, there is some global family, $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$, containing each equivalence class of $\mathcal{S}(\phi_0)$ so that $\Psi : \mathcal{P} \to U_r$ is a finite unramified cover. The subtleties here are about finding families for which Ψ has minimal degree. We describe this briefly in §6.3 as a model of the best we can expect.

6.3. **When the order of the monodromy group is prime to char(k).** Assume $k = \mathbb{C}$. Suppose $\gamma : [0,1] \to U_r$ is a path starting at $\boldsymbol{z}_0$ and ending at $\boldsymbol{z}_1$. Then analytic continuation along γ produces a canonical cover $\phi_1 : X_1 \to \mathbb{P}^1_z$ with branch point set $\boldsymbol{z}_1$.

By attaching extra data to the point $\boldsymbol{z}_0$, homotopy classes $\pi_1(U_r, \boldsymbol{z}_0)$ of closed paths γ based at $\boldsymbol{z}_0$ are isomorphic to the Hurwitz monodromy quotient H_r of the braid group B_r on r strings ([Fr77, §4], [Vö, Chap. 10], [BF, §3]). This creates a formula for describing all the covers at the end of closed paths starting at (X_0, ϕ_0). Let $U_{\boldsymbol{z}}$ be $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$. For any $z_0 \in U_{\boldsymbol{z}}$, consider $\pi_1(U_{\boldsymbol{z}}, z_0)$. Denote generators for H_r by $Q_1, \dots, Q_{r-1}$. Suppose $(g_1, \dots, g_r) \in G^r$ are any generators of a group G, in some order in a collection of conjugacy classes $\mathrm{C}_1, \dots, \mathrm{C}_r$ ($g_i \in \mathrm{C}_{\pi(i)}$, for some $\pi \in S_r$, $i = 1, \dots, r$). Assume further the elements are subject to the *product condition*: $g_1 \cdots g_r = 1$.

Lemma 6.1. *Let $\gamma_1, \dots, \gamma_r$ be classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$: each homotopic to a path that loops once clockwise around z_i, nonintersecting except at the endpoints and homotopic as a set to piecewise differential paths that leave z_0 clockwise. The homotopy classes $[\gamma_1], \dots, [\gamma_r]$ generate $\pi_1(U_{\boldsymbol{z}}, z_0)$ freely modulo the one relation $[\gamma_1] \cdots [\gamma_r] = 1$. Then, (X_0, ϕ_0) corresponds canonically to a (conjugacy class of) subgroup(s) H_0 of finite index in $\pi_1(U_{\boldsymbol{z}}, z_0)$. The group represented by $\pi_1(U_{\boldsymbol{z}}, z_0)$ acting on the cosets of H_0 is the monodromy group of the cover. The image g_i of $[\gamma_i]$, $i = 1, \dots, r$ gives $g_1, \dots, g_r$ as a branch cycle description of the cover, determining the cover up to equivalence as a cover of $\mathbb{P}^1_z$ modulo the action of $N_{S_n}(G)$ acting by conjugation.*

The action of H_r on these equivalence classes gives branch cycle descriptions $\boldsymbol{g}'$ for all covers $(X_{\boldsymbol{g}'}, \phi_{\boldsymbol{g}'})$ at the end points of paths in $\pi_1(U_r, \boldsymbol{z}_0)$.

There are various equivalences, on covers, depending on the moduli task they serve [BF, §3.5]. Here we record only an equivalence of $\phi : X \to \mathbb{P}^1_z$ with ϕ_0 if there exists an isomorphism $\psi : X_0 \to X$ with $\phi \circ \psi = \phi_0$. Grothendieck's Theorem extends to say that something like this holds in positive characteristic. We assume the definition field k is the algebraic closure of a finite field for simplicity.

Proposition 6.2. *There is a lift (X_0', ϕ_0') of (X_0, ϕ_0) to the Witt vectors of k. Further, if p does not divide $|G|$, for any element of H_r, there is a p-adic analytic continuation of (X_0', ϕ_0') to each element of $\{(X_{\boldsymbol{g}'}, \phi_{\boldsymbol{g}'})\}_{(\boldsymbol{g}')Q, Q \in H_r}$ so that the reduction map is injective on these equivalence classes, producing covers in the Nielsen class of (X_0, ϕ_0) having the same branch points as ϕ_0 (*[Fri95a, App. E]*, detailed proof in* [We98]*).*

6.4. **Galois closure construction when ramification is tame.** As a convenience related to our main applications, assume curve covers are covers of $\mathbb{P}^1_z$.

