

MODULI RELATIONS BETWEEN ℓ -ADIC REPRESENTATIONS AND THE REGULAR INVERSE GALOIS PROBLEM

MICHAEL D. FRIED

ABSTRACT. There are two famous Abel Theorems. Most well-known, is his description of “abelian (analytic) functions” on a one dimensional compact complex torus. The other collects together those complex tori, with their prime degree isogenies, into one space. Riemann’s generalization of the first features his famous Θ functions. His deepest work aimed at extending Abel’s second theorem; he died before he fulfilled this.

That extension is often pictured on complex higher dimension torii. For Riemann, though, it was to spaces of Jacobians of compact Riemann surfaces, W , of genus \mathbf{g} , toward studying the functions $\varphi : W \rightarrow \mathbb{P}_z^1$ on them. Data for such pairs (W, φ) starts with a monodromy group G and conjugacy classes \mathbf{C} in G . Many applications come from putting all such covers attached to (G, \mathbf{C}) in natural – *Hurwitz* – families.

We *connect* two such applications: The *Regular Inverse Galois Problem* (**RIGP**) and *Serre’s Open Image Theorem* (**OIT**). We call the connecting device *Modular Towers* (**MTs**). Backdrop for the **OIT** and **RIGP** uses Serre’s books [Se68] and [Se92]. Serre’s **OIT** example is the case where **MT** levels identify as modular curves.

With an example that isn’t modular curves, we explain conjectured **MT** properties – generalizing a Theorem of Hilbert’s – that would conclude an **OIT** for all **MTs**. Solutions of pieces on both ends of these connections are known in significant cases.

CONTENTS

1. Introduction	3
1.1. Preliminaries on the RIGP and the OIT	3
1.2. Key words and modular curve comparison	4
1.2.1. Hurwitz spaces	5
1.2.2. Towers of reduced Hurwitz spaces	5
1.2.3. Computations and the case $r = 4$	6
1.3. Beyond modular curves	7
1.3.1. Involution realizations of dihedral groups	7
1.3.2. An abstract for [Fr20]	8
1.3.3. Defining MTs	9
1.3.4. MTs ; the remaining sections	11
2. Changing from isogonies to sphere covers	13
2.1. Part I: Data for sphere covers	13

2.1.1.	Branch points and local monodromy	14
2.1.2.	Algebraic Geometry	15
2.1.3.	Analytic Geometry	16
2.2.	Part II: Braids and deforming covers	18
2.2.1.	Dragging a cover by its branch points	18
2.2.2.	Braids and equivalences	19
2.2.3.	Genus formula: $r = 4$	20
2.3.	Main Example	21
2.3.1.	A lift invariant	21
2.3.2.	A_4 , $r = 4$	21
2.3.3.	The sh -incidence matrix	22
3.	ℓ -Frattini covers and MT s	23
3.1.	Universal ℓ -Frattini covers	24
3.2.	Universal Frattini cover	25
3.3.	Using ${}_{\ell}\tilde{G}_{\text{ab}}$ and ${}^1_{\ell}G$	27
3.3.1.	${}_{\ell}\tilde{G}_{\text{ab}}$ lattice quotients	27
3.3.2.	Nielsen classes of ℓ -Frattini lattices	29
3.3.3.	Normal ℓ -Sylow and other cases	29
3.4.	Test for a nonempty MT	30
3.4.1.	The obstructed component criterion	30
3.4.2.	Obstructed A_n components	31
3.4.3.	Finishing Ex. 2.16	32
3.5.	Archetype of the ℓ -Frattini conjectures	33
4.	Monodromy and ℓ -adic representations	35
4.1.	ℓ -adic representations	35
4.1.1.	Geometric monodromy of a MT	35
4.1.2.	Using $G \mapsto G^{\text{n-lm}}$	36
4.2.	MT s and the OIT generalization	38
4.3.	Using eventually ℓ -Frattini	39
4.4.	RIGP and Faltings	39
4.4.1.	ℓ' RIGP	39
4.4.2.	GL_2 toughest point	40
5.	The path to M(odular)T(ower)s	41
5.1.	Historical motivations	42
5.1.1.	Guides	42
5.1.2.	Settings for the RIGP and acknowledgements	43
5.2.	What Gauss told Riemann about Abel's Theorem	44
5.2.1.	Early OIT and MT relations	45

5.2.2.	Competition between algebra and analysis	47
5.2.3.	Profinite: Frattini and Grothendieck	49
5.3.	The TimeLine of the MT program	51
5.3.1.	Organization	51
5.3.2.	Lessons from Dihedral groups Before '95	53
5.3.3.	Constructions and Main Conjectures from 1995 to 2004	61
5.3.4.	Progress on the MT conjectures and the OIT	66
	References	74

2000 Mathematics Subject Classification: Primary 11F32, 11G18, 11R58; Secondary 20B05, 20C25, 20D25, 20E18, 20F34.

Keywords and Phrases: Braid group, Serre's Open Image Theorem, Modular Towers, ℓ -adic representations, moduli of covers, Hurwitz spaces, Hilbert's Irreducibility, Regular Inverse Galois Problem, upper half-plane quotients, Falting's Theorem.

1. INTRODUCTION

Denote the Riemann sphere uniformized by an inhomogeneous variable z , as in a first course in complex variables, by $\mathbb{P}_z^1 = \mathbb{C} \dot{\cup} \{\infty\}$. Indeed, §2.1 actually starts there, and explains geometric monodromy groups from their contrasting algebraic and analytic approaches; ours is analytic.

§5.3 – The TimeLine of the **MT** program – is the most unusual in the paper. It is a list of extended abstracts of papers, showing how the material of the rest of the sections can fit together into a coherent program. Here is the format for historical references, as applied to $\bullet[\text{Se68}]\bullet$, indicating Serre's book. The reference year telegraphs that it is in §5.3.2, pre-1995 material. There a reader will find a \star -display, here $\star[\text{Se68}]\star$, elaborating on how Serre's book relates to our topics.

1.1. Preliminaries on the RIGP and the OIT. §1.2 ties together three acronyms, the subjects of this paper: **RIGP** (Regular Inverse Galois Problem), **OIT** (Open Image Theorem) and **MT** (Modular Towers). The inverse Galois problem asks whether any finite group G is the (Galois) group of some Galois field extension F/\mathbb{Q} . That is, G is a quotient of the absolute Galois group, $G_{\mathbb{Q}}$ of the rational numbers.

The *regular* (and much stronger) version replaces F/\mathbb{Q} with $F^*/\mathbb{Q}(z)$ where $F^* \cap \bar{\mathbb{Q}} = \mathbb{Q}$, with $\bar{\mathbb{Q}}$ the algebraic closure of \mathbb{Q} . Despite being stronger,

the regular version points to explicit spaces for a given G on which to look for \mathbb{Q} points giving **RIGP** solutions. Prop. 1.1, [FrV91, Main Thm], uses key definitions for how the *monodromy version* of the **RIGP** works:

Inner and absolute Nielsen classes (2.7); Hurwitz spaces
Princ. 2.7; fine moduli •[Sh64]•. An exposition expanding
Serre’s book on the **RIGP** already appeared in [Fr94].

Proposition 1.1. *For each Nielsen class $Ni(G, \mathbf{C})$ with inner classes having fine moduli, there is a variety $\mathcal{H}(G, \mathbf{C})^{\text{in}}$ whose K points – $\mathcal{H}(G, \mathbf{C})^{\text{in}}(K)$, K a number field – correspond to **RIGP** realizations over K in the Nielsen class •[Se92] and [Fr94]•.*

It has often succeeded – say, by using Thm. 2.10 as in the §2.3 example – in giving solutions. The connection is by applying Hilbert’s Irreducibility Theorem. Indeed, one view of our **OIT** generalization is that it aims at a very strong version of Hilbert’s Theorem (see §1.3.3).

Serre’s version of the **OIT** started in [Se68]. Each formulation – including Thm. 5.12 which considers a collection of modular curve towers, each corresponding to a prime ℓ – compares the arithmetic monodromy group of a j -line point in a tower with the geometric monodromy group of the tower. Serre’s case is the model. •[Fr78]• uses it in an exposition guiding what we can expect at this time from our conjectures.

This paper is a guide to an in-progress book [Fr20].¹ §1.3 gives an abstract of [Fr20], and then how subsections introduce **MTs** as formed from Hurwitz spaces and Frattini covers. §4 has a dictionary on *Nielsen classes* – descriptions of sphere covers – with their braid action, for quickly defining **MTs** and where the ℓ -adic representations come into this. Our running example, starting in §2.3 is explicit on these. By this paper’s end we see how the principle conjectures apply to it.

The several conjectures about **MTs** have a division between them: Some go with the **RIGP**, others with the **OIT**.

We label the main conjectures accordingly:
Main **RIGP** Conj. 5.8 and Main **OIT** Conj. 4.2.

1.2. Key words and modular curve comparison. Among finite groups G , excluding G nilpotent, there are a finite number of natural projective sequences of groups covering G for which a positive answer to a simple **RIGP** question produces **MTs** with particular diophantine properties. They carry ℓ -adic representations allowing a conjectural version of a diophantine statement generalizing Serre’s original **OIT**.

¹It is also an expansion of a previous conference proceedings article.

1.2.1. *Hurwitz spaces.* §2 starts with 1-variable rational functions (as in junior high). With these we introduce *Nielsen classes* attached to (G, \mathbf{C}) : with $\mathbf{C} = \{C_1, \dots, C_r\}$ a collection of $r = r_{\mathbf{C}}$ conjugacy classes in G . Denote the lcm of the orders of elements in \mathbf{C} by $N_{\mathbf{C}}$.

For nontriviality, we assume the collection of all elements in \mathbf{C} generate G . A natural *braid group action* on a Nielsen class gives us *Hurwitz spaces*: moduli spaces of covers of \mathbb{P}_z^1 .

Consider U_r , subsets of r distinct unordered elements $\{\mathbf{z}\} \subset \mathbb{P}_z^1$. It is also

$$(1.1) \quad \begin{array}{l} \text{projective } r\text{-space } \mathbb{P}^r \text{ minus its } \textit{discriminant locus } (D_r). \\ \text{If the Nielsen class is non-empty, then each Hurwitz} \\ \text{space component is an unramified cover of } U_r. \end{array}^2$$

Significantly, as in our examples, Hurwitz spaces can have more than one component. That distinguishes the tower of Hurwitz spaces from a **MT** lying on such a tower. Usually, however, components are either conjugate (under $G_{\mathbb{Q}}$ action) or they have moduli properties that distinguish them. Separating these two situations is a major issue with the **OIT**.

1.2.2. *Towers of reduced Hurwitz spaces.* In our examples we easily see when a collection of conjugacy classes is generating. It can be harder to decide for given $r = r_{\mathbf{C}}$ whether a Nielsen class is nonempty. In particular, for any prime ℓ dividing $|G|$, where G has no \mathbb{Z}/ℓ quotient, forming **MTs** starts with considering choices of r *generating* conjugacy classes, whose elements have ℓ' (prime to ℓ) order.

That gives a *canonical* tower of Hurwitz spaces over a base Hurwitz space (§3). Points on each level represent equivalence classes of covers of \mathbb{P}_z^1 , with monodromy group a cover of G and data given by \mathbf{C} .

Powers, ℓ^{k+1} , of ℓ reference general tower levels, \mathcal{H}_k , $k \geq 0$, all sharing similar *moduli* properties. Our emphasis is on relating the **OIT** and the **RIGP**. So, we usually – unless otherwise said – concentrate on these \mathbb{P}_z^1 cover equivalences §2.2.2:

$$(1.2) \quad \begin{array}{l} \mathcal{H}(G, \mathbf{C}) \text{ denotes } \textit{inner} \text{ classes; for comparing with modular} \\ \text{curves } \mathcal{H}(G, \mathbf{C})^{\text{rd}} \text{ denotes inner } \textit{reduced} \text{ classes.} \end{array}$$

We often speak of both types without extra decoration, except when it is crucial to include reduced equivalence. That comes from extending the $\text{SL}_2(\mathbb{C})$ action in (1.14) to include the Hurwitz space itself.

² \mathbb{P}^r is the symmetric product on \mathbb{P}_z^1 . The discriminant locus is all of the z_1, \dots, z_r with $z_i = z_j$ for some $i \neq j$.

1.2.3. *Computations and the case $r = 4$.* Reduced equivalence cuts down the (complex) dimension of the Hurwitz spaces by 3. For *all* 1-dimensional, reduced examples, the *normalization* of \mathbb{P}_j^1 in the function field of a reduced Hurwitz space component \mathcal{H}^{rd} , gives a unique projective (nonsingular) curve cover $\bar{\mathcal{H}}^{\text{rd}} \rightarrow \mathbb{P}_j^1$.³ Cusps on this space are, on one hand, the points over ∞ . Significantly for computations:

(1.3) we associate to a cusp an orbit in a *reduced Nielsen class* of a cusp group, Cu_4 , (1.15). Cusp widths are ramification indices over $j = \infty$. They are also lengths of the cusp orbits.

Each component of \mathcal{H}_k is an upper half-plane quotient ramified over $\mathbb{P}_j^1 \setminus \{\infty\}$ at 0, 1. Thm. 2.10 efficiently computes genres, cusps and their widths, of each compactified component.⁴

MTs generalize modular curve towers. As is well-known, modular curves are moduli spaces for elliptic curves and their torsion, defined by *congruence subgroups* of $\text{SL}_2(\mathbb{Z})$. Less well known, they are the image – under a map that is an isomorphism on the underlying spaces, losing none of the abelian variety data – of moduli for *certain* covers of \mathbb{P}_z^1 .

Our illustrating example is Prop. 2.15. Here G is the alternating group, A_4 , of degree 4. We use this for comparing with modular curves and illustrating the formulation of **MTs**, and braid and ℓ -adic actions.

For a given union of Hurwitz space components, $\mathcal{H}' \leq \mathcal{H}(G, \mathbf{C})$, we can often compute the following.

Definition 1.2. *Moduli definition field:* a minimal field extension $\mathbb{Q}_{\mathcal{H}'}$, of \mathbb{Q} contained in the field of definition of any object representing $\mathbf{p} \in \mathcal{H}'$.⁵

When, $\mathcal{H}' = \mathcal{H}(G, \mathbf{C})$, $\mathbb{Q}_{\mathcal{H}'} = \mathbb{Q}_{G, \mathbf{C}}$ is an explicit cyclotomic field by the *branch cycle lemma* (**BCL** Thm. 5.20).

When *fine moduli* holds, the residue class field $\mathbb{Q}_{\mathcal{H}'}(\mathbf{p})$ of \mathbf{p} is a field of definition of a representing object. Yes, unlike modular curves there can be – significantly – more than one component as in §2.3. There are cases with fine moduli for one component, but not the other; fine moduli for an inner space may be lost by using reduced equivalence.

³The same normalization process works for $r > 4$, but the reduced Hurwitz spaces may be singular, even though normal, and the target space is J_r (§1.14).

⁴In our examples beyond dihedral groups, congruence subgroups don't give these moduli space upper half-plane quotients.

⁵This is a field of definition of the whole structure space representing the moduli problem, including the collection of families of representing objects.

1.3. Beyond modular curves. §1.3.2 uses Serre's two books as a backdrop for the basic goal, relating the **RIGP** and the **OIT** by outlining [Fr20]. §1.3.3 shows how we introduce properties of **MTs**.

Before that, §1.3.1 shows how finding *involution realizations of dihedral groups* maps to finding cyclotomic points on hyperelliptic jacobians. The moduli problems slightly differ. Yet, this gives an isomorphism of the underlying spaces for significant **MTs** to hyperelliptic jacobians.

1.3.1. Involution realizations of dihedral groups. Many people study modular curves, and their higher dimensional variant, spaces of hyperelliptic jacobians. Hurwitz spaces are less known and more general.

As a prelude to Frattini ideas generalizing these classical problems, we connect this section's title to cyclotomic points on hyperelliptic jacobians.

Ex. 1.3, overlapping with Ex. 1.11, even before the formal definition of Hurwitz spaces, §2.2.2 introduces the (connected) levels of **MTs** that will help understand them in general.

Example 1.3. Consider $G_{k+1} = D_{\ell^{k+1}} = \mathbb{Z}/\ell^{k+1} \times^s \mathbb{Z}/2$ the order $2\ell^{k+1}$ dihedral group, with ℓ an odd prime.⁶; C is the involution conjugacy class of G_{k+1} . Consider the following two collections (formal definitions of the words appear in §2.1.3).

(1.4a) Connected covers $\varphi : W \rightarrow \mathbb{P}_z^1$ of degree ℓ^{k+1} ramified (only) at $r \geq 4$ (branch) points of \mathbb{P}_z^1 , with *branch cycles* elements of C .

(1.4b) Replace φ in (1.4a) by $\hat{\varphi} : \hat{W} \rightarrow \mathbb{P}_z^1$, Galois with group $D_{\ell^{k+1}}$ with its induced cover $\hat{W}/\langle h \rangle \rightarrow \mathbb{P}_z^1$, $h \in C$.

With $\mathbf{C} = \mathbf{C}_{2r}$ (r repetitions of C ; r must be even), the label for these collections as complex analytic spaces is $\mathcal{H}(D_{\ell^{k+1}}, \mathbf{C}_{2r})^{\text{abs}}$ (resp. $\mathcal{H}(D_{\ell^{k+1}}, \mathbf{C}_{2r})^{\text{in}}$) using the respective (2.15) equivalences (and Ex. 1.11). \triangle

Both spaces in (1.4) have fine moduli, and therefore, if K is a number field, a K point on either space produces a representing cover.

Definition 1.4. Call $\hat{\varphi}$ from (1.4b) an *involution realization* (C -realization) over a number field K , if $\hat{\varphi}$ corresponds to $\hat{\mathbf{p}} \in \mathcal{H}(D_{\ell^{k+1}}, \mathbf{C}_{2r})^{\text{in}}(K)$.

For inner Hurwitz spaces, where the representing object includes an isomorphism up to conjugacy with the Galois group, the following makes sense. Choose $\hat{\varphi}$ from (1.4b). Mod out by the normal subgroup U generated by an element of order ℓ^{k+1} in G_{k+1} .

⁶All primes eventually count; but here we stay with ℓ odd.

That gives an unramified cover $\hat{W} \rightarrow \hat{W}/U \stackrel{\text{def}}{=} X$; $\varphi_X : X \rightarrow \mathbb{P}_z^1$ is a hyperelliptic cover – of genus $\mathbf{g}_r = \frac{r-2}{2}$ – of \mathbb{P}_z^1 , ramified at r points. Denote the divisor classes of degree t on X by $\text{Pic}^t(X)$, and the ℓ^{k+1} torsion points on $\text{Pic}^0(X)$ by $T_{\ell^{k+1}}(X)$.

Denote a copy of \mathbb{Z}/ℓ^{k+1} on which $G_{\mathbb{Q}}$ acts as if it was the multiplicative group generated by $e^{2\pi i/\ell^{k+1}}$ by $\mu_{\ell^{k+1}}$. By replacing $G_{\mathbb{Q}}$ by G_K , K a number field, we can use the same notation. Prop. 1.5 is [DFr94, Lem. 5.3].

Proposition 1.5. *The set of involution realizations of $D_{\ell^{k+1}}$ over K associated to a fixed X injects into the G_K equivariant injections from $\mu_{\ell^{k+1}}$ into $T_{\ell^{k+1}}(X)$. It is onto, if in addition $\text{Pic}^1(X)$ has a K point.⁷*

Relating all algebraic points on both spaces requires considering all number fields K . Then, to get an isomorphism of the underlying spaces,

$$(1.5) \quad \begin{array}{l} \text{equivalence hyperelliptic covers } \varphi_X : X \rightarrow \mathbb{P}_z^1 \text{ and } \varphi_{X'} : X' \rightarrow \mathbb{P}_z^1 \\ \text{if they differ by composing } \varphi_X \text{ and } \varphi_{X'} \text{ by an } \alpha \in \text{PSL}_2(\mathbb{C}). \end{array}$$

This is reduced equivalence. Also, consider the crucial **RIGP** case, $K = \mathbb{Q}$. This says involution realizations of dihedral groups over \mathbb{Q} give *cyclotomic* points on hyperelliptic jacobians. Then, for a given X , if $\text{Pic}^1(X)$ is isomorphic to $\text{Pic}^0(X)$ over \mathbb{Q} , the converse is true.

Question 1.6. Fix ℓ and r . We ask these two questions for each $k \geq 0$.

(1.6a) Is there a C-realization $\hat{\varphi}_k : \hat{W}_k \rightarrow \mathbb{P}_z^1$ over \mathbb{Q} in $\text{Ni}(D_{\ell^{k+1}}, \mathbf{C}_{2^r})^{\text{in}}$?

(1.6b) Is there a $\mu_{\ell^{k+1}}$ point on a hyperelliptic jacobian of dimension \mathbf{g}_r ?

Prop. 1.5 says “Yes” to (1.6a) implies “Yes” to (1.6b).

(1.7) Impossibility of (1.6a) is the dihedral version of the Main
RIGP Conj. 5.8. Impossibility of (1.6b) is the hyperelliptic
version of the *Torsion Conjecture* 5.7 on abelian varieties.

•[FrK97]• and •[CaD08]• have a version that applies to any (G, ℓ) with ℓ -perfect (Def. 1.8) G . Further, the latter gives good reasons to explicitly relate Hurwitz spaces and classical spaces as does Quest. 1.6. That includes our **OIT** case in •[FrH20]•.

1.3.2. *An abstract for [Fr20].*

Monodromy, ℓ -adic representations and the Inverse Galois Problem

⁷The curve X naturally embeds in $\text{Pic}^1(X)$. To get a K embedding in $\text{Pic}^0(X)$ requires translation by a K divisor class of degree 1.

This book connects to, and extends, key themes in two of Serre’s books:

- (1.8a) *Topics in Galois theory* (the original and enhanced reviews [Fr94], and in French, translated by Pierre Dèbes [D95]); and
- (1.8b) *Abelian ℓ -adic representations and Elliptic Curves* (see the 1990 review by Ken Ribet [Ri90]).

Its theme is to relate ℓ -adic representations, as in generalizing Serre’s **OIT**, and the **RIGP** using **MTs**. Indeed, Galois theory/cohomology interprets many problems, not just the **RIGP**, along the way.

[Fr20, §4] explains **MTs** (started in 1995) as a program motivated by such a relationship. [Fr20, §1 and §2] includes exposition tying research threads prior to **MTs**. Especially, it recasts [FrV91] and [FrV92] to modernize investigating moduli definition fields of components of Hurwitz spaces vis-a-vis lift invariants with examples.

Then, we interpret the **OIT** in a generality not indicated by Serre’s approach. [Fr20, §5] uses one example – in that all modular curves are one example – clearly not of modular curves. This explains why our approach to (families of) covers of the projective line can handle a barrier noted by Grothendieck to generalizing the **OIT**. [Fr20, §3] joins the 1st and last 3rd of the book, in an approach to the lift invariant and Hurwitz space components based on the *Universal Frattini cover* of a finite group.

The *lift invariant* (Def. 2.13) is a tool that applies to many aspects of Hurwitz spaces. That includes giving distinguishing characteristics to components and cusps. For example:

- (1.9a) The criterion for nonempty **MTs** §3.4.
- (1.9b) Classifying cusps and components on which they lie; geometrically separating them for $G_{\mathbb{Q}}$ action (5.24) in •[Fr06] and [CaTa09] •.
- (1.9c) Using (1.9b) and §4.1 ℓ -adic representations, for labeling abelian variety collections a’ la the Torsion Conjecture •[CaD08] •.

There are still unsolved problems. Many, though, as listed in Rem. 5.25 on Thm. 5.24, have precise formulations and corroborating evidence.

1.3.3. *Defining MTs.* Ex. 1.3, even before §2.2 constructs Hurwitz spaces, gave an example on how the **RIGP** fits with properties of some classical spaces. In the same style we give ingredients from which we form **MTs**.

For a given finite group G , as with the **RIGP**, start with a set of generating conjugacy classes **C** (or a related collection of such).

Definition 1.7. Call a cover of groups, $\psi : H \rightarrow G$, *Frattini* if, given a subgroup $H^* \leq H$ with $\psi(H^*) = G$, then $H^* = H$. It is ℓ -Frattini if in addition the kernel is an ℓ -group.⁸

Definition 1.8. If $\ell || G|$, then G is ℓ -perfect if it has no \mathbb{Z}/ℓ quotient.

For the related **OIT**, we build on a (assumed nonempty) Hurwitz space $\mathcal{H}(G, \mathbf{C})^{\text{in}}$ formed in §2.2. Initial ingredients for forming **MTs**:

$$(1.10) \quad \begin{array}{l} \text{a prime } \ell || G|, \text{ as in §3.1, for which} \\ \text{elements of } \mathbf{C} \text{ are } \ell', \text{ and } G \text{ is } \ell\text{-perfect.} \end{array} \quad 9$$

Then, as in §3.3, there are a *finite number* (with indexing set I) of projective sequences $\{ {}_i G_k \}_{k=0}^{\infty}$, $i \in I$, ${}_i G_0 = G$, with limit ${}_i \tilde{G} = \lim_{\infty \leftarrow k} {}_i G_k$ having the following properties.

- (1.11a) ${}_i \tilde{\psi} : {}_i \tilde{G} \rightarrow G$ is an ℓ -Frattini cover; and
- (1.11b) $\ker({}_i \tilde{\psi})$ is a $\mathbb{Z}_{\ell}[G]$ module, free of finite \mathbb{Z}_{ℓ} rank ${}_i m$; with
- (1.11c) ${}_i m$ the same as the vector space dimension of the common $\mathbb{Z}/\ell[G]$ module $\ker({}_i G_{k+1} \rightarrow G) / \ker({}_i G_k \rightarrow G)$, $k \geq 0$.

There is an $i_{\text{mx}} \in I$ for which

$$\text{each } {}_i \tilde{\psi} : {}_i \tilde{G} \rightarrow G \text{ is a quotient of } {}_{i_{\text{mx}}} \tilde{\psi} : {}_{i_{\text{mx}}} \tilde{G} \rightarrow G.$$

§3.3 denotes ${}_{i_{\text{mx}}} \tilde{G}$ by ${}_{\ell} \tilde{G}_{\text{ab}}$; ${}_{i_{\text{mx}}} m$ is a characteristic number attached to (G, ℓ) . For $i \in I$, form a tower of Hurwitz spaces ${}_i \mathbb{H} = \{ \mathcal{H}({}_i G_k, \mathbf{C})^{\text{in}} \}_{k=0}^{\infty}$.

For a given i , regard a component \mathcal{H}' associated to the Nielsen class braid orbit \mathcal{O}' on $\mathcal{H}({}_i G_k, \mathbf{C})^{\text{in}}$ as a level k vertex of a graph. Attach an arrow from \mathcal{H}' to $\mathcal{H}'' \leftrightarrow \mathcal{O}''$ on $\mathcal{H}({}_i G_{k+1}, \mathbf{C})^{\text{in}}$ if the homomorphism ${}_i G_{k+1} \rightarrow {}_i G_k$ induces a map (surjection) $\mathcal{O}'' \rightarrow \mathcal{O}'$.

Components form a *component tree* on ${}_i \mathbb{H}$.

Definition 1.9. A **MT** for $i \in I$ is a (nonempty) projective system of absolutely irreducible components on ${}_i \mathbb{H}$ with a unique vertex on each level of the ${}_i \mathbb{H}$ component tree.

§3.4 gives an if and only if criterion for ${}_i \mathbb{H}$ to be nonempty (at all levels).

Lemma 1.10. *From the Tychonoff Theorem,¹⁰ conclude that if ${}_i \mathbb{H}$ passes this criterion, there is at least one **MT** on ${}_i \mathbb{H}$.*

⁸Called a Frattini p -cover in [FrJ86, Rem. 22.11.9]₂.

⁹Dropping this ℓ' assumption is sometimes important, but we leave that out here.

¹⁰This is as applied to using Falting's Theorem as in §4.4.2.

Ex. 3.4.1 lists cases explaining such phenomena as in [Se92, Chap. 9].

For $i \in I$, the notation shows \mathbf{C} interprets meaningfully in ${}_iG_k$, $k \geq 0$. Given (G, ℓ, \mathbf{C}, i) , there is a related Nielsen limit class:

$$(1.12) \quad \begin{array}{l} \text{Ni}(G^{\text{n-lm}}, \mathbf{C}) \text{ referring to the } \textit{Nielsen limit} \text{ group,} \\ G^{\text{n-lm}}, \text{ giving a final step in the } \mathbf{OIT}. \end{array}$$

Again: In the direction $(G, \mathbf{C}) \Rightarrow (G^{\text{n-lm}}, \mathbf{C})$, \mathbf{C} didn't change. We explain how $G^{\text{n-lm}}$ works in §4.1.2, comparing with its arising in Serre's \mathbf{OIT} as in Ex. 1.11 and in Rem. 5.26 for our example.¹¹

Example 1.11 (Modular curves). The \mathbf{MT} hyperelliptic jacobian case in Ex.1.3 for $r = 4$ is the isomorphism of the levels in the respective two sequences $\{\mathcal{H}(D_{\ell^{k+1}}, \mathbf{C}_{2^4})^{\text{in,rd}}\}_{k=0}^\infty$ and $\{X_1^0(\ell^{k+1})\}_{k=0}^\infty$. The superscript 0 denotes restricting the classical modular curve X_1 to $\mathbb{P}_j^1 \setminus \{\infty\}$.

The corresponding $G^{\text{n-lm}}$ in (1.12) is $(\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/2$ [Fr20, Chap. 6 §3]. The modular curves $X_0(\ell^{k+1})$ appear in a similar sequence, with those spaces identified with $\{\mathcal{H}(D_{\ell^{k+1}}, \mathbf{C}_{2^4})^{\text{abs,rd}}\}_{k=0}^\infty$,¹² including the variant for the general case of even r in Ex. 1.3. \triangle

1.3.4. \mathbf{MT} s; *the remaining sections*. \mathbf{MT} levels ($k \geq 0$) are moduli of \mathbb{P}_z^1 covers, up to precise equivalences §2.2.2. Computations use an *automatic braid action* on Nielsen classes §2.2, starting with *Orbit principle* (1.13).

$$(1.13) \quad \begin{array}{l} \text{Absolutely irreducible components of a Hurwitz space} \leftrightarrow \\ \text{braid } (\S 2.2) \text{ orbits on the corresponding Nielsen class } \text{Ni}(G, \mathbf{C}). \end{array}$$

The §2.3 example shows precisely how computations work, to interpret the major problems. §3 defines the moduli spaces of the levels of a \mathbf{MT} for a given (G, \mathbf{C}) , starting with defining the *Universal Frattini cover*, \tilde{G} , of G and \tilde{G}_{ab} , its *abelianized* version. This produces a tower of Hurwitz spaces (from Princ. 2.7) in the notation of (4.7).

Def. 3.14 gives the key – *eventually ℓ -Frattini* – for formulating the \mathbf{OIT} conjecture, Conj. 4.2. §4.3 explains why that generalizes Serre's \mathbf{OIT} for modular curves. Also, its resemblance to a grand version of *Hilbert's Irreducibility Theorem*.

Basic assumptions. The Hurwitz spaces, $\mathcal{H}(G, \mathbf{C})$, will be reduced versions of inner equivalence classes of covers in the Nielsen class. Denote projective r -space by \mathbb{P}^r . For a given \mathbf{MT} its whole system of tower levels covers a natural configuration space.

¹¹Often both the \mathbf{RIGP} and the \mathbf{OIT} consider reduced Nielsen classes. For one, that suits the moduli problems as in Ex.1.3.

