PAUL L. BAILEY DISSERTATION.

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UNIVERSITY OF CALIFORNIA,
IRVINE

Incremental Ascent of a Modular Tower via Branch Cycle Designs

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Paul L. Bailey

Dissertation Committee:
Professor Michael Fried, Chair
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2002

The dissertation of Paul L. Bailey
is approved and is acceptable in quality
and form for publication on microfilm:

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University of California, Irvine
2002

Dedication

This effort is dedicated to the memory of my father, Clark J. Bailey.

Contents

Acknowledgements

I would like to thank my lovely wife Marizza and four wonderful daughters Jessica, Maia, Sabrina, and Korina, for enduring my stress and distractedness throughout this process. I also thank my mother Alice, sister Clair, and niece Christy for their encouragement.

There are several mathematicians whose role in my education was indispensable, and whose influence contributed to this dissertation. I had many enlightening conversations with Darren Semmen, a fellow graduate student. Ali Nessin, Ron Stern, and Bruce Bennett introduced me to the beautiful intricacies of group theory, algebraic topology, and algebraic geometry. Yaacov Kopeliovich encouraged me to investigate arithmetic geometry.

My sincerest thanks to my advisor, Mike Fried, the most creative mathematician known to me, and with whom it was a great pleasure to explore new ideas.

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Abstract

Incremental Ascent of a Modular Tower via Branch Cycle Designs

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Let G be a finite group and let $\boldsymbol{C}$ be an r-tuple of conjugacy classes from G which generate G. The *reduced Hurwitz space* $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ parameterizes weak equivalence classes of ramified covers of the Riemann sphere $\mathbb{P}^1$ with ramification in $\boldsymbol{C}$. If the rank r is four, the reduced space is a Riemann surface. The modular curves $Y_1(n)$ are such spaces, with $G = D_n$ and $\boldsymbol{C}$ four conjugacy classes of involutions.

We ask three modest questions regarding reduced rank four Hurwitz spaces:

1) How many components are there?

2) What are the genera of the components?

3) What are the fields of definition of the components?

Let p be a prime which divides the order of G. The *universal elementary p-Frattini cover* ${}_p^1\tilde{G} \to G$ is versal for Frattini covers of G with elementary p-group kernel. Inductively define ${}_p^{k+1}\tilde{G} = {}_p^1({}_p^k\tilde{G})$. If p does not divide the orders of the elements in $\boldsymbol{C}$, these conjugacy classes lift uniquely to ${}_p^k\tilde{G}$, producing a sequence of Riemann surfaces

$$\ldots \mathcal{H}({}_p^{k+1}\tilde{G}, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{H}({}_p^{k}\tilde{G}, \boldsymbol{C})^{\mathrm{in,rd}} \to \cdots \to \mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{J}_4,$$

which is called a *Modular Tower*, and is denoted by $\mathbf{MT}_p(G, \boldsymbol{C})$; this generalizes towers of modular curves. Understanding a Modular Tower requires combining knowledge of the base space and techniques of lifting information.

Certain configurations of the branch points give *Harbater-Mumford* covers, which are necessarily defined over $\mathbb{R}$, producing real points on the Hurwitz space. If $p = 2$, these are the only points which lie in projective systems of real points up the tower, and lay at the center of computations.

Given a ramified cover, we develop its *Nielsen graph*, which dictates which covers can factor through the given one. Classical generators for the base space of the cover lift to an embedded realization of the graph in the covering space; this is a *branch cycle design*, and it produces classical generators for the covering space. Using branch cycle designs as platforms and real points as ladders, we ascend to the first level of the Modular Tower $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$, and answer some of the questions posed above.

Preface

We briefly outline the contents of this dissertation. The first three chapters are an overview of the foundations for the later results, emphasizing the covering theory and group theory which are our main tools. The next three chapters introduce details into previously published material, and build tools for the last three chapters, which consist of examples.

Chapter I reviews the three basic mathematical categories which are related by Riemann's Existence Theorem and which produce Hurwitz spaces. These are topological covers, ramified covers, and function field extensions. This section is provided to fix notation and emphasis.

Chapter II discusses braid groups and constructs Hurwitz spaces as the natural covers produced by representations of these groups, emphasizing the interplay between group actions and topology. Reduction of Hurwitz spaces is discussed, specifically in the case of rank four Nielsen classes. This chapter draws from [**Fr77**], [**Fr87**], and [**FV91**].

Chapter III defines the notion of Modular Towers, which were introduced in [**Fr95**] and further explored in [**FK97**] and [**BF02**]. Included in this chapter are brief explanations of profinite groups and Frattini covers, as well as a motivational section on moduli spaces of elliptic curves.

Chapter IV relates certain Nielsen tuples to ramified covers defined over $\mathbb{R}$ and real points on Hurwitz spaces. The initial formulas from [**DF90**] and [**DF94**] are refined for use modulo reduction. The key role of Harbater-Mumford tuples and the prime $p = 2$ begins to take shape.

Chapter V introduces Nielsen graphs and branch cycle designs which produce algorithms for splicing ramified covers. This is the main tool applied to an example in Chapter VII.

Chapter VI reviews and expands upon the group theory of $\frac{1}{2}\tilde{A}_5$ and $\frac{1}{2}\tilde{A}_4$, including a discussion of the automorphisms and spin covers of $\frac{1}{2}\tilde{A}_4$.

Chapter VII investigates the Modular Tower $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$ by incrementally ascending to level one and beyond with the use of branch cycle designs. We show that there are two Harbater-Mumford components, each of genus one.

Chapter VIII draws conclusions from Chapter VII. We find the j-invariants for the Harbater-Mumford components, and discuss the absolute space, obstruction, and real points.

Chapter IX describes GAP programs used to verify our computations. We discuss the Modular Tower $\mathbf{MT}_2(A_5, \boldsymbol{C}_{5^2_\pm})$, and point out its striking similarity to $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$.

CHAPTER I

Ramified Covers

1. Group Actions

1.1. Group Actions.

1.1.1. *Group Actions.* A *group action* is a function $G \times X \to X$, where G is a group and X is a set, such that $1 \cdot x = x$ for every $x \in X$ and $(g_1 g_2)x = g_1(g_2 x)$ for every $g_1, g_2 \in G$ and every $x \in X$. This induces a group homomorphism $\tau : G \to \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ denotes the group of permutations of X, via $\tau_g(x) = gx$, where we write τ_g instead of $\tau(g)$. The *kernel* of the action is

$$\ker(\tau) = \{g \in G \mid gx = x \text{ for all } x \in X\}.$$

The action is *faithful* if for every distinct $g, h \in G$ there exists $x \in X$ such that $gx \neq hx$, that is, when the kernel is trivial. In this case, induced homomorphism $\tau : G \to \mathrm{Sym}(X)$ is injective, so G acts as a subgroup of $\mathrm{Sym}(X)$.

The action is *transitive* if for every $x, y \in X$ there exists $g \in G$ such that $gx = y$. This is equivalent to the condition that for every $x \in X$, the map $G \to X$ given by $g \mapsto gx$ is surjective. Thus if G acts transitively on a finite set X, then $|G| \geq |X|$.

The action is *free* if for every distinct $g, h \in G$ and every $x \in X$ we have $gx \neq hx$. This is equivalent to the condition that for every $x \in X$, the map $G \to X$ given by $g \mapsto gx$ is injective. Thus if G acts freely on a finite set X, then $|G| \leq |X|$. We note that free actions are faithful.

The action is *regular* if it is transitive and free, in which case $|G| = |X|$.

The *orbit* of $x \in X$ under the action of G is

$$\mathrm{Orb}_G(x) = \{y \in X \mid gx = y \text{ for some } g \in G\}.$$

The orbits under the action of G partition the set X, and G acts transitively on each orbit.

The *stabilizer* of $x \in X$ under the action of G is

$$\mathrm{Stb}_G(x) = \{g \in G \mid gx = x\}.$$

This is a subgroup of G. If $gx = y$, then $\mathrm{Stb}_G(x) = g^{-1}\mathrm{Stb}_G(y)g$, so the stabilizers of points in an orbit are conjugate subgroups of G. The intersection of the stabilizers of the points in an orbit is a normal subgroup of G, because it is the kernel of the action of the group on that orbit.

Let $Y \subset X$. The *setwise stabilizer* of Y is the stabilizer of Y under the induced action of G on $\mathcal{P}(X)$. The *pointwise stabilizer* of Y is the set of elements of G which fix every point in Y; this is the intersection of the one point stabilizers for all the points in Y. The pointwise stabilizer of X is the kernel of the action.

1.1.2. *Morphisms of Group Actions.* Let G be a group acting on sets X and Y. A *morphism* between these actions consists of a function $f : X \to Y$ such that $f(gx) = gf(x)$. This produces the category of actions by G, and defines equivalence as isomorphisms in this category.

Let $x \in X$ and $U = \mathrm{Stb}_G(x)$, and let G/U denote the left coset space of U in G. Then G acts on G/U by left multiplication. There is a bijective correspondence between G/U and the points in $\mathrm{Orb}_G(x)$, given by $gU \mapsto gx$. This produces an equivalence between the actions of G on G/U and $\mathrm{Orb}_G(x)$. In this context, regular actions are those given by the action of G on itself by left multiplication.

1.1.3. *Opposite Groups.* Given a group G, construct the *opposite group* G^{opp} as the group with the same set as G but with multiplication $*$ given by $g_1 * g_2 = g_2 g_1$. Define a function from $\omega : G \to G^{\mathrm{opp}}$ by $\omega : g \mapsto g^{-1}$; then $\omega(g_1 g_2) = g_2^{-1} g_1^{-1} = g_1^{-1} * g_2^{-1} = \omega(g_1)\omega(g_2)$, so ω is an isomorphism.

An *antihomomorphism* between group G and H is a function $\alpha : G \to H$ such that $\alpha(g_1 g_2) = \alpha(g_2)\alpha(g_1)$. The identity map on the set G gives an antihomomorphism from G to G^{opp}. An antihomomorphism $G \to H$ may always be factored as the identity antihomomorphism from G to G^{opp} followed by a homomorphism from G^{opp} to H.

1.1.4. *Right Actions.* A *right action* (X, G) of a group G on a set X as a function $X \times G \to X$ satisfying $x \cdot 1 = x$ and $x(g_1 g_2) = (xg_1)g_2$. In this case the induced function $G \to \mathrm{Sym}(X)$ is an antihomomorphism. What we previously defined to be action, we now call *left action*, and an action is either a left or right action. All concepts we discussed regarding left actions have direct analogs for right actions.

It is not practical to consider only left or only right actions. For example, conjugation appears naturally on the right. For $g, h \in G$, let $h^g = g^{-1}hg$. Then $h^{g_1 g_2} = (h^{g_1})^{g_2}$, giving a right action and justifying the exponential notation.

For pedagogical reasons which will reveal themselves later, we let S_n denote the group of permutations of $\mathbb{N}_n = \{1, \ldots, n\}$, which we compose from left to right. Thus $S_n = \mathrm{Sym}(\mathbb{N}_n)^{\mathrm{opp}}$, and S_n acts on the right of $\mathbb{N}_n$.

1.2. Permutation Representations.

1.2.1. *Permutation Representations.* A *permutation representation* of a group G is a group homomorphism $\rho : G \to S_n$ for some positive integer n. We call n the *degree* of the representation. This produces a right action of G on $\mathbb{N}_n$. Two permutation representations of the same group are *equivalent* if they differ by an inner automorphism of S_n; that is, if they are equivalent as actions.

An *enumeration* of a finite set X is a bijective function $\epsilon : X \to \mathbb{N}_n$. Note that ϵ induces an antiisomorphism $\epsilon_* : \mathrm{Sym}(X) \to S_n$. If a group G acts on X on the right, let $\tau : G \to \mathrm{Sym}(X)$

be the associated antihomomorphism and set $\rho = \epsilon_* \circ \tau$; then ρ is a permutation representation of G, which is independent of ϵ up to equivalence. If G acts on the left, we obtain in this manner an antihomomorphism $G \to S_n$, which may be called an *antirepresentation.* Thus we may study group actions on finite sets by studying S_n, as is convenient to do.

Let G be a group and let $U \leq G$ of finite index n. Enumerate the right cosets of U in G so that $U \mapsto 1$. The right action of G induces a permutation representation $\rho_U : G \to S_n$, such that the stabilizer of 1 in $\rho_U(G)$ is the image of U. Now

$$g \in \ker(\rho_U) \Leftrightarrow Uhg = Uh \Leftrightarrow hgh^{-1} \in U \Leftrightarrow g \in U^h$$

for all $h \in G$. The *core* of U in G, denoted $K_G(U)$, is the intersection of all of the conjugates of U in G, and it is the kernel of the permutation representation. We say that U is *coreless* in G if $K_G(U)$ is trivial.

Suppose $U_1, U_2 \leq G$ produce equivalent permutation representations. Then there exists $\sigma \in S_n$ such that $\sigma \circ \rho_{U_1} = \rho_{U_2}$. Then U_1 is the stabilizer of 1σ and U_2 is the stabilizer of 1 under the right action of G on $\mathbb{N}_n$ induced by ρ_{U_2}; thus U_1 and U_2 are conjugate subgroups of G. This describes a bijective correspondence between the following sets:

(1) equivalence classes of transitive faithful permutation representations of G of degree n;

(2) conjugacy classes of coreless subgroups of G of index n.

1.2.2. *Centralizers of Permutation Representations.* Let $\rho : G \to S_n$ be a permutation representation; this gives a right action of G on $X = \mathbb{N}_n$. An automorphism of this action consists of a bijective function $\alpha : X \to X$ such that $\alpha(xg) = \alpha(x)g$. Let $A \leq \mathrm{Sym}(X)$ denote the set of such automorphisms; the left action of A on X produces an antirepresentation $\zeta : A \to S_n$. Then $\zeta(A) = C_{S_n}(\rho(G))$.

