# REPORT ON "BIVARIATE POLYNOMIAL MAPPINGS ASSOCIATED WITH SIMPLE COMPLEX LIE ALGEBRAS,"JNT-D-16-00062, BY ÖMER KÜCÜKSAKALLI

## 1. What the paper is about

My summary statement §3 states that I think very well of this paper: in both its technical skill and its considerable imagination. Like many journals J. of Number Theory has many submissions. So I explain in §2 what will draw readers to this paper, which I refer to as [Kü16], expecting the author will also take the comments to heart.

1.1. **Literature Prelude.** A key to the paper is the definition of *integrable polynomial*

$$f : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n] :$$

Where there is

$$g : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n] \text{ with which } f \text{ compositionally commutes,}$$

and for which the collections of their iterates are disjoint.

The author has results relating the property of a polynomial being integrable to it being exceptional: $f : V \to W$ a cover (finite flat morphism of normal varieties, likely not étale), over a number field $K$, is exceptional if $f$ maps one-one on points over $\infty$-ly many residue class fields $\boldsymbol{p}$ of the ring of integers of $K$. We would have to have $V = W$ (but the cover has degree exceeding 1) to use the notion of integrability, in particular covering the one variable cases where $V = \mathbb{P}^1$ (or $\mathbb{A}^1$).

There is some literature the author doesn't seem to know: [Fr78], [GMS02] and [Fr05]. The author's publication [Kü14] likely is previously given in [Fr78, §2], for which more complete contexts are given in [GMS02] and [Fr05], under the rubric of the role of *exceptionality*. In this literature $V$ and $W$ are often, but not necessarily, projective varieties. In this paper they are both $\mathbb{A}^n$. The adjustments to go between them are easy. The case in [FL86, p. 142] is almost identical to the argument of [Kü16, §1] for the Lie algebras designated $A_n$, except the former uses projective varieties ($(\mathbb{P}^1)^n$ and $\mathbb{P}^n$, say in diagram (2.1)) instead of affine space.

From knowledge of these papers, one can see that integrability of a rational function is allied, but is not implied, by exceptionality. The converse, however, seems to have attracted the author. He limits himself to integrable polynomials associated to complex Lie Algebras $\mathcal{G}$, whose rank is $n = 2$. His polynomials, $\mathcal{P}_{\mathcal{G}} = \{P^k_{\mathcal{G}}(\boldsymbol{x})\}_{k=1}^{\infty}$, have the form of a formal group law stated in Thm. 0.1. Then, $P^k_{\mathcal{G}}(\boldsymbol{x}) \circ P^\ell_{\mathcal{G}}(\boldsymbol{x}) = P^{k\ell}_{\mathcal{G}}(\boldsymbol{x})$, making their integrability obvious.

1.2. **What he proves about exceptionality.** Consider a cover $f : V \to W$. Denote those primes $\boldsymbol{p}$ of the ring of integers $\mathcal{O}_K$ for which $f : V(\mathcal{O}/\boldsymbol{p}) \to W(\mathcal{O}/\boldsymbol{p})$ is one-one by $E'_{f,K}$. If $|E'_{f,K}| = \infty$, refer to $f$ as exceptional over $K$. There is a better set, $E_{f,K}$: primes $\boldsymbol{p}$ for which $f \mod \boldsymbol{p}$ is one-one on infinitely many *extensions* of the residue class field of $\boldsymbol{p}$. There are three points to keep in mind about exceptionality [Fr05, §3].

- (1.1a) $E'_{f,K} \setminus E_{f,K}$ is a finite set and whether $\boldsymbol{p} \in E_{f,K}$ is governed by the value of the Frobenius attached to $\boldsymbol{p}$ in the Galois closure of a *non-trivial* extension $L_{f,K}/K$.
- (1.1b) (1.1a) holds precisely in the following sense: to give a one-one map on the basis of the Chebotarev value is included under a notion called *monodromy precision* when it holds, as it does with exceptional covers [Fr05, Cor. 3.6]).
- (1.1c) The primes in $E_f$, even when $W = V = \mathbb{P}^1$ are rarely obvious, but $E_{f,K}$ will never be *almost all residue class fields*: infinitely many primes of $K$ missing from $E_{f,K}$).

When a prime degree 1-variable rational function $f$ (not given by a change of variable of cyclic or Chebychev polynomials) over $K$ is exceptional, then $E_f$ in (1.1c) consists of arithmetic progressions of primes ($K$ usually containing a complex quadratic extension of $\mathbb{Q}$) thanks to the *theory of complex multiplication*. When, however, an exceptional $f$ corresponds to multiplication by a prime $p$ ($p^2$ degree isogeny of elliptic curves without complex multiplication), $E_f$ does not consist of unions of arithmetic progressions of primes. The description of the primes of $E_f$ would amount to an extension of the $\mathrm{GL}_2$ case of Serre's open image theorem [Fr05, §6.1–6.3].

In [Kü16], $K = \mathbb{Q}$, and the field $L/\mathbb{Q}$ in question is a cyclotomic field. But there is nothing obvious about it since the author finds $E_f$ by a geometric process for each rank 2 lie algebra $\mathcal{G}$.