¹²Here $N_{S_{\ell^{k+1}}, \mathbf{C}}(D_{\ell^{k+1}}) = \mathbb{Z}/\ell^{k+1} \times^s (\mathbb{Z}/\ell^{k+1})^*$.

- (1.14) Inner space target: $U_r \stackrel{\text{def}}{=} \mathbb{P}^r$ minus its *discriminant locus*.
 Add reduced equivalence: the target space is $U_r/\text{SL}_2(\mathbb{C}) \stackrel{\text{def}}{=} J_r$.

Then, J_4 is the classical j -line, minus $\{\infty\}$. We put $\{\infty\}$ back whenever we need it. §2.3 has one of the collection of examples with results toward the main conjecture of this paper. All examples start with assumptions about (G, \mathbf{C}) – using the **BCL** – to assure any $G_{\mathbb{Q}}$ orbit of points over $\bar{\mathbb{Q}}$ on a Hurwitz space remains on the Hurwitz space.

In addition to Orbit Principle (1.13), we can *usually* give geometric names to $G_{\mathbb{Q}}$ orbits of absolutely irreducible components on the levels of a **MT** (starting with level 0) based on Cusp Principle (1.15). This uses a subgroup – the cusp group Def. 2.9 – of the braid group.

- (1.15) Cusp group orbits on a braid orbit \leftrightarrow cusps on the component;
 the form of cusp orbit elements distinguish braid orbits (5.24).

Typically we can geometrically distinguish different components in a Nielsen class, corresponding to additional moduli properties recognized by $G_{\mathbb{Q}}$.

§4.4.1 is on a theme – ℓ' **RIGP** applied to a finite group G . This generalizes for each G how Ex. 1.3 connects dihedral groups and cyclotomic points on hyperelliptic Jacobians.

§4.4.2 is on the difficulties the **RIGP** causes for the **OIT**. Especially around the hardest case in Serre’s **OIT** requiring Falting’s Theorem for its completion. That was many years after Serre’s program started.

The modular curve $X_1(\ell) \rightarrow \mathbb{P}_j^1$, attached to $(D_\ell, \mathbf{C}_{2^4}, \ell)$, with one component, includes cusps we call *Harbater-Mumford* (**HM**; Def. 5.16). So, we label the unique component as **HM**. Other types of distinguishing cusps appear as (G, \mathbf{C}) changes, as in our examples.

§5 gives background, motivation and the conjectures driving [Fr20]. §5.1.1 shows how the modular curve case expanded beyond dihedral groups.

§5.2 lists a series of problems with a large literature outside the considerations of modular curves that suddenly had a connection to them using parts of Serre’s version of the **OIT**.

§5.3 goes through three phases of the **MT** timeline. This led to the main conjectures by which the **RIGP** formulates entirely in terms of Hurwitz spaces, finally tying to a general version of the **OIT**.

The history: §5.3.2 is prior to 1995, §5.3.3 goes to 2004 with the main **RIGP** conjectures on **MTs**. Then the progress on cusps, and Frattini ideas led to connecting the **RIGP** to the **OIT**.

§5.3.4 discusses papers in the period of formulating the generalization of the **OIT**. This continues Ex. 2.3, as Thm. 5.24, describing level 0 of a **MT** beyond Serre's **OIT**. This example is of **MT** levels for all primes ℓ where the number of components increases both with ℓ and, for each ℓ , with the level k . Yet the *tree structure* (Def. 1.9) of those components remains coherent from level to level based on the lift invariant.

ℓ -Frattini covers arises in two profinite ways. Consider a **MT**, $\mathbb{H} = \{\mathcal{H}_k \rightarrow J_r\}_{k=0}^\infty$: a projective sequence of absolutely irreducible Hurwitz space components covering J_r , attached to (G, \mathbf{C}, ℓ) , $r = r_{\mathbf{C}}$ (as in §1.3.3) satisfying c(1.10).

- (1.16a) The construction of \mathbb{H} comes from the universal abelianized ℓ -Frattini cover, ${}_\ell G_{\text{ab}}$, of G .
- (1.16b) The guiding **MT** conjecture is that the sequence of geometric monodromy groups of \mathbb{H} is eventually ℓ -Frattini.

2. CHANGING FROM ISOGONIES TO SPHERE COVERS

§2.1 goes from rational function covers of \mathbb{P}_z^1 to any covers of the sphere to produce all we need. The starting objects are *Nielsen classes* attached to (G, \mathbf{C}) with G a finite group, and $r = r_{\mathbf{C}}$ conjugacy classes, \mathbf{C} , in G . Thereby we introduce the basic moduli – reduced *Hurwitz* – spaces, $\mathcal{H}(G, \mathbf{C})^{\text{rd}}$.

MTs generalizes modular curves as in Ex. 1.3, starting from essentially any finite group G and prime ℓ for which G is ℓ -perfect (Def. 1.8). This dihedral case is discussed in \bullet [DFr94] \bullet and surveyed in [Fr94] (reprinted in several languages by Serre).

That generalization starts with an **RIGP** statement with no reference to **MTs**. Yet, it forces the existence of **MTs** using a diophantine conjecture (no rational points at high levels), that generalizes to **MTs** (Main **RIGP** Conj. 5.8) that there are no points on high modular curve levels.

We can compute properties of the tower levels from assiduous use of an Artin braid group quotient: the *Hurwitz monodromy group* H_r (and its significant subgroups). §2.2 defines **MTs**.

§2.3 is an explicit example on using Nielsen classes, labeling cusps, and computing genres using Thm. 2.10, as alluded to in §1.2.3 (when $r = 4$) for computing properties of reduced Hurwitz spaces as covers of \mathbb{P}_j^1 .

2.1. Part I: Data for sphere covers. Here is notation for a rational function $f \in \mathbb{C}(w)$: $f : \mathbb{P}_w^1 \rightarrow \mathbb{P}_z^1 : w \mapsto f(w) = z$:

$$f = f_1(w)/f_2(w), \text{ with } (f_1, f_2) = 1, n = \deg(f) = \max(\deg(f_1), \deg(f_2)).$$

For example: $\frac{w^3+1}{w^5-3}$ has degree $\deg(f) = 5$.

2.1.1. *Branch points and local monodromy.* Branch points are places z' where the distinct $w' \mapsto z'$, $w'_1, \dots, w'_{t_{z'}}$, have cardinality $t_{z'} < n$. Take $z' = 0$. For simplicity assume

no $w' = \infty$ ($\deg(f_1) \geq \deg(f_2)$), and the f_i s have leading coefficient 1.

We write solutions, w , of $f(w) = z$, as analytic functions in a recognizable variable. For $1 \leq k \leq t$, write $f(w) = (w - w'_k)^{e_k} m_k(w)$ with $m_k(w'_k) \neq 0$. Form an expression in the variable $z^{1/e_k} = u_k(z)$:

$$w_k(z^{1/e_k}) \stackrel{\text{def}}{=} w'_k + u_k(z) + a_2 u_k(z)^2 + a_3 u_k(z)^3 \cdots = w_k(u_k(z)), k = 1, \dots, t_{z'}.$$

Substitute $w_k(z^{1/e_k}) \mapsto w$ in $f(w) = z$. Look at leading powers of $u_k(z)$ on the left and on the right. They are equal. Now solve inductively for a_2, a_3, \dots , so the left side is identically equal to z . Easily verify these.

(2.1a) The result for $w_k(z^{1/e_k})$ is analytic in a neighborhood of $u_k(z) = 0$.

(2.1b) With $\zeta_m = e^{2\pi\sqrt{-1}/m}$, substitution(s) $u_k(z) \mapsto \zeta_{e_k}^j u_k(z)$,

$j = 1, \dots, e_k$, give e_k distinct solutions $w \in \mathbb{C}((u_k(z)))/\mathbb{C}((z))$.

Take $\bar{e} \stackrel{\text{def}}{=} \bar{e}_{z'} = \text{lcm}(e_1, \dots, e_{t_{z'}})$. Now, put those solutions together.

Write all $w_k(\zeta_{e_k}^j z^{1/e_k})$ s, $k = 1, \dots, t_{z'}$, as power series in $z^{1/\bar{e}}$:

Substitute the obvious power of $z^{1/\bar{e}}$ for each $u_k(z)$.

This gives n distinct solutions, $L_{z'}$, of $f(w) = z$ in the field $\mathbb{C}((z^{1/\bar{e}}))$ and a natural permutation on $L_{z'}$ from the substitution

$$(2.2) \quad \hat{g}_{z'} : z^{1/\bar{e}} \mapsto e^{2\pi i/\bar{e}} z^{1/\bar{e}}.$$

This gives an element in S_n (the identify, if all e_k s are 1), the symmetric group, on the letters $L_{z'}$.¹³

Do this for each branch point, z'_1, \dots, z'_r , to get $\mathbf{g} \stackrel{\text{def}}{=} (g_1, \dots, g_r)$.

(2.3a) How can we compare entries of \mathbf{g} , by having them all act on one set of symbols¹⁴ rather than on r different sets, L_{z_1}, \dots, L_{z_r} ?

(2.3b) With success on (2.3a), denote the group $\langle \mathbf{g} \rangle$ the \mathbf{g} generate by G_f . Of what use is it and \mathbf{g} ?

(2.3c) Was there anything significant about using rational functions?

¹³It abuses notation by still having $z' = 0$, but we are about to drop that by using z_1, \dots, z_r for a labeling of the branch points.

¹⁴When it is convenient we will take these symbols to be $1, \dots, n$.

Below I answer questions (2.3). In §2.4, with those answers, as a preliminary, we extend to do the following.

(2.4) Invert this, to get $\mathbf{g} \mapsto f = f_{\mathbf{g}}$, regardless of the branch points.¹⁵

I produce G_f , up to isomorphism as a subgroup of S_n , answering (2.3a), first using *Algebraic Geometry*, then using *Analytic Geometry*.

2.1.2. *Algebraic Geometry.* Take any cover, $\varphi : W \rightarrow \mathbb{P}_z^1$, not necessarily rational given by f above. We introduce a compact Riemann surface cover, $\hat{\varphi} : \hat{W} \rightarrow \mathbb{P}_z^1$, the *Galois closure* of φ , minimal with these properties:

(2.5a) $\hat{\varphi}$ factors through φ ; and

(2.5b) it is a *Galois cover* of \mathbb{P}_z^1 .

The group of those automorphisms is G_{φ} . The phrase *Galois cover* in (2.5b) means having $\deg(\hat{\varphi})$ automorphisms commuting with $\hat{\varphi}$.

Form the fiber product of φ , $\deg(\varphi) = n$ times (assume $n > 1$):

$$(\mathbb{P}_w^1)^{(n)}_{\varphi} \stackrel{\text{def}}{=} \{(w_1, \dots, w_n) \in (\mathbb{P}_w^1)^n \mid \varphi(w_1) = \dots = \varphi(w_n)\} \text{ over } \mathbb{P}_z^1.$$

The resulting object is *singular* if two coordinates, w'_k, w'_l , lying over the same branch point z_i , have both $e_k > 1$ and $e_l > 1$. To see this, project onto the (k, l) coordinates through the point $(w'_k, w'_l) \in \mathbb{P}_w^1 \times_{\mathbb{P}_z^1} \mathbb{P}_w^1$. Around (w'_k, w'_l) , this space (with its map φ) is locally analytically isomorphic to

$$\{(w_k, w_l) \in D_{w'_k=0} \times_{D_{z'=0}} D_{w'_l=0} \mid w_k^{e_k} - w_l^{e_l} = 0\} \rightarrow D_{z'=0}.$$

The *Jacobian criterion* reveals the singularity at $(0, 0)$: both partials $\frac{\partial}{\partial w_k}, \frac{\partial}{\partial w_l}$ of $w_k^{e_k} - w_l^{e_l}$ are 0 at $(0, 0)$. Indeed, normalizing this as a cover of $D_{z=0}$ results in $\gcd(e_k, e_l)$ copies of the cover $w \mapsto w^{\text{lcm}(e_k, e_l)}$.¹⁶

Denote the normalization by $*W_{\varphi}^{\{n\}}$. The normalization may have several components. One for certain is the *fat diagonal*,

$$\Delta_{\varphi}^{\{n\}} = \text{closure of the locus where 2 or more of those } w_i \text{ s are equal.}$$

Remove the components of $\Delta_{\varphi}^{\{n\}}$. On the result, there is a natural action of S_n , by permuting those distinct w_i s, that extends to the whole normalized (since 1-dimensional, nonsingular) $*W_{\varphi}^{\{n\}}$.

If $*W_{\varphi}^{\{n\}}$ is irreducible, then it is a Galois over \mathbb{P}_z^1 with group S_n . If it is *not* irreducible, consider a component, $\tilde{\varphi} : \tilde{W}_{\varphi} \rightarrow \mathbb{P}_z^1$. Then,

$$G_{\varphi} = \{g \in S_n \mid g \text{ preserves } \tilde{W}_{\varphi}\}, \text{ geometric monodromy of } \varphi: |G_{\varphi}| = \deg(\tilde{\varphi}).$$

¹⁵Even if we restricted to rational functions, this inversion is nontrivial.

¹⁶A disk, $D_{z'=0}$, around $z' = 0$, is a convenient open set, as we see in §2.2. Technically, that means someone has selected a metric on, say, \mathbb{P}_z^1 .

Denote the conjugacy class of $g \in G_\varphi$ by C_g . Though $\mathbf{g} = (g_1, \dots, g_r)$ still depends on how we labeled points over branch points, this approach does define the conjugacy classes C_{g_i} , $i = 1, \dots, r$, in G_φ as follows. Consider maps of the function field of W_φ into $\mathbb{C}(((z - z_i)^{1/\bar{e}_i}))$ fixed on $\mathbb{C}((z - z_i))$ and restrict the automorphism \hat{g}_{z_i} of (2.2) to the image of that embedding.

For many purposes, this construction is inadequate to that of §2.1.3. Since, however, it is algebraic, it gives another group we need.

Definition 2.1. If the cover φ is defined over a field K , then $\hat{G}_{\varphi, K} \stackrel{\text{def}}{=} \hat{G}_\varphi$, the *arithmetic monodromy* of φ (over K), is defined exactly as above, except take $\hat{\varphi} : \hat{W} \rightarrow \mathbb{P}_z^1$ to be a component defined over K .

That is, if we started with \tilde{W} , a geometric component, then we would take for $\hat{\varphi} : \hat{W} \rightarrow \mathbb{P}_z^1$, the union of the conjugates $\tilde{\varphi}^\sigma : \tilde{W}^\sigma \rightarrow \mathbb{P}_z^1$, $\sigma \in G_K$. This would be defined and irreducible over K .

2.1.3. Analytic Geometry. For $g \in S_n$ with t disjoint cycles, $\text{ind}(g) = n - t$ is its *index*. For \mathbf{C} , r conjugacy classes – some may be repeated, count them with multiplicity – in a group G , use $\mathbf{g} \in G^r \cap \mathbf{C}$ to mean an r -tuple \mathbf{g} has entries in *some* order (with correct multiplicity) in \mathbf{C} . We denote the group the entries generate by $\langle \mathbf{g} \rangle$.

Example 2.2. If $G = S_4$, and $\mathbf{C} = C_2 C_3$ consists of two repetitions each of the class, C_2 , of 2-cycles, and the class, C_3 , of 3-cycles, then both

$${}_1\mathbf{g} = ((12), (234), (34), (134)) \text{ and } {}_2\mathbf{g} = ((234), (13), (13), (324))$$

are in $(S_4)^4 \cap \mathbf{C}$. For both (2.7a) holds; only for ${}_2\mathbf{g}$ does (2.7b) hold. \triangle

Our next approach to the Galois closure, based on Thm. 2.3, gives us a better chance to answer questions (2.3a)-(2.3c). To simplify notation, unless otherwise said, always make these two assumptions.

(2.6a) Conjugacy classes, $\mathbf{C} = \{C_1, \dots, C_r\}$, in G are *generating*.

(2.6b) G is given as a transitive subgroup of S_n .

Meanings: (2.6a) \implies the full collection of elements in \mathbf{C} generates G ; and (2.6b) \implies the cover generated by \mathbf{g} is connected. Even with (2.6a), it may be nontrivial to decide if there is $\mathbf{g} \in G^r \cap \mathbf{C}$ that generates.

Thm.2.3 is a version of *Riemann's Existence Theorem* (RET), done in detail in [Fr80]. A less complete version is in [Vo96].

Theorem 2.3. Assume $\mathbf{z} \stackrel{\text{def}}{=} z_1, \dots, z_r \in \mathbb{P}_z^1$ distinct. Then, some degree n cover $\varphi : W \rightarrow \mathbb{P}_z^1$ with branch points \mathbf{z} , and $G = G_\varphi \leq S_n$ produces classes \mathbf{C} in G , if and only if there is $\mathbf{g} \in G^r \cap \mathbf{C}$ with these properties:

(2.7a) $\langle \mathbf{g} \rangle = G$ (generation); and

(2.7b) $\prod_{i=1}^r g_i = 1$ (product-one).

Indeed, r -tuples satisfying (2.7) give all possible Riemann surface covers – both up to equivalence (see §2.2.2) – with these properties.

Refer to one of those covers attached to \mathbf{g} as $\varphi_{\mathbf{g}} : W_{\mathbf{g}} \rightarrow \mathbb{P}_z^1$.

(2.8) The genus $\mathbf{g}_{\mathbf{g}}$ of $W_{\mathbf{g}}$ appears in $2(\deg(\varphi) + \mathbf{g}_{\mathbf{g}} - 1) = \sum_{i=1}^r \text{ind}(g_i)$.

The set of \mathbf{g} satisfying (2.7) are the Nielsen classes associated to (G, \mathbf{C}) . A Riemann surface $W_{\mathbf{g}}$ is isomorphic to \mathbb{P}_w^1 over \mathbb{C} if and only if formula (2.8) – Riemann-Hurwitz – gives $\mathbf{g}_{\mathbf{g}} = 0$.

Denote $\mathbb{P}_z^1 \setminus \{\mathbf{z}\}$ as $U_{\mathbf{z}}$ and choose $z_0 \in U_{\mathbf{z}}$. Thm. 2.3 follows from existence of classical generators of $\pi_1(U_{\mathbf{z}}, z_0)$. These are paths $\mathcal{P} = \{P_1, \dots, P_r\}$ on $U_{\mathbf{z}}$ based at z_0 , of form $\lambda_i \circ \rho_i \circ \lambda_i^{-1}$ with these properties.

(2.9a) ρ_i s are non-intersecting clockwise loops around the respective z_i s.

(2.9b) The λ_i s go from z_0 to a point on ρ_i .

(2.9c) Otherwise there are no other intersections.

(2.9d) The $\lambda_1, \dots, \lambda_r$ emanate clockwise from z_0 .

Suppose a cover, $\varphi : W \rightarrow \mathbb{P}_z^1$, has a labeling of the fiber w_1^*, \dots, w_n^* over z_0 . Then, analytic continuation of a lift, $P_{k,i}^*$, of P_i , starting at w_k^* will end at a point, say $w_{(k)g_i}^*$, on $k \in \{1, \dots, n\}$.

This produces the permutations g_1, \dots, g_r satisfying (2.7). This results from knowing (2.9) implies P_1, \dots, P_r are generators of $\pi(U_{\mathbf{z}}, z_0)$, and

(2.10) they have product 1 and *no other relations*.

From (2.10), mapping $P_i \mapsto g_i$, $i = 1, \dots, r$, produces a permutation representation $\pi(U_{\mathbf{z}}, z_0) \rightarrow G \leq S_n$.

From the theory of the fundamental group, this gives a degree n cover $\varphi^0 : W^0 \rightarrow U_{\mathbf{z}}$. Completing the converse to Thm. 2.3 is not immediate. You must fill in the holes in φ_0 to get the desired $\varphi : W \rightarrow \mathbb{P}_z^1$. A full proof, starting from [Ahl79], is documented in [Fr80, Chap. 4].

Definition 2.4. Given (G, \mathbf{C}) , the set of \mathbf{g} satisfying (2.7) is the *Nielsen class* $\text{Ni}(G, \mathbf{C})^\dagger$, with \dagger indicating an equivalence relation referencing the permutation representation $T : G \rightarrow S_n$.

A cover doesn't include an ordering of its branch points. Adding such would destroy most applications number theorists care about. This makes sense of saying *a cover is in the Nielsen class* $\text{Ni}(G, \mathbf{C})^\dagger$.

Remark 2.5 (Permutation notation). Our usual assumptions start with a faithful transitive permutation representation $T : G \rightarrow S_n$ with generating conjugacy classes \mathbf{C} , from which we may define a Nielsen class $\text{Ni}(G, \mathbf{C})^\dagger$. Except in special cases, the notation for T indicates the symbols on which elements $T(g)$, $g \in G$, acts are $\{1, \dots, n\}$. Example: Denote the stabilizer of the symbol 1 by $G(T, 1)$ or just $G(1)$ if T is understood.

Or, if T comes from a cover $\varphi : W \rightarrow \mathbb{P}_z^1$, denote the permutation representation by T_φ even when applied to \hat{G}_φ . Then, denote the group of \hat{W}/W by $\hat{G}_\varphi(1)$. That indicates it is the subgroup stabilizing the integer 1 in the representation, and $G_\varphi(1) = \hat{G}_\varphi(1) \cap G_\varphi$.

2.2. Part II: Braids and deforming covers. Take a basepoint ${}_0\mathbf{z}$ of U_r , and denote $\pi_1(U_r, {}_0\mathbf{z})$, the *Hurwitz monodromy group*,¹⁷ by H_r . A *Hurwitz space* is a cover of U_r that parametrizes all covers in a Nielsen class. §2.2.1 explains how it comes from a representation of H_r on a Nielsen class.

§2.2.2 lists the equivalences we use, and then the action of H_r on Nielsen classes that defines Hurwitz spaces. Then, §2.2.3 gives the formula for computing the genus of reduced Hurwitz spaces when $r = 4$.

2.2.1. Dragging a cover by its branch points. Here is the way to think of forming a Hurwitz space. Start with ${}_0\varphi : {}_0W \rightarrow \mathbb{P}_z^1$, with branch points ${}_0\mathbf{z}$, classical generators ${}_0\mathcal{P}$ and (branch cycles) ${}_0\mathbf{g} \in \text{Ni}(G, \mathbf{C})$.

Drag ${}_0\mathbf{z}$ and ${}_0\mathcal{P}$, respectively, to ${}_1\mathbf{z}$ and ${}_1\mathcal{P}$ along any path B in U_r . With no further choices, ${}_t\mathcal{P} \mapsto {}_0\mathbf{g}$ by ${}_tP_i \mapsto g_i$, $i = 1, \dots, r$, forms a trail of covers ${}_t\varphi : {}_tW \rightarrow \mathbb{P}_z^1$, $t \in [0, 1]$, with respect to the same ${}_0\mathbf{g}$ along the path indicated by the parameter.

This produces a collection of \mathbb{P}_z^1 covers of cardinality $|\text{Ni}(G, \mathbf{C})^\dagger|$ over each $\mathbf{z} \in U_r$. This forces upon us a decision: when to identify two covers as equivalent. For B closed, denote the homotopy class $[B]$ as $q_B \in H_r$.

Principle 2.6. *For B a closed path, we can identify branch cycles ${}_1\mathbf{g}$ for the cover ${}_1f : {}_1W \rightarrow \mathbb{P}_z^1$ lying at the end of the path, relative to the original classical generators ${}_0\mathcal{P}$ from ${}_0\mathcal{P} \mapsto ({}_0\mathbf{g})q_B^{-1}$.*

Here are key points going back to [Fr77, §4].

(2.11a) *Endpoint of the Drag:* A cover at the end of B is still in $\text{Ni}(G, \mathbf{C})^\dagger$; it depends only on the homotopy class of B with its ends fixed.

¹⁷Other call this the sphere braid group. I first used it in 1968 while at the Institute for Advanced Study, when I discovered Hurwitz used it for a special case of what I generalized. It was a monodromy group in my work, and no topologist corrected me.

(2.11b) H_r orbits: (Irreducible) components of spaces of covers in $\text{Ni}(G, \mathbf{C})^\dagger$
 \leftrightarrow braid (H_r) orbits.

Whatever the application, we must be able to identify the Galois closure (§2.1.2) of the cover. The key ambiguity is in labeling $\mathbf{w}^* = \{w_1^*, \dots, w_n^*\}$, points lying over z_0 . Changing that labeling changes $T : G \rightarrow S_n$. A slightly subtler comes from changing z_0 . There is a distinction between them. Changing z_0 to z_0^* is affected by rewriting the z_i -loops as

$$(2.12) \quad \lambda^* \circ \lambda \circ \rho \circ \lambda^{-1} \circ (\lambda^*)^{-1}, \text{ with } \lambda^* \text{ a path from } z_0^* \text{ to } z_0.$$

2.2.2. *Braids and equivalences.* In this paper we primarily need the H_r , $r = r_{\mathbf{C}}$, action on $\text{Ni}(G, \mathbf{C})$.

(2.13a) Two elements generate this H_r action:

$$\begin{aligned} q_i &: \mathbf{g} \stackrel{\text{def}}{=} (g_1, \dots, g_r) \mapsto (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_r); \\ \mathbf{sh} &: \mathbf{g} \mapsto (g_2, g_3, \dots, g_r, g_1) \text{ and } H_r \stackrel{\text{def}}{=} \langle q_2, \mathbf{sh} \rangle \text{ with} \\ &\quad \mathbf{sh} q_i \mathbf{sh}^{-1} = q_{i+1}, \quad i = 1, \dots, r-1. \end{aligned}$$

(2.13b) From braids, B_r , on r strings, $H_r = B_r / \langle q_1 \cdots q_{r-1} q_{r-1} \cdots q_1 \rangle$.

The case $r = 4$ in (2.13a) is so important in examples, that in *reduced* Nielsen classes, we refer to q_2 as the *middle twist*. As usual, in notation for free groups modulo relations, (2.13b) means to mod out by the normal subgroup generated by the relation $q_1 \cdots q_{r-1} q_{r-1} \cdots q_1 = R_H$.

Principle 2.7. *From (2.13), we get a permutation representation of H_r on $\text{Ni}(G, \mathbf{C})^\dagger$. Given \dagger , that gives a cover $\Phi \stackrel{\text{def}}{=} \Phi^\dagger : \mathcal{H}(G, \mathbf{C})^\dagger \rightarrow U_r$. The Hurwitz space of \dagger -equivalences of covers.*

Elements in $\langle q_1 \cdots q_{r-1} q_{r-1} \cdots q_1 \rangle$ have this effect :

$$(2.14) \quad \begin{aligned} \mathbf{g} &\in \text{Ni}(G, \mathbf{C}) \mapsto \mathbf{g} \mathbf{g} g^{-1} \text{ for some } g \in G. \text{ Indeed, for} \\ \mathbf{g} &\in \text{Ni}(G, \mathbf{C}), \{(\mathbf{g}) q^{-1} R_H q \mid q \in B_r\} = \{g^{-1} \mathbf{g} \mathbf{g} \mid g \in G\}. \end{aligned}$$

Denote the subgroup of the normalizer, $N_{S_n}(G)$, of G in S_n that permutes a given collection, \mathbf{C} , of conjugacy classes, by $N_{S_n}(G, \mathbf{C})$. Circumstances dictate when we identify covers $\varphi_u : {}_u W \rightarrow \mathbb{P}_z^1$, $u = 0, 1$, branched at $0\mathbf{z}$, obtained from any one cover using the dragging-branch-points principle. Two equivalences that occur on the Nielsen classes:

(2.15a) *Inner*: $\text{Ni}(G, \mathbf{C})^{\text{in}} \stackrel{\text{def}}{=} \text{Ni}(G, \mathbf{C})/G$ corresponding to (2.14); and

(2.15b) *Absolute*: $\text{Ni}(G, \mathbf{C})^{\text{abs}} \stackrel{\text{def}}{=} \text{Ni}(G, \mathbf{C})/N_{S_n}(G, \mathbf{C})$.

One might regard *Inner* (resp. *Absolute*) equivalence as *minimal* (resp. *maximal*). Act by H_r on either equivalence (denoted by a \dagger superscript).

Definition 2.8 (Reduced action). A cover $\varphi : W \rightarrow \mathbb{P}_z^1$ is *reduced* equivalent to $\alpha \circ \varphi : W \rightarrow \mathbb{P}_z^1$ for $\alpha \in \text{PSL}_2(\mathbf{C})$.

Also, α acts on $\mathbf{z} \in U_r$ by acting on each entry. Def. 2.8 extends to any cover $\Phi^\dagger : \mathcal{H}(G, \mathbf{C})^\dagger \rightarrow U_r$, giving a *reduced* Hurwitz space cover:

$$(2.16) \quad \Phi^{\dagger, \text{rd}} : \mathcal{H}(G, \mathbf{C})^{\dagger, \text{rd}} \rightarrow U_r / \text{PSL}_2(\mathbb{C}) \stackrel{\text{def}}{=} J_r.$$

2.2.3. Genus formula: $r = 4$. Identify $U_4 / \text{PSL}_2(\mathbb{C})$ with $\mathbb{P}_j^1 \setminus \{\infty\}$. A reduced Hurwitz space of 4 branch point covers is a natural j -line cover. As in §1.2.3, that completes to $\overline{\mathcal{H}}(G, \mathbf{C})^{\dagger, \text{rd}} \rightarrow \mathbb{P}_j^1$ ramified over $0, 1, \infty$.

Definition 2.9. Denote $\langle q_1 q_3^{-1}, \mathbf{sh}^2 \rangle$ by \mathcal{Q}'' . The *cuspidal group*, Cu_4 , is the group that \mathcal{Q}'' and q_2 generate.

Cusps on the projective non-singular completion of the Hurwitz space over $\infty \in \mathbb{P}_j^1$ have a purely combinatorial definition.¹⁸

$$(2.17) \quad \begin{array}{l} \text{As in (1.15), they correspond to } \text{Cu}_4 \text{ orbits on} \\ \text{reduced Nielsen classes } \text{Ni}(G, \mathbf{C})^{\dagger, \text{rd}} \stackrel{\text{def}}{=} \text{Ni}(G, \mathbf{C})^\dagger / \mathcal{Q}'' . \end{array}$$

[BFr02, §4.2] proves the Thm. 2.10 formula using the (2.17) definitions.

Theorem 2.10. Suppose a component, $\overline{\mathcal{H}'}$, of $\overline{\mathcal{H}}(G, \mathbf{C})^{\dagger, \text{rd}}$ corresponds to an H_r orbit, O , on the Nielsen classes $\text{Ni}(G, \mathbf{C})^{\dagger, \text{rd}}$. Then, points of ramification, respectively over $0, 1, \infty$, of $\overline{\mathcal{H}'} \rightarrow \mathbb{P}_j^1$ correspond to

disjoint cycles of $\gamma_0 = q_1 q_2$, $\gamma_1 = q_1 q_2 q_1$, $\gamma_\infty = q_2$ acting on O .

The genus, $g_{\overline{\mathcal{H}'}}$, of $\overline{\mathcal{H}'}$, a la Riemann-Hurwitz, appears from

$$2(|O| + g_{\overline{\mathcal{H}'}} - 1) = \text{ind}(\gamma_0) + \text{ind}(\gamma_1) + \text{ind}(\gamma_\infty).$$

Remark 2.11. Notice that $\gamma_1 \gamma_2 \gamma_3 = 1$ (product-one) is a consequence of the Hurwitz braid relation

$$q_1 q_2 \cdots q_{r-1} q_{-1} q_{r-2} \cdots q_1 = R_H$$

combined for $r = 4$ with modding out by $q_1 = q_3$. Also, that immediately gives $\gamma_0^3 = 1$ in its action on reduced Nielsen classes. Hint: Use also the braid relations $q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1}$.