Let $a \in A$ and suppose $ax = x$ for some $x \in X$; then $axg = xg$ for all $g \in G$; since G is transitive, $a = \mathrm{id}_X$. Therefore the action of A is free.

Let $a \in A$ and $x \in X$; since G is transitive, there exists $g \in G$ such that $ax = xg$. Let $u \in U$; then $x = axug^{-1} = xgug^{-1}$, so $gug^{-1} \in U$ and $g \in N = N_G(U)$. Define $\nu : N \to A$ by $g \mapsto a$, where $ax = xg$. This is well-defined because A is free, and is surjective because G is transitive. Moreover, it is an antihomomorphism with kernel U. Thus $\zeta \circ \nu : N \to C_{S_n}(\rho(G))$ is a homomorphism with kernel U, and $N_G(U)/U \cong C_{S_n}(\rho(G))$. In particular, if $U \triangleleft G$, then it is the kernel of the action and $G/U \cong C_{S_n}(G)$; this realizes $C_{S_n}(G)$ as the opposite group of G/U.

1.2.3. *Normalizers of Permutation Representations.* Let $G \leq S_n$ so that G acts on the right of $X = \mathbb{N}_n$. Conjugation in S_n of G by $N_{S_n}(G)$ induces an antihomomorphism $\psi : N_{S_n}(G) \to \mathrm{Aut}(G)$ whose kernel is $C_{S_n}(G)$.

Assume that G acts regularly on X; we have $G \cong C_{S_n}$. Selection of $x \in X$ induces a bijective correspondence between G and X by $g \mapsto xg$, which in turn induces a left action of $\mathrm{Aut}(G)$ on X

by $\xi(xg) = x(\xi(g))$ for $\xi \in \mathrm{Aut}(G)$, giving an antihomomorphism $\rho : \mathrm{Aut}(G) \to S_n$. The image of ρ acts on G by conjugation, in the manner that $\mathrm{Aut}(G)$ acts on G. Thus $\rho(\mathrm{Aut}(G)) \leq N_{S_n}(G)$, and $\psi \circ \rho = \mathrm{id}_{\mathrm{Aut}(G)}$; that is, ρ is a section of ψ, which reveals $N_{S_n}(G)$ to be a semidirect product, $N_{S_n}(G) \cong C_{S_n}(G) \rtimes \rho(\mathrm{Aut}(G)) \cong G \rtimes \mathrm{Aut}(G)$.

2. Topological Covers

2.1. Topological Covers.

2.1.1. *Topological Covers.* A *topological cover* is a continuous function $\varphi : Y \to X$ between topological spaces with the property that every point in X has a neighborhood U whose preimage is the disjoint union of countably many components which are mapped homeomorphically onto U by φ. We will assume that X is Hausdorff, locally compact, locally path connected, and locally simply connected; these are the conditions under which covering theory works best. It follows that Y also has these properties.

We call X the *base space* and Y the *covering space*. A topological cover is an open map, that is, it sends open subsets of Y to open subsets of X. The fiber over a point in X is a discrete subspace of Y. Every fiber has the same cardinality; this cardinality is called the *degree* of the cover, and is denoted $\deg(\varphi)$. The cover is said to be *finite* if it has finite degree. The cover is said to be *connected* if the covering space is connected, whence the base space is connected.

2.1.2. *Path Lifting.* Let X be topological space and let $\gamma : I \to X$ be a path in X, where $I = [0, 1] \subset \mathbb{R}$ is the closed unit interval. We denote the homotopy class of γ by $[\gamma]$; when speaking of paths, we always mean fixed endpoint homotopy. Let $\varphi : Y \to X$ be a topological cover. Select y in the fiber over $\gamma(0)$. Then there is a unique path $\tilde{\gamma} : I \to Y$ such that $\tilde{\gamma}(0) = y$ and $\varphi \circ \tilde{\gamma} = \gamma$; this is the lift of γ to y.

Let W be a topological space and let $F : I \times W \to X$ be a homotopy of $F(0, w)$ to $F(1, w)$. Suppose there exists a map $g : W \to Y$ such that $\varphi \circ g = F(0, w)$. For $w \in W$, let $\gamma_w : I \to X$ be given by $\gamma_w(t) = F(t, w)$. Uniquely lift γ_w to Y starting at $g(w)$. This produces the unique continuous map $G : I \times W \to Y$ such that $G(0, w) = g(w)$ and $\varphi \circ G = F$. Thus each homotopy lifts uniquely, and in particular, homotopic paths in X lift to homotopic paths in Y.

Let $y_0 \in Y$ such that $\varphi(y_0) = x_0$. Consider the map $\varphi_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ given by $[\tilde{\gamma}] \mapsto [\varphi \circ \tilde{\gamma}]$. Let $\tilde{\gamma}$ be a loop in Y based at y_0 which represents an element of the kernel. Its image γ is homotopic to the constant x_0, and this homotopy lifts so that $\tilde{\gamma}$ is homotopic to the constant y_0. Thus φ_* is injective. If Y is connected, there is a path in Y between any two points in the fiber over x_0, and this path maps to a loop in X. Thus the image group $\varphi_*(\pi_1(Y, y_0))$ depends on the choice of y_0 only up to conjugation in $\pi_1(X, x_0)$.

2.1.3. *Morphisms of Topological Covers.* We have interest in two types of morphisms for covers of a given base space X.

A *strong morphism* from $\varphi_1 : Y_1 \to X$ to $\varphi_2 : Y_2 \to X$ is a surjective continuous function $\nu : Y_1 \to Y_2$ such that $\varphi_1 = \varphi_2 \circ \nu$. The condition that X is locally simply connected ensures that ν is a covering map. Two covers which are isomorphic in the category of topological covers and strong morphisms are called *strongly equivalent* (or simply *equivalent*). The group of strong automorphisms of a topological cover $\varphi : Y \to X$ is denoted $\mathrm{Aut}(\varphi)$.

A *weak morphism* from $\varphi_1 : Y_1 \to X$ to $\varphi_2 : Y_2 \to X$ is a continuous function $\nu : Y_1 \to Y_2$ together with an automorphism $\mu : X \to X$ such that $\mu \circ \varphi_1 = \varphi_2 \circ \nu$. Two covers which are isomorphic in the category of topological covers and weak morphisms are called *weakly equivalent.*

2.2. Group Actions on Topological Covers.

2.2.1. *Automorphism Action.* Let $\varphi : Y \to X$ be a topological cover. Then $\mathrm{Aut}(\varphi)$ acts on Y. Let $\alpha \in \mathrm{Aut}(\varphi)$. Then α is completely determined by its effect on a single point. This follows from unique path lifting thusly: suppose we know that $\alpha(y_1) = y_2$. Let $y \in Y$ and let γ be a path from y_1 to y. Drop γ to X and lift it to y_2. The endpoint is now $\alpha(y)$. Thus $\mathrm{Aut}(\varphi)$ acts freely on Y.

Let $x_0 \in X$ and $F = \varphi^{-1}(x_0)$. The set F is stabilized by the action of $\mathrm{Aut}(\varphi)$, so $\mathrm{Aut}(\varphi)$ acts on F, and this action is free. Thus $|\mathrm{Aut}(\varphi)| \leq |F| = \deg(\varphi)$.

2.2.2. *Discrete Actions.* Let Y be a topological space and let G be a group which acts continuously on Y; that is, we have a homomorphism $G \to \mathrm{Aut}(Y)$, where $\mathrm{Aut}(Y)$ is the group of homeomorphisms from Y onto itself. We say that the action is *discrete* if for every $y \in Y$ there exists a neighborhood U of y such that for every $g \in G$, either $g = 1$ or $gU \cap U = \varnothing$. A continuous action by a finite group is discrete if and only if every orbit is a discrete subset. Discrete actions are necessarily free, and in particular are faithful.

Let $\bar{Y} = Y/G$ be the quotient space of Y under a discrete action by G, and let $\varphi : Y \to \bar{Y}$ be the quotient map. The discreteness condition guarantees that φ is a topological cover. Moreover, G acts regularly (transitively and freely) on the fibers, so $|G| = \deg(\varphi)$. Furthermore, every element of the image of G in $\mathrm{Aut}(Y)$ is an automorphism of φ, so $G \cong \mathrm{Aut}(\varphi)$.

Let $\psi : Y \to X$ be a topological cover, and let $G \leq \mathrm{Aut}(\psi)$; then G acts discretely on Y, producing a cover $\xi : Y \to \bar{Y}$ with $\mathrm{Aut}(\xi) = G$. The points of $\bar{Y}$ are elements of Y which lie in a single orbit of $\mathrm{Aut}(\xi)$; since automorphisms preserve fibers, there is a map $\varphi : \bar{Y} \to X$ mapping the orbit $\bar{y}$ of y to $\varphi(y)$. Thus $\psi = \varphi \circ \xi$.

2.2.3. *Monodromy Action.* Let $\varphi : Y \to X$ be a topological cover, and let $x_0 \in X$. Let $F = \varphi^{-1}(x_0)$ be the fiber over x_0. Then $\pi_1(X, x_0)$ acts on F through path lifting; since we concatenate paths from left to right, this action is naturally from the right, given by setting $x[\gamma]$ equal to the endpoint of the unique lift of γ to x (since γ is a loop, this endpoint is in the fiber over x_0). We refer to this action as the *monodromy action.* It gives us a group antihomomorphism $\pi_1(X, x_0) \to \mathrm{Sym}(F)$.

Assume that Y is connected. In this case, the monodromy action is transitive. The stabilizer of a point $y_0 \in F$ is the set of all homotopy classes of loops in X which lift to loops at y_0; that is, the stabilizer is $\varphi_*(\pi_1(Y, y_0))$. Thus $\deg(\varphi) = [\pi_1(X, x_0) : \varphi_*(\pi_1(Y, y_0))]$.

The kernel of the action on the orbit of y_0 is the core of $\varphi_*(\pi_1(Y, y_0))$ in $\pi_1(X, x_0)$; that is, it is the intersection of all its conjugates. The *monodromy group* of φ is

$$\mathrm{Mon}(\varphi) = \pi_1(X, x_0)/K_{\pi_1(X,x_0)}(\varphi_*(\pi_1(Y, y_0))),$$

together with the action of this group on F.

Let $\epsilon : F \to \mathbb{N}_n$ be an enumeration of the fiber, where $n = \deg(\varphi)$. Denote $\epsilon^{-1}(i)$ by y_i. Composing our action with ϵ, we obtain a permutation representation $T_\varphi : \pi_1(X, x_0) \to S_n$. A different enumeration of the fiber will give a conjugate image in S_n. We have $\ker(T_\varphi) = K_{\pi_1(X,x_0)}(\varphi_*(\pi_1(Y, y_0))$, so the image of T_φ is isomorphic to $\mathrm{Mon}(\varphi)$, and may also be referred to as the monodromy group of the cover.

Since the automorphism group acts freely on fibers and the monodromy group acts transitively, we see that $|\mathrm{Aut}(\varphi)| \leq \deg(\varphi) \leq |\mathrm{Mon}(\varphi)|$. The automorphism group of the monodromy action is canonically identified with the automorphism group of the cover. Let G be the image in S_n of the monodromy representation of the fundamental group, and let A be the image in S_n of the automorphism group. We have $C_{S_n}(G) = A$. This says more than that the automorphism group is isomorphic to $C_{S_n}(G)$; the identification of A with $C_{S_n}(G)$ explicitly detects the action of an automorphism on a fiber.

2.3. Normal Covers.

2.3.1. *Extension of Monodromy Action.* Let $\varphi : Y \to X$ be a topological cover, and let $x_0 \in X$ and $y_0 \in \varphi^{-1}(x_0)$. Then the induced homomorphism $\varphi_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective. Suppose we select a different basepoint $y_1 \in \varphi^{-1}(x_0)$ for Y. A path from y_0 to y_1 drops to a loop γ in X, and we have $\varphi_*(\pi_1(Y, y_1))^\gamma = \varphi_*(\pi_1(Y, y_0))$. Moreover, if φ_1 and φ_2 are equivalent covers, their corresponding subgroups in $\pi_1(X, x_0)$ are conjugate.

Now we examine when we can extend the action of $\pi_1(X, x_0)$ on the fiber over x_0 to an action on all of Y. We attempt to define a right action of $\pi_1(X, x_0)$ on Y as follows. Let γ be a loop based at x_0. Let $y \in Y$ and let α be a path from y_0 to y. Let $\tilde{\gamma}$ be the lift of γ to y_0; the endpoint of $\tilde{\gamma}$ is $y_1 = y_0[\gamma]$. Lift the path $\varphi(\alpha)$ to a path starting at y_1, and denote the endpoint of this lift by $y[\gamma]_\alpha$. If β is a different path from y_0 to y, then $\alpha\beta^{-1}$ is a loop based at y_0. Let $\lambda = \varphi(\alpha\beta^{-1})$ be the projection of this loop to X and then lift this projection to a path $\tilde{\lambda}$ based at y_1; then $y[\gamma]_\alpha = y[\gamma]_\beta$ if and only if $\tilde{\lambda}$ is a loop at based at y_1. This happens if and only if $\tilde{\gamma}\tilde{\lambda}\tilde{\gamma}^{-1}$ is a loop based at y_0. Projecting this to X, and noting that every loop at y_0 is homotopic to a loop through y, we see that this happens for all pairs of paths (α, β) exactly when γ normalizes $\varphi_*(\pi_1(Y, y_0))$.