Assume the parameter space for any family in our examples is normal and connected. In the tame case, Hurwitz families simplify because the Galois closure group does not change in the family. In fact, if $\mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ is any family of covers of the z-line, with each member $\mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ (for $\boldsymbol{p} \in \mathcal{P}$) of the family tamely ramified and having r geometric branch points, then the geometric Galois group of the Galois closure of members in the family is constant. Still, there is interesting monodromy in considering it (§6.4.2). In tamely ramified situations we have canonical conjugacy classes attached to monodromy around branch points of a cover.

6.4.1. *Galois closure of a single tame cover.* Use notation from [BF, §3.1.3] for a cover $\phi : X \to \mathbb{P}^1_z$ of degree $n = \deg(\phi)$ with $(G, \mathbf{C})$ and a permutation representation $T_\phi = T : G \to S_n$ attached. The Galois closure of ϕ over any defining field for (X, ϕ) has a geometric formulation.

Take the fiber product $X^{(n)} \stackrel{\text{def}}{=} X^{(n)}_\phi \stackrel{\text{def}}{=} X^{(n)}_{\mathbb{P}^1_z}$ of ϕ, n times. Points on $X^{(n)}$ consist of n-tuples $(x_1, \dots, x_n)$ of points on X satisfying $\phi(x_1) = \phi(x_2) = \cdots = \phi(x_n)$. This variety will be singular around n-tuples where x_i and x_j are both ramified through ϕ. Replace $X^{(n)}$ by its normalization to make it now a non-singular cover. Retain the notation $\phi^{(n)} : X^{(n)} \to \mathbb{P}^1_z$. Then, $X^{(n)}$ has components where at least two of the coordinates are identical, the *fat diagonal*.

Remove components of this fat diagonal to give X^*. Over the algebraic closure X^* has as many components as $(S_n : G)$. List one of these components over $\bar{K}$ as $X^\dagger$. The stabilizer in S_n of $X^\dagger$ is a conjugate of G. Normalize by choosing $X^\dagger$ so the stabilizer is actually G. Now, choose any K component $\hat{X}$ of X^* containing $X^\dagger$. Then, $\hat{\phi} : \hat{X} \to \mathbb{P}^1_z$ is Galois (over K) with group $\hat{G} \le N_{S_n}(G, \mathbf{C})$ having G as a subgroup. (The notation $N_{S_n}(G, \mathbf{C})$ is for normalizing of G in S_n consisting of elements permuting the conjugacy classes in $\mathbf{C}$.) Also, $\hat{\phi}$ has the same conjugacy classes $\mathbf{C}$ attached to the branch points $\boldsymbol{z}$ and it factors through ϕ (project on any coordinate of $X^{(n)}$). The Galois cover $\hat{X} \to X$ has group $\hat{G}(1) = \hat{G}(T, 1)$ where T is the coset representation of G on $G(1)$.

6.4.2. *Galois closure of a family of tame covers.* Let $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ be a family of tame degree n covers as above over a field k. Use exactly the same construction as above to form the Galois closure of Φ: Normalize a component $\hat{\mathcal{T}}$ of the n-fold fiber product $\mathcal{T}^{(n)} \stackrel{\text{def}}{=} X^{(n)}_\phi \stackrel{\text{def}}{=} X^{(n)}_{\mathcal{P} \times \mathbb{P}^1_z}$ of the fat diagonal of Φ. Next let $\hat{\mathcal{P}}$ be the integral closure of $\mathcal{P}$ in the function field of $\hat{\mathcal{T}}$. This gives a family $\hat{\Phi} : \hat{\mathcal{T}} \to \hat{\mathcal{P}} \times \mathbb{P}^1_z$. The Stein Factorization Theorem says $\hat{\Phi}$ is a normal variety. The value of the construction is immediate.

