ℓ-adic Representations and the Regular Inverse Galois Problem

Oberwolfach, April 15-21, 2018: Mike Fried, Emeritus UC Irvine

PART 1: Nielsen classes and Galois closures of covers over **C**

1.a. Algebraic/Geometric Galois closure of $f: W \to \mathbb{P}^1_z$.

- $\hat{f}: \hat{W} \to \mathbb{P}^1_z$ factors through f; and it is a *Galois cover*. The group of those automorphisms is G_f.
- Construction from the fiber product of f, $\deg(f) = n$ times

$$(W)^{(n)}_f \stackrel{\text{def}}{=} \{(w_1, \dots, w_n) \in (W)^n \mid f(w_1) = \cdots = f(w_n)\} \text{ over } \mathbb{P}^1_z.^0$$

- Normalize, ${}^*W^{\{n\}}_f$ (nonsingular), in its function field; remove *fat diagonal* $\Delta^{\{n\}}_f$. Result has transitive S_n action.

Take $\tilde{f}: \tilde{W}_f \to \mathbb{P}^1_z$, a component.

1.b. Conjugacy classes and Nielsen classes. Denote the conjugacy class of $g \in G_f$ by C_g.

- *Geometric monodromy* group: $G_f = \{g \in S_n | g \text{ preserves } \tilde{W}_f\}$.

  Automatically $|G_f| = \deg(\tilde{f}) \implies$ Galois.
- Each *branch point* z_i of f gives a conjugacy class $\mathrm{C}_i = \mathrm{C}_{g_i}$ in G_f, $i = 1, \ldots, r$. It doesn't pick out g_i *uniquely*.
- Also gives *arithmetic monodromy*, $\hat{G}_f$:

  Consider components defined over a field K.[1]

Analytic Geometry: Invert $f \mapsto \boldsymbol{g}$

Nielsen Classes, $\mathrm{Ni}(G, \mathbf{C})$:

- $f : W \to \mathbb{P}^1_z, \deg(f) = n$; r distinct branch points $\boldsymbol{z}$; $(G = G_f \le S_n, \mathbf{C}) \Leftrightarrow \boldsymbol{g} \in G^r \cap \mathbf{C}$ so the following hold.
- **Generation** – $\langle \boldsymbol{g} \rangle = G$; and **Product-one** – $\prod_{i=1}^r g_i = 1$.
- Nielsen classes of $(G, \mathbf{C})$ covers (both sides up to equivalence):

  Branch cycles: $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}) \Leftrightarrow f_{\boldsymbol{g}} : W_{\boldsymbol{g}} \to \mathbb{P}^1_z$.
- Genus $\mathbf{g}_{\boldsymbol{g}}$ of $W_{\boldsymbol{g}}$: $2(\deg(f) + \mathbf{g}_{\boldsymbol{g}} - 1) = \sum_{i=1}^r \mathrm{ind}(g_i).^2$

- $G \leq S_n$ a group, $\mathbf{C}$, r conjugacy classes: $g \in S_n$ with t disjoint cycles, its *index* is $\mathrm{ind}(g) = n-t$.
- Notation: For $\mathbf{C}$, r conjugacy classes – may be repeated.

  $\boldsymbol{g} \in G^r \cap \mathbf{C}$ means an r-tuple $\boldsymbol{g}$ has entries in *some* order in $\mathbf{C}$. Denote group the entries of $\boldsymbol{g}$ generate by $\langle \boldsymbol{g} \rangle$.
- Basic Assumptions: G is a transitive subgroup of S_n. Conjugacy classes, $\mathbf{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_r\}$, in G are *generating*.

Dragging a cover by its branch points

- r unordered $\{\boldsymbol{z}\} \subset \mathbb{P}^1_z$ points, minus locus where two meet: $U_r =$ projective r-space $\mathbb{P}^r$ minus its *discriminant locus* (D_r).
- Start: ${}_0f : {}_0W \to \mathbb{P}^1_z \Leftrightarrow {}_0\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, branch points ${}_0\boldsymbol{z}$,

  Classical generators ${}_0\boldsymbol{P}$ of $\pi_1(\mathbb{P}^1_z \setminus \{\boldsymbol{z}_0\}, z_0)$,