Thus $N_{\pi_1(X,x_0)}(\varphi_*(\pi_1(Y,y_0)))$ acts continuously on Y, and this action preserves fibers of φ, thus inducing an antihomomorphism

$$f : N_{\pi_1(X,x_0)}(\varphi_*(\pi_1(Y,y_0))) \to \mathrm{Aut}(\varphi).$$

Let $\zeta \in \mathrm{Aut}(\varphi)$ and $y_1 = \zeta(y_0)$. Then ζ_* embeds $\pi_1(Y,y_0)$ into $\pi_1(Y,y_1)$. Since ζ commutes with φ, we have $\varphi_*(\pi_1(Y,y_0)) \leq \varphi_*(\pi_1(Y,y_1))$, and since these groups are conjugates, they are equal. Thus if γ is a loop in X such that $y_0[\gamma] = y_1$, we see that γ normalizes $\varphi_*(\pi_1(Y,y_0))$; such a γ exists because Y is connected. So $f([\gamma])$ is an element of $\mathrm{Aut}(\varphi)$ which sends y_0 to y_1, and since ζ is determined by its effect on a single point, $f(\gamma) = \zeta$. Thus f is surjective.

The kernel of f is the set of homotopy classes of loops which have trivial action on φ; that is, $\ker(f) = \varphi_*(\pi_1(Y,y_0))$. Therefore,

$$\mathrm{Aut}(\varphi) \cong N_{\pi_1(X,x_0)}(\varphi_*(\pi_1(Y,y_0)))/\varphi_*(\pi_1(Y,y_0)).$$

2.3.2. *Normal Covers.* A topological cover $\varphi : Y \to X$ is called *normal* if $\varphi_*(\pi_1(Y,y_0)) \triangleleft \pi_1(X,x_0)$. In this case, $\pi_1(X,x_0)$ acts on the fiber over x_0 through the full automorphism group of the cover, and the kernel of the action is the image of the covering space's fundamental group in the base space's fundamental group. This action is transitive and free (i.e. regular), so $|\mathrm{Aut}(\varphi)| = \deg(\varphi)$. Thus normal covers are also called *regular.* A normal subgroup is its own core, so $|\mathrm{Mon}(\varphi)| = \deg(\varphi)$, and $\mathrm{Aut}(\varphi) \cong \mathrm{Mon}(\varphi)$. We summarize this information.

Let $\varphi : Y \to X$ be a finite connected cover, and let $y_0 \in Y$ and $x_0 = \varphi(y_0)$. The following conditions are equivalent:

- **(1)** $\varphi^*(\pi_1(Y,y_0)) \triangleleft \pi_1(Y,y_0)$;
- **(2)** $\mathrm{Aut}(\varphi)$ acts transitively on the fiber over x_0;
- **(3)** $\mathrm{Mon}(\varphi)$ acts freely on the fiber over x_0;
- **(4)** $|\mathrm{Aut}(\varphi)| = \deg(\varphi)$;
- **(5)** $|\mathrm{Mon}(\varphi)| = \deg(\varphi)$;
- **(6)** $\mathrm{Aut}(\varphi) \cong \mathrm{Mon}(\varphi)$;
- **(7)** $\xi : Y/\mathrm{Aut}(\varphi) \to X$ is a homeomorphism;
- **(8)** if one lift of a loop is closed, then all lifts are closed.

2.3.3. *Covers from Fundamental Subgroups.* Let X be a topological space. The fact that the fundamental group of the covering space embeds in the fundamental group of the base space produces a function from equivalence classes of covers of X to conjugacy classes of subgroups of the fundamental group of X. We now demonstrate an inverse to this function.

Let $H \leq \pi_1(X,x_0)$ be any subgroup. We construct a cover $\varphi : X_H \to X$ and a point $y_0 \in \varphi^{-1}(x_0)$ such that $\varphi_*(\pi_1(X_H,y_0)) = H$.

Let $\lambda(X, x_0)$ be the set of all paths in X based at x_0 modulo fixed endpoint homotopy. Define an equivalence relation on $\lambda(X, x_0)$ by stating that $[\gamma_1] \sim [\gamma_2]$ if $[\gamma_1\gamma_2^{-1}] \in H$. Set X_H equal to the set of equivalence classes.

Let $[\gamma] \in X_H$ and let $U \subset X$ be a simply connected open neighborhood of $\gamma(1)$. Let $D([\gamma], U)$ denote the set of equivalence classes of paths of the form $\gamma\alpha$, where α is a path in U based at $\gamma(1)$. Define a topology on X_H by taking all sets of this form as a basis. Note that in this topology, $D([\gamma], U)$ is homeomorphic to U.

Define a function $\varphi : X_H \to X$ by setting $\varphi([\gamma])$ equal to the endpoint of γ. This is well defined, continuous, and is a topological cover. The degree of φ is the index in $\pi_1(X, x_0)$ of H. Finally, let y_0 be the equivalence class of the constant path at x_0. This process inverts the function $\varphi \mapsto \varphi_*(\pi_1(Y, y_0))$, yielding a correspondence between the following sets:

(1) equivalence classes of covers of X;

(2) conjugacy classes of subgroups of the fundamental group of X.

The set of all equivalence classes of covers of X is partially ordered by $\varphi \leq \psi$ if there exists ξ such that $\psi = \xi \circ \varphi$. The *normal closure* of a cover $\varphi : Y \to X$ is a cover $\hat{\varphi} : \hat{Y} \to X$ which is a minimal normal cover of X which factors through φ. It is the cover which corresponds to the core of $\varphi_*(\pi_1(Y, y_0))$ in $\pi_1(X, x_0)$. Thus φ and $\hat{\varphi}$ have isomorphic monodromy groups, which are in turn isomorphic to the automorphism group of $\hat{\varphi}$.

The *universal cover* of X is the cover which corresponds to the identity in $\pi_1(X, x_0)$. Its fundamental group is trivial, that is, it is *simply connected.* It is versally repelling in the category of covers of X. All covers of X can be retrieved (up to equivalence) through acting on the universal cover by subgroups of the fundamental group of the base space, which is the automorphism group of the universal cover.

2.4. Static Covers.

2.4.1. *Static Covers.* A *static cover* of a space X with group G is a normal topological cover $\varphi : Y \to X$ together with a group isomorphism $\tau : G \to \mathrm{Aut}(\varphi)$. We use this definition to construct a category of covers whose objects have limited automorphisms.

The group G acts on Y via τ; this is a discrete action whose corresponding quotient cover is equivalent to φ. On the other hand, suppose that G acts discretely on a space Y; this entails a homomorphism $\tau : G \to \mathrm{Aut}(Y)$. Let X be the quotient space and let φ be the quotient map; we obtain a static cover (φ, τ).

2.4.2. *Morphisms of Static Covers.* Let $(\varphi_1 : Y_1 \to X, \tau_1)$ and $(\varphi_2 : Y_2 \to X, \tau_2)$ be static covers of X by G. A *morphism* from the first to the second consists of a continuous function $\xi : Y_1 \to Y_2$ with $\varphi_1 = \varphi_2 \circ \xi$, such that $\tau_2 = \xi_* \circ \tau_1$, where $\xi_* : \mathrm{Aut}(\varphi_1) \to \mathrm{Aut}(\varphi_2)$ is the isomorphism given by $\alpha \mapsto \xi \circ \alpha \circ \xi^{-1}$. This creates the category of static covers of X by G. Any morphism in this category is necessarily an isomorphism.

Let (φ, τ) be an static cover and let $\xi \in \mathrm{Aut}(\varphi, \tau)$ be an automorphism in this category. Then $\xi \in \mathrm{Aut}(\varphi)$, and $\tau = \xi_* \circ \tau$, so ξ_* is trivial. Since ξ_* is left conjugation by ξ, we have $\xi \in Z(\mathrm{Aut}(\varphi))$. Thus a static cover has no nontrivial automorphisms if its group is centerless.

2.4.3. *Static Covers and Outer Automorphisms.* Let $\varphi : Y \to X$ be a cover and let $\tau_1, \tau_2 : G \to \mathrm{Aut}(\varphi)$ be isomorphisms. Then $\tau_1^{-1} \circ \tau_2 \in \mathrm{Aut}(G)$, and $(\varphi, \tau_1) \cong (\varphi, \tau_2)$ as static covers if and only if $\tau_1^{-1} \circ \tau_2 \in \mathrm{Inn}(G)$. Given a cover φ, selection of a specific $\tau_1 : G \to \mathrm{Aut}(\varphi)$ produces a bijection between isomorphism classes of static covers $\{[\varphi, \tau]\}$ and outer automorphisms $\mathrm{Out}(G)$ given by $[\varphi, \tau] \mapsto \tau_1^{-1} \circ \tau$.

2.4.4. *Functors between Static Cover Categories.* We would like to extend this category to allow morphisms between static covers with varying groups. Unfortunately, in killing the automorphism group of the cover, we have precluded doing this in any canonical manner. However, we can do the following. Let $f : H \to G$ be a fixed surjective group homomorphism, and let $(\psi : Z \to X, \upsilon : H \to \mathrm{Aut}(\psi))$ be a static cover. Let $K = \ker(f) \leq H$; then K acts discretely on Z, producing a normal covers $\xi : Z \to Y$ and $\varphi : Y \to X$, where $Y = Z/K$. Automorphisms of ψ descend to well defined automorphisms of φ, giving a surjective homomorphism $\rho : \mathrm{Aut}(\psi) \to \mathrm{Aut}(\varphi)$ with kernel $\mathrm{Aut}(\xi) = \upsilon(K)$. This in turn produces a well-defined discrete action $\tau : G \to \mathrm{Aut}(\varphi)$ given by $\tau(f(h)) = \rho(\upsilon(h))$. Thus f induces a functor from the category of static covers with group H to the category of static covers with group G.

3. Ramified Covers

3.1. Ramified Covers.

3.1.1. *Ramified Covers.* A *ramified cover* is a nonconstant morphism between compact connected Riemann surfaces. Let $\varphi : Y \to X$ be a ramified cover. The image of φ is open by the Open Mapping Theorem, and it is also closed because Y is compact and X is Hausdorff. Since X is connected, the image of φ must be all of X, so φ is surjective. Let $x_0 \in X$ and let $F = \varphi^{-1}(x_0)$; the Identity Theorem implies that F is a discrete subset of Y, so F is finite because Y is compact.

Although the value of the derivative of φ is not well-defined, the order of its vanishing at a given point in Y is well-defined. Since Y is compact, the Identity Theorem implies that there are only finitely many points on Y where the derivative vanishes. These points in Y are called the *ramification points* of the cover. Their images are called the *branch points* of the cover; let $\mathrm{Bpt}(\varphi)$ denote the set of branch points.

Let Δ be the open unit disk in $\mathbb{C}$. Let $y_0 \in Y$ and $x_0 = \varphi(y_0)$. There exists charts $\kappa_V : V \to \Delta$ and $\kappa_U : U \to \Delta$ around y_0 and x_0 with $\varphi(V) = U$ and $\kappa_V(y_0) = \kappa_U(x_0) = 0$. We may choose κ_V and κ_U such that $\kappa_U \circ \varphi \circ \kappa_V^{-1}(z) = z^e$ for all $z \in \Delta$ and some positive integer e. We see that $e > 1$ if and only if y_0 is a ramification point; we call e the *ramification index* of y_0, and denote this number by $e(y_0)$. In particular, the map φ is e to 1 in a deleted neighborhood of y_0.

3.1.2. *Corresponding Topological Covers.* Let $\varphi : Y \to X$ be a ramified cover. Let $B = \mathrm{Bpt}(\varphi)$ and $R = \varphi^{-1}(B)$. Set $Y^\circ = Y \smallsetminus R$, $X^\circ = X \smallsetminus B$, and $\varphi^\circ = \varphi \restriction_{Y^\circ}$. Then $\varphi^\circ : Y^\circ \to X^\circ$ is a topological cover. The *degree* of φ is the degree of φ°, and is denoted by $\deg(\varphi)$. We see that for any $x \in X$, we have $\deg(\varphi) = \sum_{y \in \varphi^{-1}(x)} e(y)$.

Let $\varphi : Y \to X$ be a finite topological cover, where $X^\bullet$ is a compact connected Riemann surface, B is a finite subset of $X^\bullet$, and $X = X^\bullet \smallsetminus B$. Thus X has a complex structure, and we obtain charts on Y by composing charts on X with φ; this produces a unique complex structure on Y such that the map φ is holomorphic. Let $x_0 \in B$, and consider a chart $\kappa : U \to \Delta$ with $\kappa(x_0) = 0$. The preimage $\varphi^{-1}(U)$ consists of finitely many connected components which are homeomorphic to punctured disks. Use a quotient construction to fill in these punctures to obtain a Riemann surface $Y^\bullet$; φ uniquely extends to a morphism $\varphi^\bullet : Y^\bullet \to X^\bullet$.

3.1.3. *Morphisms of Ramified Covers.* Let $\psi : Z \to X$ and $\varphi : Y \to X$ be ramified covers. The *strong morphism* from ψ to φ is a nonconstant morphism of Riemann surfaces $\xi : Z \to Y$ such that $\psi = \varphi \circ \xi$. In this case, ξ is also a ramified cover. This defines equivalence of ramified covers of X.

Let B be a finite subset of X. The map $\varphi^\circ \mapsto \varphi$ produces a bijective correspondence between the following sets:

(1) equivalence classes of topological covers of X° of degree n;

(2) equivalence classes of ramified covers of X with branch points in B of degree n.

Let $\varphi : Y \to X$ be a ramified cover. An *automorphism* of φ is an isomorphism from φ to itself. The set of all automorphisms of φ is denoted by $\mathrm{Aut}(\varphi)$. If $\alpha \in \mathrm{Aut}(\varphi)$, the α restricts to $\alpha^\circ \in \mathrm{Aut}(\varphi^\circ)$. If $\beta \in \mathrm{Aut}(\varphi^\circ)$, then β extends uniquely to $\beta^\bullet \in \mathrm{Aut}(\varphi)$. Thus $\mathrm{Aut}(\varphi) \cong \mathrm{Aut}(\varphi^\circ)$.

We say that φ is a *normal* ramified cover if φ° is a normal topological cover. The Galois correspondence of finite topological covers now carries over into the realm of ramified covers.