Remark 2.12 (Braid orbits). Thm. 2.10 shows that identifying braid orbits, O in the Nielsen class is crucial. As in Rem. 5.25 on the **MT** from the level 0 Nielsen class (5.2c) as ℓ changes in [FrH20] or [Fr20, §5].

¹⁸Nielsen classes allow precise definitions of cusps for all $r \geq 4$. For $r > 4$, the cuspidal group is generated just by q_j , $j = 2$. Relating this to cusps on, say, Siegel Upper-half spaces, hasn't yet been elaborated.

2.3. Main Example. We now do one example – using the definitions in (2.17) – to illustrate the Thm. 2.10 genus calculation. [BFr02] is our main source for the theory and other examples illustrating, purposely chosen to show on one full example of a **MT**. Something one might care about if these were modular curves; though they are not.

Level 0 of the **MT** for that example is designated $\text{Ni}(A_5, \mathbf{C}_{3^4})$. It has just one braid orbit, unlike that of §2.3.1 which has two.

2.3.1. A lift invariant. Before using general Frattini covers (Def. 1.7) in §3.2, suppose $\psi : H \rightarrow G$ is a *central* Frattini cover: $\ker(\psi)$ in the center of H . The most famous Frattini *central* extension (kernel in the center of the covering group) arises in *quantum mechanics* from the spin cover, $\psi : \text{Spin}_n \rightarrow O_n(\mathbb{R})$, $n \geq 3$, of the orthogonal group. Regard $\ker(\psi)$ as $\{\pm 1\}$. The natural permutation embedding of A_n in O_n induces the

Frattini cover $\psi : \text{Spin}_n \rightarrow A_n$, abusing notation a little.

A braid orbit O of $\mathbf{g} = (g_1, \dots, g_r) \in \text{Ni}(A_n, \mathbf{C})$, with \mathbf{C} conjugacy classes consisting of odd-order elements, passes the (spin) lift invariant test if the natural (one-one) map $\text{Ni}(\text{Spin}_n, \mathbf{C}) \rightarrow \text{Ni}(A_n, \mathbf{C})$ has image containing \mathbf{g} . Each g_i lifts to a same-order element $\tilde{g}_i \in \text{Spin}_n$.

Definition 2.13 (Lift invariant). Then, $s_{\text{Spin}_n/A_n}(O) \stackrel{\text{def}}{=} \prod_{i=1}^r \tilde{g}_i \in \ker(\psi)$. Generally, for ℓ -perfect G and ℓ' conjugacy classes \mathbf{C} :¹⁹ for $s_{H/G}$ on a braid orbit O on $\text{Ni}(G, \mathbf{C})$ substitute $A_n \rightarrow G$ and $\text{Spin}_n \rightarrow H$ in $s_{\text{Spin}_n/A_n}(O)$.

One result: If covers in $\text{Ni}(A_n, \mathbf{C})$ have genus 0, then $s_{\text{Spin}_n/A_n}(O)$ depends only on $\text{Ni}(A_n, \mathbf{C})$, not on O , and there is an explicit computation for it.

Example 2.14. For $n = 4$, there are two classes of 3-cycles, $\mathbf{C}_{\pm 3}$, but just one for $n \geq 5$. For $\mathbf{g} \in \text{Ni}(A_n, \mathbf{C}_{3^{n-1}})$, $n \geq 5$, $n-1$ repetitions of \mathbf{C}_3 , $s_{\text{Spin}_n/A_n}(\mathbf{g}) = (-1)^{n-1}$. For $n = 4$, the only genus 0 Nielsen classes of 3-cycles are $\text{Ni}(A_4, \mathbf{C}_{+3^3})$ and $\text{Ni}(A_4, \mathbf{C}_{-3^3})$, and the lift invariant is -1.

The short proof of [Fr10, Cor. 2.3] is akin to the original statements I made to Serre for [Se90a]. △

2.3.2. A_4 , $r = 4$. This is the most natural case not included in Ex.2.14, and it is level 0 for $\ell = 2$ of our illustration for generalizing Serre's **OIT** • [FrH20] •. The Nielsen class is $\text{Ni}(A_4, \mathbf{C}_{+3^2-3^2})^{\text{in,rd}}$ with conjugacy classes a rational union. From the **BCL** (Thm. 5.10) the Hurwitz space has moduli definition field \mathbb{Q} . Here is what to expect.

¹⁹You can drop both assumptions, as in [FrV91, App.], but the definition is trickier.

- (2.18a) The Hurwitz space has two components, labeled \mathcal{H}_0^\pm , that we will see clearly using the **sh**-incidence matrix (2.19) and \bullet [BFr02] \bullet .
- (2.18b) The Hurwitz spaces have fine moduli, but neither component has fine *reduced* moduli (criterion of [BFr02, Prop. 4.7]).²⁰
- (2.18c) The Spin lift invariant Def. 2.13 separates the components and each component has genus 0 and a characteristic cusp type.
- (2.18d) Neither component is a modular curve, but we can compute their arithmetic and geometric monodromy as j -line covers.

Comment on (2.18b): Fine moduli for inner Hurwitz spaces here comes from A_4 having no center. Checking fine moduli on a *reduced* space braid orbit O has two steps [BFr02, §4.3.1]: \mathcal{Q}'' must act as a Klein 4-group (called b(irrational)-fine moduli); and neither γ_0 nor γ_1 has fixed points (on O).

Comments on (2.18c): Thm. 5.18 uses the spin lift invariant. Thm. 5.24 uses a *Heisenberg* lift invariant for $\ell \neq 2$ prime.

2.3.3. *The sh-incidence matrix.* Subdivide $\mapsto \text{Ni}(A_3, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$ using sequences of conjugacy classes $\mathbf{C}_{\pm 3}$; $q_1 q_3^{-1}$ and **sh** switch these rows:

$$\begin{array}{ccc} [1] & + - + - & [2] & + + - - & [3] & + - - + \\ [4] & - + - + & [5] & - - + + & [6] & - + + - \end{array}$$

The rest of this example displays the two H_r orbits on the Nielsen classes, and the geniuses of their corresponding Hurwitz space components. Here is the **sh**-incidence matrix notation for cusps, labeled $O_{i,j}^k$: k is the cusp width, and i, j corresponds to a labeling of orbit representatives. \bullet [BFr02, §2.10] \bullet says much more about the **sh**-incidence matrix, which works for all $r \geq 4$, an example of which we now present. \bullet [BFr02] \bullet says more about other examples on which it has been used.

It has appeared where braid orbits would have been otherwise difficult to either compute or to display.²¹ Its entries are

$$(2.19) \quad |O \cap (O')\mathbf{sh}|, \text{ with } (O, O') \text{ cusp orbits. Read cusp widths by adding entries in a given row of each block.}$$

Consider $g_{1,4} = ((1\ 2\ 3), (1\ 3\ 4), (1\ 2\ 4), (1\ 2\ 4))$. Its γ_∞ orbit $O_{1,4}^4$ is what Thm. 5.24 calls *double identity* (repeated elements in positions 3 and 4).²² There are also two other double identity cusps with repeats in positions 2

²⁰Both components have moduli definition field \mathbb{Q} as in Def. 1.2 for many reasons. The easiest: If they were conjugate, $G_{\mathbb{Q}}$ would preserve their degrees over \mathbb{P}_j^1 .

²¹The point is that the display is illuminating. I remain leary of relying on **GAP** or some other computer program without a corroborating proof.

²²Its shift gives a cusp of type o-2' (5.24).

and 3, denoted $O_{3,4}^1$ and $O_{3,5}^1$. The following elements are in a Harbater-Mumford component (5.16).

$$\begin{aligned} \text{H-M rep.} \mapsto \mathbf{g}_{1,1} &= ((1\ 2\ 3), (1\ 3\ 2), (1\ 3\ 4), (1\ 4\ 3)) \\ \mathbf{g}_{1,3} &= ((1\ 2\ 3), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 2)) \\ \text{H-M rep.} \mapsto \mathbf{g}_{3,1} &= ((1\ 2\ 3), (1\ 3\ 2), (1\ 4\ 3), (1\ 3\ 4)) \end{aligned}$$

Ni_0^+ Orbit	$O_{1,1}^4$	$O_{1,3}^2$	$O_{3,1}^3$	Ni_0^- Orbit	$O_{1,4}^4$	$O_{3,4}^1$	$O_{3,5}^1$
$O_{1,1}^4$	1	1	2	$O_{1,4}^4$	2	1	1
$O_{1,3}^2$	1	0	1	$O_{3,4}^1$	1	0	0
$O_{3,1}^3$	2	1	0	$O_{3,5}^1$	1	0	0

Proposition 2.15. *On $\text{Ni}(\text{Spin}_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$ (resp. $\text{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$) H_4/\mathcal{Q}'' has one (resp. two) orbit(s). So, $\mathcal{H}(\text{Spin}_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$ (resp. $\mathcal{H}(A_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$) has one (resp. two) component(s), $\mathcal{H}_{0,+}$ (resp. $\mathcal{H}_{0,+}$ and $\mathcal{H}_{0,-}$).*

Then, $\mathcal{H}(\text{Spin}_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$ maps one-one to $\mathcal{H}_{0,+}$ (though changing A_4 to Spin_4 give different moduli). The compactifications of $\mathcal{H}_{0,\pm}$ both have genus 0 from Thm. 2.10 (Ex. 2.16).

The diagonal entries for $O_{1,1}^4$ and $O_{1,4}^4$ are nonzero. In detail, however, γ_1 (resp. γ_0) fixes 1 (resp. no) element of $O_{1,1}$, and neither of γ_i , $i = 0, 1$, fix any element of $O_{1,4}^4$. Ex. 2.16 uses (2.19) for the cusp widths.

Example 2.16 (Compute the genus). Use $(\gamma_0, \gamma_1, \gamma_\infty)$ from the **sh**-incidence calculation in Prop. 2.15. Denote their restrictions to lifting invariant +1 (resp. -1) orbit by $(\gamma_0^+, \gamma_1^+, \gamma_\infty^+)$ (resp. $(\gamma_0^-, \gamma_1^-, \gamma_\infty^-)$).

Read indices of + (resp. -) elements from the Ni_0^+ (resp. Ni_0^-) matrix block: Cusp widths over ∞ add to the degree $4+2+3 = 9$ (resp. $4+1+1 = 6$) to give $\text{ind}(\gamma_\infty^+) = 6$ (resp. $\text{ind}(\gamma_\infty^-) = 3$).

As γ_1^+ (resp. γ_1^-) has 1 (resp. no) fixed point and γ_0^\pm have no fixed points, $\text{ind}(\gamma_1^+) = 4$ (resp. $\text{ind}(\gamma_1^-) = 3$) and $\text{ind}(\gamma_0^+) = 6$ (resp. $\text{ind}(\gamma_0^-) = 4$). From

$$\begin{aligned} 2(9 + g_+ - 1) &= 6 + 4 + 6 = 16 \text{ and} \\ 2(6 + g_- - 1) &= 3 + 3 + 4 = 10. \end{aligned}$$

the genus of $\bar{\mathcal{H}}_{0,\pm}$ is $g_\pm = 0$. \triangle

3. ℓ -FRATTINI COVERS AND MTs

§3.1 is an overview of the universal Frattini cover of a finite group G . §3.2 introduces the universal ℓ -Frattini cover of ℓ -perfect G .

§3.3 focuses on ${}_\ell\tilde{\psi}_{G_{\text{ab}}} : {}_\ell\tilde{G}_{\text{ab}} \rightarrow G$, the abelianized ℓ -Frattini cover. That gives **MTs**, a source of ℓ -adic representations attached to (G, ℓ, \mathbf{C}) with \mathbf{C} consisting of generating ℓ' conjugacy classes in G .

§3.4 explains the cohomological obstruction to **MT** levels being nonempty. That includes taking *lattice* quotients of ${}_{\ell}\tilde{\psi}_{G_{\text{ab}}}$. § 3.4.3 continues Ex. 2.16 as a case included in \bullet [CaD08] \bullet . §3.5 starts the motivation from [Se68] for the eventually ℓ -Frattini Definition.

3.1. Universal ℓ -Frattini covers. To realize the relation between the **RIGP** and diophantine questions about classical spaces, we have an aid in a geometric approach to groups that naturally correspond to points on such spaces. We use these definitions:

(3.1a) Def. 1.8: Finite group G is ℓ -perfect.

(3.1b) Def. 1.7: Frattini cover $\psi : H \rightarrow G$ of profinite groups: ℓ -Frattini if $\ker(\varphi)$ is a pro- ℓ group.

There is a Universal profinite group, \tilde{G} , for the Frattini covering property. Further, for each prime ℓ dividing $|G|$, there is a universal ℓ -Frattini cover ${}_{\ell}\tilde{\psi} : {}_{\ell}\tilde{G} \rightarrow G$ for the Frattini property with kernel an ℓ group, and by modding out by the commutator of $\ker({}_{\ell}\tilde{\psi})$, an abelianized version ${}_{\ell}\tilde{\psi}_{\text{ab}} : {}_{\ell}\tilde{G}_{\text{ab}} \rightarrow G$. A characteristic sequence $\{{}_{\ell}^k G_{\text{ab}}\}_{k=0}^{\infty}$ of quotients of ${}_{\ell}\tilde{G}_{\text{ab}}$ canonically defines a series of moduli space covers of $\mathcal{H}(G, \mathbf{C})^{\text{rd}}$ when the elements of \mathbf{C} are ℓ' (prime to ℓ).

This is elementary and in \bullet [Fr95] \bullet , [Fr20, App. B] and reviewed in §3.2. With the assumptions of §1.3.3 – G is ℓ -perfect and \mathbf{C} consists of $r = r_{\mathbf{C}}$ ℓ' conjugacy classes of G , with $\text{Ni}(G, \mathbf{C})$ nonempty – form a profinite version of Nielsen classes, $\text{Ni}({}_{\ell}\tilde{G}_{\text{ab}}, \mathbf{C})$.

Classes of \mathbf{C} lift to classes of same order elements – so we don't change the notation – of the universal abelianized ℓ -Frattini cover ${}_{\ell}\tilde{\psi}_{\text{ab}} : {}_{\ell}\tilde{G}_{\text{ab}} \rightarrow G$, whose kernel is a finite rank $\mathbb{Z}_{\ell}[G]$ module.

Then H_r extends to this Nielsen class. A **MT** for (G, \mathbf{C}, ℓ) consists of a profinite H_r orbit. Denote the collection of these by $\mathcal{F}_{G, \mathbf{C}, \ell} \stackrel{\text{def}}{=} \mathcal{F}_{G, \mathbf{C}, \ell, {}_{\ell}\tilde{\psi}_{\text{ab}}}$ (§3.3.2). The homomorphism ${}_{\ell}\tilde{\psi}_{\text{ab}}$ has at least one proper ℓ -Frattini lattice quotient $L^{\star} \rightarrow G^{\star} \xrightarrow{{}_{\ell}\tilde{\psi}^{\star}} G$ of ${}_{\ell}\psi_{\text{ab}}$ (Def. 3.4) attached to each quotient of the characteristic ℓ -Frattini module attached to (G, ℓ) .

To each lattice quotient the same definition, conjectures and variants on properties applies. §4.1 explains how ℓ -adic representations appear from these **MTs**. In cases the **MTs**, $\mathcal{F}_{G, \mathbf{C}, \ell, {}_{\ell}\tilde{\psi}^{\star}}$, for proper quotients are variants on classical spaces. \bullet [CaD08] \bullet expands on this.

A cohomological condition, Thm. 3.13, checks precisely for when $\mathcal{F}_{G, \mathbf{C}, \ell, \ell} \tilde{\psi}^*$ is nonempty.²³ •[Fr20]• relates Frattini covers and the Inverse Galois Problem. This paper describes Serre's original **OIT** in §5.3.2 in the discussion of •[Se68]• and •[Fr78]•, updated from the original papers with references to [Fr05], [GMS03] and [Se97b]. Especially, this gives background on the problems that connected our Hurwitz space approach to the **OIT**.

[Fr20] is complete on the Universal Frattini cover itself, and especially the role of the *lift invariant*. Nontrivial lift invariants arise from what group theorists call representation covers of G . They are also a detectible subset of central (in the notation of Def. 1.7, when $\ker(\psi)$ is in the center of H) Frattini covers. This set of ideas gives the main information we require to understand the **MT** levels, components, their cusps and why they are appropriate for generalizing the OIT.

3.2. Universal Frattini cover. We start by putting a nilpotent tail on any finite group, producing for any G , even $G = A_5$ myriad extensions by, say, 2-groups, none of which have ever been realized as Galois groups. Let $\psi_i : H_i \rightarrow G$, $i = 1, 2$, be Frattini covers (Def. 1.7).

Lemma 3.1. *A minimal (not necessarily unique) subgroup $H \leq H_1 \times_G H_2$ that is surjective to G , is a Frattini cover of G that factors surjectively to each H_i . Thus, Frattini covers of G form a projective system. From their definition, taking a Frattini cover of a group preserves the rank.*

Proof. The projection $\text{pr}_i : H \leq H_1 \times_G H_2 \rightarrow H_i$, makes $\text{pr}_i(H)$ a subgroup of H_i mapping surjectively to G . As ψ_i is Frattini, $\text{pr}_i(H) = H_i$, $i = 1, 2$. \square

Also, Frattini covers of perfect groups are perfect. Key for Frattini covers is that $\ker(\psi)$ is nilpotent [FrJ86, Lem. 20.2]₁ or [FrJ86, Lem. 22.1.2]₂.

Write $\ker(\psi) = \prod_{\ell|G} \ker(\psi)_\ell$ indicating the product is over its ℓ -Sylows. For each ℓ , quotient by all Sylows for primes other than ℓ dividing $\ker(\psi)$. Thus form ${}_\ell\psi : {}_\ell H \rightarrow G$.

The fiber product of the ${}_\ell H$ s over G equals H . [Fr20, App. B] discusses elementary structural statements about the construction of \tilde{G} below. Some version of these appear in [FrJ86, Chap. 22]₂.

Definition 3.2. This produces a profinite cover, the *Universal Frattini cover*, $\tilde{\psi}_G : \tilde{G} \rightarrow G$. Similarly,

$$\tilde{\psi}_{G_{\text{ab}}} : \tilde{G}/[\ker(\tilde{\psi}_G), \ker(\tilde{\psi}_G)] \stackrel{\text{def}}{=} \tilde{G}_{\text{ab}} \rightarrow G \text{ (resp. } {}_\ell\tilde{\psi}_{G_{\text{ab}}} : {}_\ell\tilde{G}_{\text{ab}} \rightarrow G)$$

is the *Universal Abelianized Frattini* (resp. ℓ -Frattini) cover of G .

²³The criterion is from the *lift invariant*, Def. 2.13.

Then, $\tilde{\psi}_G$ is a minimal projective object in the category of profinite groups covering G . So, given any profinite group cover $\psi : H \rightarrow G$, some homomorphism $\tilde{\psi}_{G,H}$ to H factors through ψ . If ψ is a Frattini cover, then $\tilde{\psi}_{G,H}$ must be a Frattini cover, too.

If $\text{rk}(G) = t$, construct \tilde{G} using a pro-free group, \tilde{F}_t , on the same (finite) number of generators. Sending its generators to generators of G gives a cover, $\tilde{F}_t \rightarrow G$. Then, $\tilde{G} \leq \tilde{F}_t$ is minimal (closed) among covers of G .

This, though is nonconstructive. It uses the Tychynoff Theorem: a nested sequence of closed subgroups of \tilde{F}_t covering G has non-empty intersection covering G . That also explains why it wasn't sufficient to replace \tilde{F}_t by the free (rather than pro-free) group on t generators.

The following quotients of \tilde{G} are more accessible. They result from decomposing the (pro-)nilpotent kernel $\ker(\tilde{\psi})$ into a product of its ℓ -Sylows.

Definition 3.3. For each prime $\ell \mid |G|$, there is a profinite Frattini cover ${}_{\ell}\tilde{\psi}_G : {}_{\ell}\tilde{G} \rightarrow G$ with $\ker({}_{\ell}\tilde{\psi}_G)$ a profree pro- ℓ group of finite rank, $\text{rk}({}_{\ell}\tilde{G})$. There are similar such covers with ${}_{\ell}\tilde{\psi}_{G_{\text{ab}}}$ replacing ${}_{\ell}\tilde{\psi}_G$.

The Frattini subgroup of an ℓ -group H is the (closed) subgroup generated by ℓ -th powers and commutators from H . Denote it by ${}_{\text{fr}}H$.

Consider the kernel of the short exact sequence $\ker_0 \rightarrow {}_{\ell}\tilde{G} \rightarrow G$. Recover a cofinal family of finite quotients of ${}_{\ell}\tilde{G}$ as follows. Mod out by successive ℓ -Frattini subgroups of *characteristic kernels* of ${}_{\ell}\tilde{G}$:

$$(3.2) \quad \ker_0 > {}_{\text{fr}}\ker_0 \stackrel{\text{def}}{=} \ker_1 \geq \cdots \geq {}_{\text{fr}}\ker_{k-1} \stackrel{\text{def}}{=} \ker_k \dots$$

Denote ${}_{\ell}\tilde{G}/\ker_k$ by ${}^k_{\ell}G$, and $\ker_k/\ker_{k'}$ by ${}_{\ell}M_{k,k'}$ or $M_{k,k'}$ for $k' \geq k$.

$$(3.3) \quad \text{Especially, } {}_{\ell}M_G \stackrel{\text{def}}{=} {}_{\ell}M_{0,1} \text{ is the characteristic } \mathbb{Z}/\ell[G] \text{ module.}$$

Denote its dimension, $\dim_{\mathbb{Z}/\ell}({}_{\ell}M_G)$ by ${}_{\ell}m_G$.

Given (G, \mathbf{C}, ℓ) , define the Nielsen classes ${}_{\ell}\text{Ni}(G, \mathbf{C})$ of a **MT** in a profinite way that extends that of an ordinary Nielsen class (Def. 2.4).

Denote the free group $\pi(U_{\mathbf{z}_0}, z_0)$ modulo inner automorphisms, by $\mathcal{G}_{\mathbf{z}_0}$. Then, consider $\psi_{\mathbf{g}} : \mathcal{G}_{\mathbf{z}_0} \rightarrow G$ given by mapping classical generators (2.9), \mathcal{P} , to the branch cycles $\mathbf{g} \in \text{Ni}(G, \mathbf{C})$ as given in §2.2.1.

Now form all homomorphisms $\psi_{\tilde{\mathbf{g}}} : \mathcal{G}_{\mathbf{z}_0} \rightarrow {}_{\ell}\tilde{G}$ through which $\psi_{\mathbf{g}}$ factors, indicating images of the classical generators, \mathcal{P} , by $\tilde{\mathbf{g}}$, that satisfy this additional condition:

$$(3.4) \quad \tilde{g}_i \text{ has the same order as } g_i, 1, \dots, r.$$

From Schur-Zassenhaus, as ${}_{\ell}\tilde{G} \rightarrow G$ has kernel an ℓ -group, this defines the conjugacy class of \tilde{g}_i uniquely. With no loss, also label it C_i , $i = 1, \dots, r$.

This makes sense of writing $\text{Ni}({}_\ell\tilde{G}, \mathbf{C})^\dagger$ (or $\text{Ni}({}_\ell\tilde{G}_{\text{ab}}, \mathbf{C})^\dagger$) with \dagger any one of the equivalences we have already discussed in §2.2.2. As previously H_r acts on the Nielsen classes. To define the Nielsen class levels, mod out successively, as in Def. 3.2, on ${}_\ell\tilde{G}$ by the characteristic kernels of (3.2). Then, H_r acts compatibly on these canonical towers of Nielsen classes:

$$(3.5) \quad \begin{aligned} &\text{forming a Hurwitz space sequence } \mathbb{H}({}_\ell\tilde{G}, \mathbf{C}) = \{\mathcal{H}({}_\ell^k G, \mathbf{C})^\dagger\}_{k=0}^\infty, \\ &\quad \text{with a natural map from level } k+1 \text{ to level } k. \\ &\text{Similarly for } \mathbb{H}({}_\ell\tilde{G}_{\text{ab}}, \mathbf{C}) = \{\mathcal{H}({}_\ell^k G_{\text{ab}}, \mathbf{C})^\dagger\}_{k=0}^\infty, \text{ the Hurwitz} \\ &\quad \text{space tower of the maximal lattice referred to in §1.3.3.} \end{aligned}$$

3.3. Using ${}_\ell\tilde{G}_{\text{ab}}$ and ${}^1_\ell G$. We now have a general situation for any (G, \mathbf{C}, ℓ) in which natural ℓ -adic representations arise from points on a tower of Hurwitz spaces relating the **RIGP** and the **OIT**. §3.3.1 adds the notion of (ℓ -Frattini) lattice quotients (of the maximal one). §3.3.2 produces their associated **MTs**, whose levels can be (variants on) classical spaces. Then, §3.3.3 notes that G with normal ℓ -Sylow gives very small such quotients.

3.3.1. ${}_\ell\tilde{G}_{\text{ab}}$ lattice quotients. Assume the standard properties (1.11) for (G, ℓ, \mathbf{C}) : G is ℓ -perfect and \mathbf{C} consists of ℓ' conjugacy classes.

Consider any short exact sequence $L^\star \rightarrow G^\star \xrightarrow{\psi^\star} G$

$$(3.6) \quad \text{with } \ker(\psi^\star) = L^\star \text{ a } \mathbb{Z}_\ell \text{ lattice and } \psi^\star \text{ an } \ell\text{-Frattini cover.}$$

Take $G_k = G^\star / \ell^k \ker(\psi^\star)$ for the analog of (3.5): a tower of Hurwitz spaces $\mathbb{H}(G^\star, \mathbf{C})$ from the Nielsen class sequence $\{\text{Ni}(G_k, \mathbf{C})^{\text{in}}\}_{k=0}^\infty$.

$$(3.7) \quad \begin{aligned} &\text{Since } \psi^\star \text{ is } \ell\text{-Frattini, there is a surjection } {}_\ell G_{\text{ab}} \xrightarrow{\mu^\star} G^\star \\ &\text{inducing a surjection } {}_\ell M_G \rightarrow \ker(G_1 \rightarrow G_0) \stackrel{\text{def}}{=} M_\psi^\star = M^\star. \end{aligned}$$

Definition 3.4. Refer to ψ^\star as a *lattice quotient* of ${}_\ell\tilde{\psi}_{G_{\text{ab}}}$ – with target M^\star . Speak of a **MT** on it or on its corresponding Nielsen class sequence.²⁴

Consider two lattice quotients of ${}_\ell G_{\text{ab}} \xrightarrow{j\mu^\star} {}_j G^\star \xrightarrow{j\psi^\star} G$ as above, with targets ${}_j M^\star$, and respective Hurwitz space towers ${}_j \mathbb{H}$, $j = 1, 2$.

Lemma 3.5. *Assume restricting ${}_2\mu^\star$ to $\ker({}_1\psi^\star)$ surjects onto $\ker({}_2\psi^\star)$. If Main **RIGP** conj. 5.8 holds for each **MT** on ${}_2\mathbb{H}$, then it does so for each **MT** on ${}_1\mathbb{H}$.*

Proof. The assumptions give $\mu_{2,1} : {}_1 M^\star \rightarrow {}_2 M^\star$, a surjection. Consider a **MT**, ${}_1\mathbb{H}$, on $\{{}_1\mathcal{H}_k\}_{k=0}^\infty$. Denote the corresponding Nielsen class orbit for ${}_1\mathcal{H}_k$ by ${}_1O_k \leq \text{Ni}(G_{k,1}, \mathbf{C})^\dagger$.

²⁴When the context is clear we use shortenings of the name ℓ -Frattini lattice quotient to ℓ -lattice quotient or just lattice quotient.

Inductively, from $\ker(\mu_{2,1})$, form a $\mathbb{Z}/\ell[G_{k,1}]$ module $M(k)^\star$ so that with $G_{k,1}/M(k)^\star = G_{k,2}$ this produces a braid orbit ${}_2O_k \leq \text{Ni}(G_{k,2}, \mathbf{C})^\dagger$ for a **MT** on ${}_2\mathbb{H}$.

These maps are natural and they map K points to K points, for K any number field. By assumption K points disappear at high levels of any **MT** on ${}_2\mathbb{H}$. Therefore they must also on ${}_2\mathbb{H}$. \square

Statements (3.8) on ${}_ellM_G$ – using considerable modular representation theory – give some sense of tools at our disposal for using Lem. 3.5. They are respectively [Fr20, Chap. 3, Prop. 1.26] (or [Fr95, Proj. Indecomp. Lem. 2.3] with help from [Se88]) and [Fr20, Chap. 3, Prop. 1.27] (or [Fr95, Prop. 2.7]). [Fr20] collects these tools under four *ℓ -Frattini principles*. That material also has detailed explanations of the cohomology involved.

(3.8a) It is indecomposable (if not then its summands would be obvious examples of M^\star) and $\dim_{\mathbb{Z}/\ell}(H^2(G, M_G)) = 1$ (see Lem. 3.6).

(3.8b) Describing it requires having explicitly only the projective indecomposables belonging to the *principal* block representations.²⁵

Consider any (non-trivial) $\mathbb{Z}/\ell[G]$ quotient M' of ${}_ellM_G$, with kernel $K_{M'}$. Since any quotient of ${}_ell\tilde{\psi}_{\text{ab}}$ mapping through G is a Frattini cover, therefore giving ${}_ell\psi_{M'} : {}_ellG/K_{M'} \stackrel{\text{def}}{=} {}_ellG_{M'} \rightarrow G$ is an ℓ -Frattini cover. We say it is *unique* if the following \mathbb{Z}/ℓ module has dimension 1 (see Rem. 3.7):

$$(3.9) \quad H^2(G, M') = \text{Ext}_{\mathbb{Z}/\ell[G]}^2(\mathbf{1}, M') \text{ [Be91, p. 70].}$$

Lemma 3.6. *As above, there is a short exact sequence*

$$(3.10) \quad L_{M'} \rightarrow {}_ell\tilde{G}_{M', \text{ab}} \xrightarrow{{}_ell\tilde{\psi}_{M'}} G^{26}$$

satisfying (3.11) for $k \geq 0$.²⁷

(3.11a) ${}_ell\tilde{\psi}_{M'}$ factors through ${}_ell\psi_{M'}$

(3.11b) $\ell^k L_{M'}/\ell^{k+1} L_{M'} \cong M'$ as a $\mathbb{Z}/\ell[G]$ module.

If M' is an indecomposable $\mathbb{Z}/\ell[G]$ module, then $L_{M'}$ is an indecomposable $\mathbb{Z}_\ell[G]$ module [Be91, Thm. 1.9.4].

Proof. For (3.10), inductively form ${}_ell^k\psi_{M'} : {}_ell^kG_{M'} \rightarrow G$ as a quotient of ${}_ell^k\psi_{\text{ab}} : {}_ell^kG_{\text{ab}} \rightarrow G$.

²⁵ $\mathbb{Z}/\ell[G]$ decomposes as a sum of indecomposable 2-sided ideals (blocks) corresponding to writing 1 as a direct sum of primitive central idempotents. The block “containing” the identity representation is the principle block [Be91, §6.1].

²⁶Referencing (3.10) just by M' is a simplification of notation, since M' can appear, in cases, as a quotient of ${}_ellM_G$ several ways. Usually this won’t be a problem.

²⁷•[CaD08]• gives a good set of these – appropriate analogs of Ex. 1.3 and Thm. 5.24 – sufficient to test the conjectures.