  $\pi_1(U_r, {}_0\boldsymbol{z}) \stackrel{\mathrm{def}}{=} H_r$, the *Hurwitz monodromy group*.
- Drag ${}_0\boldsymbol{z}$ and ${}_0\boldsymbol{P}$ to ${}_1\boldsymbol{z}$ along any path B in U_r.
  Form a trail of covers ${}_tf : {}_tW \to \mathbb{P}^1_z$ using ${}_t\mathcal{P} \mapsto {}_0\boldsymbol{g}$, $t \in [0, 1]$.
- For B closed, ${}_1f$ depends only on $[B] \in H_r$.[3]

For B closed, $[B] = q_B \in H_r$ computing ${}_1\mathcal{P} \to {}_0\boldsymbol{g}$ as ${}_0\mathcal{P} \to ({}_0\boldsymbol{g})q_B^{-1}$

- Normalizer of G in S_n, permutating $\mathbf{C}$: $N_{S_n}(G, \mathbf{C})$.

$$q_i: \ \boldsymbol{g} \stackrel{\text{def}}{=} (g_1, \dots, g_r) \mapsto (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_r);$$
$$\mathbf{sh}: \ \boldsymbol{g} \mapsto (g_2, g_3, \dots, g_r, g_1); \mathbf{sh}\, q_i\ \mathbf{sh}^{-1} = q_{i+1},\ i \bmod r-1.$$
$$R = q_1 \cdots q_{r-1} q_{r-1} \cdots q_1.$$

  - ⋆ Generators of H_r: $H_r = B_r/\langle R\rangle, H_r \stackrel{\text{def}}{=} \langle q_2, \mathbf{sh}\rangle$
- *Inner equivalence*: R forces $\text{Ni}(G, \mathbf{C})/G = \text{Ni}(G, \mathbf{C})^{\text{in}}$.[4a]
- *Absolute equivalence*: $\text{Ni}(G, \mathbf{C})/N_{S_n}(G, \mathbf{C}) = \text{Ni}(G, \mathbf{C})^{\text{abs}}$.

1.c. Hurwitz space components: † is an equivalence Genus of reduced Hurwitz spaces when $r=4$: $\mathcal{Q}'' \stackrel{\text{def}}{=} \langle q_1 q_3^{-1}, \mathbf{sh}^2\rangle.^{4b}$

Fundamental group H_r acts on $\text{Ni}(G,\mathbf{C})^\dagger \Leftrightarrow \mathcal{H}(G,\mathbf{C})^\dagger \to U_r$.

- *H_r orbits*: Components of $\mathcal{H}(G,\mathbf{C})^\dagger \Leftrightarrow H_r$ orbits on $\text{Ni}(G,\mathbf{C})^\dagger$.
- *Reduced equivalence*: $f \sim \alpha \circ f : W \to \mathbb{P}^1_z$ for $\alpha \in \mathrm{PSL}_2(\mathbb{C})$.

  $\mathcal{H}(G,\mathbf{C})^\dagger/\mathrm{PSL}_2 \stackrel{\text{def}}{=} \mathcal{H}(G,\mathbf{C})^{\dagger,\mathrm{rd}} \to U_r/\mathrm{PSL}_2 = J_r$.

  Reduced classes for $r=4$:

  $$\text{Ni}(G,\mathbf{C})^{\dagger,\mathrm{rd}} \stackrel{\text{def}}{=} \text{Ni}(G,\mathbf{C})^\dagger/\mathcal{Q}''.$$

  - $\mathcal{H}(G,\mathbf{C})^{\dagger,\mathrm{rd}} \to J_4 = \mathbb{P}^1_j \setminus \{\infty\}$ that completes to $\overline{\mathcal{H}}(G,\mathbf{C})^{\dagger,\mathrm{rd}} \to \mathbb{P}^1_j$ ramified over 0 (order 3), 1 (order 2), ∞.

A genus formula for $r = 4$

Thm (Riemann-Hurwitz for j-line covers)

Component $\overline{\mathcal{H}}'$, of $\overline{\mathcal{H}}(G, \mathbf{C})^{\dagger,\mathrm{rd}} \leftrightarrow$ braid orbit, O, on Ni$(G, \mathbf{C})^{\dagger,\mathrm{rd}}$. Ramification, respectively over $0, 1, \infty$, of $\overline{\mathcal{H}'} \to \mathbb{P}^1_j \leftrightarrow$

disjoint cycles of $\gamma_0 = q_1 q_2, \gamma_1 = q_1 q_2 q_1, \gamma_\infty = q_2$ (cusps).