Define a *weak morphism* of the ramified covers $\psi : Z \to X$ and $\varphi : Y \to X$ to be a pair (ξ, α), where $\xi : Z \to Y$ and $\alpha : X \to X$ are morphisms of Riemann surfaces with $\alpha \circ \psi = \varphi \circ \xi$. This defines weak equivalence of covers of X.

3.1.4. *Riemann-Hurwitz Formula.* The *Riemann sphere* is $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. This is complex projective one space, which we sometimes view in homogeneous coordinates $[x, y]$, where $\mathbb{C}$ is identified with $\{[x, 0]\}$ and ∞ is identified with $[0, 1]$. We may use a subscript, such as $\mathbb{P}^1_x$, to indicate the coordinate system on $\mathbb{P}^1$.

Let $\varphi : Y \to X$ be a ramified cover of degree n; we may compute the genus of Y from the genus of X and the ramification of φ as follows. Recall that the *Euler characteristic* of X, denoted $\chi(X)$, is the number of faces minus the number of edges plus the number of vertices of any triangulation of X. Select a triangulation of X which includes all branch points as vertices. The preimages of the faces determine a triangulation of Y. Each edge and face on X lifts to n distinct edges and faces on Y. However, if $v \in X$ is a vertex over which ramification occurs, the number of lifts of v is less than

n by the extent of the ramification, which we express as

$$\chi(Y) = n\chi(X) - \sum \text{ ramification } ;$$

more precisely, we obtain the *Riemann-Hurwitz Formula*

$$2 - 2g_Y = n(2 - 2g_X) - \sum_{p \in Y} (e(p) - 1).$$

If $X = \mathbb{P}^1$, then $g_X = 0$; solving this for $g = g_Y$ yields *Riemann's Formula*

$$g = 1 - n + \frac{1}{2} \sum_{p \in Y} (e(p) - 1).$$

3.2. Branch Cycle Descriptions.

3.2.1. *Classical Generators.* Let X be a Riemann surface and let $x \in X$. Then x lies in a chart $\kappa : U \to \Delta$, where U is a simply connected neighborhood of x, $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$, κ is holomorphic, and $\kappa(x) = 0$. By a *circle* around x, we mean the path $\kappa^{-1}(\exp(-2\pi i t))$, for $t \in [0, 1]$. Since $\exp(-2\pi i t)$ has winding number -1 around 0, this circle proceeds in a clockwise direction around x.

Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be an ordered tuple of distinct points in X, and let $\underline{\boldsymbol{x}} = \{x_1, \ldots, x_r\}$ denote the corresponding unordered set. Let $X^\circ = X \smallsetminus \underline{\boldsymbol{x}}$, and let $x_0 \in X^\circ$. A *classical loop* in X° about $x \in X$ based at x_0 is a loop which is homotopic in X° to a loop of the form $\lambda = \alpha\delta\alpha^{-1}$, such that

- **(a)** δ is a circle around x, based at $u \in X$, which is null homotopic in $X^\circ \cup \{x\}$;
- **(b)** α is an injective path in $X^\circ \smallsetminus U$ from x_0 to u.

Suppose λ_0 is another classical loop about x. Then λ_0 is homotopic to $\alpha_0\delta\alpha_0^{-1}$ for some path α_0; if $\beta = \alpha\alpha_0^{-1}$, then λ is homotopic to $\beta\lambda_0\beta^{-1}$ in X°. Thus $[\lambda]$ is conjugate to $[\lambda_0]$ in $\pi_1(X^\circ, x_0)$.

A *bouquet* of classical loops in X° with respect $(\boldsymbol{x}, x_0)$ is a tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ of loops in X based at x_0 such that

- **(a)** λ_i is a classical loop about x_i;
- **(b)** $\lambda_i(t_1) = \lambda_j(t_2) \Rightarrow t_1, t_2 \in \{0, 1\}$ for $i \neq j$;
- **(c)** there exists a circle around x_0 which intersects each path exactly once in the given order.

Assume that $X = \mathbb{P}^1$. In this case, X° is homotopy equivalent to a disk with $r - 1$ punctures, which in turn is homotopy equivalent to the wedge sum of $r - 1$ loops. Van Kampen's Theorem implies that the fundamental group of this space is free on $r - 1$ generators. We wish to select particular generators for the fundamental group of the punctured sphere. We add one generator and one relation to the presentation.

Let $\boldsymbol{\lambda}$ be a bouquet with respect to $(\boldsymbol{x}, x_0)$. The homotopy classes of the paths in $\boldsymbol{\lambda}$ generate the fundamental group of X°, and the concatenation of the paths in $\boldsymbol{\lambda}$ is null homotopic in X, so the product of their homotopy classes is trivial. Moreover, $\pi_1(X^\circ, x_0)$ is freely generated by $[\lambda_1], \ldots, [\lambda_r]$ modulo the relation that their product is trivial, where $[\lambda_i]$ is the homotopy class of λ_i. We call these homotopy classes *classical generators* for $\pi_1(X^\circ, x_0)$. Thus a *classical tuple* with respect to $(\boldsymbol{x}, x_0)$ is a tuple of classical generators based at x_0, $[\boldsymbol{\lambda}] = ([\lambda_1], \ldots, [\lambda_r])$, where $\boldsymbol{\lambda}$ is a bouquet.

3.2.2. *Branch Cycle Descriptions.* Let $\varphi : Y \to \mathbb{P}^1$ be a connected ramified cover of degree n, and let $\varphi^\circ : Y^\circ \to X^\circ$ be the corresponding topological cover. Select a basepoint x_0 in X° and set $F = \varphi^{-1}(x_0)$. Let $\epsilon : F \to \{1, \ldots, n\}$ be an enumeration of the fiber over x_0. The action of $\pi_1(X^\circ, x_0)$ on F produces a homomorphism $T_\varphi : \pi_1(X^\circ, x_0) \to S_n$ via ϵ. Let G be the image of this homomorphism; that is, G the monodromy group of the cover. Since Y is connected, G is transitive. A different choice of ϵ will produce a homomorphism which differs from T_φ by an inner automorphism of S_n; thus G is well-defined up to permutation equivalence.

Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be the branch points of φ, so that $X^\circ = \mathbb{P}^1 \setminus \underline{\boldsymbol{x}}$. Let $\boldsymbol{\lambda}$ be a bouquet with respect to x_0 and $\boldsymbol{x}$. Set $g_i = T_\varphi([\lambda_i]) \in S_n$. Then $\{g_1, \ldots, g_r\}$ generates G, and $\Pi_{i=1}^r g_i = 1$. Each g_i describes the action of λ_i on the fiber over x_0 via path lifting, and is a product of disjoint cycles in S_n. If we include cycles of length one in this decomposition, we see that each disjoint cycle in g_i corresponds to a point in the fiber over the i^{th} branch point, and the length of the disjoint cycle gives the ramification index. In this way $\boldsymbol{g} = (g_1, \ldots, g_r)$ describes the ramification of φ. We call $\boldsymbol{g}$ the *branch cycle description* of the cover φ with respect to $\boldsymbol{\lambda}$.

3.2.3. *Nielsen Tuples.* Let H be a group and let $\boldsymbol{g} = (g_1, \ldots, g_r) \in H^r$. Let $\langle \boldsymbol{g} \rangle = \langle g_1, \ldots, g_r \rangle$ denote the subgroup of H generated by the entries in $\boldsymbol{g}$, and let $\Pi \boldsymbol{g} = \Pi_{i=1}^r g_i$ denote their product, in the order given.

A *Nielsen tuple* of degree n and rank r is a tuple $\boldsymbol{g} = (g_1, \ldots, g_r) \in S_n^r$ satisfying

(a) $\langle \boldsymbol{g} \rangle = G$ is a transitive subgroup of S_n;

(b) $\Pi \boldsymbol{g} = 1$.

Let $\boldsymbol{h} = (h_1, \ldots, h_s)$ and $\boldsymbol{g} = (g_1, \ldots, g_r)$ be Nielsen tuples with of rank n and m, respectively, so that $H = \langle \boldsymbol{h} \rangle \leq S_n$ and $G = \langle \boldsymbol{g} \rangle \leq S_m$. A *morphism* from $\boldsymbol{h}$ to $\boldsymbol{g}$ is a function $f : \mathbb{N}_n \to \mathbb{N}_m$ which induces a homomorphism $f_* : H \to G$ which sends h_i to g_i. This necessitates that $n \geq m$ and that f is surjective. In particular, if $n = m$, f must be bijective and f_* is given by conjugation in S_n; in this case $\boldsymbol{h}$ and $\boldsymbol{g}$ are *equivalent*. We obtain the category of Nielsen tuples such that equivalence is isomorphism in this category.

Now suppose that we wish to construct a cover of $\mathbb{P}^1$ with specified ramification. Select branch points $\boldsymbol{x}$, a base point x_0 not among them, and a bouquet $\boldsymbol{\lambda}$ with respect to $(\boldsymbol{x}, x_0)$. Select a Nielsen tuple $\boldsymbol{g}$. Let $X = \mathbb{P}^1 \setminus \underline{\boldsymbol{x}}$. Map $[\lambda_i]$ to g_i to obtain a homomorphism $T : \pi_1(X, x_0) \to S_n$ with image $G = \langle \boldsymbol{g} \rangle$. Let U be the stabilizer of 1 in G; then $T^{-1}(U)$ is a subgroup of $\pi_1(X, x_0)$. Let $\varphi : Y \to X$ be the topological cover which corresponds to this subgroup. By filling in the missing points, one obtains a ramified cover $\varphi^\bullet : Y^\bullet \to \mathbb{P}^1$. Up to equivalence, this produces an inverse to the process of obtaining a Nielsen tuple from a ramified cover. Thus a classical tuple produces a bijective correspondence between the following sets:

(1) equivalence classes of covers of $\mathbb{P}^1$ ramified over $\underline{\boldsymbol{x}}$ of degree n;

(2) equivalence classes of Nielsen tuples of rank r and degree n.

3.2.4. *Conjugacy Classes.* Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover and let $x, x_0 \in \mathbb{P}^1$ distinct. Let λ be a classical loop about x based at x_0, and let $\mathrm{Con}_x(\varphi)$ denote the conjugacy class of λ in $\mathrm{Mon}(\varphi)$. This conjugacy class is independent of the choice of x_0 and λ. Moreover, this definition makes sense for every point in $\mathbb{P}^1$, not just the branch points. Indeed, the invariant $\mathrm{Bpt}(\varphi)$ can be recovered from the set $\{\mathrm{Con}_x(\varphi) \mid x \in \mathbb{P}^1\}$ as those points x with nontrivial $\mathrm{Con}_x(\varphi)$. Although the conjugacy class is unambiguous in $\mathrm{Mon}(\varphi)$, the same cannot be said of its image under a permutation representation; the conjugacy class of the image depends on the enumeration of the fiber.

3.3. Meromorphic Functions.

3.3.1. *Meromorphic Functions.* Let X be a compact Riemann surface. A *meromorphic function* on X is a holomorphic function from X to $\mathbb{P}^1_x$ which is not constantly ∞. Such a function is either constant or surjective. Let $\mathrm{Mer}(X)$ denote the set of meromorphic functions on X. Let $*$ denote addition or multiplication in $\mathbb{C}$ and define $*$ in $\mathrm{Mer}(X)$ by

$$(f * g)(x) = \lim_{\substack{y \to x \\ f(y), g(y) \neq \infty}} f(y) * g(y).$$

This gives $\mathrm{Mer}(X)$ the structure of a field, into which $\mathbb{C}$ embeds as the constant functions. We call $\mathrm{Mer}(X)$ the *function field* of X.

Let $f \in \mathrm{Mer}(X)$ be a nonconstant function; then f is a ramified cover of $\mathbb{P}^1_x$. The *zeros* of f are the points in the fiber over 0 and the *poles* of f are the points in the fiber over ∞. The *order* of a zero or a pole is its ramification index. The number of zeros equals the number of poles, when counted with multiplicity; this number is the degree of the ramified cover.

3.3.2. *Endomorphisms of the Riemann Sphere.* A meromorphic function on the Riemann sphere is an endomorphism, so the set of endomorphisms $\mathrm{End}(\mathbb{P}^1_x)$ equals $\mathrm{Mer}(\mathbb{P}^1_x) \cup \{\infty\}$ as a set, but becomes a monoid under composition. Let $\mathrm{Hol}(\mathbb{P}^1_x) = \mathrm{End}(\mathbb{P}^1_x)^*$ denote the group of holomorphic isomorphisms from $\mathbb{P}^1_x$ to itself. We determine the field $\mathrm{Mer}(\mathbb{P}^1_x)$ and the group $\mathrm{Hol}(\mathbb{P}^1_x)$.

Let $f \in \text{Mer}(\mathbb{P}^1_x)$, with zeros $a_1, \ldots, a_n$ and poles $b_1, \ldots, b_n$. Set

$$g(x) = \frac{\Pi_{i=1}^n (x - a_i)}{\Pi_{j=1}^n (x - b_j)}.$$

Then f/g is a function without zeros or poles, which is constant by the Open Mapping Theorem. Thus $f = ag$ for some $a \in \mathbb{C}$, and f is a rational function. Therefore

$$\text{Mer}(\mathbb{P}^1_x) = \mathbb{C}(x),$$

where $\mathbb{C}(x)$ denotes the quotient field of the polynomial ring $\mathbb{C}[x]$.