Use the universal property of ${}_{\ell}^{k+1}\psi_{\text{ab}} : {}_{\ell}^{k+1}G_{\text{ab}} \rightarrow G$ for ℓ -Frattini covers of G with exponent ℓ^{k+1} kernel: This factors through ${}_{\ell}^k\psi_{M'}$ as

$$\psi'' : {}_{\ell}^{k+1}\psi_{\text{ab}} : {}_{\ell}^{k+1}G_{\text{ab}} \rightarrow {}_{\ell}^kG_{M'}.$$

Continue inductively, assuming we have ${}^kK_{M'}$, the kernel of ${}_{\ell}^kG_{\text{ab}} \rightarrow {}_{\ell}^{k-1}G_{M'}$. Then, on ${}_{\ell}^{k+1}G_{\text{ab}}$, mod out by $\ell\psi''^{-1}({}^kK_{M'}) \stackrel{\text{def}}{=} {}^{k+1}K_{M'}$ to form ${}_{\ell}^{k+1}G_{M'} \rightarrow G$.

For the final profinite group cover given by ${}_{\ell}\tilde{\psi}_{M'}$, take the projective limit of these group covers of G . \square

Remark 3.7. The uniqueness definition in (3.9) extends to consider all related coefficients, \mathbb{Z}/ℓ^k and \mathbb{Z}_{ℓ} . We think ${}_{\ell}^1\psi_{M'}$ in Prop. 3.5 is unique for Frattini covers of G with kernel M' if and only if (3.10) is unique with properties (3.11a) and (3.11b). The only if part is easy, but at this time we don't have a complete proof of the other direction.

3.3.2. Nielsen classes of ℓ -Frattini lattices. Apply the formation of Nielsen classes, as in Thm. 2.3 with the assumptions of §1.3.3, now though to a profinite version, $\text{Ni}({}_{\ell}\tilde{G}_{\text{ab}}, \mathbf{C})$. Then H_r extends to this Nielsen class.

A **MT** for (G, \mathbf{C}, ℓ) consists of a profinite H_r orbit. Denote the collection of these by $\mathcal{F}_{G, \mathbf{C}, \ell} \stackrel{\text{def}}{=} \mathcal{F}_{G, \mathbf{C}, \ell, {}_{\ell}\tilde{\psi}_{\text{ab}}}$. The same constructions works for

$$(3.12) \quad \begin{array}{l} \text{any proper } \ell\text{-Frattini lattice quotient } L^* \rightarrow G^* \xrightarrow{{}_{\ell}\tilde{\psi}^*} G, \text{ with} \\ \text{the corresponding } {}^kG^* \text{ from modding out on } G^* \text{ by } \ell^k L^*. \end{array}$$

§4.1 explains how ℓ -adic representations appear from these **MTs**. For many proper lattice quotients, elements of $\mathcal{F}_{G, \mathbf{C}, \ell, {}_{\ell}\tilde{\psi}^*}$ are variants on classical spaces as started in Ex. 1.3, §3.3.3 and expanded on in •[CaD08]•.

3.3.3. Normal ℓ -Sylow and other cases. We get a fairly small ℓ -Frattini lattice quotient with $G = N \times^s H$, a normal ℓ -Sylow, N ($(|H|, \ell) = 1$). Then, the construction above for ${}_{\ell}M_G$ feels much less abstract.

That is because ${}_{\ell}\tilde{G}$ is just $\tilde{N} \times^s H$ with \tilde{N} the pro- ℓ , pro-free group on the same number of generators as has N . The point is the action of H extends to \tilde{N} [FrJ86, Prop. 22.12.2]²⁸.

Denote the Frattini subgroup of N by ${}_{\text{fr}}N$ (as in (3.2)) and its quotient $N/{}_{\text{fr}}N$ by $N' = (\mathbb{Z}/\ell)^t$ ($t = \text{rank}(N)$). Then, $N \rightarrow N'$ extends to $N \times^s H \rightarrow N' \times^s H$, and then to $\mu' : {}_{\ell}\tilde{N} \times^s H \rightarrow N' \times^s H$.

Both Lem. 3.8 (which follows from the above) and Ex. 3.9 apply to the §5.3.4 main example.

²⁸That makes it sound explicit, but extension of that action is abstract.

Lemma 3.8. *As ${}_{\ell}\tilde{N} \times^s H$ is ℓ -projective, μ' extends to*

$$\mu : (\mathbb{Z}/\ell^2)^t \times^s H \rightarrow (\mathbb{Z}/\ell)^t \times^s H, \text{ a Frattini cover.}$$

Conclude: ${}_{\ell}M_G$ has $\ker(\mu) = (\mathbb{Z}/\ell)^t \stackrel{\text{def}}{=} M'$ as a quotient. So, from Lem. 3.6, ${}_{\ell}\tilde{\psi}_{\text{ab}} : {}_{\ell}G_{\text{ab}} \rightarrow G$ has (3.10) lattice quotient with $L_{M'} = (\mathbb{Z}_{\ell})^t$.

Ex. 3.9 – giving what appears to be a very simple ℓ -Frattini lattice quotient – shows we can isolate out the role of the Schur multiplier of G , as giving a lift invariant. A list of its appearances is in (5.12). Explicit examples of this are in Prop. 2.15 and in Thm. 5.24 in §5.3.4. These appearances extend all the way up the levels of that **MT** (not in this paper).

Example 3.9 (Schur multipliers). Again G is ℓ -perfect. Consider when $\psi_H : H \rightarrow G$ is a central ℓ -Frattini cover. Then, $\ker(\psi_H)$ is a quotient of ${}_{\ell}M_G$. The ℓ -Frattini quotient lattice in (3.6) then has lattice kernel in the center of ${}_{\ell}\tilde{G}_{M',\text{ab}}$. These **MTs** based on ℓ' classes **C** have two features.

The components at higher levels (if they pass the obstruction test of Thm. 3.13) would have the same underlying spaces as at level 0. As in the discussion, however, following Def. 1.2, the *moduli definition field test* for realization of representing covers at that level would not directly work. These would definitely not be fine moduli spaces. \triangle

Remark 3.10 (**MT** levels and being centerless). If G is centerless and ℓ -perfect, then ${}^k_{\ell}G$ and ${}^k_{\ell}G_{\text{ab}}$ are also [BFr02, Prop. 3.21]. The significance with inner Hurwitz spaces: this is the criterion for fine moduli. Ex. 3.9 shows this may not hold for ℓ -Frattini lattice quotients attached to G .

3.4. Test for a nonempty MT. Again, G is ℓ -perfect and we have ℓ' conjugacy classes **C**. §3.4.1 gives the main criterion for non-empty **MTs**. Notice, a' la Rem. 3.10 that, even if G is centerless, Thm. 3.13 deals with a group that has a center.

Subsections §3.4.2 and §3.4.3 give examples of obstructed braid orbits (Hurwitz space components). They also give examples of computing for reduced **MT** levels for $r = 4$ with genera > 1 .

3.4.1. The obstructed component criterion. This section shows just how important is modular representations in understanding geometric properties of **MT** levels. [Fr06] (and [Fr20]) show how this works for cusp analysis. Denote the maximal central ℓ -Frattini extension by ${}_{\ell}\alpha : {}_{\ell}G^{\natural} \rightarrow G$.²⁹

²⁹This is a cover with finite kernel.

Lemma 3.11. *If each level of $\mathbb{H}(\ell\tilde{G}_{\text{ab}}, \mathbf{C})$ is nonempty, then there is at least one **MT** on $\mathbb{H}(\ell\tilde{G}_{\text{ab}}, \mathbf{C})$.*

Proof. Producing such a **MT** is equivalent to forming

$$(3.13) \quad \begin{array}{l} \text{branch cycles } \mathcal{G} \stackrel{\text{def}}{=} \{ {}_k\mathbf{g} \in \text{Ni}({}_\ell^k G_{\text{ab}}, \mathbf{C}) \}_{k=0}^\infty \text{ such that} \\ \text{the natural map } {}_\ell^{k+1} G_{\text{ab}} \rightarrow {}_\ell^k G_{\text{ab}} \text{ maps } {}_{k+1}\mathbf{g} \text{ to } {}_k\mathbf{g}, k \geq 0. \end{array}$$

Use the axiom of choice (Tychonoff Theorem) as in §4.4.2. Some nonempty chain hits each **MT** level. \square

Definition 3.12. Taking the braid orbits of the elements in \mathcal{G} gives a **MT** through the braid orbit of ${}_0\mathbf{g}$. If ${}_0\mathbf{g} \in O$ we say this is a **MT** through O . If there is no **MT** through O , then O is *obstructed*.

Theorem 3.13. *There is a **MT** on $\mathbb{H}(\ell\tilde{G}_{\text{ab}}, \mathbf{C})$ through O if and only if the natural map $\text{Ni}({}_\ell G^\natural, \mathbf{C}) \rightarrow \text{Ni}(G, \mathbf{C})$ has image ${}_0\mathbf{g}$.³⁰*

Now replace ${}_\ell\tilde{G}_{\text{ab}} \rightarrow G$ by any ℓ -Frattini lattice $\psi^ : G^* \rightarrow G$ and ${}_\ell G^\natural \rightarrow G$ by the maximal central ℓ -Frattini extension of G that is a quotient G^* . Then, the same statement guarantees a **MT** on ψ^* through O .*

Proof. Assume an ℓ -Frattini cover $H'' \rightarrow G$ that factors through $H' \rightarrow G$, with $M' = \ker(H'' \rightarrow H')$ an irreducible $\mathbb{Z}/\ell[H']$ module.

$$(3.14) \quad \begin{array}{l} [\text{FrK97, Obst. Lem. 3.2}]: \text{ then } \text{Ni}(H'', \mathbf{C}) \rightarrow \text{Ni}(H', \mathbf{C}) \\ \text{surjects on all braid orbits, unless } M' = \mathbf{1}_{H'}. \text{ In that case} \\ \text{it surjects only on braid orbits with lift invariant 1.} \end{array}$$

[Fr06, Lem. 4.9] shows (3.14) holds with G_{k+1} (resp. G_k) replacing H'' (resp. H') and with the maximal quotient of $\ker(G_{k+1} \rightarrow G_k)$ on which G_k acts trivially replacing G^\natural . Then, [Fr06, Lem. 4.14] shows this holds for the sequence of Hurwitz spaces defined by ${}_\ell\tilde{G}_{\text{ab}} \rightarrow G \rightarrow G_0$ using G^\natural .

The last paragraph uses ℓ -Poincaré duality, a' la [Se97a, I.4.5]. Except, instead of a pro- ℓ group, ${}_\ell\tilde{G}_{\text{ab}} \rightarrow G$ has G at its head [We05, Prop. 3.2]. We have only to adjust to the appropriate test on the central extension when applied to the ℓ -Frattini lattice quotient. \square

3.4.2. Obstructed A_n components. Use the *spin* cover, $\alpha_n : \text{Spin}_n \rightarrow A_n$, $n \geq 4$ and its lift invariant (§2.3.1). Denote the genus of covers in $\text{Ni}(A_n, \mathbf{C}_{3^r})$ – adjusting for the two conjugacy classes C_\pm for $n = 4$ – by $\mathbf{g}_{n,r}$.

$$(3.15a) \quad \mathbf{g}_{n,n-1} = 0: \text{ There is one Hurwitz space component with lift invariant } (-1)^{n-1} \text{ starting with } -1 \text{ at } n = 4.$$

³⁰Computing Schur multipliers is hard. So, a criterion for a component, assuring without computation, that the lift invariant is always trivial is valuable. The generalization, Def. 5.20, of Harbater-Mumford components, Def. 5.16, gives exactly that.

(3.15b) $\mathbf{g}_{n,r} > 0$: The two Hurwitz space components, $\mathcal{H}_{n,r}^\pm$, are separated by their lift invariants.

(3.15c) In $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{in}} \rightarrow \mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{abs}}$, all $r \geq n-1$, each image component has only one preimage.

Here is the rephrasing of this in obstructed components for the full ℓ -Frattini lattice (quotient) ${}_\ell\tilde{G}_{\text{ab}} \rightarrow G$. The unique component in (3.15a) is obstructed if and only if n is even. The component $\mathcal{H}_{n,r}^+$ (resp. $\mathcal{H}_{n,r}^-$) in (3.15b) is unobstructed (resp. obstructed).

§3.4.3 finishes two issues with Ex. 2.16. Here we consider both the *full* ℓ -Frattini lattice ${}_2\tilde{G}_{\text{ab}} \rightarrow G$, and the *minimal* ℓ -Frattini lattice (§3.3.3):

$$(3.16) \quad (\mathbb{Z}_2)^2 \times {}^s\mathbb{Z}/3 \rightarrow (\mathbb{Z}/2)^2 \times {}^s\mathbb{Z}/3, \text{ the case } \ell = 2 \text{ of Thm. 5.24.}$$

For each lattice quotient L^\star , we want a **MT** level value (k_{L^\star}), so any reduced Hurwitz space component at that level has genus > 1 .³¹

3.4.3. *Finishing Ex. 2.16.* The Main **MT** Conj. 5.8 [Fr06, §5] has the crucial hypothesis for giving a lower bound on the genus of high levels of a **MT**. It is the existence of ℓ -cusps – basically that ℓ divides the cusp widths – on each component (5.24).

Here $\ell = 2$: Each component has 2-cusps: respectively $O_{1,1}^4$ and $O_{1,4}^4$. Both inner reduced components have genus 0 (Prop. 2.15). So, we need the argument of \bullet [Fr06] \bullet to conclude Main **RIGP** Conj. 5.8 – as in discussing Falting’s in §4.4.2 – for each lattice L^\star to get the (higher) level, k_{L^\star} , at which component genres rise beyond 1.

[BFr02, Thm. 9.1] did exactly that for the full lattice of $(A_5, \mathbf{C}_{3^4}, \ell = 2)$, listing the genus’s at level 1 as 9 and 12. The characteristic module for both A_4 and A_5 for $\ell = 2$ is a copy of $(\mathbb{Z}/2)^5$; action of A_5 giving that of A_4 by restriction to the subgroup [Fr95, Prop. 2.4].

The rest of [BFr02, §9.1] outlines this for the full lattice for A_4 . [FrH20] does this for the complete series of groups in Thm. 5.24, including A_4 , and the minimal lattice. Here, as often, the smaller lattice quotient gives more components, and trickier issues in bounding the genus.

Now consider the lattice in Lem. 3.8 and obstruction for the two $(A_4, \mathbf{C}_{\pm 3^2}, \ell = 2)$ level 0 components. The lift invariant separated the two components. Yet, neither is obstructed: $(\mathbb{Z}_2)^2 \times {}^s\mathbb{Z}/3 \rightarrow (\mathbb{Z}/2)^2 \times {}^s\mathbb{Z}/3$ does not factor through $\text{Spin}_4 \rightarrow A_4$.

³¹Putting aside the problem of an explicit Falting’s result: At which tower level the finitely many K (number field) points actually disappear?

3.5. Archetype of the ℓ -Frattini conjectures. This section reminds of many reasons for using proper ℓ -Frattini lattice quotients (§3.3.1) of ${}_{\ell}\tilde{G}_{\text{ab}} \rightarrow G$: Easier group theory; reflection on the main (**RIGP** and **OIT**) conjectures; and the classical connections.

Many of our examples are the §3.3.3 type. Here though we give the example that arises in considering Serre's **OIT**. Simultaneously, it inspired the most significant Def. 3.14. It also gives proper ℓ -Frattini lattice quotients that aren't from §3.3.3.

Definition 3.14. Call a sequence of finite group covers

$$\cdots \rightarrow H_{k+1} \rightarrow H_k \rightarrow \cdots \rightarrow H_1 \rightarrow H_0 = G$$

eventually Frattini (resp. eventually ℓ -Frattini) if there is a k_0 for which $H_{k_0+k} \rightarrow H_{k_0}$ is a Frattini (resp. ℓ -Frattini) cover for $k \geq 0$.

If the projective limit of the H_k s is \tilde{H} , then we say it is eventually Frattini since the same holds for any cofinal sequence of quotients. Note: any open subgroup of \tilde{H} will also be eventually Frattini (resp. ℓ -Frattini).

Take $G = \text{PSL}_2(\mathbb{Z}/\ell)$, ${}_{\ell}M_G$ its characteristic ℓ -Frattini module (3.3). Here is a relevant commutative diagram for this, for Serre's setup, with Ad_3 indicating the 2×2 trace 0 matrices.

$$(3.17) \quad \begin{array}{ccccc} \text{Ad}_3(\mathbb{Z}_{\ell}) & \longrightarrow & \text{SL}_2(\mathbb{Z}_{\ell}) & \xrightarrow{{}_{\ell}\tilde{\psi}} & \text{SL}_2(\mathbb{Z}/\ell) \\ \text{Id} \downarrow & & {}_{\ell}\tilde{\alpha} \downarrow & & {}_{\ell}\alpha \downarrow \\ \text{Ad}_3(\mathbb{Z}_{\ell}) & \longrightarrow & \text{PSL}_2(\mathbb{Z}_{\ell}) & \xrightarrow{{}_{\ell}\tilde{\psi}} & \text{PSL}_2(\mathbb{Z}/\ell) \end{array}$$

The cover ${}_{\ell}\alpha$ is a Frattini extension, of degree 2 (say, [Fr20, Chap. 6, Lem. 3.1]). Though not for $\ell \geq 3$ an ℓ -Frattini extension, it connects the upper and lower rows of (3.17) on their respective Frattini conclusions.

Proposition 3.15. *The natural cover $\text{SL}_2(\mathbb{Z}/\ell^{k+1}) \rightarrow \text{SL}_2(\mathbb{Z}/\ell)$ is an ℓ -Frattini cover for all k if $\ell > 3$. For $\ell = 3$ (resp. 2),*

$$\text{SL}_2(\mathbb{Z}/\ell^{k+1}) \rightarrow \text{SL}_2(\mathbb{Z}/\ell^{k_0+1}), \quad k \geq k_0 \text{ where } k_0 = 1 \text{ (resp. 2),}$$

is the minimal value for which these are Frattini covers.

For all ℓ , $\text{PSL}_2(\mathbb{Z}_{\ell}) \rightarrow \text{PSL}_2(\mathbb{Z}/\ell)$ is eventually ℓ -Frattini.

For $\ell > 3$, $\text{Ad}_3(\mathbb{Z}/\ell)$ is a quotient of ${}_{\ell}M_{\text{PSL}_2(\mathbb{Z}/\ell)}$, but it is not for $\ell = 2$ or 3. Further, for no ℓ is it the whole module. Therefore, for $\ell > 3$, the lower row of (3.17) is a proper ℓ -Frattini lattice quotient of ${}_{\ell}\tilde{G}_{\text{ab}}$.

Proof. The first sentence is [FrJ86, Cor. 22.13.4]₂, from [Se68, Lem. 3, IV-23] which has exercises that the same statement and proof applies to $\text{SL}_d(\mathbb{Z}/\ell)$. Here we do those exercises for just $\ell = 3$.

First, we trim the treatment of [FrJ86, Cor. 22.13.4]₂. Use $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Following [FrJ86, p. 532]₂, add $C = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ to give three independent generators of $\text{Ad}_3(\mathbb{Z}/\ell)$, all with square 0: every $u \in \text{Ad}_3(\mathbb{Z}/\ell)$ is a sum of square 0 elements.

Our induction hypothesis is $H \leq \text{SL}_2(\mathbb{Z}/\ell^{k+1}) \rightarrow \text{SL}_2(\mathbb{Z}/\ell^k)$ maps surjectively. We need to show, for $u \in \text{Ad}_3(\mathbb{Z}/\ell)$, there is $h \in H$ of form $1 + \ell^k u$. For this, the induction gives $h_0 \in H$ and $v \in \text{Ad}_3(\mathbb{Z}/\ell)$ with $h_0 = 1 + u\ell^{k-1} + v\ell^k$ for some $v \in \text{Ad}_3(\mathbb{Z}/\ell)$. Here are the remaining steps.

(3.18a) Binomially expand $h = (h_0)^\ell$ to see it is $1 + u\ell^k \pmod{\ell^{k+1}}$ unless $k = 1$ when it is $1 + u\ell + \ell(u + v\ell)^2(\bullet) + (u + v\ell)^\ell$ with $\bullet \in \mathbb{Z}$.

(3.18b) If $u^2 = 0$ (and $k = 1$), the result is $1 + \ell u \pmod{\ell^2}$, if $(u + v\ell)^\ell \equiv 0 \pmod{\ell^2}$. For $\ell > 3$ this is numerically easy to see.

(3.18c) Write $u \in \text{Ad}_3(\mathbb{Z}/\ell)$ as a sum of squares $u = \sum_{i=1}^t u_i$. From (3.18b) with $h_i = 1 + u_i \ell \in H$, then $\prod_{i=1}^t h_i = 1 + u\ell \pmod{\ell^2}$.

Then, (3.18c) concludes the induction argument, for the first sentence.

Now we do the conclusion for $\ell = 3$. [Se68, IV-28, Exer. 3] asks to show that $\text{SL}_2(\mathbb{Z}/3^2) \rightarrow \text{SL}_2(\mathbb{Z}/3)$ is not Frattini. We say it purely cohomomologically. Then, $\mu \in H^2(\text{SL}_2(\mathbb{Z}/3), \text{Ad}_3(\mathbb{Z}/3))$ defines the cohomology class of this extension [Nor62, p. 241].

For any cohomology group, $H^*(G, M)$, with M a $\mathbb{Z}/\ell[G]$ module, restrict to an ℓ -Sylow $P_\ell \leq G$. This gives an isomorphism onto the G invariant elements of $H^*(P_\ell, M)$ [Br82, III. Prop. 10.4]. So, μ splits if μ_ℓ splits.

There is an element, g_3 , of order 3 in $\text{SL}_2(\mathbb{Z}) - \text{PSL}_2(\mathbb{Z})$ is well-known to be freely generated by an element of order 3 and an element of order 2 – and so in $\text{SL}_2(\mathbb{Z}/3^2)$. This element of order 3 – given, say, by $A' = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}$ – generates a 3-Sylow in $\text{SL}_2(\mathbb{Z}/3)$. Conclude that μ_3 splits.

Denote the conjugacy class of A' by C_3 . Its characteristic polynomial is $x^2 + x + 1$. Any lift of any non-trivial element in

$$\ker(\text{SL}_2(\mathbb{Z}/3^2) \rightarrow \text{SL}_2(\mathbb{Z}/3)) = \text{Ad}_3(\mathbb{Z}/3)$$

is an element of order 3^2 in this Frattini cover

$$\ker(\text{SL}_2(\mathbb{Z}/3^3) \rightarrow \text{SL}_2(\mathbb{Z}/3)) = (\mathbb{Z}/3^2)^3 \rightarrow (\mathbb{Z}/3)^3.$$

So, the extension $\text{SL}_2(\mathbb{Z}/3^3) \rightarrow \text{SL}_2(\mathbb{Z}/3^2)$ certainly does not split.

[Se68, IV-28, Exer. 1.b] states that $\text{SL}_2(\mathbb{Z}/3^{k+2}) \rightarrow \text{SL}_2(\mathbb{Z}/3^2)$, $k \geq 0$, is Frattini. Take $v \in \ker(\text{SL}_2(\mathbb{Z}/3^2) \rightarrow \text{SL}_2(\mathbb{Z}/3))$.

For $\tilde{v} \in \ker(\mathrm{SL}_2(\mathbb{Z}/3^3) \rightarrow \mathrm{SL}_2(\mathbb{Z}/3))$ lifting v , \tilde{v}^3 identifies with v , but in $\ker(\mathrm{SL}_2(\mathbb{Z}/3^3) \rightarrow \mathrm{SL}_2(\mathbb{Z}/3^2))$. From that stage, any subgroup mapping onto $\mathrm{SL}_2(\mathbb{Z}/3)$ has the kernel in it.

We are done except for showing $\mathrm{Ad}_3(\mathbb{Z}/\ell)$ is not the whole characteristic ℓ -Frattini module: Rem. 3.16. \square

Remark 3.16. Take $G = \mathrm{SL}_2(\mathbb{Z}/5)$. [Fr95, §II.F] shows ${}_5M_{\mathrm{SL}_5(\mathbb{Z}/5)}$ has Loewy display $M = \mathrm{Ad}_3(\mathbb{Z}/5) \rightarrow \mathrm{Ad}_3(\mathbb{Z}/5)$. (Applying Shapiro's Lemma, inducing the identify from the action of a D_5 in A_5 on $\mathbf{1}$, $\dim_{\mathbb{Z}/\ell}(H^2(G, M)) = 1$ (see Lem. 3.6). So ${}_5M_{\mathrm{SL}_5(\mathbb{Z}/5)}$ is not $\mathrm{Ad}_3(\mathbb{Z}/5)$. This argument can work for all $\ell \geq 5$ [Fr20, Chap. 6 §1.6] based on [Fr02b, §2.2.2].

[FrJ86, §22.14]₂ notes diagram (3.17) isn't the universal ℓ -Frattini cover of $\mathrm{PSL}_2(\mathbb{Z}/\ell)$. That, here, is beside the point: an ℓ -Frattini lattice quotient can only be the whole universal ℓ -Frattini cover when ${}_\ell M_G$ has dimension 1, since the lattice and all its finite index subgroups have bounded rank. The ℓ -Sylow of the universal ℓ -Frattini cover is a pro-free pro- ℓ group. From Schreier's Theorem [FrJ86, Prop. 17.5.7]₂, if its rank exceeds 1, the rank of its open subgroups grows with their indices.

4. MONODROMY AND ℓ -ADIC REPRESENTATIONS

§4.1 gives notation for how ℓ -Frattini lattice quotients give the ℓ -adic representations of the title. §4.2 formulates the **MT** main **OIT** conjectures based on Def. 3.14 – an eventually ℓ -Frattini sequence. (In Prop. 3.15, $\ell = 3$ is an example from Serre's case of the **OIT**.)

§4.3 shows how we use eventually ℓ -Frattini, including its Hilbert's Irreducibility aspects. In §4.4, the **RIGP** and the **OIT** problems interact on a **MT**. That includes discussing Serre's most difficult case: a model for how Falting's Theorem enters.

4.1. ℓ -adic representations. As usual, K is a number field. Also, as usual, assume \mathbf{C} is a rational union of conjugacy classes. From the **BCL** Thm. 5.20, the moduli definition field of the Hurwitz spaces – though maybe not **MT** levels on them – is \mathbb{Q} .

4.1.1. Geometric monodromy of a MT. Consider the Nielsen classes referenced by an ℓ -Frattini lattice quotient, $L^* \rightarrow G^* \xrightarrow{\tilde{\psi}^*} G$ of ${}_\ell \tilde{G}_{\mathrm{ab}} \rightarrow G$ in §3.3.1. We denoted the collection of H_r orbits – the actual **MTs** – by $\mathcal{F}_{G, \mathbf{C}, \ell, \ell \tilde{\psi}^*}$. Indicate one of these, say O , as a sequence of H_r orbits $\{O_k\}_{k=0}^\infty$ naturally mapped from level $k+1$ to level k analogous to (3.13).

From this (§2.2.2 or (2.16)) comes a canonical Hurwitz space tower,

$$(4.1) \quad \{\mathcal{H}(G_k, \mathbf{C})\}_{k=0}^\infty; \text{ the } (G, \mathbf{C}, \ell, L^\star) \text{ tower.}$$

Denote the collection of projective sequences of points

$$\bar{\mathbf{p}}' = \{\mathbf{p}'_k \in \mathcal{H}(G_k, \mathbf{C})^\dagger\}_{k=0}^\infty \text{ lying above } \mathbf{p}'_0 \text{ on the tower by } \bar{\mathbf{S}}_{\mathbf{p}'_0}.^{32}$$

Abusing notation slightly, also denote the similar sequences of points above $J' \in J_r$ by $\bar{\mathbf{S}}_{J'}$.

(4.2a) That the extension $L^\star \rightarrow G^\star \xrightarrow{\psi^\star} G$ is ℓ -Frattini is precisely why the construction of this Nielsen class tower is canonical.

(4.2b) A $\bar{\mathbf{p}}' \in \bar{\mathbf{S}}_{\mathbf{p}'_0}$ represents a surjection in $\text{Hom}_{\mathbb{Z}_\ell[G]}(H_1(\hat{W}_{\mathbf{p}'_0}, \mathbb{Z}_\ell), L^\star)$.

Ex. 1.3 is a particular example of this situation. All the spaces above have canonical polarizations, and therefore quasi-projective structures. That implies if, $\mathbf{p}_0 \in \mathcal{H}(G_0, \mathbf{C})(K)$ (resp. $J' \in J_r(K)$), then G_K acts on $\bar{\mathbf{S}}_{\mathbf{p}'_0}$ (resp. $\bar{\mathbf{S}}_{J'}$). It also acts on the **MT**s on the Hurwitz space tower.

Generalizing Ex. 1.3 starts with (4.3) and completes in §4.1.2. Use generic notation, $W_{\mathbf{p}} \rightarrow \mathbb{P}_z^1$, for a cover represented by $\mathbf{p} \in \mathcal{H}(G_0, \mathbf{C})^\dagger$.

(4.3a) Running over $\mathbf{p} \in \mathcal{H}(G_0, \mathbf{C})^\dagger$, create a total family of varieties with fibers $\text{Pic}^u(W_{\mathbf{p}})$, $u = 1, 2$, over $\mathcal{H}(G_0, \mathbf{C})^\dagger$.

(4.3b) Interpret elements of (4.2b) as subspaces, with an appropriate G action, on the $\mathbb{Z}_\ell[G]$ Tate module, $T_{\mathbf{p}}$, of $\text{Pic}(W_{\mathbf{p}})$.³³

(4.3c) Identify H_r action on Nielsen classes, and their orbits as **MT**s.

4.1.2. *Using* $G \mapsto G^{\text{n-lm}}$. §4.2 formulates the Main **OIT MT** conjectures starting from (4.2b). As in Thm. 5.24, it starts with the display from the **sh**-incidence matrix (as in §2.3.3). It relies on identifying geometric monodromy groups attached to the levels of a **MT**.

Suppose for each **MT** we are given \mathbf{p}' , a base point on a component of $\mathcal{H}(G_0, \mathbf{C})^\dagger$. How can we

$$(4.4) \quad \text{translate (4.3c) into } H_r \text{ acting on each Tate module, } T_{\mathbf{p}'}?$$

The point of the Nielsen class $(G^{\text{n-lm}}, \mathbf{C})$ from (G, \mathbf{C}) is to produce a recognizable group within which we can see the Hurwitz monodromy action, and thereby label the geometric (and arithmetic) monodromy groups

³²For the **RIGP** we often take \dagger equivalence to be inner, reduced. Here though in comparing with the **OIT** we need absolute equivalence, too.

³³We are sloughing off capturing that G^\star is a Frattini extension of G .

attached to a specific **MT**. Even in recognizing this in Serre's case, the process is illuminating. What makes it work is using the relation between inner and absolute classes given in (5.6) in §5.2.1.³⁴

A forerunner application in (5.13) [FrV92] uses this Inner-Absolute sequence for a presentation of $G_{\mathbb{Q}}$. That goes from (G, \mathbf{C}) to a new (G^*, \mathbf{C}^*) using its absolute (rather than inner) Hurwitz space. The result: $N_{S_N}(G^*, \mathbf{C}^*)$ – as in (2.15) – contains every outer automorphism of G^* .

Changing the Nielsen class to that of the limit group $G^{\text{n-lm}}$, moves all the H_r action into outer automorphisms of $G^{\text{n-lm}}$ as in (4.5). Substituting $G \mapsto G^{\text{n-lm}}$, keeps \mathbf{C} the same, using that $G^{\text{n-lm}}$ covers G . Then, canonically lift the classes \mathbf{C} to $G^{\text{n-lm}}$, and relate corresponding absolute and inner Nielsen classes.