Genus of $g_{\bar{\mathcal{H}}'}$, appears from the formula

$$2(|O| + g_{\bar{\mathcal{H}}'} - 1) = \mathrm{ind}(\gamma_0) + \mathrm{ind}(\gamma_1) + \mathrm{ind}(\gamma_\infty).^5$$

Genus of components for $\text{Ni}_4 \stackrel{\text{def}}{=} \text{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$.

- $A_4 = (\mathbb{Z}/2)^2 \times^s \mathbb{Z}/3 \le A_5$, $\mathbf{C}_{\pm 3^2}$ is two repetitions of the two classes of 3-cycles in $\mathbb{Z}/3$
- Thus, from the **BCL** the classes are a rational union

  $\Leftrightarrow$ the Hurwitz spaces are defined over $\mathbb{Q}$.

  - ${}_{\mathbf{c}}O^k_{i,j}$ indicates a γ_∞ orbit; k is the cusp width.

  sh-incidence matrix entries are $|{}_{\mathbf{c}}O^k_{i,j} \cap ({}_{\mathbf{c}}O^{k'}_{i',j'})\mathbf{sh}|$.
- Next a list of γ_∞ (cusp) orbits that show this **sh**-incidence matrix has two blocks.

Each cusp ${}_{\mathbf{c}}O^k_{i,j}$ has $\boldsymbol{g}$ with entries $\{g, g^{-1}\}$ or $\{g, g\}$ resp. **HM** in Ni_0^+, or **D**(ouble) **I**(dentity) in Ni_0^-.[6]

Ni_0^+ Orbit	${}_{\mathbf{c}}O^4_{1,1}$	${}_{\mathbf{c}}O^2_{1,2}$	${}_{\mathbf{c}}O^3_{1,3}$
${}_{\mathbf{c}}O^4_{1,1}$	1	1	2
${}_{\mathbf{c}}O^2_{1,2}$	1	0	1
${}_{\mathbf{c}}O^3_{1,3}$	2	1	0

Ni_0^- Orbit	${}_{\mathbf{c}}O^4_{2,1}$	${}_{\mathbf{c}}O^1_{2,2}$	${}_{\mathbf{c}}O^1_{2,3}$
${}_{\mathbf{c}}O^4_{2,1}$	2	1	1
${}_{\mathbf{c}}O^1_{2,2}$	1	0	0
${}_{\mathbf{c}}O^1_{2,3}$	1	0	0

- ${}_{\mathbf{c}}O^4_{1,1} = (g_{1,1})^{q_2,\bullet}, \boldsymbol{g}_{1,1} = ((1\,2\,3),(1\,3\,2),(1\,3\,4),(1\,4\,3))$
  ${}_{\mathbf{c}}O^3_{1,3} = (\boldsymbol{g}_{1,3})^{q_2,\bullet}, \boldsymbol{g}_{1,3} = ((1\,2\,3),(1\,3\,2),(1\,4\,3),(1\,3\,4))$.
- ${}_{\mathbf{c}}O^4_{2,1} = (g_{2,1})^{q_2,\bullet}, \boldsymbol{g}_{2,1} = ((1\,2\,3),(1\,3\,4),(1\,2\,4),(1\,2\,4))$.
  ${}_{\mathbf{c}}O^1_{2,2}$ and ${}_{\mathbf{c}}O^1_{2,3}$ seeded by **DI**s repeated in positions 2 and 3.

Read genus and *reduced fine moduli*[7] from the blocks Denote $(\gamma_0, \gamma_1, \gamma_\infty)$ on Ni_0^+ (resp. Ni_0^-) orbit by $(\gamma_0^+, \gamma_1^+, \gamma_\infty^+)$(*resp.* $(\gamma_0^-, \gamma_1^-, \gamma_\infty^-)$).

- Fixed points of γ_0 and γ_1 appear on the diagonal. Diagonal entries for ${}_{\mathbf{c}}O_{1,1}^4$ and ${}_{\mathbf{c}}O_{2,1}^4$ are nonzero;

  γ_1 (resp. γ_0) fixes 1 (resp. no) element of ${}_{\mathbf{c}}O_{1,1}$.