Proposition 6.3. *The fibers $\hat{\Phi}_{\hat{\boldsymbol{p}}} : \hat{\mathcal{T}}_{\hat{\boldsymbol{p}}} \to \mathbb{P}^1_z$ for $\hat{\boldsymbol{p}} \in \hat{\mathcal{P}}$ are each absolutely irreducible Galois covers with Galois group G (independent of $\hat{\boldsymbol{p}}$), with cover and automorphisms all defined over $K(\hat{\boldsymbol{p}})$. The natural map $\hat{\mathcal{P}} \to \mathcal{P}$ is an étale cover.*

Comments on the construction. This is a geometric rephrasing of the construction of [FV, Thm. 1] for producing the *inner Hurwitz space* of Galois covers associated to an *absolute Hurwitz space* of covers with no automorphisms [BF, §3.5].

If the characteristic of k is 0, Riemann's existence Theorem says the group $G_{\hat{\boldsymbol{p}}}$ attached to the cover at $\hat{\boldsymbol{p}}$ is locally constant. While the reason is utterly simple — using Riemann's Existence Theorem — it is profound in its failure without it. Local constantness of the Galois closure group for the fibers follows immediately if absolutely irreducible components of fibers of $\mathcal{T}^{(n)} \to \mathcal{P} \times \mathbb{P}^1_z$ have locally constant degree over $\mathbb{P}^1_z$.

This is, however, exactly what fails for the same construction with wildly ramified covers. The actual argument goes like this [Fr77, Lem. 1.2]. It suffices in the complex topology to show $G_{\hat{\boldsymbol{p}}}$ is locally constant as a function of $\hat{\boldsymbol{p}}$. Take a small complex neighborhood of a point, so the branch points of the covers do not wander far. Let z_0 (fixed) be distinct from the collection $\boldsymbol{z}(\hat{\boldsymbol{p}})$ of r branch points for the cover associated with $\hat{\boldsymbol{p}}$ in a neighborhood of $\hat{\boldsymbol{p}}_0$. Take an ordering of these to be $z(\hat{\boldsymbol{p}})_1, \dots, z(\hat{\boldsymbol{p}})_r$.

Consider a collection of classical generating paths $\gamma_1, \dots, \gamma_r$ to give generators for the fundamental group $\pi_1(U_{\hat{\boldsymbol{z}}(\boldsymbol{p}_0)}, z_0)$ as in [BF, §1.2] or §6.3. For $\hat{\boldsymbol{p}}$ in a suitably small neighborhood of $\hat{\boldsymbol{p}}_0$, the *same* set of paths $\gamma_1, \dots, \gamma_r$ gives classical generators for the fundamental group of $\pi_1(U_{\hat{\boldsymbol{p}}}, z_0)$. Homotopy classes $[\gamma_1]_{\hat{\boldsymbol{p}}}, \dots, [\gamma_r]_{\hat{\boldsymbol{p}}}$ of these paths in $U_{\hat{\boldsymbol{p}}}$ freely generated the fundamental group modulo the relation $[\gamma_1]_{\hat{\boldsymbol{p}}} \cdots [\gamma_r]_{\hat{\boldsymbol{p}}} = 1$. So, we may canonically identify the fundamental groups $\pi_1(U_{\boldsymbol{z}(\hat{\boldsymbol{p}})}, z_0)$ in a neighborhood of $\hat{\boldsymbol{p}}_0$ with a fixed group Π.

Finally, the covers $\hat{\Phi}_{\hat{\boldsymbol{p}}}$ come canonically from a normal subgroup H of Π obtained by mapping the generators $[\gamma_1],\dots,[\gamma_r]$ of Π to fixed elements of G. So, the group $G_{\hat{\boldsymbol{p}}}$ is constant.

It is Grothendieck's Theorem ([Gr], [Mu]) together with Artin-approximation that allows lifting affine neighborhoods of any $\hat{\boldsymbol{p}} \in \hat{\mathcal{P}}$ to characteristic 0. This reverts the local constantness under the assumption of tame ramification of the group to the same statement in characteristic 0. □

We have just shown it makes sense with tame covers to consider the Galois closure of covers in a family. Even over $\mathbb{C}$, it is valuable to consider covers that may not be Galois, for comparing what happens in going to the Galois closure of the covers in a given family. Here we conclude by reminding of the easiest examples distinguishing between families of tame covers in characteristic p and those in characteristic 0 when this (common) Galois closure group has order divisible by p.