Then, in applications the image of the H_r action interprets inside the symplectic group given as automorphisms of $H_1(W_{\mathbf{p}}, \mathbb{C})$, with its canonical pairing. In this process, the underlying inner (reduced) Hurwitz spaces in a **MT** all come out to be J_r . So, we can consider **MT** fibers over $J' \in J_r(\bar{\mathbb{Q}})$, just as he did on projective systems of order ℓ^{k+1} on the elliptic curve with j -invariant $j' \in \mathbb{P}_j^1 \setminus \{\infty\}$.

Example 4.1. Ex. 1.11 included an example for the modular curves denoted X_1 and X_0 as a special case of the mapping from inner to absolute Hurwitz spaces as in (5.6). [Fr20, Chap. 6, §3.3.1] shows how it works with limit group $G_{k+1}^{\text{n-lm}} = (\mathbb{Z}/\ell^{k+1})^2 \times^s \mathbb{Z}/2$, corresponding to $\mathbf{C} = \mathbf{C}_{2^4}$.

(4.5a) We construct $\text{Ni}(G_{\ell^{k+1}}^{\text{n-lm}}, \mathbf{C}_{2^4})$ from $\text{Ni}(D_{\ell^{k+1}}, \mathbf{C}_{2^4})$, with the H_4 action easily read off from this.

(4.5b) Apply $\mathbf{sh}, q_2 \in H_4$ directly to $\text{Ni}(G_{\ell^{k+1}}^{\text{n-lm}}, \mathbf{C}_{2^4})$. They interpret on $\text{SL}_2(\mathbb{Z}/\ell^{k+1})$ from the action on generators of $\text{SL}_2(\mathbb{Z}_{\ell})$.

From this, the geometric monodromy of

$$\bar{\mathcal{H}}(G_{\ell^{k+1}}^{\text{n-lm}}, \mathbf{C}_{2^4})^{\text{in,rd}} \rightarrow \mathbb{P}_j^1 = \bar{\mathcal{H}}(G_{\ell^{k+1}}^{\text{n-lm}}, \mathbf{C}_{2^4})^{\text{abs,rd}} \text{ is } \text{SL}_2(\mathbb{Z}/\ell^{k+1})/\{\pm 1\}.$$

That the arithmetic monodromy group is $\text{GL}_2(\mathbb{Z}/\ell^{k+1})/\{\pm 1\}$ interprets using the *Weil pairing* along the Hurwitz space giving

$$H^1(W_{\mathbf{p}}, \mathbb{Z}_{\ell}) \times H^1(W_{\mathbf{p}}, \mathbb{Z}_{\ell}) \rightarrow H^2(W_{\mathbf{p}}, \mathbb{Z}_{\ell}) \text{ coming from roots of 1.}$$

This is the extension of constants of \bullet [Fr78] \bullet . **MTs** gets it from the Heisenberg lift invariant as in Thm. 5.24 [Fr20, Chap. 6, Prop. 3.7]. \triangle

³⁴This is counterintuitive: The **RIGP** ends up about inner equivalence. How can it be that the **OIT** benefits from absolute equivalence?

4.2. MTs and the OIT generalization. Let us backtrack to (4.3c). This amounts to finding projective sequences, O , of braid orbits

$$(4.6) \quad \begin{aligned} & \{\text{Ni}_k \leq \text{Ni}(G_k, \mathbf{C})\}_{k=0}^\infty, k \geq 0: \\ & \psi_{k+1,k} : \text{Ni}_{k+1} \mapsto \text{Ni}_k \pmod{\ker(G_{k+1} \rightarrow G_k)}, k \leq 1. \end{aligned} \quad 35$$

Restricting the braid action (§2.2.2), this gives a projective sequence of finite morphisms of reduced and absolutely irreducible components:

$$(4.7) \quad \cdots \rightarrow \mathcal{H}_{k+1} \xrightarrow{\Psi_{k+1,k}} \mathcal{H}_k \rightarrow \cdots \rightarrow \mathcal{H}_1 \xrightarrow{\Psi_{1,0}} \mathcal{H}_0 \xrightarrow{\Psi_0} J_r.$$

Ex. 2.3 and Thm. 5.24 show the main computational techniques for distinguishing braid orbits:

$$(4.8) \quad \begin{aligned} & \text{the lift invariant and the shift-incidence matrix. Unless} \\ & \text{there at least 2 HM components, these have distinguished} \\ & \text{all components with their (moduli) definition fields.} \end{aligned} \quad 36$$

Again, using the **BCL** criterion for (G, \mathbf{C}) we assume $G_{\mathbb{Q}}$ acts on the set of **MTs**. Consider the $G_{\mathbb{Q}}$ orbit ${}_{\mathbb{Q}}O$ of O , with the level k component orbit denoted ${}_{\mathbb{Q}}\mathcal{H}_k$. For each k , (as in §2.1.2) consider the respective geometric and arithmetic monodromy groups of

$$\psi_k : \mathcal{H}_k \rightarrow J_r \text{ and of } {}_{\mathbb{Q}}\psi_k : {}_{\mathbb{Q}}\mathcal{H}_k \rightarrow J_r.$$

Denote the respective projective limits of these sequences by \mathcal{G}_O and ${}_{\mathbb{Q}}\hat{\mathcal{G}}_O$.

Use the notation, $\bar{\mathbf{S}}_{\mathbf{p}'}$ (resp. $\bar{\mathbf{S}}_{J'}$) for projective sequences of points, on O , over \mathbf{p}'_0 (resp. J') from §4.1.1 Consider the *decomposition group*, $\mathcal{G}_{\bar{\mathbf{p}}_{J'}}$, of $\bar{\mathbf{p}}_{J'} \in \bar{\mathbf{S}}_{J'}$: the projective limit of the groups of the Galois closures of $\mathbb{Q}_{\mathcal{H}_k}(\mathbf{p}'_k)/\mathbb{Q}_{\mathcal{H}_k}$. For all $\bar{\mathbf{p}}_{J'}$ these groups are conjugate inside ${}_{\mathbb{Q}}\hat{\mathcal{G}}_O$. Denote their isomorphism class by the symbol $\mathcal{G}(\bar{\mathbf{S}}_{J'})$.

Recall Def. 3.14 for an eventually ℓ -Frobenius sequence.

Conjecture 4.2 (Main **OIT** Conj.). With the notation above:

(4.9a) The geometric monodromy G_O of O is an eventually ℓ -Frobenius sequence; and

(4.9b) for each J' , $\mathcal{G}(\bar{\mathbf{S}}_{J'}) \cap \mathcal{G}_O$ is eventually ℓ -Frobenius.

Thm. 5.24 is evidence that (4.9) is true, beyond Serre's modular curve **OIT** case: as in Prop. 3.15. List (5.11) uses (4.9) language on Serre's **OIT**.

³⁵Ex. 1.11 has at all levels just one braid orbit, while other examples don't. So, O is one of many possible **MTs**.

³⁶§5.3.3 has discussion of **HM**-components, Def. 5.16, and their generalization. These components are transparent to obstruction (§3.4.1), but pose problems if there is more than one at a given level.

4.3. Using eventually ℓ -Frattini. For K a number field and $\hat{\psi} : \hat{W} \rightarrow \mathbb{P}_z^1$ a finite Galois cover K , with arithmetic monodromy group $G_{\hat{\psi}}$, denote the decomposition group of $z' \in \mathbb{P}_z^1(K)$ by $D_{z'}$. Here is a form of *Hilbert's Irreducibility Theorem* (**HIT**) [FrJ86, Chap. 12–13]₂.

$$(4.10) \quad \text{For a dense set of } z' \in \mathbb{P}_z^1(K), G_{\hat{\psi}} = D_{z'}.$$

Dense can mean by almost any measure, including Zariski dense, or p -adically dense for p a prime.

Proposition 4.3. *Use the k_0 from Def. 3.14. Combining **HIT** for the cover $\mathbb{Q}\psi_{k_0}$ and (4.9a) implies, for a dense set of $J' \in J_r(K)$, $\mathbb{Q}\hat{\mathcal{G}}_O = \mathcal{G}(\bar{\mathcal{S}}_{J'})$.*

Starting from \bullet [Se68] \bullet , [Fr20, Chap. 6 §3] reshapes it for **MTs**. [Se68] concentrates on two types of fibers over $j' \in U_j \stackrel{\text{def}}{=} \mathbb{P}_j^1(\bar{\mathbb{Q}}) \setminus \{\infty\}$.

(4.11a) Those called complex multiplication (CM) coming from j -invariants of elliptic curves with nontrivial rings of endomorphisms (Def. 5.11).

(4.11b) Those for j' that are not algebraic integers.

A promised Tate paper never materialized; it suggested *all* non-CM fibers (not just those in (4.11b)) would give fibers over j' of

$$(4.12) \quad \begin{array}{l} \text{GL}_2 \text{ type: } G_{\mathbb{Q}(j')} \text{ acts on lines of the elliptic curve 1st} \\ \ell\text{-adic cohomology as an open subset of } \text{GL}_2(\mathbb{Z}_\ell). \end{array}$$

Later Serre papers asked how (4.11b) varies with ℓ . For example:

Proposition 4.4. *For $j' \in U_j(\mathbb{Q})$ and almost all ℓ , $G_{\mathbb{Q}(j')} = \text{GL}_2(\mathbb{Z}_\ell)$.*

Also, there exist j' for which equality holds for all ℓ .

See Thm. 5.12 and Ogg's examples at the end of \bullet [Fr78] \bullet . That is appropriate for the part of Thm. 5.24 we display, too. In our case there are more than two types of fibers over $j' \in U_j(\bar{\mathbb{Q}})$. Yet, for almost all ℓ , they are all the expected general type.

4.4. RIGP and Faltings. Up to this point, the central object has been a Nielsen class. §4.4.1 asks a question without referring to any specific conjugacy classes, nor anything about **MTs**. Yet, the answer forces existence of **MTs**. §4.4.2 explains how Falting's Theorem engages the most difficult aspect of Serre's case of the **OIT**.

4.4.1. ℓ' RIGP. As in §3.3.1, consider $L^\star \rightarrow G^\star \xrightarrow{\tilde{\psi}^\star} G$, an ℓ -Frattini lattice quotient of ${}_\ell\tilde{G}_{\text{ab}} \rightarrow G$. Prop. 4.5 considers the **RIGP** for the collection $\mathcal{G}_{G,\ell,L^\star} \stackrel{\text{def}}{=} \{\psi_k : G_k \rightarrow G\}_{k=0}^\infty$ of corresponding ℓ -Frattini covering groups.

Call a regular realization ℓ' if it occurs with **C** consisting of ℓ' classes.

Proposition 4.5. *For any ℓ -perfect G , assume we have \mathbb{Q} regular realizations of each G_k (say, in $\text{Ni}(G, \mathbf{C}_k)$, $k \geq 0$). Assume also that $r_{\mathbf{C}_k}$ is bounded independent of k by B .*

*Then, there exists a **MT** for $\tilde{\psi}^*$ and some specific ℓ' classes \mathbf{C} with \mathbb{Q} points at every level \bullet [FrK97] \bullet .*

That is, if the Main **RIGP MT** Conjecture is true, ℓ' regular realization of all groups in $\mathcal{G}_{G,\ell}$ requires increasingly large numbers of branch points.

The easiest case, D_ℓ (ℓ odd), has two contrasting situations.

(4.13a) There are regular realizations of all the $\mathcal{G}_{D_\ell, \ell, \mathbb{Z}_\ell}$.

(4.13b) Beyond the few cases known by Mazur's Theorem (see \bullet [Fr78] \bullet), no one has produced any ℓ' realizations of the $\mathcal{G}_{D_\ell, \ell, \mathbb{Z}_\ell}$.

Regular realizations for (4.13a) are given in 1st year algebra. The branch point number increases with k .

We already discussed (4.13b): ℓ' regular realizations correspond to cyclotomic torsion points on hypelliptic Jacobians in §1.3.1.

Also, there are versions – Thm. 5.17 – of both parts of (4.13) for all the finite groups we have considered.

4.4.2. *GL₂ toughest point.* As in (4.2b), start from a point at level 0, \mathbf{p}'_0 , corresponding to a Galois cover, $\hat{\varphi}_{\mathbf{p}'_0} : \hat{W}_{\mathbf{p}'_0} \rightarrow \mathbb{P}_z^1$. Denote the complete collection of projective sequences over \mathbf{p}'_0 on, say, a **MT** by $\bar{\mathbf{S}}_{\mathbf{p}'_0}$.

To simplify relating the **RIGP** and the analog of Serre's case, assume $\mathbb{Q}_{\mathcal{H}_0} = \mathbb{Q}$ and $k_0 = 0$ in (1.12). This gives a $G_{\mathbb{Q}}$ action on $\bar{\mathbf{S}}_{\mathbf{p}'_0}$; each element gives a copy of the $\mathbb{Z}_\ell[G]$ module L^* §4.1.1 as a subspace on $H^1(\hat{W}_{\mathbf{p}'_0}, \mathbb{Z}_\ell)$. The ℓ -adic action maps $G_{\mathbb{Q}}$ into the group permuting such subspaces.

For the GL₂ type given by (4.11b), Serre used Tate's ℓ -adic elliptic curve with j -invariant not an ℓ -adic integer (discussion of (5.11b)). From *wild ramification* on the \mathbb{Z}/ℓ torsion on the Jacobian of the Tate elliptic curve at j' comes a crucial piece of $X_1(\mathbb{Z}/\ell) \rightarrow \mathbb{P}_z^1$ geometric monodromy.

Using the Frattini property of (4.9a), and the entwining of the two Nielsen classes of Ex. 1.11, that concluded Serre's result for such j' .

For general **MTs**, the analog of Serre's toughest GL₂ case – those not in (4.11b) – comes from $\mathbf{p}'_0 \in \mathcal{H}_0(\mathbb{Q})$. If fine moduli holds,

this automatically gives an ℓ' \mathbb{Q} regular realization in $\text{Ni}(G, \mathbf{C})$.

Completing Serre's **OIT** awaited Falting's Theorem [Fa83]. For K a number field, Conj. 5.8 says high **MT** levels have no K points (§5.3.3). Both \bullet [Fr06] and [CaTa09] \bullet used Faltings in their versions for $r = r_{\mathbf{C}} = 4$.

Faltings says, when the genus of a **MT** level exceeds 1, it has only finitely many rational points. If they exist (off the cusps) at each level – applying the Tychonoff Theorem – some subset of them would be part of a projective system of K points on the **MT**.

This contradicts Weil’s Theorem on Frobenius action on the first ℓ -adic cohomology. That is, the Frobenius on the corresponding subspace ($\equiv L^*$ as in (4.2b)) would be trivial, not of absolute value $q^{\frac{1}{2}}$) when you reduce the tower modulo a *good prime* $p|q$: p is good if p doesn’t divide the order of G (a consequence of [Gr71]).

[Fr06] used properties of cusps in the genus formula of Thm. 2.10, as in Ex. 2.3. See the discussion with ℓ -lattice quotients in § 3.4.3 for a particular case. While this is explicit for $r = 4$ as to the level where the genus rises, Faltings isn’t explicit for rational points disappearing at higher levels.

The general case using the entwining of Nielsen classes of (1.12) – as in the example below Thm. 5.24 – benefits from both [Fr06] and [CaTa09].

Completing the conjectured **OIT** requires a useful listing of the eventually ℓ -Frattini sequences in the geometric monodromy of the **MT**. Also, finding which are achieved in (4.9b) as decomposition groups for some values of $j' \in \mathbb{P}_z^1(\bar{\mathbb{Q}})$. [Fr20] considers this, for being as explicit as Serre’s **OIT** – as in the steps in (5.29) – only for the case of Thm. 5.24.

Consider an ℓ' **RIGP** realizations of an ℓ -Frattini cover $G_k \rightarrow G$ over a number field K . This corresponds to a K point \mathbf{p}_k on a **MT**.

Conclusion: showing the $J' \in J_r(K)$ below \mathbf{p}_k satisfies the **OIT** is akin to Serre’s toughest **OIT** case, requiring extending Falting’s Theorem.

5. THE PATH TO **M**(ODULAR)**T**(OWER)S

§5.1 emphasizes historical examples from the case $r = 4$ (four branch point covers) that motivated this paper as a prelude to [Fr20]. Considerable motivation came from interacting with Serre. This plays on modular curve virtues, as being upper half-plane quotients, even though congruence subgroups do not define the **MT** levels except when G is close to dihedral.

§5.2 elaborates on “What Gauss told . . .” a hidden history that has obscured nonabelian aspects of **R**(iemann)’s **E**(xistence)**T**(heorem). The universal Frattini cover of G allows launching into such non-abelian aspects, though [Fr20] takes a middle road.

Applications extending the **OIT** and using ℓ -adic monodromy come at the book’s end. The **RIGP** and its interpretations by rational points on Hurwitz spaces comes at the book’s beginning. The middle joins these, using the universal Frattini cover \tilde{G} , and braid orbits on Nielsen classes.

Suppose, you accede to taking on a serious simple group (say, A_5 , and a very small ℓ -Frattini cover (for the prime $\ell = 2$). For example, as in [Se92, Chap. 9], the short exact sequence $0 \rightarrow \mathbb{Z}/2 \rightarrow \text{Spin}_5 \rightarrow A_5 \rightarrow 1$. Then you might want to recognize the sequence for the abelianized 2-Frattini cover

$$(5.1) \quad 0 \rightarrow (\mathbb{Z}_2)^5 \rightarrow {}_2\tilde{A}_{5,\text{ab}} \rightarrow A_5 \rightarrow 1 \text{ [Fr95, Prop. 2.4]}$$

(and its characteristic quotients) as quite a challenge at the present time.³⁷ Even according to Conj 5.15 if you only had to find *any* number field K for which all those characteristic quotients have **RIGP** realizations over K . The Prop. 4.5 question without the bound B .

§5.3, the longest in this paper, explains the logic of the book, with extended abstracts on several papers on **MTs**, starting from 1995.

5.1. Historical motivations. §5.1.1 alludes to my interactions with Serre on two seemingly disparate topics, at far separated periods. It is meant to show how those topics came together. The brief §5.1.2 acknowledges two referees. It also connects to recent work that harkens to Serre's [Se92] where he notes there was an alternative **RIGP** approach.

5.1.1. *Guides.* This section is about on I learned from four examples.

(5.2a) Abel's spaces as level 0 of a **MT** classically denoted $\{X_0(\ell^{k+1})\}_{k=0}^\infty$.

(5.2b) The **MT** from the Nielsen class $\text{Ni}(A_5, \mathbf{C}_{3^4})$ and the prime $\ell = 2$.

(5.2c) The **MT** system, from the Nielsen classes $\text{Ni}((\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{3^4})$, running over primes ℓ , as my foray into an **OIT** beyond Serre's.³⁸

(5.2d) Relating $\{X_1(\ell^{k+1})\}_{k=0}^\infty$ and $\{X_0(\ell^{k+1})\}_{k=0}^\infty$ as a special case of a general relation between *inner* and *absolute* Hurwitz spaces.

In (5.2a) and (5.2d), there is a parameter k indicating a tower level. Since in these two cases, the tower levels are traditionally related to a power of a prime ℓ , I assume k as a level requires no more explanation.

My initial relation with the **OIT**, during my first decade as a mathematician, was based on (5.2a) on several stages using practical problems with considerable literature, on which this paper elaborates. The two series in (5.2d) are reasonably considered the mainstays of modular curves.

My interactions with Serre on [Se90a] and [Se90b], before they were written, related to my review of [Se92] (see [Fr94]), before it appeared caused me to go deeply into (5.2b). [Fr90], that Serre saw me present in Paris in 1988, had him write to me asking – essentially – for the lift invariant formula for the families of genus 0 covers in the Nielsen classes $\text{Ni}(A_n, \mathbf{C}_{3^{n-1}})$, $n \geq 4$.

³⁷ $\text{Spin}_5 \rightarrow A_5$ is the smallest nontrivial quotient of $\frac{1}{2}A_5 \rightarrow A_5$, as in (3.2).

³⁸What to do about $\ell = 3$, is tricky. (See •[FrH20] starting with (5.27).) •

For that reason, I have alluded to [Fr12] in the discussions. Especially, applied to a *braid orbit* on a Nielsen class,

the idea of the *braid lift invariant* from a *central Frattini cover*.

The comparison between general and central Frattini covers of a finite group appears in many places to interpret **MTs**.

I aimed with (5.2c) to show commonalities and differences from the source of Serre's **OIT** (5.2a). Especially I refer to what works for modular curves. Also, to what I learned that applies even to modular curves, though not previously observed, or the approach is different/illuminating. That is the concluding topic of [Fr20]. Therefore I am brief on it here, merely recording some of its results that show what is new from an example that goes beyond Serre's **OIT**.

5.1.2. *Settings for the **RIGP** and acknowledgements.* If **RIGP** realizations of G exist, where are they? From [FrV92] (and related), such must correspond to \mathbb{Q} points on Hurwitz spaces, with Main **RIGP** Conj. 5.8 a tool to understand that. We have three small subsections.

*Missing topic from **RIGP** vs IGP discussions:* Why has the **RIGP** (combined with Hilbert's Irreducibility Theorem) been more successful in finding **RIGP** realizations of groups? [NScW00] proves that all solvable groups are Galois groups, but not with **RIGP** realizations.

Something akin to **RIGP** realizations have now been formed for supersolvable groups [HWi20].³⁹ They use spaces that may not be rational varieties, but satisfy *weak-weak approximation*: (say, over \mathbb{Q}) weak approximation for finite sets of primes outside some specific finite set of primes.

[Se92, §3.5 and §3.6] shows that varieties, Z , with a weak-weak approximation in place of \mathbb{P}_z^1 suffice for Hilbert's irreducibility in this sense:⁴⁰

If $\varphi : W \rightarrow Z$ is a Galois cover with group G over \mathbb{Q} , then a dense set in $Z(\mathbb{Q})$ has irreducible φ fibers (producing G as a Galois group) [Ek90].

Instead of Noether covers, they embed G into $\mathrm{SL}_N(\mathbb{Q})$. Instead of directly using weak-weak approximation, they use *unramified Brauer-Manin conditions*. Prob. 5.1 gives two questions for this approach.

Problem 5.1. Imitate Lem. 5.22 with $\tilde{B} \times^s H$ where B instead of abelian is an ℓ -group, and \tilde{B} is the lattice quotient from Lem. 3.8. Assume H is

³⁹A subseries of groups $G > G_1 > G_2 > \cdots > G_t > \{1\}$ each of index a prime in the previous, but unlike solvable, each is normal in G .

⁴⁰A unirational variety – image of a projective space – was conjectured by Colliot-Thelene to have weak-weak approximation \implies Noether covers $\mathbb{A}^n \rightarrow \mathbb{A}^n/G$, $G \leq S_n$, would have an Hilbertian property.

weak-weak style regularly realized. Also, how would you relate to Nielsen classes in this weak-weak regular style?

What about homological stability: It played a big role in [FrV91] and [FrV92] where its use was subtle, in relating absolute and inner classes (a footnote in §4.1.2), and it will be in [Fr20]. This is the story of making changes in one of the two parameters in a Nielsen class, $\text{Ni}(G, \mathbf{C})$. §4.1.2 discussed fixing \mathbf{C} and changing G to $G^{\text{n-lm}}$.

Homological stability is a consideration when fixing G and letting \mathbf{C} change by increasing the multiplicity of appearance of (all) classes in \mathbf{C} , thereby getting a sequence of new conjugacy classes in G .

[FrV91, App.] found there is a k_0 with dimensions of $\{H^0(\mathcal{H}(G, \mathbf{C}_k))\}_{k \geq k_0}^\infty$ all the same: the order of a quotient of the Schur multiplier of G .⁴¹

§3.4.2 for A_n is a special case. Actually, [FrV92] used the distinguished component, wherein the lift invariant is trivial. We didn't write out the full theorem into print at the time. Since then [Sa19] has.

We suspected stable homotopy would continue to higher cohomology, giving a stable $H^1(\mathcal{H}(G, \mathbf{C}_k))$ on the components with trivial lift invariant – under the same high multiplicity condition on the \mathbf{C}_k s. This would make its ℓ -adic cohomology useful. That has been fulfilled by [EVW19] when G is dihedral.⁴²

This paper had two referees: Both helped the author make the best of its first versions. The first referee – known to me – more familiar with the area made suggestions, including assuring inclusion of topics related to work of himself and his cowriters. The second referee had serious reorganization suggestions that shortened the original abstract, then expanded the introduction to point to the definitions and §5.3 abstracts that would help guide the reader: **Thank you to both!**

5.2. What Gauss told Riemann about Abel's Theorem. The title is the same as that of the paper [Fr02]. It must be shocking that most upper half-plane quotients – j -line covers ramified over $0, 1, \infty$ – are not modular curves, or that they are related to practical problems.

One lesson from ${}_\ell\mathcal{X}_0 \stackrel{\text{def}}{=} \{X_0(\ell^{k+1})\}_{k=0}^\infty$ came from Galois. He computed the geometric monodromy of $X_0(\ell) \rightarrow \mathbb{P}_j^1$ (over \mathbb{C} ; $k = 0$), finding it to be $\text{PSL}_2(\mathbb{Z}/\ell)$ which is simple for $\ell \geq 5$. As an example of his famous theorem:

⁴¹Saying this precisely requires the lift invariant definition without the assumption on \mathbf{C} in Def. 2.13.

⁴²Actually, their condition was that G is a group of order congruent to 2 mod 4 and they have used the conjugacy classes \mathbf{C} to be supported in the unique class of involutions.

radicals don't generate the algebraic functions describing modular curve covers of the j -line.

5.2.1. *Early OIT and MT relations.* Recall Ex. 1.3. The **MT** description of $X_0(\ell)$ in (5.2a): they are *absolute* – see (2.15b) – reduced Hurwitz spaces $\mathcal{H}(D_{\ell^{k+1}}, \mathbf{C}_{24})^{\text{abs,rd}}$. There were two stages in this recognition.

Stage 1: [Fr78, §2] starts the observation of the close relation between Serre's **OIT** and the description of *Schur covers* $f \in \mathbb{Q}(w)$. Those are rational functions $f \in \mathbb{Q}(w)$ for which:

$$(5.3) \quad f : \mathbb{P}_w^1(\mathbb{Z}/p) \rightarrow \mathbb{P}_z^1(\mathbb{Z}/p) \text{ is 1-1 for } \infty\text{-ly many } p.$$

Similarly, over any number field K replace $\mathbb{Q}(w)$ by $K(w)$, and \mathbb{Z}/p by residue class fields $\mathcal{O}_K/\mathfrak{p} = \mathbb{F}_{\mathfrak{p}}$:

$$(5.4) \quad f \text{ is 1-1 on } \mathbb{P}_w^1(\mathbb{F}_{\mathfrak{p}}) \text{ for } \infty\text{-ly many } \mathfrak{p}.$$

The Galois closure of such an f may only be defined over a proper extension \hat{K}/K .⁴³ Indeed, for f to be a Schur cover over K , we must have $\hat{K} \neq K$.

Use the permutation notation of Rem. 2.5, $T_f : \hat{G}_f \rightarrow S_n$ and the respective stabilizers of 1 by $\hat{G}_f(1)$, $G_f(1)$.

To prevent accidents, define the exceptional set (for the Schur property):

$$\text{Exc}_{f,K} = \{\mathfrak{p} \mid f \text{ is one-one on } \infty\text{-ly many extensions of } \mathbb{F}_{\mathfrak{p}}\}.$$

If f_i , $i = 1, 2$, are exceptional over K , then

$$(5.5) \quad \text{so is } f_1 \circ f_2, \text{ if } |\text{Exc}_{f_1,K} \cap \text{Exc}_{f_2,K}| = \infty.$$

That is, $f_1 \circ f_2 = f$ is a decomposition of f over K . The condition that f is *indecomposable* over K is that $T_f : \hat{G}_f \rightarrow S_n$ is *primitive*: There is no group H properly between $\hat{G}_f(1)$ and \hat{G}_f .

Problem 5.2 (Schur Covers). Explicitly describe *exceptional* f *indecomposable* over K .

Thm. 5.3 connects exceptional f with the **OIT**: exceptionality relates the arithmetic and geometric monodromy of the covers from f .

Theorem 5.3. *The following is equivalent to exceptionality.*

There exists $\hat{g} \in \hat{G}_f(1)$, such that each orbit of $\langle G_f(1), \hat{g} \rangle$ on $\{2, \dots, n\}$ breaks into (strictly) smaller orbits under $G_f(1)$ [Fr78, Prop. 2.1].

Thm. 5.3 holds for essentially any cover (absolutely irreducible over K ; the sphere need not be the domain) [Fr05, Prop. 2.3]. This application of a wide ranging Chebotarev density theorem is a case of *monodromy precision*.

⁴³Extension of constants as in Ex. 4.1.

Usually a *Chebotarev density* application in the \implies direction isn't so precise. Yet, here instead of saying f is *almost* one-one, it implies it is exactly one-one for ∞ -ly many \mathbf{p} .

[Fr05] expanded on situations giving monodromy precision. This is as an improvement on the intricate industry refining the appropriate error term in the Riemann hypothesis over finite fields.

To relate to the **OIT** for rational f , it turned out enough to concentrate on two cases with ℓ prime: $\deg(f) = \ell$, or ℓ^2 . The next step was to describe those Nielsen classes that produce the exceptional f . [Fr78, Thm. 2.1] lists the Nielsen classes (of genus 0 covers) for $\deg(f) = \ell$ that satisfy these conditions, noting a short list of 3-branch point cases, with a main case of $r = 4$ branch points forming one connected family.

Thereby, it identifies $X_0(\ell^{k+1})$ (resp. $X_1(\ell^{k+1})$) as reduced absolute (resp. inner) Hurwitz spaces as in (5.2d). This was a special case of [Fr78, §3], the *extension of constants* rubric for covers, by going to their Galois closure.

This sequence encodes the moduli interpretation Inner to Absolute:

$$(5.6) \quad \mathcal{H}(G, \mathbf{C})^{\text{in}} \rightarrow \mathcal{H}(G, \mathbf{C})^{\text{abs}} \rightarrow U_r$$

and its expansion to total spaces (over $U_r \times \mathbb{P}_z^1$) as in [FrV91, Thm. 1]. This relates G_f (geometric) and \hat{G}_f (arithmetic) monodromy, as in §2.1.

[Fr78, §2] shows, for prime degree ℓ rational functions, identifying Schur covers is essentially equivalent to the theory of complex multiplication. Further, from that theory, we may describe $\text{Exc}_{f,K}$ as an explicit union of arithmetic progressions, thereby allowing testing the condition (5.5).

Describing prime-squared degree exceptional rational functions interprets the GL_2 part of Serre's **OIT**, as in §4.3 [Fr05, §6.1–§6.3]. We state the main point, over \mathbb{Q} , again using precision. This also fits the inner-absolute Hurwitz space relation above by using the limit Nielsen class $\text{Ni}((\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/2, \mathbf{C}_{2^4})$ Ex. 4.1, still a modular curve case. The Nielsen class collection $\{\text{Ni}((\mathbb{Z}/\ell^{k+1})^2 \times^s \mathbb{Z}/2, \mathbf{C}_{2^4})\}_{k=0}^\infty$ identifies with a modular curve tower [Fr05, Prop. 6.6].