  Neither of γ_i, $i = 0, 1$, fix any element of ${}_{\mathbf{c}}O_{2,1}^4$.
- Cusp widths over ∞ add to the degree 9 (resp. 6) to give

  $$\mathrm{ind}(\gamma_0^+) = 6,\ \mathrm{ind}(\gamma_1^+) = 4,\ \mathrm{ind}(\gamma_\infty^+) = 6$$
  $$\mathrm{ind}(\gamma_0^-) = 4,\ \mathrm{ind}(\gamma_1^-) = 3,\ \mathrm{ind}(\gamma_\infty^-) = 3.$$
- The genus of $\bar{\mathcal{H}}_{0,\pm}$ is $g_\pm = 0$:

  $$2(9+g_+-1) = 6+4+6 = 16 \text{ and } 2(6+g_--1) = 4+3+3 = 10.$$

PART 2: The **R**(egular)**I**(nverse)**G**(alois)**P**(roblem) and **M**(odular) **T**(ower)s

- *Frattini cover*: $\psi : H \to G$; if $H^* \leq H$ and $\psi(H^*) = G \Longrightarrow H^* = H$.

  ℓ-perfect: $\ell || G|$, but G has no $\mathbb{Z}/\ell$ quotient.
- Problem: Most groups are not like simple or solvable.
- Example: Take G, ℓ-perfect and centerless: $\exists \nu(G,\ell) > 0$ (> 1, outside supersolvable) and an extension

$$1 \to (\mathbb{Z}_\ell)^{\nu(G,\ell)} \to {}_\ell\tilde{G}_{\text{ab}} \xrightarrow{{}_\ell\tilde{\psi}_{\text{ab}}} G \to 1 :$$ [8]

  ${}_\ell\tilde{G}_{\text{ab}}$ universal for ℓ-Frattini covers of G with abelian kernel.
- Subex.: Even for $G = A_5$, and where $v(G,2) = 5$, for no $k > 0$ has ${}_2\tilde{A}_{5,\text{ab}}/2^k \ker({}_\ell\tilde{\psi}_{\text{ab}}) = {}^k_2A_5$ been realized over $\mathbb{Q}$.[9]

Which is more *important/serious/...*?: Cases similar to $(G,\ell)=(D_\ell,\ell=\ell)$ or to $(A_5,\ell=2)$. **MT** definition

- Assume r conjugacy classes, $\mathbf{C}$ of G; elements of order ℓ'.

  Schur-Zassenhaus lifts these classes uniquely to ${}_\ell\tilde{G}_{\rm ab}$.
- Makes sense of ${}_\ell\mathbb{H}(G,\mathbf{C})^{\rm in,rd} \stackrel{\rm def}{=} \{\mathcal{H}({}^k_\ell G,\mathbf{C})^{\rm in,rd}\}_{k=0}^\infty$.

  MT: Projective sequence of components on ${}_\ell\mathbb{H}(G,\mathbf{C})^{\rm in,rd}$.[9]
- ${}_\ell\mathbb{H}(D_\ell,\mathbf{C}_{2^4})^{\rm in,rd}=\{X_1(\ell^{k+1})\}_{k=0}^\infty$ for ℓ odd.
- Example: ${}_2\tilde{A}_4$ is the pullback of $A_4\le A_5$ to ${}_2\tilde{A}_5$.

  $(\mathbb{Z}_2)^5$: As A_4(but not A_5) module has $(\mathbb{Z}_2)^2$ as a quotient.

  $\mathcal{H}(A_5,\mathbf{C}_{3^4})^{\rm in,rd}$ has one genus 0 component.[10]

Why $\mathcal{H}(A_4, \pm C_{3^2})^{\rm in,rd}$ has two components
Commutator central ext. $\psi : H \to G$: $\ker(\psi) = [H, H] \cap \ker(\psi)$.