Example 6.4 (Dihedral covers)**.** Consider *involution* covers where G is a dihedral group D_{p^n} (group of order $2p^n$) with p an odd prime. The precise canonical description of the fine moduli space of called $\mathcal{H}(D_{p^n},\mathbf{C}_{2^{2l}})$ is in [DF, §5.2]. The gist is that each point $\boldsymbol{p} \in \mathcal{H}(D_{p^n},\mathbf{C}_{2^{2l}})$ corresponds to a cyclic unramified cover $\hat{X}_{\boldsymbol{p}} \to Y_{\boldsymbol{p}}$ having degree p^n with $Y_{\boldsymbol{p}}$ hyperelliptic of genus $g \stackrel{\text{def}}{=} l-1$.

The collection of involution automorphisms α of $\hat{X}_{\boldsymbol{p}}$ lifting the hyperelliptic involution ι of $Y_{\boldsymbol{p}}$ form a conjugacy class in D_{p^n}. The quotient cover $\hat{X}_{\boldsymbol{p}}/\alpha \to Y_{\boldsymbol{p}}/\iota = \mathbb{P}^1_z$ has (geometric) Galois closure group D_{p^n}. Let $N(n,l)$ be the number of $\mathbb{Z}/p^n$ submodules M in $(\mathbb{Z}/p^n)^{2g}$ for which $(\mathbb{Z}/p^n)^{2g}/M$ is cyclic of order p^n. As in §6.2, U_{2l} is the space of distinct, unordered collections of $2l$ points in $\mathbb{P}^1_z$. The Hurwitz space $\mathcal{H}(D_{p^n},\mathbf{C}_{2^{2l}})_{\mathbb{C}}$ is a proper cover of U_{2l} of degree $N(n,l)$.

In, however, characteristic p, while some of the same analysis works the degree is not correct. As previously let $\boldsymbol{z}(\boldsymbol{p}) \in U_{2l}$ be the branch point locus of the cover associated to $\boldsymbol{p}$. Depending on what is $Y_{\boldsymbol{p}}$, one must replace the $\mathbb{Z}/p^n$ module $(\mathbb{Z}/p^n)^{2g}$ by $(\mathbb{Z}/p^n)^{t_{\boldsymbol{z}(\boldsymbol{p})}}$ with $t_{\boldsymbol{z}(\boldsymbol{p})}$, as $\boldsymbol{p}$ varies, including every integer between 0 and g. So, computing the degrees of $\mathcal{H}(D_{p^n},\mathbf{C}_{2^{2l}}) \to U_{2l}$ over particular hyperelliptic curve euivalence classes requires finding $\mathbb{Z}/p^n$ modules M in a $\mathbb{Z}/p^n$ module isomorphic to $(\mathbb{Z}/p)^t$ with $t \le g$. Achieved values of t are all integers between 0 and g.

6.5. **Configuration spaces for wild covers and isotriviality.** This section has the structure of a research program. There has been little development of global families of covers mixing tame and wild ramification. The goal is to use data about local ramification to glean the nature of the totality of covers with such local ramification. The successful attack on understanding such covers of [Ha] and [Ra] has produced covers with specific groups through Abhyankar's conjecture. Their method however, leaves much about the covers a mystery, including their genuses and their relation to other covers. Phenomena about local ramification groups attached to a cover that appear in applying this paper are known to few. This includes even their application to families of covers ramified at one point only of the projective line. We present here an outline of what to expect in the wild case.

Let $\mathcal{P}(\mathcal{R})$ be the versal family attached to the ramification data $\mathcal{R}$. For the versal family $\mathcal{P}(\mathcal{R})$, the Galois closure groups of the local fields change in the family. This means the Galois closure groups of the covers in a family could change, even without moving the branch points. For simplicity again assume k is algebraically closed.