## 2. Properties of particular integrable polynomials

The following commutative diagram is intrinsic to [FL86]. It comes from the defining Chebychev polynomial relation $T_{1,k}(\frac{\sigma_i+\sigma_i^{-1}}{2}) = \frac{\sigma_i^k+\sigma_i^{-k}}{2}$. The author uses a change of variables $T_{k,1}(2x)/2 = D_k(x)$. Without the Chebychev part, it is tantemount to the similar observations of [FL86, p. 142] and [Kü16, §1] I alluded to above.

$$
\begin{array}{ccc}
\prod_{i=1}^{n+1}\mathbb{P}^1_{\sigma_i} & \xrightarrow{\nu_k = k\text{-th power}} & \prod_{i=1}^{n+1}\mathbb{P}^1_{u_i} \\
\big\downarrow \pi_{n+1} & & \big\downarrow \pi_{n+1} \\
\prod_{i=1}^{n+1}\mathbb{P}^1_{u_i} & \xrightarrow{D_{k,n+1}} & \prod_{i=1}^{n+1}\mathbb{P}^1_{u_i} \\
\big\downarrow \bmod S_{n+1} & & \big\downarrow \bmod S_{n+1} \\
\mathbb{P}^{n+1} & \xrightarrow{T_{n+1,k}} & \mathbb{P}^{n+1}
\end{array}
\tag{2.1}
$$

The horizontal line at the bottom is defined by the elementary symmetric function theorem. Since the functions that appear at the bottom of the right side are symmetric functions in the variables $\sigma_1+\sigma_1^{-1},\dots,\sigma_{n+1}+\sigma_{n+1}^{-1}$, their coordinates are given by a polynomial – denoted here by $T_{n+1,k}$ – in the elementary symmetric functions in $\sigma_1+\sigma_1^{-1},\dots,\sigma_{n+1}+\sigma_{n+1}^{-1}$. Label diagram (2.1) as $\mathcal{D}_k$. It makes sense to juxtapose $\mathcal{D}_{k_2}$ to the right of $\mathcal{D}_{k_1}$, into one large commutative diagram which we denote by $\mathcal{D}_{k_1}\cdot\mathcal{D}_{k_2}$.

**Corollary 2.1.** *Juxtapose $\mathcal{D}_{k_1}$ and $\mathcal{D}_{k_2}$ in the opposite order as $\mathcal{D}_{k_2}\cdot\mathcal{D}_{k_1}$. Conclude that*

$$T_{n+1,k_1}\circ T_{n+1,k_2} = T_{n+1,k_2}\circ T_{n+1,k_1} \text{ as a consequence of } x^{k_1\cdot k_2} = x^{k_2\cdot k_1}.$$

That is, for each fixed $n$, these Chebychev polynomial generalizations $T_{n+1,k}$ satisfy the integrable polynomial condition of the author. Now I will tie this in with what the author does.

(2.2a) [Kü16,§2] (resp. [Kü16,§3] computes explicitly using recursion relations from [Wi88] that the bivariate map $\mathcal{B}_k$ (resp. $\mathcal{G}_k$) for Lie Algebra $B_2$ (resp. $G_2$) fits in place of $T_{2,k}$ (resp. $T_{3,k}$).

(2.2b) Then, [Kü16, §2 and §3] go through extensive geometric arguments first computing the number of fixed points of his bivariate polynomials (resp. [Kü16, Thm. 2.3 and Thm. 3.4]) in both cases $k^2$.

(2.2c) Finally, those two sections conclude with when they give bivariate maps respectively on $(\mathbb{F}_q)^2$ and $(\mathbb{F}_q)^3$ : respectively, when $(k, q^4-1) = 1$ and $(k, q^6-1) = 1$.

Then, by comparing the primes of exceptionality for the $\mathcal{A}_n$ s with with those of $\mathcal{B}_k$ s he sees they don't match up for the conjecture of [LW72] to be correct.

Here are comments on (2.2).

(2.3a) Comment on (2.2a): He left off the $_k$ on $\mathcal{B}$ in the diagram on p. 5. He also says the commutativity follows from [Kü16, Thm. 0.1], but the argument below (2.1) does that.

(2.3b) Comment on (2.2b): This is the longest part of the paper. Still, it is part of the tradition of this area, for which he gives appropriate sources. It is interesting in its own right.

(2.3c) Comment on (2.2c): Despite what I say in (2.3b), the cyclotomic fixed points in (2.2b) and the final result in (2.2c) suggest that the field $L_{f,K}$, and so the description of $E_{f,K}$

where $f$ runs over the $T_{n+1,k}$ of (2.1) should fit as a natural generalization of the condition $(q^2-1,k)=1$ from [Fr70] quoted by [Kü16].

## 3. Summary Statement

I think the paper is excellent. If published, it should inspire others (or the author) to consider which of the integrable functions from Lie Algebras are naturally associated to the commutative diagram (2.1), without reliance on the functional equations of [Wi88]. Ditto for finding arguments detecting the primes of exceptionality attached to $\mathcal{B}_k$ (resp. $\mathcal{G}_k$) directly from its associated (cyclotomic) field denoted $L_{K,\mathcal{B}_k}$ (resp. $L_{K,\mathcal{G}_k}$) as in (1.1a).

Still, there is no reason the author should have to consider these topics at this moment. Again, I highly recommend publication.

## 4. Bibliography

### References


[Fr78] M. Fried, *Galois groups and Complex Multiplication*, T.A.M.S. **235** (1978) 141–162.

[FL86] ——— and R. Lidl, *On Dickson polynomials and Rédei Functions*, contributions to General Algebra **5**, Salzburg Conference, Mai 29–June 1, 1986, Stuffgart.

[Fr05] ———, *The place of exceptional covers among all diophantine relations*, J. Finite Fields 11 (2005) 367–433, arXiv:0910.3331v1 [math.NT]

[GMS02] R. Guralnick, P. Müller and J. Saxl, *The rational function analogue of a question of Schur and exceptionality of permutations representations*, Memoirs of the AMS 162–773 (2003), ISBN 0065-9266.

[LW72] R. Lidl, C. Wells, *Chebyshev polynomials in several variables*. J. Reine Angew. Math. **255** (1972), 104–111.

[Wi88] W. D. Withers, *Folding polynomials and their dynamics*, Amer. Math. Monthly **95** (1988), no. 5, 399–413.