- Commutator Central extension ψ is automatically Frattini.[11]
- Universal central extension, $\hat{A}_4$, of A_4 is pullback of
  $$A_4 \le A_5 \text{ to } \{\pm 1\} \to \mathrm{SL}_2(\mathbb{Z}/5) \to \mathrm{PSL}_2(\mathbb{Z}/5) = A_5.$$
- $O_{\boldsymbol{g}}$ braid orbit *Lift invariant*, $\hat{\boldsymbol{g}} \in \mathrm{SL}_2(\mathbb{Z}/5)^4 \cap \mathbf{C}_{\pm 3^2}$ over $\boldsymbol{g}$:
  $$s_\psi(O_{\boldsymbol{g}}) = \prod_{i=1}^r \hat{g}_i \stackrel{\rm def}{=} \prod(\hat{\boldsymbol{g}}).$$
  If $\boldsymbol{g} \in \mathrm{Ni}_0^+(A_4, \mathbf{C}_{\pm \mathbf{C}^2})$ an **HM** rep., then $s_\psi(O_{\boldsymbol{g}}) = +1$.[12]
- Lift invariant correspondence: $\mathrm{Ni}(A_4, \mathbf{C}_{+3^3}) \Leftrightarrow \mathrm{Ni}^-(A_4, \mathbf{C}_{\pm 3^2})$
  $$\boldsymbol{g}' = (g_1', g_2', g_3') \Leftrightarrow ((g_1')^{-1}, g_2', g_3', (g_1')^{-1}) \text{ (\textbf{DI} rep.)}$$
  Then, $s_\psi(O_{\boldsymbol{g}'}) = s_\psi(O_{1,4\boldsymbol{g}}) = -1$. *Fried-Serre formula*: Covers in $\mathrm{Ni}(A_4, \mathbf{C}_{+3^3})^{\rm abs}$ have genus 0, so lift invariant is $(-1)^3$.[13]

Relation to the **RIGP** over $\mathbb{Q}$: Dihedral groups, ℓ odd

- If $B > 0$ and $\exists$ a $\mathbb{Q}$ regular realization of $D_{\ell^{k+1}}$ for each $k \geq 0$

  with $\leq B$ branch points, then (with Pierre Dèbes):

- **Thm:** $\exists r \leq B$ with $\mathbb{Q}$ points on every ${}_{\ell}\mathbb{H}(D_\ell, \mathbf{C}_{2^r})$ **MT** level: $\Leftrightarrow \exists d < \frac{B-2}{2}$ and an ℓ^{k+1} cyclotomic point on a hyperelliptic jacobian (over $\mathbb{Q}$; varying with k) of dimension d, $k \geq 0$.
- Torsion Conjecture: This is not possible.
- *B*-**free Conjecture**: Don't stipulate any B.
  For each ℓ^{k+1} there is such a point for some $d = d_{k,\ell}$.

  Yet, no one has found them beyond $r = 4$ and $\ell = 7$.

Analog of D_{ℓ^∞} for G, an ℓ-perfect group $(\mathbb{Z}_\ell)^{\nu(G,\ell)} \to {}_\ell\tilde{G}_{\rm ab} \xrightarrow{\ell\tilde{\psi}} G$, $\ker({}_\ell\tilde{\psi})$ a $\mathbb{Z}_\ell[G]$ module.

- If $B > 0$ and $\exists$ $\mathbb{Q}$ regular realization with $\leq B$ branch points

  $$\text{of } {}^k_\ell G = \tilde{G}_{\rm ab}/\ell^k \ker({}_\ell\tilde{\psi}) \text{ for each } k:$$

- **Thm:** $\Leftrightarrow \exists$ $r < B$ conjugacy classes $\mathbf{C}$, ℓ', and a **MT**

  $$\{\mathcal{H}_k\}_{k=0}^\infty \leq {}_\ell\mathbb{H}(G, \mathbf{C}) \text{ with } \mathcal{H}_k(\mathbb{Q}) \neq \emptyset, k \geq 0.$$

- **Main Conj:** High **MT** levels have *general type* + no $\mathbb{Q}$ points.
- **Thm:** (Fried, Cadoret-Tamagawa)True for $r = 4$.
  Cadoret-Dèbes: Torsion Conj. $\Longrightarrow$ $\mathbb{Q}$ statement of Main Conj.[14]

PART 3: The **O**(pen) **I**(mage) **T**(heorem) and **MT**s
3a. Eventually ℓ-Frattini sequences and a weak **OIT**.

- *Eventually (ℓ)-Frattini* sequence of group covers, $\{H_k\}_{k=0}^{\infty}$ $\exists k_0$ with $H_{k_0+k} \to H_{k_0}$ Frattini (resp. ℓ-Frattini) for $k \geq 0$.
- **OIT Conj.**: For a **MT** $\{\mathcal{H}_k\}_{k=0}^{\infty} \leq {}_{\ell}\mathbb{H}(G, \mathbf{C})$, $\Phi_k : \mathcal{H}_k \to J_r$, with $H_k = G_{\Phi_k}$ (geom. monodromy of Φ_k), then

$${}_{\ell}G_j(\Phi) \stackrel{\text{def}}{=} \lim_{\infty \leftarrow k} H_k \text{ is eventually } \ell\text{-Frattini.}^{15}$$

- **Weak OIT Conclusion**: Then, the *decomposition group* ${}_{\ell}\hat{G}_{j'}(\Phi)$ of a *general* $j' \in \bar{\mathbb{Q}}$ equals ${}_{\ell}\hat{G}_j(\Phi)$ (arith. mon.).