Proposition 6.5. *There is a stratification $\mathcal{X}$ of $\mathcal{P}(\mathcal{R})$ into finitely many constructible subsets with the following property. Suppose $\boldsymbol{p} \in X \in \mathcal{X}$. If $L_{\boldsymbol{p}}$ is the extension of $k((x))$ attached to $\boldsymbol{p}$ then the Galois closure group for $L_{\boldsymbol{p}}/k((x))$ depends only on X.*

Outline of proof. Apply the fiber product construction of Galois closure in §6.4.2 to the cover $Y \to \mathcal{P}(\mathcal{R})$ of Thm. 4.4. The construction works though the strong conclusions are reserved for the tame case. This gives $\hat{Y} \to Y \to \mathcal{P}(\mathcal{R})$. On $Y \to \mathcal{P}(\mathcal{R})$ we have a section $\psi : \mathcal{P}(\mathcal{R}) \to Y$ through the defining totally ramified place. The procedure for the stratification is similar to the Galois stratification procedure of [FJ, Chp. 25.1-25.4]. Start the stratification over a Zariski open subset U of $\mathcal{P}(\mathcal{R})$ that is flat or where even $\hat{Y}_U \to U$ is like projection from a hypersurface for the map $U \times \mathbb{A}^1 \to U$ by projection on the first factor.

We have only to algebraically describe the locus $\boldsymbol{p} \in U$ where the degree of a ramified point of $\hat{Y}_{\boldsymbol{p}}$ over $\boldsymbol{p}$ is maximal. That gives one element X (open in U) of the stratification $\mathcal{X}$. Then, restrict $Y \to \mathcal{P}(\mathcal{R})$ over the complement of X and continue inductively according to the dimension of the components. The distinction with [FJ, Chp. 25.1-25.4] is we are dealing with an inertia group here rather than a decomposition group. Full applications, however, require discussing the full decomposition group, and starting over a finite field. □

Use Prop. 6.5 to stratify the configuration space $S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i})$ from Prop. 6.7 for a similar Galois closure statement. Further, it may be necessary in many cases to do so to speak confidently of a family of Galois covers.

There is a finite cover of the parameter space of the family inducing a map to the configuration space $S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i})$ (Prop. 6.7). It is a subtlety that we must pull back to a finite cover. It is conspicuously an intrinsic part of the configuration space properties what type of pullback is required, based on p-developed polynomials. So, this property should eventually have a more refined statement. Prop. 6.8 shows the map to the configuration space is constant if and only if the family is isotrivial. The proof is an outline based on the following result of Garuti [Ga]. As noted earlier our notational simplification has Y in this statement equal to $\mathbb{P}^1$.

Theorem 6.6. *Let Y_k be a smooth projective curve over an algebraically closed field k of positive characteristic p. Let $f_k : X_k \to Y_k$ be a finite Galois covering of Y_k, with Galois group G. Let R be a characteristic zero, complete discrete valuation ring, of residue field k, and fix a smooth lifting Y of Y_k to R. Then after a finite extension R'/R, there exists a generically étale covering of proper normal R'-curves $f' : Y' \to Y' = Y \times_R R'$ of Galois group G such that the special fibre X'_k can only have cusps as singular points and admits a G-equivariant morphism of normalisation $X_k \to X'_k$ which is an isomorphism outside the ramified points.*

Proposition 6.7. *Suppose $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ is a family of covers of type $\{\mathcal{R}_{j,i}\}_{(j,i)\in I_{\boldsymbol{u}}}$. Then, there is a finite cover $\mu : \mathcal{P}' \to \mathcal{P}$ so that with $\mathcal{T}' = \mathcal{T} \times_{\mathcal{P}} \mathcal{P}'$ and $\Phi' = (\Phi, \mu)$, the pullback $\Phi' : \mathcal{T}' \to \mathcal{P}' \times \mathbb{P}^1_z$ has the following property. There is a morphism of quasiprojective varieties $\Psi' : \mathcal{P}' \to S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i})$ where for $\boldsymbol{p}' \in \mathcal{P}'$, the fiber $\Phi_{\boldsymbol{p}'} : \mathcal{T}'_{\boldsymbol{p}'} \to \mathbb{P}^1_z$ has ramification type $\Psi'(\boldsymbol{p}')$ (Def. 4.7).*