For a given ℓ , from Serre's (eventual, Thm.5.12) version of the **OIT** we conclude this. If the elliptic curve E (say, over \mathbb{Q}) has a GL_2 j -invariant, $j_E = j_0$, then the corresponding degree ℓ^2 rational function f_{j_0, ℓ^2} has arithmetic/geometric monodromy group quotient $\text{GL}_2(\mathbb{Z}/\ell)/\{\pm 1\} = G(\mathbb{Q}_{j', \ell^2}/\mathbb{Q})$ for all primes $\ell \geq \ell_0$ for some ℓ_0 dependent on j_0 [Se97b]. For a Galois extension \hat{L}/\mathbb{Q} , use $\text{Fr}_{L,p}$ for the Frobenius (conjugacy class) at p .

Proposition 5.4. *Given any $\ell \geq \ell_0$ as above, $\text{Exc}_{f_{j', \ell^2}}$ is the set of p so that the group $\langle -1, \text{Fr}_{\mathbb{Q}_{j', \ell^2, p}} \rangle$ acts irreducibly on $(\mathbb{Z}/\ell)^2$. This is always infinite.*

Proof. The classical Chebotarev density theorem implies $\text{Exc}_{f_{j', \ell^2}}$ is infinite if any element of $\text{GL}_2(\mathbb{Z}/\ell)$ acts irreducibly on $(\mathbb{Z}/\ell)^2$. For example, on the degree 2 extension \mathbb{F}_{ℓ^2} of $\mathbb{Z}/\ell = \mathbb{F}_\ell$, multiply by a primitive generator α of $\mathbb{F}_{\ell^2}/\mathbb{F}_\ell$ to get an invertible 2×2 matrix with no invariant subspace. \square

The GL_2 case is vastly different from the CM case in that the exceptional set described in Prop. 5.4 is definitely not a union of arithmetic progressions. [Fr05, §6.3.2] relates to [Se81] on using the (conjectural) Langlands program to consider these exceptional sets.

Guralnick-Müller-Saxl [GMS03] show that – excluding those above – other indecomposable Schur covers by rational functions, are *sporadic*. That is, they correspond to points on a finite set of Hurwitz space components.

Definition 5.5 (Named Nielsen Classes). Use the respective names CM and GL_2 for the Nielsen classes $\text{Ni}(D_\ell, \mathbf{C}_{24})^{\dagger, \text{rd}}$ and $\text{Ni}((\mathbb{Z}/\ell)^2 \times {}^s\mathbb{Z}/2, \mathbf{C}_{24})^{\dagger, \text{rd}}$ (or to the whole series with $\ell^{k+1}, k \geq 0$ in place of ℓ), with \dagger referring to either inner or absolute equivalence, and the relation between them.

Stage 2 discussion, and its relation to the **OIT**, couldn't happen until there was a full formulation of **MTs**, the topic of §5.3. Still, a transitional phase after Stage 1 occurred with dihedral groups and the space of hyperelliptic jacobians as in §1.3.1 and §5.2.3.

The **MT** free question of Prop. 4.5 shows how **MTs** for each finite group G and ℓ -perfect prime of G generalizes what the same question applied to dihedral groups posed for hyperelliptic jacobians. This – expanded in • [CaD08] • – shows the **MT** project, through the **RIGP**, tying to classical considerations.

5.2.2. *Competition between algebra and analysis.* The full title of [Fr02, §7] is *Competition between algebraic and analytic approaches*. This subsection consists of brief extracts from it and [Fr02, §10]. §7 was gleaned partly from [Ne81], and my own observation of [Ahl79] and [Sp57]. §10 was a personal “modern” dealing with the common ‘appreciation’ of mathematical genius.

Riemann's early education [Fr02, §7.1]: Riemann was suitable, as no other German mathematician then, to effect the first synthesis of the “French” and “German” approaches in general complex function theory.

Competition between Riemann and Weierstrass [Fr02, §7.2] and [Ne81, p. 93]: In 1856 the competition between Riemann and Weierstrass became

intense, around the solution of the *Jacobi Inversion problem*. Weierstrass consequently withdrew the 3rd installment of his investigations, which he had in the meantime finished and submitted to the Berlin Academy.

Soon after Riemann died [Fr02, §7.3] and [Ne81, p. 96]: After Riemann's death, Weierstrass attacked his methods often and even openly. Curiously, the unique reference in Ahlfor's book was this:

Without use of integration R.L.Plunkett proved the continuity of the derivative (BAMS65,1959). E.H. Connell and P. Porcelli proved existence of all derivatives (BAMS 67, 1961). Both proofs lean on a topological theorem due to G. T. Whyburn.

This is an oblique reference to to Riemann's use of *Dirichlet's Principle* for constructing the universal covering space of a Riemann surface.

There is a complication in analyzing Neuenschwanden's thesis that this resulted in mathematicians accepting Riemann's methods. How does this event resurrect the esteem of Riemann's geometric/analytic view?

Final anecdote [Fr02, §10]: While at the Planck Institute in Bonn, to give talks in the early 21st Century, I visited Martina Mittag, a humanities scholar, who had earlier visited UC Irvine. In private conversation she railed that mathematicians lacked the imagination of humanities scholars. Yet, she was vehement on the virtues of Einstein.

I explained that Einstein was far from without precedent; that we mathematicians had geniuses of his imaginative. My example was Riemann: I called him the man who formed the equations that gave Einstein his scalar curvature criterion for gravity: his thesis, and admittedly not my expertise. "Mike," she said, "You're just making that up! Who is Riemann?"

I took the *R* book in her (German) encyclopedia series from the shelves on her walls, without the slightest idea of what I would find. Opening to Riemann, I found this [in German] in the first paragraph:

Bernhard Riemann was one of the most profound geniuses of modern times. Notable among his discoveries were the equations that Einstein later applied to general relativity theory.

In modern parlance, what Gauss explained to Riemann was what – when I was young – were called the *cuts*. These are always displayed with pictures that are impossible – not hard, but rather *cannot* exist, as in [Fr80, Chap. 4, §2.4] under the title "Cuts and Impossible Pictures."

The pictures usually apply to covers that are cyclic, degree 2 or 3, as in [Con78, p. 243], which, though, is excellent in many ways for students not comfortable with algebra. What Riemann learned, again in modern parlance, is that you don't need – explicitly – the universal covering space, nor a subgroup of its automorphism group, to produce covers.⁴⁴

5.2.3. *Profinite: Frattini and Grothendieck.* I gleaned §5.2.2 from reading long ago (from [Sp57]), that Riemann's θ s, in a sense defined Torelli space, the period matrix cover of the moduli space, $\mathcal{M}_{\mathbf{g}}$, of curves of genus \mathbf{g} . This codifies the integrals that Riemann used to introduce one version of moduli of curves. [Sp57] does explain fundamental groups. Yet, it always relies on universal covering spaces.

One famous theorem is that the universal covering space of $\mathcal{M}_{\mathbf{g}}$ is a (simply-connected, Teichmüller) ball: It is contractible. Many beautiful pictures of fundamental domains come from this.

This paper (and [Fr20]) uses Hurwitz spaces, to investigate the place of one group, G , at a time. Yet, it considers the full gamut of its appearances through varying conjugacy classes defining covers with G as monodromy. In doing that it replaces universal covering spaces with the *Universal Frattini* cover $\tilde{G} \rightarrow G$, and its abelianized version $\tilde{G}_{\text{ab}} \rightarrow G$ (as in §3.1).

It introduces new tools, albeit with a topological component: the *lift invariant* (as in Thm. 3.13) and the **sh**-*incidence matrix* (as in §2.3.3 with applications such as Prop. 2.15 et. al.). These tie directly to group cohomology/modular representations. Thus by-passing relevant but famously difficult problems like dealing with non-congruence subgroups.

The relation between $\mathcal{M}_{\mathbf{g}}$ and Hurwitz spaces starts by realizing that the latter generalizes the former, adding data that divides the former into smaller pieces. Using that division effectively does not require you must know all finite groups (or even all simple groups).

[Fr20, App. B] has a section on how each problem appears to have its own appropriate finite groups, based on a well-known paradigm – *the Genus 0 problem*. This is commentary on [Fr05], a guide inspired by solving problems like *Schur's*, *Davenport's* and *Schinzel's*, that came from the middle of the 20th century or before.

There are ∞ -ly many spaces $\mathcal{M}_{\mathbf{g}}$ (resp. $\mathcal{H}(G, \mathbf{C}, \ell)$) indexed by \mathbf{g} (resp. (G, \mathbf{C}, ℓ)) in each case. The indexing seems more complicated in the

⁴⁴As in §1.2.3 remarks that for $r = 4$, you aren't using congruence subgroups to form the reduced Hurwitz spaces for $r = 4$.

Hurwitz case. Yet, in the former case, you suspect they should all fit together. Grothendieck's famous *Teichmüller* group attempted to gather their presence together into one profinite group with an hypothesis that he was describing $G_{\mathbb{Q}}$.

That created quite an industry. Still, [FrV92] showed that the Hurwitz space approach was up to the challenge of describing properties of $G_{\mathbb{Q}}$ that most mathematicians can understand. For example (5.7).

Theorem 5.6. *We may choose a(n infinite) Galois algebraic extension L/\mathbb{Q} so that $G_{\mathbb{Q}}$ has a presentation (see also Conj. 5.13):*

$$(5.7) \quad 1 \rightarrow F_{\omega} = G_L \rightarrow G_{\mathbb{Q}} \rightarrow \Pi_{n=2}^{\infty} S_n \stackrel{\text{def}}{=} \mathcal{S}_{\infty} \rightarrow 1$$

That is $G_{\mathbb{Q}}$, has a product of S_n s as a quotient (the Galois group of L/\mathbb{Q}) with the kernel a pro-free group on a countable set of generators.

This overview result hid that these were practical techniques giving a new context connecting classical problems to the **RIGP**. We illustrated that first with Thm. 4.5 connecting involution realizations of dihedral groups with torsion points on hyperelliptic jacobians – as developed in §1.3.1 and Prop. 1.5 with its direct connection to Conj. 5.7.

Conjecture 5.7. Torsion Conjecture: There is a negative conclusion to statement (1.7) on existence of a \mathbb{Q} cyclotomic point of order ℓ^{k+1} , for each k , among all hyperelliptic jacobians of any fixed dimension d .

B -free Conjecture: Without any bound B , for each ℓ^{k+1} there is a \mathbb{Q} cyclotomic point on some hyperelliptic jacobian, corresponding to a $(D_{\ell^{k+1}}, \mathbf{C}_{2^r})$ (r dependent on ℓ^{k+1}) **RIGP** involution realization.

Despite the last part of Conj. 5.7, no one has found those **RIGP** *involution* realizations beyond $r = 4$ and $\ell = 7$. The theme of [Fr94, §7] – using this paper's notation – still seems reasonable. For any prime $\ell \geq 3$, as in §5.2.1, and given a choice, you should rather

regularly realize the *Monster* than the collection $\{D_{\ell^{k+1}}\}_{k=0}^{\infty}$.⁴⁵

Thm. 4.5 generalizes this consideration to all ℓ -perfect finite groups (for example the A_5 , $\ell = 2$ case of (5.1)). Then, the negative conclusion using the Torsion conjecture generalization of Conj. 5.7 as in •[CaD08] •. This leads to progress, Thm. 5.9, on Conj. 5.8.

⁴⁵referring to the famous *Monster* simple group.

Conjecture 5.8 (Main **RIGP** Conj.). High **MT** levels have *general type* and no \mathbb{Q} points.⁴⁶

Theorem 5.9. *Conj. 5.8 is true for $r = 4$, where \mathcal{H}_k s are upper half-plane quotients. Thm. 2.10 is a tool for showing the genus rises with k .*

For K a number field, concluding in Conj. 5.8 that high **MT** levels have no \mathbb{Q} points is of significance only if there is a uniform bound on the definition fields of the **MT** levels. Therefore distinguishing between towers with such a uniform bound, and figuring the definition field as the levels grow if there is no uniform bound, is a major problem.

Our approach allows us to compute, and to list properties of **MT** levels. This is progress in meeting Grothendieck's objection that jacobian correspondences impossibly complicate generalizing Serre's **OIT**.

In our [Fr20, §5] example, that complication is measured by the appearance of distinct Hurwitz space components. The *lift invariant* accounts for most. Still, others pose a problem at this time – we know them, but not their moduli definition fields, as *Harbater-Mumford* components.

That problem occurs because there is more than one with the same 0 lift invariant as discussed around Prop. 2.15 and Thm. 5.24. As in Thm. 2.10, we know their braid orbits on the Nielsen class; modular curves and complex multiplication are not a guide.

5.3. The TimeLine of the MT program. After a prelude we have divided this section into three subsections:

- §5.3.2, prior to 1995;
- §5.3.3, the next decade of constructions/main conjectures; and
- §5.3.4, progress on the Main **OIT** 4.2 and **RIGP** 5.8 conjectures.

5.3.1. Organization. Each TimeLine item connects to a fuller explanation of the history/significance of a paper's contributions.

We trace the **RIGP** literature, starting with the definition of Nielsen classes (Def. 2.4 and Thm. 2.3), then going to **MT** conjectures as in §1.1.

Notation reminder: $\text{Ni}(G, \mathbf{C})$ referencing (unordered) conjugacy classes $\mathbf{C} = \{C_1, \dots, C_r\}$ of a finite group G . **R**(iemann)-**H**(urwitz) (2.8) gives the genus $\mathbf{g} \stackrel{\text{def}}{=} \mathbf{g}_{\mathbf{g}}$ of a sphere cover corresponding to $(g_1, \dots, g_r) \in \text{Ni}(G, \mathbf{C})$.

The *Branch Cycle Lemma* (**BCL**) ties moduli definition fields (Def. 1.2) of covers (and their automorphisms) to branch point locations. A whole

⁴⁶*General type* means that some multiple of the *canonical bundle* gives a projective embedding of the variety.

section in [Fr20, Chap. 2 §4] is taken with the **BCL** for good reasons. There is nothing else quite like it in most moduli space thinking.

Especially, it gives the precise moduli definition field of Hurwitz *families* for $\text{Ni}(G, \mathbf{C})$ for any equivalences (as in §2.2.2). While this is key for number theory (on the **RIGP**, and generalizing Serre's **OIT**), we emphasize one easily stated corollary of Lem. 5.14.

Theorem 5.10. *The total space of an inner (even as a reduced) Hurwitz space, $\mathcal{H}(G, \mathbf{C})^{\text{in}}$ together with its extra structure as a moduli space of \mathbb{P}_z^1 covers, is a cyclotomic field given in the response to (5.18c) as (5.20).⁴⁷ In particular, an inner Hurwitz (moduli) space structure is defined over \mathbb{Q} if and only if*

$$\mathbf{C}^u = \mathbf{C} \text{ for all } u \in (\mathbb{Z}/N_{\mathbf{C}})^*: \mathbf{C} \text{ is a rational union.}^{48}$$

For this reason we use rational unions of conjugacy classes in all examples. Individual **MTs** have an attached prime (denoted p in the early papers, but ℓ here because of the latest work).⁴⁹

We support the somewhat abstract description of a **MT** in §1.3.3 with many examples. **MTs** come with what we call the usual **MT** conditions:

(5.8a) Each has an attached group G , and a collection of r conjugacy classes, \mathbf{C} in G with ℓ' elements (of orders prime to ℓ).

(5.8b) Further, G is ℓ -perfect: ℓ divides $|G|$, but G has no surjective homomorphism to \mathbb{Z}/ℓ .

For G a dihedral group, with ℓ odd and $r = 4$, we are in the case of *modular curve towers*. So, **MTs** generalizes modular curves towers. Since there are so many ℓ -perfect groups, the generalization is huge.

Conditions to form a **MT** begin with an ℓ -Frattini lattice quotient §3.3.1 of the universal (abelianized) ℓ -Frattini cover ${}_{\ell}\tilde{\psi} : {}_{\ell}\tilde{G}_{\text{ab}} \rightarrow G$ of G §3.1. Given such a lattice quotient, ℓ' conjugacy classes \mathbf{C} of G and a braid orbit O on a Nielsen class $\text{Ni}(G, \mathbf{C})$, there is a succinct cohomology test, Thm. 3.13, for existence of a **MT** – and therefore of its corresponding projective sequence of nonempty absolutely irreducible varieties above O .

When we started this project, we considered our Main **RIGP** conjecture 5.8 only on the maximal lattice quotient. With, however, solid evidence

⁴⁷Total space means to includes a representing family of covers $\mathcal{T} \rightarrow \mathcal{H}(G, \mathbf{C})^{\text{in}} \times \mathbb{P}_z^1$ in the case of fine moduli, where there is one, and standard generalizations of this.

⁴⁸Here \mathbf{C}^u means to put each element of \mathbf{C} to the power u .

⁴⁹It is insufficient to say that the Hurwitz space is defined over a given field. Examples both old and new, [Fr20, Chap. 2 §4.3], include those with components isomorphic as covers of U_r , but inequivalent as moduli carrying family structures.

from $(A_5, \mathbf{C}_{3^4}, \ell = 2)$, and related examples (§2.3.1), among good reasons for opening up the territory to all lattice quotients, we could then include diophantine connections to classical spaces. Our main examples do just that, thus benefitting the classical spaces, too.

MT data passing the lift invariant test gives an infinite (projective) system of nonempty levels. Using reduced equivalence of \mathbb{P}_z^1 covers, as in Ex. 2.3, each level, \mathcal{H}'_k , has a projective normalized compactification $\bar{\mathcal{H}}'_k$. This $r-3$ dimensional algebric variety covers the compactification of J_r , a quotient of an open subset of \mathbb{P}^r by $\mathrm{SL}_2(\mathbb{C})$. For $r = 4$, this is the classical j -line, \mathbb{P}_j^1 .

Here is a paraphrase of Main **RIGP** Conj. 5.8:

(5.9a) High tower levels have general type; and

(5.9b) even if all levels have a fixed definition field K , finite over \mathbb{Q} , still K points disappear (off the cusps) at high levels.

Bringing particular **MTs** alive plays on *cusps*, as do modular curves. Cusps already appeared in Thm. 2.10 as the disjoint cycles of γ_∞ (corresponding to the points over ∞ on the j -line). §5.3.4 of our TimeLine precisely compares **MT** cusps with those of modular curve towers, and consequences of this.

It also discusses two different methods giving substantial progress on the Main Conjectures. The argument of §4.4 – dependent on Falting’s Theorem – shows why (5.9a) implies (5.9b) when $r = 4$.

The **sh**(ift)-incidence matrix graphical device, used in the table above Prop. 2.15, displays these cusps, and the components – corresponding to blocks in the matrix – in which they fall.

Several papers emphasize that cusps for Hurwitz spaces often have extra structure – meaningful enough to suggest special names for them – that come from group theory in ways that doesn’t appear in the usual function theory approach to cusps. The use of the names *Harbater-Mumford* and *double identity* cusps in Thm. 5.24 are examples of these.

We emphasize two connections between abelian varieties and **MTs**: The **S**(trong) **T**(orsion) **C**(onjecture) •[CaD08]• and ℓ -adic representations §4.1. Here is the main **MT** device for applications to the **RIGP** and the **OIT**.

Give meaningful labels to cusps on a component
of the space of covers of \mathbb{P}_z^1 defined by a Nielsen class.

5.3.2. *Lessons from Dihedral groups Before ’95.* This section goes from well-known projects to their connection with the **MT** program. The references to Serre’s work was around two very different types of mathematics:

His **OIT**, with its hints of a bigger presence of Hilbert’s Irreducibility Theorem, and his desire to understand the difficulty of regularly realizing the Spin cover of A_n . This section concludes with \bullet [DFr94] \bullet when the project divided into two branches.

The arithmetic concentrated in the hands of Pierre Dèbes and his collaborators Cadoret, Deschamps and Emsalem. The structure of particular **MTs** – based on homological algebra and cusp and component geometry of the spaces – follows my papers and my relation to Bailey and Kopeliovic with influence from the work discussed with Liu-Osserman and Weigel. Effect of quoted work of Ihara, Matsumoto and Wewers, all present at my first **MT** talks, is harder to classify.

\star [Sh64] \star : I studied this during my two year post-doctoral 67–69 at IAS (the *Inst. for Advanced Study*). Standout observation: Relating a moduli space’s properties to objects represented by its points, through the *Weil co-cycle condition*. That lead to *fine moduli* conditions on absolute, inner and reduced Hurwitz spaces (resp. [Fr77, Thm. 5.1] and [BFr02, §4.3]).

Having fine moduli gives positive solutions for a group G toward the **RIGP** from rational points on inner moduli spaces. Use the notation for the stabilizer of the integer 1 in the representation T given by Rem. 2.5. Here are those respective conditions (as in (2.15)):

(5.10a) Absolute equivalence: Given $T : G \rightarrow S_n$, as a subgroup of G , $G(T, 1)$ is its own normalizer.

(5.10b) Inner equivalence: G is centerless.

Refer to (G, T) satisfying (5.10a) as *self-normalizing*. Fine moduli gives a (unique) total space \mathcal{T} , over $\mathcal{H} \times \mathbb{P}_z^1$, with \mathcal{H} the Hurwitz space. This represents covers corresponding to $\mathbf{p} \in \mathcal{H}$:

$$\mathcal{T}_{\mathbf{p}} \rightarrow \mathbf{p} \times \mathbb{P}_z^1 \text{ from pullback of } \mathbf{p} \times \mathbb{P}_z^1 \text{ to } \mathcal{T}.$$

The fine moduli condition, with the addition of reduced equivalence to Nielsen classes (Def. 2.8) is in [BFr02, §4.3], as in our example Ex. 2.16.

Results: The **BCL** Thm. 5.20 and early uses starting with the solution of *Davenport’s problem* as in [Fr12, §1], for problems not previously considered as moduli-related. Later: A model for producing “automorphic functions” supporting the Torelli analogy through θ nulls on a Hurwitz space (§5.2.3) as in [Fr10, §6].

\star [Se68] \star : Serre gave one lecture on his book during my 2nd year (1968–1969) post-doctoral at IAS. His amenuenses were writing his notes. I asked

questions and interpreted the hoped for theorem – a little different than did Serre – as this. For each fixed ℓ as $j' \in \bar{\mathbb{Q}}$ varies, consider the field, ${}_{\ell}\mathcal{K}_{j'}$, generated over $\mathbb{Q}(j')$ by coordinates of any projective sequence of points

$${}_{\ell}\mathbf{x}' \stackrel{\text{def}}{=} \{x'_k \in X_0(\ell^{k+1})\}_{k=0}^{\infty} \mid \cdots \mapsto x'_{k+1} \mapsto x'_k \mapsto \cdots \mapsto j'.$$

Denote the Galois closure of ${}_{\ell}\mathcal{K}_{j'}/\mathbb{Q}(j')$ by ${}_{\ell}\hat{\mathcal{K}}_{j'}$, and its Galois group, the *decomposition group* at j' , by ${}_{\ell}\hat{G}_{j'}$.⁵⁰ Imitating the notation of the arithmetic monodromy group of a cover in Def. 2.1, denote the arithmetic monodromy group of the cover

$${}_{\ell}\varphi_{j,k} : X_0(\ell^{k+1}) \rightarrow \mathbb{P}_j^1 \text{ by } {}_{\ell}\hat{G}_{\ell\varphi_{j,k}} \stackrel{\text{def}}{=} {}_{\ell}\hat{G}_{j,k} \text{ and } {}_{\ell}\hat{G}_j \text{ its projective limit.}$$

Similarly, without the $\hat{}$, the projective limit of the geometric monodromy is ${}_{\ell}G_j$. For $j' \in \bar{\mathbb{Q}}$, in a natural way ${}_{\ell}\hat{G}_{j'} \leq {}_{\ell}\hat{G}_j$.⁵¹

Such fields don't vary smoothly: they bifurcate into two very distinctive types: CM (4.11a) from the theory of complex multiplication which takes up a great part of [Se68]; and GL_2 . The last rightly divides into two types itself: (4.11b) for j' not an algebraic integer on which Serre's book gets a grasp, and (4.12). §4.4.2 explains the extreme difference between these two, despite that in both cases ${}_{\ell}\hat{G}_j$ is open in $\text{GL}_2(\mathbb{Z}_{\ell})$.

Using the Def. 3.14 of eventually ℓ -Frobenius in Prop. 3.15 and §4.2, what made an impact on our approach from [Se68] was this.

- (5.11a) For each fixed ℓ , ${}_{\ell}G_j$ is eventually ℓ -Frobenius. Further, for $\ell > 3$, it is ℓ -Frobenius (right from the beginning).
- (5.11b) If for some prime p , $j' \in \bar{\mathbb{Q}}$ is not integral at p , then the intersection ${}_{\ell}\hat{G}_{j'} \cap {}_{\ell}G_j$ is open in ${}_{\ell}G_j$.
- (5.11c) If a given j' has complex multiplication type (Def. 5.11), then the intersection of ${}_{\ell}\hat{G}_{j'}$ with ${}_{\ell}G_j$ is eventually ℓ -Frobenius.
- (5.11d) From either (5.11b) or (5.11c), you have only to get to a value of k' with ${}_{\ell}\hat{G}_{j',k'}$ within the ℓ -Frobenius region to assure achieving an open subgroup of the respective GL_2 or CM expectation.

The group ${}_{\ell}G_j$ in (5.11a) is $\text{PSL}_2(\mathbb{Z}_{\ell})$. Serre frames his result differently, so his group is $\text{SL}_2(\mathbb{Z}_{\ell})$. The distinction is between the monodromy group view from a **MT** (§4.1.1) versus going to the ℓ -adic representation view (§4.1.2). Both are necessary for progress on the Main **OIT** Conj. 4.2.

We interpret (5.11b) as giving an ℓ -adic germ representating the moduli space – through Tate's ℓ -adically uniformized elliptic curve — around the

⁵⁰Potential confusion of notation: j here is not an index, but the traditional variable used for the classical j -line.

⁵¹The points, $\{j = 0, 1\}$ of ramification of the covers are special. We exclude them here; a more precise result (due to Hilbert) includes them as CM points, too.

(long) cusp we call Harbater-Mumford on $X_0(\ell)$. This is a model for gleaning $G_{\mathbb{Q}_\ell}$ action for j ℓ -adically “close to” ∞ .

Suppose K is a complex quadratic extension of \mathbb{Q} . The technical point of complex multiplication is the discussion of 1-dimensional characters of G_K on the \mathbb{Q}_ℓ vector space – Tate module, or 1st ℓ -adic étale cohomology – of an elliptic curve with complex multiplication by K . On the 2nd ℓ -adic étale cohomology it is the cyclotomic character; on the 1st there is no subrepresentation of any power of the cyclotomic character.

Only a part of abelian extensions of K are cyclotomic – generated by roots of 1, a result that generalizes to higher dimensional complex multiplication in [Sh64]. Much of [Se68] is taken with (5.11c). The groups there are primarily the (abelian) ideal groups of classical complex multiplication.

As [Ri90] emphasizes, Serre’s book is still relevant, especially for the role of abelian characters, those represented by actions on Tate modules (from abelian varieties), and those not.

We reference this discussion in many places below. The full (and comfortable) completion of Serre’s **OIT** awaited replacement of an unpublished Tate piece by ingredients from Falting’s Thm. [Fa83] (as in [Se97b] and the more complete discussion of §4.4).

★ [Fr78] ★ : This was the forerunner of the always present relation between absolute, $\mathcal{H}(G, \mathbf{C})^{\text{abs}}$, and inner, $\mathcal{H}(G, \mathbf{C})^{\text{in}}$, Hurwitz spaces (2.15). The latter naturally maps – via the equivalence – to the former. [Fr78, §3] – *Determination of arithmetic monodromy from branch cycles* – was based on the idea I informally call *extension of constants*.

The definition field of an absolute cover in a Hurwitz family (represented by $\mathbf{p} \in \mathcal{H}(G, \mathbf{C})^{\text{abs}}$) might require a definition field extension from going to the Galois closure of the cover. That extension comes from coordinates of any $\hat{\mathbf{p}} \in \mathcal{H}(G, \mathbf{C})^{\text{in}}$ above \mathbf{p} . [FrV91, Thm. 1] gives the standard codification of this relation (also for reduced Hurwitz spaces).

[Fr78, §2] was a special case of it, where $G = D_\ell$, ℓ odd, and $\mathbf{C} = \mathbf{C}_{3^4}$ is 4 repetitions of the involution conjugacy class. In this case, it was describing the pair of fields $(\mathbb{Q}(\mathbf{p}), \mathbb{Q}(\hat{\mathbf{p}}))$, for $\mathbf{p} \in \mathcal{H}(G, \mathbf{C})^{\text{abs}}$ and $\hat{\mathbf{p}} \in \mathcal{H}(G, \mathbf{C})^{\text{in}}$ over it. As in §5.2.1, this was the main case in describing prime degree (ℓ) rational functions having the Schur cover property (5.3).

More precisely, consider the cover $f_{\mathbf{p}} : W_{\mathbf{p}} \rightarrow \mathbb{P}_z^1$, from $\mathbf{p} \in \mathcal{H}(G, \mathbf{C})^{\text{abs}}$ with $\hat{f}_{\mathbf{p}}$ its Galois closure. Here $W_{\mathbf{p}}$ is isomorphic to \mathbb{P}_w^1 over $\mathbb{Q}(\mathbf{p})$ (because

ℓ is odd). If $\mathbb{Q}(\mathbf{p}) \neq \mathbb{Q}(\hat{\mathbf{p}})$, with extension of constants from (5.11c)

$f_{\mathbf{p}}$ is a Schur cover over $\mathbb{Q}(\mathbf{p})$ (Thm. 5.3).

If $\mathbb{Q}(\mathbf{p}) = \mathbb{Q}(\hat{\mathbf{p}})$, then $\hat{f}_{\mathbf{p}}$ is a (4 branch point) **RIGP** *involution* realization of D_ℓ over $\mathbb{Q}(\mathbf{p})$. • [DFr94] •, as in Prop. 1.5, completely classifies involution realizations of dihedral groups. This qualifies – as stated in Ques. 1.6 – as the “easiest” case of untouched problems on the **RIGP**; justifying why the **RIGP** generalizes Mazur’s results on modular curves.

From each elliptic curve over \mathbb{Q} with non-integral j -invariant, the GL_2 part of the **OIT** (5.11b), gives explicit production of Schur cover rational functions (5.3) of degree ℓ^2 , for infinitely many primes ℓ . As with the CM case, the distinction is measured by the difference between $\mathbb{Q}(\mathbf{p})$ and $\mathbb{Q}(\hat{\mathbf{p}})$ with $\hat{\mathbf{p}}$ on the inner space over \mathbf{p} in the absolute space.

When they are different, the degree ℓ^2 rational function over $\mathbb{Q}(\mathbf{p})$ decomposes, over $\mathbb{Q}(\hat{\mathbf{p}})$, into two rational functions of degree ℓ , with no such decomposition over $\mathbb{Q}(\mathbf{p})$ ([GMS03] and [Fr05, Prop. 6.6]). This is a phenomenon that cannot happen with *polynomials* of degree prime to the characteristic, a fact exploited for the Schur cover property (as in [Fr70]).

Conj. 4.2 – expressing our best guess for what to expect of an **OIT** from a **MT**, is the result of thinking how the relation between these two different Schur covers compares with Serre’s **OIT**. Especially considering what is possible to prove at this time, both theoretically and explicitly.

For example, CM cases are famously explicit. Just as, in Schur covers given by polynomials (cyclic and Chebychev), the nature of the exceptional set $\mathrm{Exc}_{f,K}$ in (5.5) is a union of specific arithmetic progressions (in ray class groups). Example use: You can decide if *compositions* of polynomials exceptionals are exceptional.

Definition 5.11. A $j' \in \bar{\mathbb{Q}}$ is *complex multiplication* if the elliptic curve with j invariant j' has a rank 2 endomorphism ring. Then, that ring identifies with a fractional ideal in a complex quadratic extension K/\mathbb{Q} .