A *linear fractional transformation* is a function $f : \mathbb{P}^1_x \to \mathbb{P}^1_x$ of the form

$$f(x) = \frac{ax + b}{cx + d}.$$

Such a function has an expression of this form with $ad - bc = 1$, which is unique up to multiplication of all coefficients by ± 1. The degree of a rational function written with relatively prime numerator and denominator is the maximum degree of these constituent polynomials. In particular, the only injective rational functions are those of degree one, that is, the linear fractional transformations. Since all morphisms from the Riemann sphere to itself are rational functions, its automorphism group is exactly the set of linear fractional transformations. Observe the action of $\text{PSL}_2(\mathbb{C})$ on $\mathbb{P}^1_x$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

Using the identification of $\mathbb{P}^1_x$ with $\mathbb{C} \cup \{\infty\}$ via $x = [x, 1]$ and $\infty = [1, 0]$, we see that if $x = \frac{x_1}{x_2}$, this acts as the function $f(x)$ above. Therefore

$$\text{Hol}(\mathbb{P}^1_x) = \text{PSL}_2(\mathbb{C}).$$

3.3.3. *Ramified Covers Produce Function Field Extensions.* Let $\varphi : Y \to X$ be a ramified cover. The function field of X naturally embeds into the function field of Y via composition; define

$$\varphi^* : \text{Mer}(X) \hookrightarrow \text{Mer}(Y) \qquad \text{by} \qquad f \mapsto f \circ \varphi.$$

Then φ^* is an injective ring homomorphism. This creates a contravariant functor from the category of compact Riemann surfaces and nonconstant morphisms into the category of fields and field embeddings.

Focus on the case that $X = \mathbb{P}^1_x$; then $\text{Mer}(X) = \mathbb{C}(x)$, and by identifying $\text{Mer}(X)$ with $\varphi^*(\text{Mer}(X))$, we obtain a field extension $\text{Mer}(Y)/\mathbb{C}(x)$; we refer to this the *function field extension* corresponding to the cover.

Let $\psi : Z \to \mathbb{P}^1_x$ and $\varphi : Y \to \mathbb{P}^1_x$ be ramified covers, and let $\xi : Z \to Y$ be a morphism of covers. The induced map $\xi^* : \text{Mer}(Y) \to \text{Mer}(Z)$ is constant on $\mathbb{C}(x)$, and so produces a morphism of function field extensions. This gives a functor from the category of ramified covers of X to the category of function field extensions of $\mathbb{C}(x)$. Next we outline an inverse to this functor.

3.3.4. *Function Field Extensions Produce Ramified Covers.* Let $E/\mathbb{C}(x)$ be a function field extension. This extension is separable over $\mathbb{C}(x)$, and so admits a primitive element; thus it is of the form $\mathbb{C}(x, f)$, where f is transcendental over $\mathbb{C}$ but algebraic over $\mathbb{C}(x)$. As such, f has a minimum polynomial over $\mathbb{C}(x)[w]$, where w transcendental over $\mathbb{C}(x)$. By clearing denominators, we find an irreducible polynomial $m(x, w) \in \mathbb{C}[x, w]$ such that $m(x, f) = 0$.

Let $V = \{(x, w) \in \mathbb{C}^2 \mid m(x, w) = 0\}$; then V is an affine set, and the ring of algebraic functions on V is isomorphic to $\mathbb{C}[x, w]/\langle m\rangle$. Since m is irreducible, V is an affine variety whose function field is the field of fractions of $\mathbb{C}[x, w]/\langle m\rangle$, which is isomorphic to $E \cong \mathbb{C}(x)[f]$. Let $\varphi : V \to \mathbb{A}^1_x$ be projection on the first coordinate. Let $T = \{v \in V \mid \frac{\partial m}{\partial x}(v) = 0 \text{ or } \frac{\partial m}{\partial w}(v) = 0\}$; then T is a finite subset of V. Set $B = \varphi(T)$, $X^\circ = X \setminus B$, $Y^\circ = Y \setminus \varphi^{-1}(\varphi(T))$, and $\varphi^\circ = \varphi\restriction_{Y^\circ}$. Apply the Implicit Function Theorem to see that $\varphi^\circ : Y^\circ \to X^\circ$ is a topological cover. This in turn produces a ramified cover $\varphi^\bullet : Y^\bullet \to \mathbb{P}^1_x$, whose corresponding function field extension is isomorphic to $E/\mathbb{C}(x)$. This outlines the bijective correspondence between the following sets:

(1) equivalence classes of ramified covers of $\mathbb{P}^1_x$;

(2) equivalence classes of function field extensions of $\mathbb{C}(x)$.

3.4. Algebraic Covers.

3.4.1. *Algebraic Models of Ramified Covers.* An *algebraic cover* of $\mathbb{P}^1$ is an algebraic function $\varphi : V \to \mathbb{P}^1$, where V is a projective curve in $\mathbb{P}^n$, and φ is given by projection onto some projective line in $\mathbb{P}^n$. Thus V is the set of zeros of some homogeneous polynomials in $n + 1$ indeterminates, and φ can be expressed as a ratio of homogeneous polynomials of the same degree.

Suppose V is the zero locus of the minimum polynomial for a function field extension, as in the subsection 3.3.4. Embed V in projective space by homogenizing the defining polynomial m; let $V^\bullet$ be the zero locus of the corresponding homogeneous polynomial. If $V^\bullet$ is singular, normalize $V^\bullet$ (see [**Sh94**] Section II.5). This produces a nonsingular curve with the same function field, embedded in projective space, together with a projection map $\varphi^\bullet : V^\bullet \to \mathbb{P}^1$ onto some $\mathbb{P}^1$ in the projective space.

If we start with a ramified cover $\varphi : Y \to \mathbb{P}^1$, this process produces an equivalent cover $V \to \mathbb{P}^1$ with an algebraic structure; call it an *algebraic model* of the ramified cover. In particular, Y is holomorphically isomorphic to V, and there exists an embedding $\xi : Y \to V$ of Y into projective space such that $\varphi \circ \xi^{-1}$ is algebraic.

3.4.2. *Fields of Definition of Ramified Covers.* Our purpose for placing an algebraic structure on a ramified cover is to have geometric access to the notion of field of definition.

Let $\varphi : V \to \mathbb{P}^1$ be an algebraic cover, and let K be a subfield of $\mathbb{C}$. We say that K is a *field of definition* for φ if the coefficients of the polynomials defining V are in K, and the coefficients of the polynomials defining φ are in K.

Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover. We say that φ *can be defined over* K, if there exists an algebraic model for φ which is defined over K. Let $\mathrm{Fld}(\varphi)$ denote the set of all subfields of $\mathbb{C}$ over which φ can be defined; then $\mathrm{Fld}(\varphi)$ is an invariant of φ. If $K \in \mathrm{Fld}(\varphi)$, this produces a well-defined meaning to being a K-point on Y, up to isomorphisms defined over K.

Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be the branch points of φ, and set $f(x) = \prod_{i=1}^{r}(x - x_r)$, with the convention that $(x - \infty) = 1$. Let K be the field generated by the coefficients of f. Thus $\underline{\boldsymbol{x}}$ is an algebraic set defined over K. Let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}$. Then φ can be defined over $\bar{K}$ (see [**Fr77**] Theorem 5.1 and [**FV91**] Section 1.5).

3.4.3. *Galois Action on Algebraic Covers.* Let $\varphi : V \to \mathbb{P}^1$ be an algebraic cover which is defined over a field K. Let $\beta \in \mathrm{Aut}(K)$ be a field automorphism. Then β acts on the defining polynomials for V and φ, producing another cover $\varphi^\beta : V^\beta \to \mathbb{P}^1$, also defined over K. If K/F is a finite normal extension, then φ is defined over F if and only if $\varphi^\beta = \varphi$ for every $\beta \in \mathrm{Gal}(K/F)$.

To see this more explicitly, extend β to an automorphism of $\mathbb{C}$, also called β. Now β acts directly on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ (fixing ∞), producing a map $\beta : \mathbb{P}^1 \to \mathbb{P}^1$; also β acts on $\mathbb{P}^n$ via application to homogeneous coordinates, producing an isomorphism $\hat{\beta} : \mathbb{P}^n \to \mathbb{P}^n$. Set $Y^\beta = \hat{\beta}(Y)$ and define the cover $\varphi^\beta : Y^\beta \to \mathbb{P}^1$ by $\varphi^\beta = \beta \circ \varphi \circ \hat{\beta}^{-1}$; this is an algebraic cover. The choice of extension for β does not effect the isomorphism class of φ^β as a cover defined over K.

3.4.4. *Galois Action on Ramified Covers.* Let $\varphi_1 : V_1 \to \mathbb{P}^1$ and $\varphi_2 : V_2 \to \mathbb{P}^1$ be algebraic covers which are equivalent as ramified covers. Then there exists a holomorphic function $\xi : V_1 \to V_2$ with $\varphi_1 = \varphi_2 \circ \xi$. A priori, ξ may be nonalgebraic; nevertheless, $\beta \in \mathrm{Aut}(K)$ induces $\xi^\beta : V_1^\beta \to V_2^\beta$ by $\xi^\beta = \beta \circ \xi \circ \beta^{-1}$, and φ_1^β is equivalent to φ_2^β via ξ^β.

Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover, and let $[\varphi]$ denote the class of covers equivalent to φ. Then $[\varphi^\beta]$ is a well-defined class of covers. Let $\mathcal{R} = \{[\varphi] \mid \varphi : Y \to \mathbb{P}^1\}$ be the set of all equivalence classes of ramified covers of $\mathbb{P}^1$, and let $\mathrm{Aut}(\mathbb{C})$ denote the group of field automorphisms of $\mathbb{C}$. Then $\mathrm{Aut}(\mathbb{C})$ acts on $\mathcal{R}$ by $\beta : [\varphi] \mapsto [\varphi^\beta]$.

3.4.5. *Fields of Moduli of Ramified Covers.* Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover. We wish to find testable sufficient conditions for φ to have a model over a field K. A necessary condition is that φ is equivalent, as a cover, to φ^β, whenever β fixes K. This implies that the branch points form an algebraic set over K.

The *field of moduli* of φ is the fixed field of the group

$$\{\beta \in \mathrm{Aut}(\mathbb{C}) \mid \varphi^\beta \text{ is equivalent to } \varphi\}.$$

If φ can be defined over its field of moduli, then φ being equivalent to φ^β implies that φ can be defined over the fixed field of β. If either φ is normal or $\mathrm{Aut}(\varphi)$ is trivial, then φ has a model over its field of moduli (see [**FV91**] Section 1.5 and [**DF94**] Sections 2.4 and 3.4).

3.5. Static Ramified Covers.

3.5.1. *Galois Covers.* Let $\varphi : Y \to \mathbb{P}^1$ be an algebraic cover and let $\beta \in \mathrm{Aut}(\mathbb{C})$. As above, $\hat{\beta} : Y \to Y^\beta$ is the restriction to Y of the action of β on projective space. Then β induces a function $\beta_* : \mathrm{Aut}(\varphi) \to \mathrm{Aut}(\varphi^\beta)$ given by $\alpha \mapsto \hat{\beta} \circ \alpha \circ \hat{\beta}^{-1}$, which is a group isomorphism.

Let F be the fixed field of β. If φ is defined over F, then $Y = Y^\beta$, which identifies $\mathrm{Aut}(\varphi)$ with $\mathrm{Aut}(\varphi^\beta)$, and $\beta_* \in \mathrm{Aut}(\mathrm{Aut}(\varphi))$. Now $\alpha \in \mathrm{Aut}(\varphi)$ is defined over F if and only if $\beta_*(\alpha) = \alpha$. View $\hat{\beta} \in \mathrm{Sym}(Y)$ and $\mathrm{Aut}(\varphi) \leq \mathrm{Sym}(Y)$; the subgroup of $\mathrm{Aut}(\varphi)$ consisting of automorphisms defined over F is $C_{\mathrm{Aut}(\varphi)}(\hat{\beta}) \leq \mathrm{Sym}(Y)$.

Let $\varphi : Y \to \mathbb{P}^1$ be a normal cover defined over a field K. We say that φ is *Galois over K* if every automorphism of φ is defined over K. This occurs exactly when $\beta^* \in \mathrm{Aut}(\mathrm{Aut}(\varphi))$ is the identity, or equivalently, $C_{\mathrm{Aut}(\varphi)}(\hat{\beta}) = \mathrm{Aut}(\varphi)$, for every $\beta \in \mathrm{Aut}(\mathbb{C}/K)$.

3.5.2. *Static Ramified Covers.* A *static ramified cover* of $\mathbb{P}^1$ with group G is a normal ramified cover $\varphi : Y \to \mathbb{P}^1$ together with an explicit isomorphism $\tau : G \to \mathrm{Aut}(\varphi)$. The corresponding topological cover is a static cover in the sense of subsection 2.4. Define morphisms for static ramified covers in the analogous fashion.

3.5.3. *Branch Cycle Descriptions of Static Ramified Covers.* Let $\boldsymbol{g}$ be a branch cycle description for a static ramified cover $(\varphi : Y \to \mathbb{P}^1, \tau : G \to \mathrm{Aut}(\varphi))$ with respect to some bouquet. Any equivalence to another static ramified cover induces an inner automorphism of G, so its branch cycle description differs from $\boldsymbol{g}$ by conjugation in G. This produces a bijective correspondence between these sets:

(1) equivalence classes of static ramified covers of $\mathbb{P}^1$ with group G;

(2) conjugacy classes of Nielsen tuples generating group G.

3.5.4. *Fields of Definition of Static Ramified Covers.* Let $(\varphi : Y \to bP^1, \tau : G \to \mathrm{Aut}(\varphi))$ be a static ramified cover. For $\beta \in \mathrm{Aut}(\mathbb{C})$, set $\tau^\beta = \beta_* \circ \tau$.

Let K be a subfield of $\mathbb{C}$. We say that (φ, τ) is *defined over* K if $\tau = \tau^\beta$ for every $\beta \in \mathrm{Aut}(\mathbb{C}/K)$. This happens exactly when φ is Galois over K.

We have an action of $\mathrm{Aut}(\mathbb{C})$ on static covers by $\beta : (\varphi, \tau) \mapsto (\varphi^\beta, \tau^\beta)$, which is well-defined on equivalence classes. The field of moduli for static covers is the fixed field of the group of field automorphisms of $\mathbb{C}$ which send (φ, τ) to an equivalent static cover. If G is centerless, then (φ, τ) has a model over its field of moduli.