3.b. Comparing Serre's system with $\mathrm{Ni}_{\ell^{k+1},3} \stackrel{\text{def}}{=} \mathrm{Ni}(G_{\ell^{k+1},3} \stackrel{\text{def}}{=} (\mathbb{Z}/\ell^{k+1})^2 \times^s \mathbb{Z}/3, \mathbf{C}_{\pm 3^2}), k \geq 0$

- In Serre's system, $\mathrm{Ni}_{\ell^{k+1},2} \stackrel{\text{def}}{=} \mathrm{Ni}(\mathbb{Z}/\ell^{k+1} \times^s \mathbb{Z}/2, \mathbf{C}_{2^4}), k \geq 0$:
  2 types of decomposition groups: CM and GL_2.[16]
- Successes of Serre's original book:
  - Interpreting complex multiplication on modular curve pts.
  - Used Tate curve at "long" cusp on $X_0(\ell)$ for conclusions on nonintegral $j' \in \bar{\mathbb{Q}}$, but complete proof needed Falting's.[17]
- *Small Heisenberg central extension* $(\mathbb{Z}/\ell^{k+1})^2 \times^s \mathbb{Z}/3$, $\ell > 3$:

$$_{\ell^{k+1}}\boldsymbol{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z}/\ell^{k+1} \right\}$$

- [FrH16,Prop. 4.18] With $\psi : {}_{\ell^{k+1}}\boldsymbol{H} \times^s \mathbb{Z}/3 \to G_{\ell^{k+1},3}$:
  lift inv. formula for $\boldsymbol{g} \in \mathrm{Ni}(\ell^{k+1}, 3)$ ($\ell = 2$: Q_8 replaces ${}_{\ell}\boldsymbol{H}$).[18]

3.c. **MT** example, circumventing a Grothendieck objection.

- Let $K_\ell = \frac{\ell \pm 1}{6}$, $\ell \equiv \mp 1 \bmod 3$, $\ell > 3$ prime. [FrH16, Thm. 5.2]:
- **Thm**[Level 0 Main Result] For $\ell > 3$ prime and level $k = 0$:
  - K_ℓ braid orbits with trivial (0) lift invariant. All **HM** orbits.
  - Braid orbits with nontrivial lift invariant consist of **DI** cusps. each such braid orbit distinguished by its lift invariant.

    All **DI** components are conjugate over $\mathbb{Q}(e^{2\pi i/\ell})$.
- **Extend OIT** ${}_\ell G_j(\Phi)$ **Conj.**: For a *decomposition group* of any $j' \in \bar{\mathbb{Q}}$ (projective sequence over $j' \in \bar{\mathbb{Q}}$)

$${}_\ell \hat{G}_{j'}(\Phi) \cap {}_\ell G_j(\Phi) \text{ is eventually } \ell\text{-Frattini.}$$

3.d. Comments on the program

- Motivic nature: Braid orbits on Hurwitz spaces break it into the Hurwitz version of *motivic* – **MT**– *pieces*. The result above shows different behaviors come with different pieces.
- [FrH16] describes all components/definition fields of **MT**s of the system $\{\text{Ni}_{\ell^{k+1},3}\}_{k=0}^{\infty}$: "Classically" interpreting how the **HM** and **DI** orbits intermix geometrically toward an **OIT**.
- The introduction noted in [Fr18] describes famous classical problems whose solutions were equivalent to *well-understood* cases in Serre's **OIT**, through Nielsen class identifications.
- An **OIT** goal aims at Hilbert's Irreducibility Theorem writ large. Making decomposition groups reflect geometry rather than accidents, as in the theory of Complex Multiplication.

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[Se97b] –, *Unpublished notes on l-adic representations*, I saw a presentation during Oct. 97 at Cal Tech; it has pieces produced by various notetakers over a long period of time. He sent me these.