The main point: G_K will respect those endomorphisms. Thereby it will limit the decomposition group of a projective system of points on the spaces $\{X_0(\ell^{k+1})\}_{k=0}^\infty$. Originally, as one of Hilbert’s famous problems, Kronecker and Weber used this situation to describe the abelian extensions of complex quadratic extensions of \mathbb{Q} .

[Se68, IV-20] concludes the proof that for j' non-integral (so not complex multiplication), the Tate curve shows there is no decomposition of the degree ℓ^2 rational function. It even gives the following result.

Theorem 5.12 (OIT strong form). *Suppose $j' \in \bar{\mathbb{Q}}$ is not a complex multiplication point. Then, not only is it a GL_2 point for any prime ℓ , but the decomposition group $G_{j'}$ is actually $\mathrm{GL}_2(\mathbb{Z}_\ell)/\{\pm 1\}$ (rather than an open subgroup of this) for almost all primes ℓ .*

Falting's theorem [Fa83] (as in [Se97b]) replaces the unpublished result of Tate. The use of Faltings in both versions of the $r = 4$ Main **MT** conjecture for **MTs** mean that both have inexplicit aspects, though the results are different on that (see \bullet [CaTa09] \bullet and \bullet [Fr06] \bullet).

So even today, being explicit on Thm. 5.12 in the Schur covering property for the GL_2 case still requires non-integral j -invariant [Fr05, §6.2.1]. [Se68, IV-21-22] references Ogg's example [O67] (or [Fr05, §6.2.2]), to give $j' \in \mathbb{Q}$ with the decomposition group $G_{j'}$ equal $\mathrm{GL}_2(\mathbb{Z}_\ell)/\{\pm 1\}$ for *all* primes ℓ .

★ [Ih86] ★ : A similar title with [Fr78] may be Shimura's influence. Both played on interpreting braid group actions, a monodromy action that captures *anabelian* data (from curves, rather than from abelian varieties). They also both thought on complex multiplication.

Ihara's paper has a moduli interpretation for how to generate the field extension using Jacobi sums derived from Fermat curves, giving the second commutator quotient of $G_{\mathbb{Q}}$; versus the first commutator quotient given (Kronecker-Weber) cyclotomic values. Abstract result from it: An interpretation of Grothendieck-Teichmüller on towers of Hurwitz spaces [IM95] (albeit, Hurwitz spaces without fine moduli properties).

★ [Se90a] ★ : At the top of §5, example (5.2b) started my interaction over this approach to the **OIT**. That expanded quickly into using the Universal Frattini cover to construct the original **MTs**.

For simplicity assume a finite group G is ℓ -perfect (5.8a). Then, the lift invariant for a prime ℓ described below comes from considering *central* Frattini extensions with ℓ group kernels. Using this gave precise statements on components of Hurwitz spaces.

The sequence (5.7) is an easy-to-state result based on this tool, giving a presentation of $G_{\mathbb{Q}}$. It also produced a simply-stated conjecture. Assume for $K \subset \bar{\mathbb{Q}}$ that G_K is a projective profinite group.⁵²

⁵²Shafarevich's conjecture is the special case that K is \mathbb{Q} with all roots of 1 adjoined.

Conjecture 5.13 (Generalizing Sharafavich’s Conjecture). Then, K is Hilbertian if and only if G_K is profree.⁵³

That sounds good. But, it didn’t lead to more general use of Nielsen classes. So, [Fr20] revamps how (5.7) arises, recasting it as a classically motivated connector between the **RIGP** and the **OIT** (as in §4.4). Historical support is below and in the next two discussions.

Now consider a cover $\varphi_{\mathbf{g}} : \mathbb{P}_w^1 \rightarrow \mathbb{P}_z^1$ representing \mathbf{g} as given by the conditions (2.7). Then, consider constructing $Z \rightarrow \hat{W} \rightarrow \mathbb{P}^1$ with \hat{W} the Galois closure of $\varphi_{\mathbf{g}}$, and $Z \rightarrow \mathbb{P}_z^1$ Galois with group Spin_n . Result: There is an unramified $Z \rightarrow \hat{W}$ if and only if \mathbf{g} is in the image of $\mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}_{3^r})$.

[Fr10, Thm. B] says, for $r \geq n$, the two braid orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ are distinguished by their lift invariants. See \bullet [Fr02b] \bullet and \bullet [We05] \bullet .

This example, including using the same naming of the same order lift class, \mathbf{C}_3 , of elements of order 3 in both A_n and Spin_n , has many of the ingredients that inspired the use of the Universal Frattini cover \tilde{G} .

The conjugacy class \mathbf{C}_3 has the same cardinality in Spin_n as it has in A_n . If we included, even once, a product of two disjoint 2-cycles as an element of the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C})$, this would kill the lift invariant. These examples are writ large in the Main Theorem of [FrV92], which [Fr20] has revamped and expanded.

Central Frattini extensions affected three kinds of results.

- (5.12a) Describing components on a Hurwitz space $\mathcal{H}(G, \mathbf{C})$ assuming, if a class appears in \mathbf{C} it does so with high multiplicity.
- (5.12b) Describing, as in Thm. 3.13, the obstruction to there being a nonempty **MT** supported by the Nielsen class $\mathrm{Ni}(G, \mathbf{C})$.
- (5.12c) As in Ex. 3.4.3, a tool for classifying cusps.

\star [Se90b] \star : A combination of this paper with [Fr10, §6] makes use of the lift invariant for any Nielsen class of odd-branched Riemann surface cover of the sphere in say, the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C})$.

It is a formula for the parity of a uniquely defined half-canonical class on any cover $\varphi : W \rightarrow \mathbb{P}_z^1$ in the Nielsen class that depends only on the spin lift invariant generalizing s_{Spin_n/A_n} defined above. From this [Fr10, §6.2] produces *Hurwitz-Torelli* automorphic functions on certain Hurwitz space components through the production of even θ -nulls.

⁵³That G_K profree implies it is Hilbertian is a consequence of a version of Chebotarev’s field crossing argument. The [FrV92] result starts with the assumption that K is $\mathbf{P}(\text{pseudo})\mathbf{A}(\text{lgebraically})\mathbf{C}(\text{losed})$.

★ [Se92] and [Fr94] ★ : Serre didn't use the braid monodromy (rigidity) method. Fried makes the connection to braid rigidity through Serre's own exercises. The difference shows almost immediately in considering the realizations of Chevalley groups of rank > 1 . Serre records just three examples of Chevalley groups of rank > 1 having known regular realizations at the publication date of his book.

[FrV91] and [FrV92] constructed, for each finite G , a centerless covering group G^* with infinitely many collections of rational conjugacy classes \mathbf{C} of G^* , also having a (faithful) representation $T^* : G^* \rightarrow S_{n^*}$. These had the following additional properties.

(5.13a) $G^*(T^*, 1)$ (stabilizing 1) is self-normalizing.⁵⁴

(5.13b) $N_{S_{n^*}}(G^*, \mathbf{C})/G^{*55}$ consists of all outer automorphisms of G^* .

(5.13c) $\mathcal{H}(G^*, \mathbf{C})^{\text{in}}$, the resulting inner Hurwitz spaces are irreducible and have definition field \mathbb{Q} .

This allowed using the Hurwitz spaces as part of a *field-crossing* argument over any

P(seudo) **A**(lgebraically) **C**(losed) field $F \subset \bar{\mathbb{Q}}$:

any absolutely irreducible variety over F has a Zariski dense set of F points. The result was that if F was also Hilbertian, then G_F is profree (see Conj. 5.13). A particular corollary was the presentation of $G_{\mathbb{Q}}$ in (5.7).

Condition (5.13a) is sufficient to say that any $K \subset \bar{\mathbb{Q}}$ point on $\mathcal{H}(G^*, \mathbf{C})^{\text{in}}$ (satisfying (5.13c)) corresponds to a K regular realization of G^* , and therefore of G . This is because G^* will have no center, the condition that the inner Hurwitz space is then a fine moduli space.

In myriad ways we can relax these conditions. Still, to use them effectively over say \mathbb{Q} requires finding \mathbb{Q} points on $\mathcal{H}(G^*, \mathbf{C})^{\text{in}}$. The usual method is to choose \mathbf{C} so that $\mathcal{H}(G^*, \mathbf{C})^{\text{in}}$ is sufficiently close to the configuration space U_r , that \mathbb{Q} points are dense in it. If $r = 4$, we may use Thm. 2.10. Compute the genus of $\mathcal{H}(G^*, \mathbf{C})^{\text{in,rd}}$, check if it has genus 0, and a degree 0 divisor of $\mathbb{Q} \implies \infty$ -ly many \mathbb{Q} points.

Soon after [FrV91], Völklein and Thompson – albeit powerful group theorists – produced abundant high rank Chevalley groups based on this method. Locating specific high-dimensional uni-rational Hurwitz spaces was the key. Examples, and the elementary uses of Riemann's Existence Theorem, abound in [Vo96].

⁵⁴As in (5.10a).

⁵⁵§2.2.2 for $N_{S_n}(G, \mathbf{C})$.

The Conway-Fried-Parker-Voelklein appendix of [FrV91] was a non-explicit method for doing that. [Fr10] shows what explicit can mean.

★ [DFr94] ★ : Ques. 1.6 gave the formulation of the Main **MT** conjecture for dihedral groups. Equivalently, if it is false, the **BCL** (Thm. 5.10) implies there is an even integer r ($\leq r^*$) and for each $k \geq 0$, a dimension $\frac{r-2}{2}$ hyperelliptic Jacobian (over \mathbb{Q}) with a $\mathbb{Q}(e^{2\pi i/\ell^{k+1}})$ torsion point, of order ℓ^{k+1} , on whose group $G_{\mathbb{Q}}$ acts as it does on $\langle e^{2\pi i/\ell^{k+1}} \rangle$.

The *Involution Realization Conjecture* says the last is impossible: There is a uniform bound as n varies on n torsion points on any hyperelliptic Jacobian of a fixed dimension, over a given number field. (The only proven case, $r = 4$, is the Mazur-Merel result bounding torsion on elliptic curves.) If a subrepresentation of the cyclotomic character occurred on the ℓ -Tate module of a hyperelliptic Jacobian (see [Se68]), the Involution Realization Conjecture would be false. This led to formulating Main **RIGP MT** conj. 5.8.

Still missing: For any prime $\ell > 2$, find cyclotomic ℓ^{k+1} torsion points on any hyperelliptic Jacobians for all (even infinitely many) values of k (§1.3.1).

5.3.3. *Constructions and Main Conjectures from 1995 to 2004.* §3.2 has the universal ℓ -Frattini cover ${}_{\ell}\psi : {}_{\ell}\tilde{G} \rightarrow G$ (Def. 3.3). It is the minimal profinite cover of G with its ℓ -Sylow a pro-free pro- ℓ group [FrJ86, Prop. 22.11.8]₂.

For P an ℓ -group, ${}_{\text{fr}}P \stackrel{\text{def}}{=} P^{\ell}[P, P]$ is its Frattini subgroup. As usual, ${}_{\text{fr}}P$ is the closed subgroup of P containing generators from the ℓ powers and commutators of P . Then, $P \rightarrow P/{}_{\text{fr}}P$ is a Frattini cover.

Recover a cofinal family of finite quotients of ${}_{\ell}\tilde{G}$ by taking $\ker_0 = \ker({}_{\ell}\psi)$ denoting the sequence of *characteristic kernels* of ${}_{\ell}\tilde{G}$ as in (3.2):

$$(5.14) \quad \ker_0 > {}_{\text{fr}}\ker_0 \stackrel{\text{def}}{=} \ker_1 \geq \cdots \geq {}_{\text{fr}}\ker_{k-1} \stackrel{\text{def}}{=} \ker_k \dots,$$

${}_{\ell}\tilde{G}/\ker_k$ by ${}^k_{\ell}G$, and the characteristic modules $\ker_k/\ker_{k'} = {}_{\ell}M_{k,k'}$, etc.

★ [Fr95] ★ : Assume generating conjugacy classes, \mathbf{C} , of G .⁵⁶ Then, with $N_{\mathbf{C}}$ the least common multiple of the orders of elements in \mathbf{C} :

$$(5.15) \quad \text{If } \ell \nmid N_{\mathbf{C}}, \text{ Schur-Zassenhaus implies the classes } \mathbf{C} \text{ lift canonically to classes of elements of the same orders in each group } {}^k_{\ell}G.$$

This opens by recasting modular curves as Hurwitz spaces of sphere covers for dihedral groups, referring to •[Fr78, §2]•. Then, applying (5.14), that any group can be used to constructed modular curve-like towers. §1.3.4 discussed the worth of this, emphasizing these:

⁵⁶The group generated by all entries of \mathbf{C} is G .

(5.16a) It works with any ℓ -perfect group G , replacing a dihedral group D_ℓ , ℓ odd and conjugacy classes \mathbf{C} satisfying (5.15); and

(5.16b) This recasts the **RIGP** and the **OIT** using points on substantive moduli spaces, allowing formulating a relation between them.

Without (5.15), there is no unique assignment of lifts of classes in \mathbf{C} to the characteristic ℓ -Frattini cover groups. Reason: If $g \in G$ has order divisible by ℓ , then the order of any lift $\tilde{g} \in {}^\ell_1 G$ is $\ell \cdot \text{ord}(g)$.

Given (5.15), we canonically form towers of Nielsen classes, and associated Hurwitz spaces, from (3.2) and their abelianizations:

$$(5.17) \quad \{\mathcal{H}({}^\ell_k G, \mathbf{C})^{\text{in}}\}_{k=0}^\infty \text{ and the abelianized version } \{\mathcal{H}({}^\ell_k G_{\text{ab}}, \mathbf{C})^{\text{in}}\}_{k=0}^\infty.$$

Originally we called these the **MTs**. Now we prefer that a **MT** – Def. 1.9 – is a projective sequence of *irreducible* components (from braid orbits on the Nielsen classes) of their respective levels.

[Fr95, Part II] describes the characteristic modules for $G = A_5$ and primes $\ell = 2, 3, 5$ dividing $|A_5| = 60$. Thereby, for these cases, it describes the tower of Nielsen classes attached to the abelianized version of (5.17).

We then required three immediate assurances.

(5.18a) That we could decide when we are speaking of a non-empty **MT**.

(5.18b) That K points on the k th tower level correspond to K regular realizations in the Nielsen class $\text{Ni}({}^\ell_k G, \mathbf{C})$ (or $\text{Ni}({}^\ell_k G_{\text{ab}}, \mathbf{C})$).

(5.18c) That we know the definition field of $\mathcal{H}({}^\ell_k G, \mathbf{C})^{\text{in}} \rightarrow U_r$ and the rest of the structure around $\mathcal{H}({}^\ell_k G, \mathbf{C})^{\text{in}}$ as a moduli space.

Comments. Response to (5.18a): The first necessary condition is that G is ℓ -perfect. Otherwise, no elements of \mathbf{C} will generate a \mathbb{Z}/ℓ image.

A much tougher consideration, though, was what might prevent finding elements $\mathbf{g} \in G^r \cap \mathbf{C}$ satisfying product-one (as in §2.1.3). Thm. 3.13, using the *lift invariant* resolves that \bullet [Fr02b] \bullet and \bullet [We05] \bullet .

Response to (5.18b): Originally I formed **MTs** to show that talking about rational points on them, vastly generalized talking about rational points on modular curve towers. Especially, that the **RIGP** was a much tougher/significant problem than usually accepted.

To assure K points on the k th level correspond to regular realizations of the Frattini cover groups, we needed the fine moduli condition that each of the ${}^\ell_k G$ s has no center. The most concise is [BFr02, Prop. 3.21]:

$$(5.19) \quad \text{If } G \text{ is centerless, and } \ell\text{-perfect, then so is each of the } {}^\ell_k G \text{ s.}$$

Response to (5.18c): The **BCL** of [Fr77, §5.1] perfectly describes moduli definition fields (Def. 1.2) of Hurwitz spaces. The result is more complicated

for absolute classes, but both results also apply to reduced classes. We have given detailed modern treatments of this now in several places including [Fr12, App. B.2]; also see [FrV91, Main Thm.] and [Vo96].

Using $N_{\mathbf{C}}$ (§1.2.1), denote the field generated over K by a primitive $N_{\mathbf{C}}$ root of 1 by $\text{Cyc}_{K, \mathbf{C}}$. Recall:

$$G(\text{Cyc}_{\mathbb{Q}, \mathbf{C}}/\mathbb{Q}) = (\mathbb{Z}/N_{\mathbf{C}})^*, \text{ invertible integers mod } N_{\mathbf{C}}.$$

Use the subgroup, $G(\text{Cyc}_{K, \mathbf{C}}/K)$, fixed on $K \cap \text{Cyc}_{\mathbb{Q}, \mathbf{C}}$, to define:

$$(5.20) \quad \begin{aligned} \mathbb{Q}_{G, \mathbf{C}} &\stackrel{\text{def}}{=} \{m \in (\mathbb{Z}/N_{\mathbf{C}})^* \mid \{g^m \mid g \in \mathbf{C}\} \stackrel{\text{def}}{=} \mathbf{C}^m = \mathbf{C}\}. \\ \mathbb{Q}_{G, \mathbf{C}, T} &\stackrel{\text{def}}{=} \{m \in (\mathbb{Z}/N_{\mathbf{C}})^* \mid \exists h \in N_{S_n}(G, \mathbf{C}) \text{ with } h\mathbf{C}^m h^{-1} = \mathbf{C}\}. \end{aligned}$$

Lemma 5.14 (Branch Cycle). *As above, then $\mathbb{Q}_{G, \mathbf{C}}$ (resp. $\mathbb{Q}_{G, \mathbf{C}, T}$) is contained in any definition field of any cover in the Nielsen class $\text{Ni}(G, \mathbf{C})^{\text{in}}$ (resp. $\text{Ni}(G, \mathbf{C}, T)^{\text{abs}} \stackrel{\text{def}}{=} \text{Ni}(G, \mathbf{C})^{\text{abs}}$ if T is understood) [Fr77, p. 62–64].*

Still, for K points to exist, there must be a component with moduli definition field K (Def. 1.2). That is a much harder problem. \square

For good reasons there can be more than one component (as in •[Se90b] • and Prop. 2.15 on [Fr10, Thm. B]) in any particular case.

The **OIT** contends with that at all levels •[FrH20] •. In lieu of Main **RIGP** Conj. 5.8 (or Conj. 5.15) for a given **MT** consider two cases.

- (5.21a) Some number field K is a moduli definition field of all tower levels.
- (5.21b) The moduli definition degrees rise with the tower levels.

Recall Main **RIGP** conj. 5.15

Conjecture 5.15. At high levels there will be no K points on a **MT**. Also, high levels will be algebraic varieties of general type. [Fr95].

•[BFr02]• inspected the case $(A_5, \mathbf{C}_{34}, \ell = 2)$. This found that significant cusps and the lift invariant revealed as much detail on this **MT** as one would expect from a modular curve tower, despite interesting differences.

Definition 5.16. An element $\mathbf{g} \in \text{Ni}(G, \mathbf{C})$ is a *Harbater-Mumford* (**HM**) representative if it has the form $(g_1, g_1^{-1}, \dots, g_s, g_s^{-1})$ (so $2s = r$). A braid orbit O is said to be **HM**, if the orbit contains an **HM** rep.

The name *Harbater-Mumford* comes from [Mu72] which used a completely degenerating curve on the boundary of a space of curves. In a sense, [Ha84], for covers, consists of a “germ” of such a construction.

[Fr95, Thm. 3.21] got the most attention by showing that if \mathbf{C} is a rational union (Thm. 5.10), then $G_{\mathbb{Q}}$ permutes the **HM** components. Further, it gave

an explicit criterion, applying to any G , for producing classes \mathbf{C} so that $\mathcal{H}(G, \mathbf{C})$ has just one **HM** component.

Thereby, it found for G , small explicit values of $r = r_{\mathbf{C}}$ with an attached **MT** and \mathbb{Q} as a moduli definition field for all **MT** levels, situation (5.21a) with the **MT** levels all **HM** components.

Motivated by this, both [DEm05] and [W98] developed a Hurwitz space Deligne-Mumford stable-compactification. This put a Harbater degeneration on its boundary. This allows a standard proof – contrasting with the group theoretic use of *specialization sequences* in Fried’s result – to show Harbater-Mumford cusps lie on Harbater-Mumford components.

Both compactifications suit the **MT** construction extending the projective systems. Further, from Grothendieck’s famous theorem, other than primes p dividing $|G|$, the **MT** system has good reduction mod p . This topic thus applies to the full Frattini cover $\tilde{G} \rightarrow G$ (rather than abelianized version) in achieving **RIGP** results over fields other than number fields. • [D06], [DDes04] and [DEm05] • continues this discussion.

★ [FrK97] ★ : Suppose G is a group with many known regular realizations. For example: A_n semidirect product some finite abelian group. (Say, a quotient of \mathbb{Z}^{n-1} on which A_n acts through its standard representation; a special case of Prop. 5.21 in •[CaD08]•). Consider, for some prime ℓ for which G is ℓ -perfect, if there are regular realizations of the whole series of ${}^k_{\ell}G$, $k \geq 0$, over some number field K .

The basic question: Could all such realizations have a uniform bound, say r^* , on numbers of branch points – with no hypothesis on the classes \mathbf{C} .

Theorem 5.17. *Such regular realizations are only possible by restricting to ℓ' classes (elements of \mathbf{C} with orders prime to ℓ). If they do occur, there must exist a **MT** over K for some one choice of $r \leq r^*$ classes, \mathbf{C} , with a K point at every level [FrK97, Thm. 4.4].*

The Thm. 5.17 conclusion is contrary to Main **MT RIGP** Conj. 5.15; proven for $r^* \leq 4$ (•[Fr06] and [CaTa09]•).

★ [BFr02] ★ : This is a book of tools that has informed all later papers on **MTs**. The thread through the book is checking phenomena on **MTs** lying over one (connected) reduced Hurwitz space: For the Nielsen class $\text{Ni}(A_5, \mathbf{C}_{3^4})$; four repetitions of the conjugacy class of 3-cycles) and the prime $\ell = 2$. It computes everything of possible comparison with modular

curves about level one (and level 0) where we take the full (not just the abelianized, Def. 3.2) universal 2-Frattini cover of A_5 .

It shows the Main **MT** Conjecture 5.15 for it: No K points at high levels (K any number field). The inner space at level 0 has one component of genus 0. Level one has two components, of genus 12 and genus 9. This concludes with a conceptual accounting of all cusps, and all real points on any **MT** over the level 0 space (none over the genus 9 component).

Many points about **MT** levels arise here; particular cases of them appear attached to the components at level 1. For example, a version of the spin cover (extending the domain of use of [Se90a]) obstructs anything beyond level 1 for the genus 9 component. Also, the argument using a prime, ℓ , of good reduction, as in §4.4 appears first here.

[BFr02, §2.10.2] introduces the *shift-incidence matrix* applying it to an early version of Ex. 2.3. This linear device detects braid orbits and organizes cusps. Recall the braid generators in (2.13).

Choose any one of the twists q_v (for $r = 4$ it suffices to choose q_2 on reduced Nielsen classes) and call it γ_∞ . Reference rows and columns of the matrix by orbits of γ_∞ on reduced Nielsen classes as O_1, \dots, O_t . Reduced Nielsen classes are special in the case $r = 4$, as in Thm. 2.10.

The (i, j) entry of the matrix is then $|(O_i)\mathbf{sh} \cap O_j|$:

Apply \mathbf{sh} to all entries of O_i , intersect it with O_j ; the (i, j) -entry is the cardinality of the result.

As in § 2.3.3, \mathbf{sh} -incidence graphically displays a **MT** component at a given level using a characteristic cusp distinguishing that component.

Since $\mathbf{sh}^2 = 1$ on reduced classes when $r = 4$, for that case the matrix is symmetric. Braid orbits correspond to matrix blocks. [BFr02, Table 2] displays the one block and the genus calculation for (A_5, \mathbf{C}_{3^4}) . Then, [BFr02, §8.5, esp. Table 4] does a similar calculation for the level 1 **MTs**, $(\frac{1}{2}A_5, \mathbf{C}_{3^4})$. (This and $\text{Ni}(A_4, \mathbf{C}_{3^4})$ are still the only cases in print going up to level 1 for the full universal ℓ -frattini cover.)

- (5.22a) There are two kinds of cusps, **HM** and near-**HM**, with near-**HM** having a special action under the complex conjugation operator.
- (5.22b) The components at level 1 have resp. genres 12 and 9 (> 1): Faltings kicks in here as in §4.4.2.
- (5.22c) Apply [DFr90]: the genus 12 (resp. 9) component has one (resp. no) component of real points.

[Fr20b] has more examples illustrating the theme of having cusp types – based on using refined aspects of G – separate components. That includes those in •[LO08] and and [Fr09] •. Recall the key issue.

When Hurwitz spaces have several components, identify moduli definition fields geometrically to recognize the $G_{\mathbb{Q}}$ action on the components.

★ [Fr02b] and [We05] ★ : [Se92, Chap. 9] added material from [Me90] on regularly realizing the $\psi_{A_n, \text{Spin}_n} : \text{Spin}_n \rightarrow A_n$ (see Ex. 2.14) cover. Stated in my language he was looking at the Nielsen class extension

$$\Phi_{A_n, \text{Spin}_n} : \text{Ni}(\text{Spin}_n, \mathbf{C}_{3^{n-1}})^{\text{in}} \rightarrow \text{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\text{in}}.$$

Theorem 5.18. [Fr10, Main Thm.] *For all n , there is one braid orbit for $\text{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\text{in}}$. For n odd, $\Phi_{A_n, \text{Spin}_n}$ is one-one, and the abelianized **MT** is nonempty. For n even, $\text{Ni}(\text{Spin}_n, \mathbf{C}_{3^{n-1}})^{\text{in}}$ is empty.*

Here is the meaning for n even. Lift the entries of $\mathbf{g} \in \text{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\text{in}}$ to same order entries in Spin_n , to get $\hat{\mathbf{g}}$. Then, the result does not satisfy product-one: $\hat{g}_1, \dots, \hat{g}_{n-1} = -1$ (the *lift invariant* in this case).

I used this to test many properties of **MTs**. Here it showed that there is a *nonempty* Modular Tower over $\text{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\text{in}}$ for $\ell = 2$ if and only if n is odd. In particular the characteristic Frattini extensions define the tower levels, but central Frattini extensions control many of their delicate properties. If you change $\mathbf{C}_{3^{n-1}}$ to \mathbf{C}_{3^r} , $r \geq n$, there are precisely two components, with one obstructed by the lift invariant, the other not.

[Fr02b, Thm. 2.8] gives a procedure to describe the ℓ -Frattini module for any ℓ -perfect G , and therefore of the sequence $\{\ell^k G_{\text{ab}}\}_{k=0}^{\infty}$. [Fr02b, Thm. 2.8] labels Schur multiplier types, especially those called *antecedent*. Example: In **MTs** where $G = A_n$, the antecedent to the level 0 spin cover affects **MT** components and cusps at all levels ≥ 1 (as in [Fr06]).

[Se97a, I.4.5] extends the classical notion of *Poincaré duality* to any pro- ℓ group. It applied to the pro- ℓ completion of $\pi_1(X)$ with X any compact Riemann surface. [We05] uses the extended notion, intended for groups that have extensions by pro- ℓ groups of any finite group.

Main Result: The universal ℓ -Frattini cover ${}_{\ell}\tilde{G}$ (and ${}_{\ell}\tilde{G}_{\text{ab}}$) is an ℓ -Poincaré duality group of dimension 2. The result was Thm. 3.13 [Fr06, Cor. 4.19]. §3.4.2 and §3.4.3 applied this $\text{Spin}_n \rightarrow A_n$, for $\text{Ni}(A_5, \mathbf{C}_{3^4})$ and $\text{Ni}(A_4, \mathbf{C}_{+3^2-3^2})$ for different ℓ -Frattini lattice quotients (Def. 3.4).

5.3.4. *Progress on the **MT** conjectures and the **OIT**.* As with modular curves, the actual **MT** levels come alive by recognizing moduli properties

attached to particular (sequences) of cusps. It often happens with **MTs** that level 0 of the tower has no resemblance to modular curves, though a modular curve resemblance arises at higher levels.

Level 0 of alternating group towers illustrate: They have little resemblance to modular curves. Yet, often level 1 starts a subtree of cusps that contains the cusp tree of modular curves. We can see this from the classification of cusps discussed in • [Fr06, §3] •.

We leave much discussion of generalizing Serre's **OIT** to [Fr20]. Still, we make one point here, based on what is in • [Se68] • and • [Fr78] •.

With \dagger either inner or absolute equivalence, it is the interplay of two Nielsen classes that gives a clear picture of the bifurcation between the two types of decomposition groups, CM and GL_2 . Those Nielsen classes are

$$(5.23) \quad \mathrm{Ni}(D_\ell, \mathbf{C}_{2^4})^{\dagger, \mathrm{rd}} \text{ and } \mathrm{Ni}((\mathbb{Z}/\ell)^2 \times {}^s\mathbb{Z}/2, \mathbf{C}_{2^4})^{\dagger, \mathrm{rd}}.$$

In the example(s) of [Fr20], the same thing happens. Of course, the j values don't have the same interpretation as for Serre's modular curve case. Further, as in • [FrH20] •, there are nontrivial lift invariants, and more complicated, yet still tractable, braid orbits.

★ [D06], [DDes04] **and** [DEm05] ★ : [D06] has expositions on [DFr94], [FrK97], [DDes04] and [DEm05] in one place.

[DDes04] assumes the Main **MT** Conjecture (5.9) is wrong: for some finite group G satisfying the usual conditions for ℓ and \mathbf{C} and some number field K , the corresponding **MT** has a K point at every level. Using [W98] compactifications of the **MT** levels, for almost all primes \mathbf{p} of K , this would give a projective system of $\mathcal{O}_{K, \mathbf{p}}$ (integers of K completed at \mathbf{p}) points on cusps. The results here considered what **MTs** (and some generalizations) would support such points for almost all \mathbf{p} using Harbater patching (from [Ha84]) around the Harbater-Mumford cusps.

We continue the concluding paragraphs of • [Fr95] • on **HM** components and cusps. [DEm05] continues the results of [DDes04]. It ties together notions of **HM** components and the cusps that correspond to them, connecting several threads in the theory. They construct, for every projective system $\{G_k\}_{k=0}^\infty$ – not just those coming from a universal Frattini cover as characteristic quotients – a tower of corresponding Hurwitz spaces, geometrically irreducible and defined over \mathbb{Q} (using the [Fr95, Thm. 3.21] criterion).

These admit projective systems of points over the Witt vectors with algebraically closed residue field of \mathbb{Z}_p , avoiding only p dividing some $|G_k|$. If you compactify the tower levels, you get complete spaces, with cusps

lying on their boundary. The **MT** approach gives precise labels to these cusps using elementary finite group theory • [Fr06] and [CaTa09] •.

★ [Fr06] and [CaTa09] ★ : We continue • [Fr02b], [We05] and [DEm05] •. [Fr06, §3.2.1] has three generic cusp types that reflect on Hurwitz space components and properties of **MTs** containing such components:

$$\ell\text{-cusps, } g(\text{roup})\text{-}\ell' \text{ and } o(\text{nly})\text{-}\ell'.$$

Modular curve towers have only the first two types, with the $g\text{-}\ell'$ cusps the special kind called shifts of **HM**.