4. Function Fields Extensions

4.1. Function Field Extensions.

4.1.1. *Functions Field Extensions.* A *function field extension* of $\mathbb{C}(x)$ is a finite extension $L/\mathbb{C}(x)$. Thus L has transcendence degree one over $\mathbb{C}$, and is a $\mathbb{C}(x)$ algebra. A *morphism* from $L_1/\mathbb{C}(x)$ and $L_2/\mathbb{C}(x)$ is a field embedding $\alpha : L_1 \to L_2$ such that $\alpha(f) = f$ for every $f \in \mathbb{C}(x)$.

It is important to note that if we view $\mathbb{C}(x)$ as the unembedded quotient field of a polynomial ring, and map the indeterminate to an element of $t \in L$ which is transcendental over $\mathbb{C}$ to construct a $\mathbb{C}(x)$ algebra, we obtain very different algebra structures depending on the choice of t. In particular, the degree of the algebraic part $[L : \mathbb{C}(t)]$ is dependent on the embedding of $\mathbb{C}(x)$ into L. In what follows, assume $x \in L$.

4.1.2. *Equivalence of Categories.* Let $\varphi : Y \to \mathbb{P}^1_x$ be a ramified cover, and let $L = \mathrm{Mer}(Y)$. The correspondence $\varphi \mapsto L/\mathbb{C}(x)$ produces a contravariant functor which is an equivalence of categories between ramified covers of $\mathbb{P}^1_x$ and function field extensions of $\mathbb{C}(x)$. In particular, $\mathrm{Aut}(\varphi) \cong \mathrm{Aut}(L/\mathbb{C}(x))$. Indeed, the map

$$\Gamma : \mathrm{Aut}(\varphi) \to \mathrm{Aut}(\mathrm{Mer}(Y)) \quad \text{given by} \quad \Gamma_\alpha(f) = f \circ \alpha^{-1},$$

where $\Gamma_\alpha : \mathrm{Mer}(Y) \to \mathrm{Mer}(Y)$ denotes the image of α in $\mathrm{Aut}(\mathrm{Mer}(Y))$, is a group antiisomorphism. Applying Galois theory to this adds that $\deg(\varphi) = [L : \mathbb{C}(x)]$.

Other invariants of φ have analogs in the category of function field extensions. The purpose of this section is to briefly describe how we intrinsically define the concepts of fields of definition and branch points for function field extensions, such that these invariants are preserved by the correspondence discussed above.

4.2. Arithmetic of Function Field Extensions.

4.2.1. *Fields of Definition of Function Field Extensions.* Let $L/\mathbb{C}(x)$ be a function field extension, and let F be a subfield of $\mathbb{C}$. We say that F is a *field of definition* for the extension, or that the extension *can be defined over* F, if there exists a primitive element for $L/\mathbb{C}(x)$ whose minimum polynomial in $\mathbb{C}(x)$ actually resides in $F(x)$. Suppose $\theta \in L$ such that $L = \mathbb{C}(x, \theta)$, whose minimum polynomial over $\mathbb{C}(x)$ is $f \in F(x)[y]$. Then θ is algebraic over $F(x)$, and $E = F(x, \theta)/F(x)$ is a finite extension. Moreover f is irreducible over $F(x)$, and $[E : F(x)] = [L : \mathbb{C}(x)]$.

Let $\mathrm{Fld}(L/\mathbb{C}(x))$ be the set of subfields of $\mathbb{C}$ over which the extension can be defined, and let $\varphi : Y \to \mathbb{P}^1_x$ be a ramified cover. Then $\mathrm{Fld}(\varphi) = \mathrm{Fld}(\mathrm{Mer}(Y)/\mathbb{C}(x))$.

4.2.2. *Regular Extensions.* Let L/F be a field extension. Let $\mathrm{alg}_L(F)$ denote the algebraic closure of F in L, that is, the set of elements of L which are algebraic over F; this is a subfield of L. We say that L/F is a *regular* extension if $\mathrm{alg}_L(F) = F$.

Let $L/\mathbb{C}(x)$ be a normal function field extension which is defined over a subfield F of $\mathbb{C}$, and let θ be a primitive element for $L/\mathbb{C}(x)$ with minimum polynomial $f \in F(x)[y]$. Let $E = F(x, \theta)$.

Then E/F is a regular extension; however, $E/F(x)$ may not be normal. Let $\hat{E}/F(x)$ be its normal closure in L, and let $\hat{F} = \mathrm{alg}_{\hat{E}}(F)$. We call $\hat{F}$ the *extension of constants* field. Clearly $\hat{E}/\hat{F}(x)$ is a normal extension, regular over $\hat{F}$. Restriction gives a map $\mathrm{Gal}(L/\mathbb{C}(x)) \to \mathrm{Gal}(\hat{E}/\hat{F}(x))$ which is an isomorphism, and we have an exact sequence

$$1 \to \mathrm{Gal}(\hat{E}/\hat{F}(x)) \to \mathrm{Gal}(\hat{E}/F(x)) \to \mathrm{Gal}(\hat{F}/F) \to 1.$$

If $\hat{F} = F$, we say that $L/\mathbb{C}(x)$ is *Galois over* F; in this case, $\mathrm{Gal}(L/\mathbb{C}(x))$ is isomorphic to the Galois group of a regular extension of F.

4.2.3. *Arithmetic versus Geometric Monodromy Groups.* Let $\varphi : Y \to \mathbb{P}^1_x$ be a normal ramified cover defined over a field F. The function field $L/\mathbb{C}(x)$ of $\hat{\varphi}$ can be defined over F; let E, $\hat{E}$ and $\hat{F}$ be as above. The topological monodromy group of φ is isomorphic to $\mathrm{Gal}(\hat{E}/\hat{F}(x))$; call this the *geometric monodromy group* of the cover. The *arithmetic monodromy group* over F of the cover is $\mathrm{Gal}(\hat{E}/F(x))$.

We can produce a geometric realization of the arithmetic monodromy group following a construction of Fried (see [**BF02**] Section 3.1.3). This construction amounts to taking the orbit of $\hat{Y}$ under the action of $\mathrm{Gal}(\hat{F}/F)$, allowing the components $\hat{Y}^\beta$ to cover $\mathbb{P}^1_x$ via $\hat{\varphi}^\beta$, for $\beta \in \mathrm{Gal}(\hat{F}/F)$. We call this the *Galois closure* of the cover over F; we obtain a disconnected cover defined over F, whose automorphism group is isomorphic to $\mathrm{Gal}(\hat{E}/F)$. The normal cover φ is Galois over F if and only if it equals its Galois closure, in which case the function field extension of φ is regular over F, and $\mathrm{Aut}(\varphi)$ is isomorphic to the Galois group of a regular extension of F.

4.3. Branch Points of Function Field Extensions.

4.3.1. *Laurent Series.* Let z be transcendental over $\mathbb{C}$. The *field of formal Laurent series* in z over $\mathbb{C}$ the field of fractions of the ring of formal power series in z over $\mathbb{C}$, given by

$$\mathcal{L}(z) = \left\{ \sum_{j=m}^{\infty} a_j z^j \mid m \in \mathbb{Z} \text{ and } a_j \in \mathbb{C} \right\}.$$

Then $\mathbb{C}(z)$ embeds in $\mathcal{L}(z)$ by expanding each rational function in its Laurent series around zero, making $\mathcal{L}(z)$ a $\mathbb{C}(z)$-algebra.

Let t be a positive integer. The map $\pi_t : \mathcal{L}(z) \to \mathcal{L}(z)$ given by $z \mapsto z^t$ is a $\mathbb{C}(z)$-algebra endomorphism with trivial kernel; thus the image $\mathcal{L}(z^t)$ is a subfield isomorphic to $\mathcal{L}(z)$. Clearly $\mathcal{L}(z)/\mathcal{L}(z^t)$ is a finite extension of degree t. Suppose that $\zeta_t \in \mathbb{C}$ is a t^{th} root of unity, and define $\mu_t : \mathcal{L}(z) \to \mathcal{L}(z)$ by $x \mapsto \zeta_t x$. Then μ_t is an automorphism of order t, which implies that $\mathcal{L}(z)/\mathcal{L}(z^t)$ is a Galois extension with cyclic Galois group generated by μ_t.

If $L/\mathcal{L}(z)$ is a finite extension of degree t, then $L = \mathcal{L}(y)$, where $y^t = z$ (see [**Vo96**]). Let $\overline{\mathcal{L}(z)}$ be a fixed algebraic closure of $\mathcal{L}(z)$, and let $\{z^{1/t} \mid t \in \mathbb{N}_+\}$ be compatible system of roots of z in $\overline{\mathcal{L}(z)}$; by compatible, we mean that $z^{1/t_1 t_2} = z^{1/t_1} z^{1/t_2}$. Then

$$\overline{\mathcal{L}(z)} = \bigcup_{t=1}^{\infty} \mathcal{L}(z^{1/t}).$$

4.3.2. *Branch Points of Function Field Extensions.* Let $a \in \mathbb{P}^1_x$ and set $z = (x - a)$ if $a \in \mathbb{C}$, or $z = \frac{1}{x}$ if $a = \infty$. The *field of formal Laurent series about* a is $\mathcal{L}_a = \mathcal{L}(z)$. For a compatible system of roots of z, set $\mathcal{P}^t_a = \mathcal{L}(z^{1/t})$. The *field of formal Puiseux expansions about* a is $\mathcal{P}_a = \cup_{t=1}^{\infty} \mathcal{P}^t_a$. We embed $\mathbb{C}(x)$ into $\mathcal{L}_a$ by expanding the rational functions in their Laurent series around a.

Let $L/\mathbb{C}(x)$ be a function field extension. Such a field extension has branch points, as we now describe. Since $\mathcal{P}_a$ is algebraically closed, we obtain an embedding $L \hookrightarrow \mathcal{P}_a$ lifting the embedding $\mathbb{C}(x) \hookrightarrow \mathcal{L}_a$, and there exists a minimal t such that $L \hookrightarrow \mathcal{P}^t_a$. We call t the *ramification index* of $L/\mathbb{C}(x)$ at a. If $t > 1$, we say that $L/\mathbb{C}(x)$ is *ramified* over a, and that a is a (nontrivial) *branch point* of $L/\mathbb{C}(x)$. Denote the set of branch points by $\mathrm{Bpt}(L/\mathbb{C}(x))$. If $\varphi : Y \to \mathbb{P}^1_x$ is a ramified cover, then $\mathrm{Bpt}(\varphi) = \mathrm{Bpt}(\mathrm{Mer}(Y)/\mathbb{C}(x))$.

Let $\hat{L}/\mathbb{C}(x)$ denote the normal closure of $L/\mathbb{C}(x)$; the branch points of $L/\mathbb{C}(x)$ and $\hat{L}/\mathbb{C}(x)$ are the same. Let b be a branch point of index t and $\iota : \hat{L}/\mathbb{C}(x) \to \mathcal{P}^t_b$ be an embedding over $\mathbb{C}(x)$. The automorphism μ_t fixes the image of $\mathbb{C}(x)$ and thus is an automorphism of $\iota(\hat{L})$. Then $\iota^{-1} \circ \mu_t \circ \iota \in \mathrm{Gal}(\hat{L}/\mathbb{C}(x))$. Any other embedding of $\hat{L}$ differs by an element of $\mathrm{Gal}(\hat{L}/\mathbb{C}(x))$, so the branch point b specifies a conjugacy class in $\mathrm{Gal}(\hat{L}/\mathbb{C}(x))$, which we denote by $\mathrm{Con}_b(\hat{L}/\mathbb{C}(x))$. Let $\hat{\varphi} : \hat{Y} \to \mathbb{P}^1_x$ be the normal closure of φ. The function Γ from subsection 4.1.2 produces an isomorphism between $\mathrm{Aut}(\hat{L}/\mathbb{C}(x))$ and the opposite group of $\mathrm{Aut}(\hat{\varphi})$, which is naturally identified with $\mathrm{Mon}(\hat{\varphi})$. Under these identifications, $\mathrm{Con}_b(\hat{\varphi}) = \mathrm{Con}_b(\hat{L}/\mathbb{C}(x))$ (see [**Vo96**] Theorem 5.9 Addendum).

One benefit of the above construction is that the Galois action on Puiseux expansions may be explicitly computed and compared to the monodromy action of the corresponding cover. We detect the Galois action by applying automorphisms to the coefficients of the power series.

4.3.3. *Branch Cycle Argument.* The branch cycle argument gives a necessary condition for a Nielsen tuple to correspond to a cover defined over $\mathbb{Q}$. Our source for this [**Fr77**] Lemma for Theorem 5.1, which acknowledges [**Fr73**] and [**Sh74**] (see also [**Vo96**] Lemma 2.8, [**Fr94**] Argument 1.2, and [**BF02**] Lemma 3.7).

If $\varphi : Y \to \mathbb{P}^1$ is a normal cover and $\beta \in \mathrm{Aut}(\mathbb{C})$, composing the identifications of $\mathrm{Mon}(\varphi)$ and $\mathrm{Mon}(\varphi^\beta)$ with $\mathrm{Aut}(\varphi)$ and $\mathrm{Aut}(\varphi^\beta)$, with the isomorphism $\beta_* : \mathrm{Aut}(\varphi) \to \mathrm{Aut}(\varphi^\beta)$, produces an isomorphism $\beta_* : \mathrm{Mon}(\varphi) \to \mathrm{Mon}(\varphi^\beta)$.