Outline of Proof. For any family $\mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ there is an attached finite cover $\mu^* : \mathcal{P}^* \to \mathcal{P}$. Each geometric point $\boldsymbol{p}^* \in \mathcal{P}^*$ corresponds to a point of $\mathcal{T}_{\mu^*(\boldsymbol{p}^*)}$ which ramifies in the corresponding cover $\mathcal{T}_{\mu^*(\boldsymbol{p}^*)} \to \mathbb{P}^1_z$. That is, pullback to $\mathcal{P}^*$ is the same as algebraically giving a labeling of the ramified points of the fibers of Φ. Therefore it is meaningfull according to Def. 4.7 to assign a precise ramification type to the fiber $\Phi^* : \mathcal{T}^*_{\boldsymbol{p}^*} \to \mathbb{P}^1_z$. Since these curve covers are nonsingular, locally (in the Zariski topology) over each branch point z_i the whole fiber is isomorphic to a closed subset of $\mathbb{A}^1 \times \mathbb{P}^1_z$. When this is the case, this is the data required to attach to the fiber over $\boldsymbol{p}^*$ a point of $S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i})$ (a repeated application of Thm. 4.4, taking each ramified point section, one at a time).

Finally, one must construct a finite cover of $\mathcal{P}^*$ so that this local hypersurface structure works globally on the pullback. □

Given the family of Prop. 6.7, refer to a finite cover $\mu : \mathcal{P}' \to \mathcal{P}$ providing the map Ψ' as a *configuration cover*.

Proposition 6.8 (Isotrivial Proposition)**.** *Let $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ be the family in Prop. 6.7. Suppose after pullback to a configuration cover $\mu : \mathcal{P}' \to \mathcal{P}$, the map Ψ' is constant. Then, after pullback to a further finite cover $\mathcal{P}'' \to \mathcal{P}'$ of $\mathcal{P}'$, the family is isomorphic to $\mathcal{P}'' \times X_0 \to \mathcal{P}'' \times \mathbb{P}^1_z$ by a map that is the identity in the first factor (the family Φ is isotrivial).*

Outline of Proof. One must show, under the constant map hypothesis, the Galois closure construction of the family gives geometrically connected fibers with constant Galois group. Then, extend Garuti's construction [Ga] to give a family version over the parameter space, so that the characteristic 0 fibers over the parameter space consist of nonsingular covers of $\mathbb{P}^1_z$ in which the branch branch points don't move and the locus of singularity in his normalization map for the special fibers is flat over the parameter space. Then, by Grothendieck's Theorem, as the branch points of the lifted family of covers don't move, the family is locally constant. This gives the isotriviality of the whole family upon suitable base change. □

6.6. **Galois closure questions.** For families of wildly ramified covers, the Galois closure groups of the local fields change. The whole point of the first part of this paper is to have a moduli space that keeps track of the isomorphism class of the local ramified fields. Since the Galois closure groups of the local fields are changing, this could mean the Galois closure groups of the covers in a family change too (even without moving the branch points). Apply Prop. 6.5 to stratify $S_{(j,i)\in I_{\boldsymbol{u}}}(\mathcal{R}_{j,i})$ and consider the case that the configuration map for the family goes into a single element of the stratification. Call such a family *local-Galois constant*.

Problem 6.9. Suppose a family is local-Galois constant. Does this imply the geometric Galois closures of the fibers in the family have constant group?

Problem 6.10. Use k-fold fiber products for the Galois closure construction, with $k \le n$ to get more more precise qualitative results telling exactly the effect of changes on the geometric fibers of the Galois closure of the family.

Arithmetic problems (over finite fields) motivate many researchers. Though most from these have little interest in a deformation approach, still such people might recognize the problems of classifying Schur (*exceptional*) covers and *Davenport pairs* ([Fr99] has a narrow historical, though broad mathematical view). Both problems demand knowledge of wildly ramified, rarely Galois, covers. Let $\phi : X \to \mathbb{P}^1$ be a cover appearing in either. Then, the serious phenomena occur when the field of definition of ϕ is one finite field k, while the definition field of the Galois closure of ϕ is a proper extension of k. We hope our techniques will allow formulating good conjectures predicting the dimension of families of covers that geometrically suit various arithmetic problems (like Schur and Davenport).

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