[Fr06, §3.2.1] develops these cusps when $r = 4$ (as alluded to in (1.9b)). For $\mathbf{g} = (g_1, g_2, g_3, g_4)$ in the cusp orbit (§1.2.3):⁵⁷

(5.24a) The cusp is $g\text{-}\ell'$ if

$$H_{1,4} = \langle g_1, g_4 \rangle \text{ and } H_{2,4} = \langle g_2, g_3 \rangle \text{ are } \ell' \text{ groups.}$$

(5.24b) It is $o\text{-}\ell'$, if $\ell \nmid g_2 g_3$ but the cusp is not $g\text{-}\ell'$.

(5.24c) It is an ℓ cusp otherwise.

These generalize to all r . For example:

Definition 5.19 ($g\text{-}\ell'$ type). For \mathbf{g} in a braid orbit O on $\text{Ni}(G, \mathbf{C})$, we say O is $g\text{-}\ell'$ if $\mathbf{g} = (g_1, \dots, g_r)$ has a partition with elements

$$P = [g_u, g_{u+1}, \dots, g_{u+u'}] \text{ (subscripts taken mod } r)$$

and $H_P = \langle g_u, g_{u+1}, \dots, g_{u+u'} \rangle$ is an ℓ' group for each partition element.

The following is in [Fr06, Prin. 3.6, Frattini Princ. 2].

Theorem 5.20. *There is a full **MT** over the Hurwitz space component corresponding to O if O contains a $g\text{-}\ell'$ representative (no need to check central Frattini extensions as in Thm. 3.13).*

The approach to more precise results has been to consider a Harbater patching converse: Identify the type of a $g\text{-}\ell'$ cusp that supports a Witt-vector realization of ${}_{\ell}\tilde{G}$.

Typically we label a braid orbit O in $\text{Ni}(G, \mathbf{C})$ by the type of cusp it contains. In actual examples, say Thm. 5.24, even these generic names get refinements where we call particular $o\text{-}\ell'$ cusps *double identity*.

Both approaches to the Main **OIT** Conj. 5.8 give its truth for $r = 4$, Thm. 5.9. Both use Falting's Theorem. First, [CaTa09], which is more general and far less explicit. As those authors say, the Main Conjecture 5.15, which Tamagawa saw at my lectures [Fr02b], motivated this.

⁵⁷The names stand, respectively, for group- ℓ' and only- ℓ .

Let $\chi : G_K \rightarrow \mathbb{Z}_p^*$ be a character, and $A[p^\infty](\chi)$ be the p -torsion on an abelian variety A on which the action is through χ -multiplication. Assume χ does not appear as a subrepresentation on any Tate module of any abelian variety (see [Se68], [DFr94] and [BFr02]). Then, for A varying in a 1-dimensional family over a curve S defined over K , there is a uniform bound on $|A_s[p^\infty](\chi)|$ for $s \in S(K)$. In particular, this gives the Main **RIGP** conj. 5.8 when $r = 4$.

It is the growth of the genus of these **MT** levels, as in [Fr06, Prop. 5.15], that assures Main **RIGP** Conj. 5.8 if a level contains at least 2 ℓ -cusps. This gives the Main **RIGP** conj. 5.8 as equivalent to (5.9a). Also, we don't know at what level the surmised finite number of rational points will become *no rational points*. If, on a **MT**, there are no ℓ -cusps at level 0, then we need [We05, Princ. 4.23] to find a level with an ℓ -cusp lying over an $o\text{-}\ell'$ cusp. [Fr20b] does a deeper analysis relating these two distinct proofs – noting **sh**-incidence matrix aspects – than we have space for here.

★ [CaD08] ★ : Abelian groups and abelian varieties are obviously related going back to Abel. Here is a positive **RIGP** result applied to $A \times^s H$ with A abelian, and H any finite group acting on A .

Proposition 5.21. *Regularly realizing H , acting on a finite abelian A , over an Hilbertian field K , extends to regularly realizing $A \times^s H$.*

This comes from the two steps in (5.25).

(5.25a) There is a regular realization of A over K .

(5.25b) Combine regular realizations of H and $A^{|H|}$ to give one of $A \wr H$, and so of $A \times^s H$.

An explicit (resp. abstract) proof of these pieces is in [FrJ86, Prop. 16.3.5]₂ (resp. [Se92, §4.2]). For (5.25a) it suffices to consider the case $A_u = \mathbb{Z}/\ell^u$, and from the **BCL** – for ℓ odd – the minimal number of possible branch points for such a realizing cover is $\ell^{u-1}(\ell - 1)$. Conclude Lem. 5.22, using these regular realizations.

Lemma 5.22. *Consider any sequence of regular realizations of $\{A_u\}_{u=1}^\infty$ over \mathbb{Q} , with corresponding classes \mathbf{C}_u , and branch point numbers r_u . Then, $r_u \mapsto \infty$. More generally, suppose H acts on $\tilde{A}' = \mathbb{Z}_\ell^m$.*

Then, this gives a projective sequence $\{A'_u \times^s H\}_{u=0}^\infty$ with regular realizations and limit $\tilde{A}' \times^s H$. Yet, branch point number goes to ∞ with u .

[CaD08] gives a new constraint for the **RIGP**. Suppose a finite group G has a regular realization over \mathbb{Q} , and P_ℓ is an ℓ -Sylow with $m = [G :$

$P_\ell]$. Then the abelianization of P_ℓ , has order ℓ^u bounded by an expression involving

$$(5.26) \quad m, \text{ the branch point number } r \text{ and the least good reduction prime } \nu \text{ of the cover.}$$

To wit: If ℓ^u is large compared to r and m , branch points of the realizing cover must coalesce modulo some prime ν ; a ν -adic measure of proximity to a cusp on the corresponding Hurwitz space.

Conj. 5.23 is a stronger version of the Main **RIGP** conj. 5.8. The parameters m and r are fixed as a function of the level k for defining a **MT**. So the Torsion Conjecture implies it, too.

Conjecture 5.23. Some expression in r and m , independent of ν , bounds ℓ^u . This follows from the Torsion Conjecture on abelian varieties.

Maybe we can turn this around. Instead of waiting for a proof of the Torsion Conjecture, use $r = 4$, as in \bullet [Fr06] and [CaTa09] \bullet via §4.1. That is, prove the Torsion Conjecture according to statement (1.9c): using cusps on a Hurwitz space to form a height on abelian varieties labeled by these cusps to test Conj. 5.23.

We remind of cases from **MTs**. Choose from §3.3.1 an ℓ -Frattini lattice quotient $L^\star \rightarrow G^\star \rightarrow G = G_0$. For the maximal quotient ${}_\ell\tilde{G}_{\text{ab}} \rightarrow G$, the rank of L^\star is necessarily ≥ 2 unless G_0 is the ℓ -supersolvable generalization of the dihedral case [GS78].

1-dimensional ℓ -Frattini lattice quotients do include the superelliptic spaces of [MaSh19]. These, though don't fit into using *all* primes ℓ as a natural generalization to Serre's **OIT**. Instead, Thm. 5.24 considers where the level 0 groups are $(\mathbb{Z}/\ell)^2 \times {}^s\mathbb{Z}/3 = {}_\ell G_0$ and the lattice rank is 2.

That is, take $L^\star \times {}^s G_0$ instead of the ℓ -Frattini cover.

For $\ell \equiv 1 \pmod{3}$, superelliptic covers for $\mathbb{Z}/3$ do appear, but not otherwise. Lem. 5.22 now gives **RIGP** realizations of all of the $(\mathbb{Z}/\ell^{k+1})^2 \times {}^s\mathbb{Z}/3 = {}_\ell G_k$ over some fixed number field K . Yet, only by increasing the branch point number. Again: The question of finding ℓ' **RIGP** realizations for them and the hyperelliptic jacobian interpretation of this as an analog of Prop. 1.5.

\star [LO08] **and** [Fr09] \star : This gave an especially good place to see the sh-incidence matrix (discussion of \bullet [BFr02] \bullet) in action on a variety of cusps with extra structure inherited from conjugacy classes in A_n .

[LO08] considered $\text{Ni}(G, \mathbf{C})^{\text{abs}}$ with these two conditions: The covers are genus 0; and \mathbf{C} is pure cycle:elements in the conjugacy classes have only

one length ≥ 2 disjoint cycle. They showed the Hurwitz space $\mathcal{H}(G, \mathbf{C})^{\text{abs}}$ has one connected component.

This overlaps with the 3-cycle result of [Fr10, Thm. 1.3], the case of four 3-cycles in A_5 (§2.3.1). [LO08, §5] gives the impression that all these Hurwitz spaces are similar, without significant distinguishing properties. [Fr09, §9], however, dispels that.

The stronger results come by considering the inner (rather than absolute) Hurwitz spaces. [Fr09, Prop. 5.15] uses the sh-incidence matrix to display cusps, elliptic fixed points, and genres of the inner Hurwitz spaces in two infinite lists of [LO08] examples. In one there are two level 0 components (conjugate over a quadratic extension of \mathbb{Q}). For the other just one.

Further, using (5.24), the nature of the 2-cusps in the **MT**s over them differ greatly. None have 2-cusps at level 0. For those with level 0 connected, the tree of cusps, starting at level 1, contains a subtree isomorphic to the cusp tree on a modular curve tower.⁵⁸

For the other list, there are 2-cusps, though not like those of modular curves. [Fr20b] includes this case as one of its **sh**-incidence matrix examples; another example where classical group theory – here from the Clifford Algebra – shows in the naming of the cusps in the **sh**-incidence display.

★ [FrH20] ★ : The culminating topic of [Fr20] is the system of **MT**s based on the Nielsen class $\text{Ni}_{\ell,3} \stackrel{\text{def}}{=} \text{Ni}((\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{+3^2-3^2})$ (as in Ex. 2.14). Notice our choice of conjugacy classes (at first in $\mathbb{Z}/3$, but extended to $\text{Ni}_{\ell,3}$) is a rational union, as given by Thm. 5.10. Using the **BCL** (Thm. 5.20) the moduli definition field of the Hurwitz spaces is \mathbb{Q} .

This is parallel to Serre’s dihedral group example. We have a series of groups $\{G_\ell\}_{\ell \text{ prime}}$ like the series of dihedral groups $\{D_\ell\}_{\ell \text{ prime}}$. This is an example of §3.3.3 with the ℓ -Sylow of G_ℓ normal. For completeness:

For a given ℓ , the group sequences appearing in the canonical ℓ -Frattini lattice quotient here are $\{G_k \stackrel{\text{def}}{=} (\mathbb{Z}/\ell^{k+1})^2 \times^s \mathbb{Z}/3\}_{k=0}^\infty$.

If we follow the general statement for a **MT** that a prime ℓ is considered only if G is ℓ -perfect, then our condition would be $(\ell, 3) = 1$, and for each $\text{Ni}_{\ell,3}$, only ℓ would be involved in the **MT**.

(5.27a) Serre includes ℓ ; how will we include $\ell = 3$?

(5.27b) Then, for each ℓ , why leave out the $\ell = 3$ in applying to the particular case of $\text{Ni}_{\ell,3}$?

⁵⁸We call this a *spire*.

The trick is this: Since in Serre's case (resp. our case), the $\mathbb{Z}/2$ (resp. $\mathbb{Z}/3$) is a splitting coming from a semi-direct product $\mathbb{Z} \times^s \mathbb{Z}/2$ (resp. $(\mathbb{Z}/2)^2 \times^s \mathbb{Z}/3$), we are free to ignore the copy of $\mathbb{Z}/2$ (resp. $\mathbb{Z}/3$) even if $\ell = 2$ (resp. 3).⁵⁹ For the same reason we ignore that prime if ℓ is not 2 (resp. 3).

Below we quote only the level 0 braid orbit description, but for all ℓ . There are several different **MTs** in our case. This doesn't occur in dihedral group cases. For one there is no central ℓ -Frattini cover of D_ℓ (for ℓ odd). So, no lift invariant occurs in the Nielsen class that starts Serre's **OIT**.

There is, though as in Thm. 5.18 in the alternating group case, though we didn't set that up with a lattice action as in this case.

Consider the matrix

$$M(x, y, z) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \text{ with inverse } \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

With R a commutative, the 3×3 *Heisenberg group* with entries in R is

$$\mathbb{H}_R = \{M(x, y, z)\}_{x, y, z \in R}.$$

[FrH20, §4.2.1] shows the $\mathbb{Z}/3$ action extends to the *small Heisenberg group* providing this Nielsen class with a non-trivial lift invariant: comments on (2.18c). Even at level 0, that separates the braid orbits with their respective lift invariants in 0 from versus in $(\mathbb{Z}/\ell)^*$.

The lift invariant values grow with the tower level, because the Heisenberg group kernel grows. This is a case of Ex. 3.9.

[FrH20, Prop. 4.18] gives a formula for the lift invariant in this case when $r = 4$, the first such formula going beyond the Nielsen classes for A_n and **C** odd order classes (as in the discussion •[Se90a]•).

Thm. 5.24 is part of [FrH20, Thm. 5.2]: level 0 components of the Hurwitz space. For $\ell > 3$ prime, define

$$K_\ell = \begin{cases} \frac{\ell+1}{6}, & \text{for } \ell \equiv -1 \pmod{3} \\ \frac{\ell-1}{6}, & \text{for } \ell \equiv +1 \pmod{3}. \end{cases}$$

Prop. 2.15 did $\ell = 2$, with subtly different group theory.

Double identity Nielsen class representatives have the form (g, g, g_3, g_4) ,

$\text{o-}\ell'$ components as in (5.24); nothing like **HM** reps (Def. 5.16).

Theorem 5.24 (Level 0 Main Result). *For $\ell > 3$ prime and $k = 0$ there are K_ℓ braid orbits with trivial (0) lift invariant. All are **HM** braid orbits.*

The remaining orbits are distinguished by having nontrivial lift invariant (in $(\mathbb{Z}/\ell)^$; each value is achieved). Each contains a double identity cusp.*

⁵⁹But we don't include $\ell = 3$ here.

Hurwitz space components for braid orbits with nontrivial lift invariant are conjugate over $\mathbb{Q}(e^{2\pi i/\ell})$. As with Serre's **OIT**, there is another Nielsen class, $\text{Ni}((\mathbb{Z}/\ell)^4 \times {}^s\mathbb{Z}/3, \mathbf{C}_{+3^2-3^2})^{\text{in,rd}}$, with limit group $G^{\text{n-lm}} = (\mathbb{Z}/\ell)^4 \times {}^s\mathbb{Z}/3$. A new phenomena occurs in Thm. 5.24:

the presence of more than one **HM** braid orbit.

As with the case $\ell = 2$, we used the **sh**-incidence matrix to see this result. Putting the higher levels of the **MT** in a graphical display is still a work in progress, but we expect to have it in [Fr20b].⁶⁰

As in Def. 5.5, use CM and GL_2 for the respective Nielsen classes $\text{Ni}(D_\ell, \mathbf{C}_{2^4})$, $\text{Ni}((\mathbb{Z}/\ell)^2, \mathbf{C}_{2^4})$ and the decorations that appear with them.

In both §5.2.1 and in §5.3.2, under \bullet [Se68] \bullet we call attention to the eventually Frattini Def. 3.14 and its abstraction for the CM and GL_2 cases in (5.11). Then, in \bullet [Fr78] \bullet we take advantage of the relation between the two distinct Nielsen classes as follows.

- (5.28a) The GL_2 covers of the j -line are related to the CM covers over the j -line, by the former being the Galois closure of the latter.
- (5.28b) For CM or GL_2 , the extension of constants indicates that we have the appropriate description of the fiber according to the **OIT**.
- (5.28c) In the GL_2 case, in referring to (5.28b), braids give the geometric elements, $\text{SL}_2(\mathbb{Z}/\ell^{k+1})/\langle \pm 1 \rangle$ in this case, of $N_{S_n}(G)/G$. That gives all the geometric monodromy of the j -line covers.

Comment on (5.28a): This was what our discussion of \bullet [Fr78] \bullet was about. From (5.28c) we “see” the GL_2 geometric monodromy. The most well-known proof of [Se68] is – our language: $\{\text{SL}_2(\mathbb{Z}/\ell^{k+1})/\langle \pm 1 \rangle\}_{k=0}^\infty$ is ℓ -Frattini (resp. eventually ℓ -Frattini) for $\ell > 3$ (resp. for all ℓ).

Such moduli spaces, affording refined ability to interpret cusps, enable objects of the style of the Tate curves (as in \bullet [Fr78] \bullet) around, say, the *Harbater-Mumford* type cusps. This is compatible with those cusps in the discussions of \bullet [Fr95, Thm. 3.21] , [W98] and [DEm05] \bullet .

As in the discussion of \bullet [Fr78] \bullet we can expect inexplicit versions of Faltings [Fa83] if we can find any version of them at this time. Also, we must ask how far into one of the eventually ℓ -Frattini strands we must go to assure the fiber over $j' \in \bar{\mathbb{Q}}$ has revealed itself?

Remark 5.25 (Rem. (2.12) Cont.). [FrH20] shows regular behavior on the Thm. 5.24 **MT** sequences above level 0. There are different types of such

⁶⁰That will also include how displays of the higher tower levels that already appear for $(A_5, \mathbf{C}_{3^4}, \ell = 2)$ in [BFR02].

towers, mostly depending on the powers of ℓ dividing lift invariants of the corresponding component sequences. Even at level 0, Thm. 5.24 shows an increasing numbers of components, many conjugate over \mathbb{Q} , as ℓ changes.

(5.29a) [FrH20] will completely display all **MT** levels.

(5.29b) What are the $G_{\mathbb{Q}}$ orbits on the **HM** components with their moduli structures (carrying families of covers) of \mathbb{P}_z^1 ?

Having several **HM** orbits in (5.29b) leaves a lift invariant puzzle. [BFr02, Ex. 9.3] has carefully diagnosed cases of this where $G = A_4$ and A_5 at level $k = 1$ of their **MTs** for $\ell = 2$. Trivial lift invariants are often valuable. They give a distinguished component, as in [FrV92]. Yet, when there is more than one, we don't yet know how to geometrically distinguish them.

Remark 5.26 (Nielsen limit group). For Thm. 5.24, the analog of §4.1.2, especially Ex. 4.1 for $G^{\text{n-lm}}$ works here, too. You construct the element in the limit group Nielsen class from those in the original Nielsen class. With that comes a natural braid action, which embeds H_4 into the symplectic group acting on $H^1(X, \mathbb{Z}_{\ell})$ of some hyperelliptic curve chosen as a base point on the Hurwitz space.

REFERENCES

- [Ahl79] L. Ahlfors, *Introduction to the Theory of Analytic Functions of One Variable*, 3rd edition, Inter. Series in Pure and Applied Math., McGraw-Hill Complex Variable, 1979.
- [BFr02] P. Bailey and M. D. Fried, *Hurwitz monodromy, spin separation and higher levels of a Modular Tower*, Arithmetic fundamental groups and noncommutative algebra, PSPUM vol. **70** of AMS (2002), 79–220. arXiv:math.NT/0104289 v2 16 Jun 2005.
- [Be91] D. J. Benson, *Representations and cohomology, I: Basic representation theory of finite groups and associative algebras*, Cambridge studies in Advanced Mathematics, Camb. Univ. Press **30** (1991).
- [Br82] K.S. Brown, *Cohomology of Groups*, Graduate texts in Mathematics, Springer-Verlag, New York, Berlin, . . . , (1982).
- [Ca06] A. Cadoret, *Harbater-Mumford subvarieties of Hurwitz moduli spaces of covers*, Math. Annalen, 2006.
- [CaD08] A. Cadoret and P. Dèbes, *Abelian obstructions in inverse Galois theory*, Manuscripta Mathematica, 128/3 (2009), 329–341.
- [CaTa09] A. Cadoret and A. Tamagawa, *Uniform boundedness of p -primary torsion of Abelian Schemes*, preprint as of June 2008.
- [Con78] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Grad. Texts in Math. **11**, Springer-Verlag, NY, Heidelberg, Berlin, 1978.
- [D95] P. Dèbes, *Compte rendu sur le livre Topics in Galois Theory de J.P.-Serre*, Gazette des mathématiciens, vol. **66**, SMF, 10/95, [traduction d'une "Book review" de M. Fried, Bull. A.M.S., vol. 30, (01/94)].
- [D06] ———, *Modular towers: Construction and diophantine questions*, Luminy Conference on Arithmetic and Geometric Galois Theory, vol. **13**, Seminaire et Congres, 2006.

- [D06a] ———, *An introduction to the modular tower program*, in Groupes de Galois arithmétiques et différentiels (Luminy 2004), eds. D. Bertrand and P. Dèbes, Séminaires et Congrès, vol. 13, SMF, 2006, 127–144. (voir aussi Modular Towers. Lecture Notes, GTEM Summer School, 09-20 June, 2008, Istanbul, Geometry and Arithmetic of Moduli Spaces of Covers.
- [DDes04] ——— and B. Deschamps, *Corps ω -libres et théorie inverse de Galois infinie*, J. für die reine und angew. Math. **574** (2004), 197–218.
- [DEm05] ——— and M. Emsalem, *Harbater-Mumford Components and Hurwitz Towers*, J. Inst. of Mathematics of Jussieu (**5/03**, 2005), 351–371.
- [DFr90] P. Dèbes and M.D. Fried, *Rigidity and real residue class fields*, Acta. Arith. **56** (1990), 13–45.
- [DFr94] ——— and ———, *Nonrigid situations in constructive Galois theory*, Pacific Journal **163** (1994), 81–122.
- [Ek90] T. Ekedahl, *An effective version of Hilbert’s irreducibility Theorem* Sem. Théorie des Nombres, Paris 1988-1989, Birkhäuser, 1990, 241–248.
- [EVW19] J. Ellenberg, A. Venkatesh and C. Westerland, *Homological Stability for Hurwitz spaces and the Cohen-Lenstra Conjecture over Function fields*, Forum of Mathematics, Camb. U. Press,
- [Fa83] G. Faltings, *Endlichkeitsätze Über Zahlkörpern*, Invent. Math. **73** (1983), 349–366.
- [Fr70] M.D. Fried, *On a Conjecture of Schur*, Michigan Math. J. Volume **17**, Issue 1 (1970), 41–55.
- [Fr77] ———, *Fields of Definition of Function Fields and Hurwitz Families and; Groups as Galois Groups*, Communications in Algebra **5** (1977), 1782.
- [Fr78] ———, *Galois groups and Complex Multiplication*, Trans.A.M.S. **235** (1978), 141–162.
- [Fr80] ———, *Riemann’s Existence Theorem: An elementary approach to moduli*, <http://www.math.uci.edu/~mfried/booklist-ret.html> takes you to an introduction to the whole topic, and to each of the chapters.
- [Fr90] ———, *Arithmetic of 3 and 4 branch point covers: a bridge provided by noncongruence subgroups of $SL_2(\mathbb{Z})$* , Progress in Math. Birkhauser 81 (1990), 77–117.
- [Fr94] ———, *Enhanced review of J.-P. Serre’s Topics in Galois Theory, with examples illustrating braid rigidity*, Recent Developments in the Galois Problem, Cont. Math., proceedings of AMS-NSF Summer Conference, Seattle **186** (1995), 1532. Briefer review—Topics in Galois Theory, J.-P. Serre, 1992, Bartlett and Jones Publishers, BAMS **30** #1 (1994), 124135. ISBN 0-86720-210-6.
- [Fr95] ———, *Introduction to Modular Towers: Generalizing dihedral group-modular curve connections*, Recent Developments in the Inverse Galois Problem, Cont. Math., proceedings of AMS-NSF Summer Conference 1994, Seattle **186** (1995), 111–171.
- [Fr02] ———, *What Gauss told Riemann about Abel’s Theorem*, resides on the Abel portion of the Danish National Academy of Science; presented in the Florida Mathematics History Seminar, Spring 2002, as part of John Thompson’s 70th birthday celebration.
- [Fr02b] ———, *Moduli of relatively nilpotent extensions*, Communications in Arithmetic Fundamental Group, Inst. of Math. Science Analysis, vol. **1267**, RIMS, Kyoto, Japan, 2002, 70–94.
- [Fr05] ———, *The place of exceptional covers among all diophantine relations*, J. Finite Fields **11** (2005) 367–433. Original pencil corrections have been incorporated in arXiv:0910.3331v1 [math.NT]
- [Fr06] ———, *The Main Conjecture of Modular Towers and its higher rank generalization*, in Groupes de Galois arithmetiques et differentiels (Luminy 2004; eds. D. Bertrand and P. Dèbes), Sem. et Congres, Vol. **13** (2006), 165–233.

- [Fr06b] ———, *Regular realizations of p -projective quotients and modular curve-like towers*, Oberwolfach report **#25**, on the conference on pro- p groups, April (2006), 64–67. Also available at the conference archive.
- [Fr09] ———, *Connectedness of families of sphere covers of A_n -type*, Preprint as of June 2008.
- [Fr10] ———, *Alternating groups and moduli space lifting Invariants*, Arxiv textbf#0611591v4. Israel J. Math. **179** (2010) 57–125 (DOI 10.1007/s11856-010-0073-2).
- [Fr12] ———, *Variables separated equations: Strikingly different roles for the Branch Cycle Lemma and the Finite Simple Group Classification*: arXiv:1012.5297v5 [math.NT] (DOI 10.1007/s11425-011-4324-4). Science China Mathematics, vol. **55**, January 2012, 1–72.
- [Fr20] ———, *Monodromy, ℓ -adic representations and the Regular Inverse Galois Problem*, <http://www.math.uci.edu/~mfried/paplist-mt/ChernBookOIT01-01-17.pdf>.
- [Fr20b] ———, *The Shift Incidence matrix and orbits of Hurwitz spaces*, in preparation.
- [FrH20] ——— and M. van Hoeij, *The small Heisenberg group and ℓ -adic representations from Hurwitz spaces*.
- [FrJ86] ——— and M. Jarden, *Field arithmetic*, Ergebnisse der Mathematik III, vol. **11**, Springer Verlag, Heidelberg, 1986 (455 pgs); 2nd Edition 2004 (780 pgs) ISBN 3-540- 22811-x. We quote here both the first and second ed., using [FrJ86]₁ and [FrJ86]₂ respectively.
- [FrK97] ——— and Y. Kopeliovic, *Applying Modular Towers to the Inverse Galois Problem*, Geometric Galois Actions II Dessins d’Enfants, Mapping Class Groups and Moduli **243**, London Mathematical Society Lecture Note series, (1997) 172–197.
- [FrV91] ——— and H. Völklein, *The inverse Galois problem and rational points on moduli spaces*, Math. Ann. **290**, (1991) 771–800.
- [FrV92] ——— and ———, *The embedding problem over an Hilbertian-PAC field*, Annals of Math **135** (1992), 469–481.
- [GS78] R. Griess and P. Schmid, *The Frattini module*, Archiv. Math. **30** (1978), 256–266.
- [Gr71] A. Grothendieck, *Revêtements étale et groupes fondamentaux*, Lect. Notes Math., vol. **224**, Berline Heidelberg New York, Springer 1971.
- [GMS03] R. Guralnick, P. Müller and J. Saxl, *The rational function analogue of a question of Schur and exceptionality of permutations representations*, Memoirs of the AMS **162** 773 (2003), ISBN 0065-9266.
- [Ha84] D. Harbater, *Mock covers and Galois extensions*, J. Algebra **91** (1984), 281–293.
- [HWi20] Y. Harpaz and O. Wittenberg, *Zro-cycles sur les espaces homognes et probleme de Galois inverse*, to appear in JAMS.
- [Ih86] Y. Ihara, *Profinite braid groups, Galois representations and complex multiplications*, Ann. of Math. (2) **123** (1986), no. 1, 43–106. MR MR825839 (87c:11055)
- [IM95] Y. Ihara and M. Matsumoto, *On Galois actions on profinite completions of braid groups*, Recent Developments in the Inverse Galois Problem, Cont. Math., proceedings of AMS-NSF Summer Conference 1994, Seattle **186** (1995), 173–200.
- [LO08] F. Liu and B. Osserman, *The Irreducibility of Certain Pure-cycle Hurwitz Spaces*, Amer. J. Math. # 6, vol. **130** (2008), 1687–1708.
- [MaSh19] A. Malmendier and T. Shaska, *From hyperelliptic to superelliptic curves*, Albanian J. Math. **13** (2019), no. 1, 107–200.
- [Me90] J.-F. Mestre, *Extensions régulières de $\mathbb{Q}(t)$ de groupe de Galois \tilde{A}_n* , J. of Alg. **131** (1990), 483–495.
- [Mu72] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, Comp. Math. **24** (1972), 129–174.

- [Ne81] E. Neuenschwanden, *Studies in the history of complex function theory II: Interactions among the French school, Riemann and Weierstrass*, BAMS **5** (1981), 87–105.
- [NScW00] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, Grundlehren der mathematischen wissenschaften **323**, Springer, 2000, Berlin.
- [Nor62] D.G. Northcott *An introduction to homological algebra*, Cambridge Univ. Press, Great Britain, 1962.
- [O67] A.P. Ogg, *Abelian curves of small conductor*, Crelles **J 226** (1967), 204–215.
- [Ri90] K. Ribet, Bull. Amer. Math. Soc. (N.S.) **22** (1990), No. 1 214–218.
- [Sa19] E. Samperton, *Schur-type invariants of branched G -covers of surfaces*, preprint.
- [Se68] J.-P. Serre, *Abelian ℓ -adic representations and elliptic curves*, New York, Benj. Publ., 1968.
- [Se81] ———, *Quelques Applications du Théorème de Densité de Chebotarev*, Publ. Math. IHES **54** (1981), 323–401.
- [Se88] ———, *Letter on Ext, Private Correspondence*, 1988.
- [Se90a] ———, *Relèvements dans A_n* , C. R. Acad. Sci. Paris **311** (1990), 477–482.
- [Se90b] ———, *Revêtements a ramification impaire et thêta-caractéristiques*, C. R. Acad. Sci. Paris **311** (1990), 547–552.
- [Se92] ———, *Topics in Galois theory*, no. ISBN #0-86720-210-6, Bartlett and Jones Publishers, notes taken by H. Darmon, 1992.
- [Se97a] ———, *Galois cohomology*, Springer-Verlag Berlin, 1997. MR1466966 (98g:12007)
- [Se97b] ———, *Unpublished notes on l -adic representations*, I saw a presentation during Oct. 97 at Cal Tech; it has pieces produced by various notetakers over a long period of time.
- [Se03] ———, *On a Theorem of Jordan*, BAMS **40** #4 (2003), 429–440.
- [Sp57] G. Springer, *Introduction to Riemann Surfaces*, 1st ed., Addison-Wesley Mathematics Series, vol. **11**, Addison-Wesley, Reading, Mass., 1957.
- [Sh64] G. Shimura, *Abelian Varieties with Complex Multiplication and Modular Functions*, Princeton U. Press, 1964 (with Y. Taniyama), latest edition 1998.
- [Th84] J.G. Thompson, *Some finite groups which appear as $Gal(L/K)$, where $K \subseteq \mathbb{Q}(\mu_n)$* , J. Alg. **89** (1984), 437–499.
- [Vo96] H. Völke, *Groups as Galois Groups*, Cambridge Studies in Advanced Mathematics **53** (1996).
- [W98] S. Wewers, *Construction of Hurwitz spaces*, Thesis, Institut für Experimentelle Mathematik **21** (1998), 179.
- [We05] T. Weigel, *Maximal ℓ -frattini quotients of ℓ -poincare duality groups of dimension 2*, volume for O.H. Kegel on his 70th birthday, Arkiv der Mathematik–Basel, 2005.

EMERITUS, UC IRVINE, 1106 W 171ST AVE, BROOMFIELD CO 80023
E-mail address: mfried@math.uci.edu