Proposition 1 (Branch Cycle Argument). *Let $\varphi : Y \to \mathbb{P}^1$ be a normal ramified cover, and let K be its field of moduli. Let $n = \deg(\varphi)$ and let $\zeta_n \in \mathbb{C}$ be a primitive n^{th} root of unity. Let $\beta \in \mathrm{Aut}(\mathbb{C}/K)$ and let $m \in \mathbb{Z}$ such that $\beta(\zeta_n) = \zeta_n^m$. Let $\beta_* : \mathrm{Mon}(\varphi) \to \mathrm{Mon}(\varphi^\beta)$ be the induced isomorphism. Then for every $b \in \mathbb{P}^1$ we have*

$$\mathrm{Con}_{\beta(b)}(\varphi^\beta) = \beta_*(\mathrm{Con}_b(\varphi))^m.$$

Proof. Let $L = \mathrm{Mer}(Y)$, and let $L^\beta = \mathrm{Mer}(Y^\beta)$. Since β fixes K, the cover φ^β is equivalent to φ, so there exists a holomorphic isomorphism $\xi : Y \to Y^\beta$ such that $\varphi = \varphi^\beta \circ \xi$. This induces a field isomorphism $L \to L^\beta$ which fixes $x \in \mathbb{C}(x) \leq L$ and extends the action of β on $\mathbb{C}$; denote this map also by β. It suffices two prove the proposition for conjugacy classes in the automorphism groups of the corresponding field extensions. Let $\mathbb{C}^\beta$ denote $\mathbb{C}$ twisted by β.

Let $g \in \mathrm{Con}_b(L/\mathbb{C}(x))$, and let $t = \mathrm{ord}(g)$. Then t divides n. Let $\zeta_t = \zeta_n^{n/t}$, so that ζ_t is a primitive t^{th} root of unity. Now $\beta(\zeta_t) = \zeta_t^m$. Clearly β extends to an isomorphism $\tilde{\beta} : \mathcal{P}_b^t \to \mathcal{P}_b^t$, given by acting on a the coefficients of a Puiseux series by β, such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\beta} & L^\beta \\ \iota\downarrow & & \downarrow\iota_\beta \\ \mathcal{P}_b^t & \xrightarrow[\tilde{\beta}]{} & \mathcal{P}_{\beta(b)}^t \end{array}$$

Apply the Galois functor (generally contravariant, but use covariant for isomorphisms) to obtain this commutative diagram:

$$\begin{array}{ccc} \mathrm{Gal}(L/\mathbb{C}(x)) & \xrightarrow{\beta_*} & \mathrm{Gal}(L^\beta/\mathbb{C}^\beta(x)) \\ \iota^*\uparrow & & \uparrow\iota_\beta^* \\ \mathrm{Gal}(\mathcal{P}_b^t/\mathcal{P}_b) & \xrightarrow[\tilde{\beta}_*]{} & \mathrm{Gal}(\mathcal{P}_{\beta(b)}^t/\mathcal{P}_{\beta(b)}) \end{array}$$

Let $g : (x-b)^{1/t} \mapsto \zeta_t(x-b)^{1/t}$ be a generator for $\mathrm{Gal}(\mathcal{P}_b^t/\mathcal{P}_b)$. For $a \in \mathbb{C}$, $g(a) = a$; thus $\beta g \beta^{-1}(a) = a$. This allows us to compute $\tilde{\beta}_*$ on $(x-\beta(b))^{1/t}$. Let $g_\beta : (x-\beta(b))^{1/t} \mapsto \zeta_t(x-\beta(b))^{1/t}$ be a generator for $\mathrm{Gal}(\mathcal{P}_{\beta(b)}^t/\mathcal{P}_{\beta(b)})$. Then

$$\begin{aligned} \tilde{\beta}_*(g)((x-\beta(b))^{1/t}) &= \tilde{\beta} g \tilde{\beta}^{-1}((x-\beta(b))^{1/t}) \\ &= \tilde{\beta} g((x-b)^{1/t}) \\ &= \tilde{\beta}(\zeta_t(x-b)^{1/t}) \\ &= \zeta_t^m(x-\beta(b))^{1/t} \\ &= g_\beta^m((x-\beta(b))^{1/t}). \end{aligned}$$

Pull these back to $\mathrm{Gal}(L/\mathbb{C}(x))$ and $\mathrm{Gal}(L^\beta/\mathbb{C}^\beta(x))$ to obtain the result. □

CHAPTER II

Hurwitz Spaces

1. Braid Groups

1.1. Configuration Spaces. Let X be a topological space and let r a positive integer. Let X^r be the cartesian product of X with itself r times, endowed with the product topology. The *pure configuration space* of X of rank r is

$$\mathcal{C}^r(X) = \{(x_1, \ldots, x_r) \in X^r \mid x_i = x_j \Rightarrow i = j\}.$$

The *pure hyperdiagonal* of X of rank r is

$$\Delta^r(X) = \{(x_1, \ldots, x_r) \in X^r \mid x_i = x_j \text{ for some } i \neq j\}.$$

Thus $\mathcal{C}^r(X) = X^r \smallsetminus \Delta^r(X)$.

The group S_r acts on X^r on the right by permuting the coordinates; for $\sigma \in S_r$, we have

$$(x_i)_j^\sigma = (x_i)_{j\sigma},$$

where $(x_i)_j \in X^r$ denotes the ordered tuple whose j^{th} entry is x_i. Let X_r denote the quotient space. This action respects the decomposition $X^r = \mathcal{C}^r(X) \cup \Delta^r(X)$. The *symmetrized configuration space* of X of rank r is

$$\mathcal{C}_r(X) = \mathcal{C}^r(X)/S_r,$$

and the *symmetrized hyperdiagonal* of X of rank r is

$$\Delta_r(X) = \Delta^r(X)/S_r,$$

each endowed with the quotient topology. Thus $\mathcal{C}_r(X) = X_r \smallsetminus \Delta_r(X)$. Points in $\mathcal{C}_r(X)$ are viewed as subsets of X of cardinality r. The action of S_r on $\mathcal{C}^r(X)$ is discrete, so we obtain a normal topological cover $\mathcal{C}^r(X) \to \mathcal{C}_r(X)$ with group S_r.

1.2. General Braid Groups. The *braid group* on r strings over X is

$$B_r(X) = \pi_1(\mathcal{C}_r(X), \underline{\boldsymbol{x}}),$$

where $\underline{\boldsymbol{x}} = \{x_1, \ldots, x_r\} \in \mathcal{C}_r(X)$ is a suitable basepoint. The *pure braid group* on r strings over X is

$$B^r(X) = \pi_1(\mathcal{C}^r(X), \boldsymbol{x}),$$

where $\boldsymbol{x} = (x_1, \ldots, x_r)$. The topological cover $\mathcal{C}^r(X) \to \mathcal{C}_r(X)$ induces an exact sequence

$$1 \to B^r(X) \to B_r(X) \to S_r \to 1;$$

in particular, $B^r(X) \triangleleft B_r(X)$. A path in $\mathcal{C}_r(X)$ permutes the points in $\{x_1, \ldots, x_r\}$, giving an action of $B_r(X)$ on $\mathbb{N}_r$. Then $B^r(X)$ is the kernel of this action, and S_r is the image.

We will be interested in the braid groups of the complex plane $\mathbb{A}^1$ and the Riemann sphere $\mathbb{P}^1$. The next few sections describe a point of view on these groups, condensed and synthesized from numerous sources, including the books [**Fr03**], [**BF02**], [**Bi75**], and [**MKS66**], the source papers [**Ar47a**], [**Bo47**], [**FV62**], [**FN62**], and additional works of Fried.

1.3. Artin Braid Group. Let $\mathcal{O}^r = \mathcal{C}^r(\mathbb{A}^1)$ and $\mathcal{O}_r = \mathcal{C}_r(\mathbb{A}^1)$. The *Artin braid group* is $B_r = B_r(\mathbb{A}^1) = \pi_1(\mathcal{O}_r)$; this is the braid group of the complex plane. The *pure* Artin braid group is $B^r = \pi_1(\mathcal{O}^r)$, and the discrete action of S_r on $\mathcal{O}^r$ produces a normal cover $\mathcal{O}^r \to \mathcal{O}_r$ with group S_r, producing an exact sequence of groups

$$1 \to B^r \to B_r \to S_r \to 1.$$

1.3.1. *Braid Generators.* Accurate identification of the braid group with a fundamental group requires selection of a basepoint. Thus let $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathcal{O}^r$, and let $\underline{\boldsymbol{x}}$ denote its image in $\mathcal{O}_r$. Generators for the braid group are formed by "twisting" adjacent points around each other. More precisely, select a paths in θ_i from x_i to x_{i+1}, and a path θ_{i+1} from x_{i+1} to x_i, such that the concatenation $\theta_i\theta_{i+1}$ is injective, has winding number -1, and is null homotopic in $\mathbb{A}^1 \setminus \{x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_r\}$. Let θ_j be the constant path at x_j, for $j \neq i, i+1$. Define a path $\boldsymbol{\theta}_i$ in $\mathcal{O}^r$ starting at $\boldsymbol{x}$ by $\boldsymbol{\theta}_i = (\theta_1, \ldots, \theta_r)$, and let $\underline{\boldsymbol{\theta}}_i$ denote the image of this path in $\mathcal{O}_r$. Then $\underline{\boldsymbol{\theta}}_i$ is a loop based at $\underline{\boldsymbol{x}}$. Let Q_i denote the homotopy class of $\underline{\boldsymbol{\theta}}_i$. Then Q_i, $i = 1, \ldots, r-1$, generate B_r freely modulo the following defining relations:

- **(B1)** $Q_iQ_j = Q_jQ_i$ for $|i-j| > 1$;
- **(B2)** $Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_{i+1}$ for $i = 1, \ldots, r-2$.

1.3.2. *Braid Action on the Fundamental Group.* If X is a topological space and $F \subset X$, let $\mathrm{Aut}(X, F)$ denote the set of homeomorphisms of X which restrict to the identity on F, considered as a topological group endowed with the compact open topology. Let $X = \mathbb{A}^1 \setminus \underline{\boldsymbol{x}}$, and let $x_0 \in X$. The Artin braid group acts on $\pi_1(X, x_0)$ in a manner we now describe. Let U be a bounded, connected, and simply connected open subset of $\mathbb{A}^1$ which contains $\underline{\boldsymbol{x}}$ but does not contain x_0, and let $F = \mathbb{A}^1 \setminus U$. Select the paths $\boldsymbol{\theta}_i$ from the previous paragraph to reside in U. The functor π_1 produces a homomorphism $\mathrm{Aut}(X, F) \to \mathrm{Aut}(\pi_1(X, x_0))$.

Define a function

$$\delta_{\boldsymbol{x}} : \mathrm{Aut}(\mathbb{A}^1, F) \to \mathcal{O}_r \quad \text{by} \quad \delta_{\boldsymbol{x}}(\xi) = \{\xi(x_1), \ldots, \xi(x_r)\},$$

where $\xi \in \mathrm{Aut}(\mathbb{A}^1, F)$. This function is continuous. Let $\alpha : I \to \mathcal{O}_r$ be a path in $\mathcal{O}_r$ starting at $\underline{\boldsymbol{x}}$. There exists a lift of α to a path $\tilde{\alpha} : I \to \mathrm{Aut}(\mathbb{A}^1, F)$, starting at the identity, such that $\delta_{\boldsymbol{x}} \circ \tilde{\alpha} = \alpha$.

Let α be a loop in $\mathcal{O}_r$ based at $\underline{\boldsymbol{x}}$ so that $[\alpha]$ is an arbitrary member of B_r. Let $\tilde{\alpha}$ be a lift of α to $\mathrm{Aut}(\mathbb{A}^1, F)$, starting at the identity, and let ξ be the endpoint of this lift. The function

$\xi : \mathbb{A}^1 \to \mathbb{A}^1$ restricts to $\xi^\circ : X \to X$, and $\xi(x_0) = x_0$. The functor π_1 produces a group isomorphism $\xi_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$. Then ξ_* is independent of the set F and the lift $\tilde{\alpha}$ chosen, producing a well-defined faithful action of B_r on $\pi_1(X, x_0)$. Since paths concatenate from left to right, this is a right action, and produces an injective antihomomorphism $B_r \to \mathrm{Aut}(\pi_1(X, x_0))$, whose image equals the image of the homomorphism $\mathrm{Aut}(X, F) \to \mathrm{Aut}(\pi_1(X, x_0))$.

1.3.3. *Braid Action on Classical Tuples.* Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be a classical tuple in $\mathbb{A}^1$ with respect to $\boldsymbol{x}$ and x_0; recall that in our lexicon, this means that the λ_i's are homotopy classes. Then $\pi_1(X, x_0)$ is a free group on r generators, freely generated by $\boldsymbol{\lambda}$. One sees that the preferred generators for B_r have the effect

$$\vec{\boldsymbol{\lambda}} Q_i = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i \lambda_{i+1} \lambda_i^{-1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_r).$$

Let $\Lambda(X, x_0)$ denote the set of all classical tuples on X with respect to x_0. If γ is a classical loop about x_i, then γ is conjugate to λ_i, and if $\boldsymbol{\gamma}$ is a classical tuple, then both $\Pi\boldsymbol{\gamma}$ and $\Pi\boldsymbol{\lambda}$ are homotopic to a loop which encircles all the points in $\underline{\boldsymbol{x}}$. Thus

$$\Lambda(X, x_0) = \{\boldsymbol{\gamma} \mid \Pi\boldsymbol{\gamma} = \Pi\boldsymbol{\lambda} \text{ and } \gamma_i \sim \lambda_{i\sigma}\},$$

where $a \sim b$ means "a is conjugate to b", and $\sigma \in S_r$ depends on $\boldsymbol{\gamma}$ but not i. The Artin braid group B_r acts regularly on $\Lambda(X, x_0)$, and this defines its image in $\mathrm{Aut}(\pi_1(X, x_0))$. We see that $\mathrm{Inn}(\pi_1(X, x_0))$ is contained in this image, and resides therein as a normal subgroup.

Let $Z = (Q_1 \cdots Q_{r-1})^r \in B_r$; this element has the effect of conjugating $\boldsymbol{\lambda}$ by $\Pi\boldsymbol{\lambda}$, and generates the center of B_r.

1.3.4. *Twist and Shift in B_r.* Call the generators $Q_i \in B_r$ the i^{th} *twist*. Define the *shift* in B_r to be the element:

$$S = \prod_{i=1}^{r-1} Q_i.$$

Then for $1 < j \leq r - 1$, we have

$$\begin{aligned}
Q_j^S &= Q_{r-1}^{-1} \cdots Q_1^{-1} Q_j Q_1 \cdots Q_{r-1} \\
&= Q_{r-1}^{-1} \cdots Q_1^{-1} Q_1 \cdots Q_{j-2} (Q_j Q_{j-1} Q_j) Q_{j+1} \cdots Q_{r-1} && \text{by } \mathbf{B1} \\
&= Q_{r-1}^{-1} \cdots Q_{j-1}^{-1} (Q_{j-1} Q_j Q_{j-1}) Q_{j+1} \cdots Q_{r-1} && \text{by } \mathbf{B2} \\
&= Q_{r-1}^{-1} \cdots Q_{j+1}^{-1} Q_{j-1} Q_{j+1} \cdots Q_{r-1} \\
&= Q_{j-1} && \text{by } \mathbf{B1}
\end{aligned}$$

Thus the standard set of generators $\{Q_i \mid i = 1, \ldots, r-1\}$ is contained in the group generated by S and Q_j, for any $1 \leq j \leq r - 1$, and in particular, $\langle S, Q_j \rangle = B_r$. In words, B_r is generated by the shift and any twist.

1.4. Hurwitz Monodromy Group. Let $\mathcal{U}^r = \mathcal{C}^r(\mathbb{P}^1)$ and $\mathcal{U}_r = \mathcal{C}_r(\mathbb{P}^1)$. The *Hurwitz monodromy group* is $H_r = B_r(\mathbb{P}^1) = \pi_1(\mathcal{U}_r)$; this is the braid group of the Riemann sphere. The *pure* Hurwitz monodromy group is $H^r = \pi_1(\mathcal{U}^r)$, and the discrete action of S_r on $\mathcal{U}^r$ produces a normal cover $\mathcal{U}^r \to \mathcal{U}_r$ with group S_r, producing an exact sequence of groups

$$1 \to H^r \to H_r \to S_r \to 1.$$

1.4.1. *Hurwitz Relation.* View $\mathbb{P}^1$ as the one point compactification of $\mathbb{A}^1$; that is, $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. We have an inclusion $\mathcal{O}^r \to \mathcal{U}^r$, which commutes with the action of the symmetric group S_r to produce the commutative diagram

$$\begin{array}{ccc} \mathcal{O}^r & \longrightarrow & \mathcal{U}^r \\ \downarrow & & \downarrow \\ \mathcal{O}_r & \longrightarrow & \mathcal{U}_r \end{array}$$

where the horizontal rows are injections and the vertical columns are normal covers with automorphism group S_r.

We interpret these spaces as follows:

- $\mathcal{O}^r$ is the parameter space of ordered r-tuples of points on $\mathbb{A}^1$;
- $\mathcal{O}_r$ is the parameter space of subsets of $\mathbb{A}^1$ with cardinality r;
- $\mathcal{U}^r$ is the parameter space of ordered r-tuples of points on $\mathbb{P}^1$;
- $\mathcal{U}_r$ is the parameter space of subsets of $\mathbb{P}^1$ with cardinality r.

The injection $\mathcal{O}_r \to \mathcal{U}_r$ induces a surjective homomorphism $B_r \to H_r$, whose kernel is the normal closure of a third defining relation for H_r:

(B3) $Q_1 \cdots Q_{r-2} Q_{r-1}^2 Q_{r-2} \cdots Q_1$.

Since relation **B3** is in B^r, there is an induced map from $B^r \to H^r$, and these groups fit into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B^r & \longrightarrow & B_r & \longrightarrow & S_r & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & H^r & \longrightarrow & H_r & \longrightarrow & S_r & \longrightarrow & 1 \end{array}$$

which is produced by the π_1 functor from the preceding diagram.

1.4.2. *Braid Action on Quotient Tuples.* Let $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathcal{O}^r$, and let $\underline{\boldsymbol{x}}$ denote its image in $\mathcal{O}_r$. Select $x_0 \in \mathbb{A}^1 \setminus \underline{\boldsymbol{x}}$, and set $F_r = \pi_1(\mathbb{A}^1 \setminus \underline{\boldsymbol{x}}, x_0)$ and $G_r = \pi_1(\mathbb{P}^1 \setminus \underline{\boldsymbol{x}}, x_0)$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be a classical tuple in $\mathbb{A}^1$ with respect to $\boldsymbol{x}$ and x_0. The inclusion $\mathbb{A}^1 \to \mathbb{P}^1$ induces a surjective homomorphism $F_r \to G_r$, whose kernel is the normal closure of $\Pi\boldsymbol{\lambda}$. Since $F_r = \langle \lambda_1, \ldots, \lambda_r \rangle$ is freely generated, G_r has presentation $\langle \lambda_1, \ldots, \lambda_r \mid \Pi\boldsymbol{\lambda} \rangle$. Since $\Pi\boldsymbol{\lambda}$ is fixed by the action of B_r on F_r, the map $F_r \to G_r$ induces an antihomomorphism $B_r \to \mathrm{Aut}(G_r)$ whose kernel is $Z(B_r)$.

1.4.3. *Hurwitz Nonaction on Classical Tuples in* $\mathbb{P}^1$. Let N_r denote the kernel of $B_r \to H_r$. Since N_r is not contained in $Z(B_r)$, an induced action of H_r on G_r is not well-defined.

Our method of defining the action of B_r on $\pi^1(\mathbb{A}^1 \smallsetminus \underline{\boldsymbol{x}}, x_0)$ uses the selection of an open set U containing $\underline{\boldsymbol{x}}$ but not x_0, which allows us to fix the basepoint through the continuous motion of $\underline{\boldsymbol{x}}$ inside U. This succeeds in that case because any loop in $\mathcal{O}_r$ based at $\underline{\boldsymbol{x}}$ is homotopic to a loop in $\mathcal{C}_r(U)$, and two loops in $\mathcal{C}_r(U)$ which are homotopic in $\mathcal{O}_r$, are homotopic via a homotopy which remains in U. In other words, the inclusion $\mathcal{C}_r(U) \to \mathcal{O}_r$ induces $\pi_1(\mathcal{C}_r, \underline{\boldsymbol{x}}) \to \pi_1(\mathcal{O}_r, \underline{\boldsymbol{x}})$ which is an isomorphism.

If we attempt to define an action of H_r on classical tuples in $\mathbb{P}^1$ in the manner of we did in $\mathbb{A}^1$, the method breaks down because, for proper simply connected open subset $U \subset \mathbb{P}^1$, the map $\pi_1(\mathcal{C}_r(U), \underline{\boldsymbol{x}}) \to \pi_1(\mathcal{U}_r, \underline{\boldsymbol{x}})$ is not injective. In $\mathbb{P}^1$, we cannot get away with fixing the basepoint, and it is this ambivalence of choice of basepoint which results in the failure of H_r to act on classical tuples in $\mathbb{P}^1$. It is well-known that the fundamental group depends on the basepoint only up to inner automorphism, which should allow us to construct a homomorphism $H_r \to \mathrm{Aut}(G_r)/\mathrm{Inn}(G_r)$. We now see that this is the best we can hope for.

1.4.4. *Hurwitz Kernel.* Let $N_r = \ker(B_r \to H_r)$. The Hurwitz relation has the effect on a classical tuple $(\lambda_1, \ldots, \lambda_r)$ of conjugating it by $(\lambda_1)^{-1}$. Thus the image of N_r in $\mathrm{Aut}(G_r)$ is $\mathrm{Inn}(G_r)$ (see chapter IV for more details). The kernel of the map $H_r \to \mathrm{Aut}(G_r)/\mathrm{Inn}(G_r)$ is $Z(H_r)$, which is cyclic of order two, generated by $Z \pmod{N_r}$ (see [**Bi75**] Lemma 4.2.3).

1.4.5. *Twist and Shift in H_r.* Let $\varphi : B_r \to H_r$ be the natural homomorphism. Following the convention of [**BF02**], we use lower case for the images of elements. Thus

- **(a)** $\varphi(Q_i) = q_i$;
- **(b)** $\varphi(S) = s = q_1 \ldots q_{r-1}$;
- **(c)** $\varphi(Z) = z = s^r$.

The inclusion $\mathbb{C} \to \mathbb{P}^1$ pushes the paths $\underline{\boldsymbol{\theta}}_1, \ldots, \underline{\boldsymbol{\theta}}_{r-1}$ discussed in subsection 1.3.1 to $\mathbb{P}^1$, and we view $q_1, \ldots, q_{r-1}$ as homotopy classes of their images.

Note that $q_1 = q_{r-1}^{s^{r-2}}$. Set $q_0 = q_1^s$. Since $s^r = z$ is the unique central involution of H_r, the order of s is $2r$. However, s^r has trivial conjugation action, so $q_0^s = q_{r-1}^{s^r} = q_{r-1}$. Thus the left conjugation action of s cyclically permutes $(q_0, q_1, \ldots, q_r)$.

1.5. Mapping Class Groups.

1.5.1. *Isotopy Class Groups.* Let X be a locally compact Hausdorff space, and let $\mathrm{Aut}(X)$ denote the set of all homeomorphisms from X to itself. Endow $\mathrm{Aut}(X)$ with the compact open topology. Then two automorphisms are homotopic if and only if they lie in the same path component of $\mathrm{Aut}(X)$.

Let $\mathrm{Iso}(X)$ denote the set of all isotopy classes of automorphisms of X. There is a natural well-defined group structure on $\mathrm{Iso}(X)$ given by

$$[f] * [g] = [f \circ g];$$

we call Iso(X) the *isotopy class group* of X. There is a natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Iso}(X)$ given by $f \mapsto [f]$, and the kernel of this map is the component of the identity in Aut(X). All the components of Aut(X) are cosets of the identity component; thus Iso(X) is the group of components of Aut(X).

1.5.2. *Mapping Class Groups.* Let X be a connected orientable manifold. The *mapping class group* of X, denoted by Map(X), is the index two subgroup of Iso(X) consisting of orientation preserving isotopy classes. Typically, X is a punctured compact Riemann surface.

Select r points $B = \{x_1, \ldots, x_r\} \subset X$. The r^{th} mapping class group of X is

$$M_r(X) = \mathrm{Map}(X \setminus B).$$

This is the group of path components of the setwise stabilizer of B among the orientation preserving members of Aut(X). Since Aut(X) is highly transitive (k-transitive for every positive integer k), the setwise stabilizers of finite sets of points of the same cardinality are conjugate, so $M_r(X)$ is well-defined up to isomorphism.

1.5.3. *Sphere Mapping Class Group.* Let M_r denote the mapping class group of a sphere with r punctures. In this paragraph, let $\mathrm{Aut}(\mathbb{C})$ and $\mathrm{Aut}(\mathbb{P}^1)$ be the groups of self homeomorphisms of $\mathbb{C}$ and $\mathbb{P}^1$, respectively. The generators $Q_1, \ldots, Q_{r-1}$ represent paths $\mathcal{O}_r$ which lift to paths in $\mathrm{Aut}(\mathbb{C})$ whose endpoints are orientation preserving self homeomorphisms which fix the set of punctures. This defines a map $B_r \to M_r$. This map is surjective, and its kernel is $Z(B_r) = \langle Z \rangle$. Similarly, the generators $q_1, \ldots, q_{r-1}$ of H_r lift to paths in $\mathrm{Aut}(\mathbb{P}^1)$, creating a surjective homomorphism $H_r \to M_r$ whose kernel is $Z(H_r) = \langle z \rangle$. Summarizing, we have

$$M_r \cong B_r/Z(B_r) \cong H_r/Z(H_r) \cong \mathrm{Aut}(\pi_1(X, x_0))/\mathrm{Inn}(\pi_1(X, x_0)),$$

where X is an r-punctured sphere.

1.5.4. *Sphere Mapping Class Cover.* Let $o : \tilde{\mathcal{O}}_r \to \mathcal{O}_r$ and $u : \tilde{\mathcal{U}}_r \to \mathcal{U}_r$ be the universal covers of $\mathcal{O}_r$ and $\mathcal{U}_r$, respectively. Each is a normal cover with respective groups B_r and H_r. Let $\underline{\boldsymbol{x}} = \{x_1, \ldots, x_r\} \in \mathcal{O}_r$; we may also view this as a point in $\mathcal{U}_r$. We have an inclusion $\mathcal{O}_r \to \mathcal{U}_r$, and viewing points in the universal covering space as homotopy classes of paths based at $\underline{\boldsymbol{x}}$, this inclusion induces a map between the covering spaces.

Let $E = o^{-1}(\underline{\boldsymbol{x}})$. Let $y \in E$, and select a tuple of classical generators to correspond to y. Since B_r acts regularly on tuples of classical generators, this creates a correspondence between E and Λ (the set of all classical tuples on $\mathbb{C} \setminus \underline{\boldsymbol{x}}$).

Let $F = u^{-1}(\underline{\boldsymbol{x}})$. Since every path in $\mathcal{U}_r$ is homotopic to a path which does not pass through a set containing ∞, the map from E to F is surjective, inducing an equivalence relation on Λ.

We may mod out the universal covers by any subgroup of the fundamental group of the base; if the subgroup is normal, the induced map from the quotient to the base is also normal. Thus set

- $\check{\mathcal{O}}_r = \tilde{\mathcal{O}}_r / Z(B_r)$;
- $\hat{\mathcal{O}}_r = \tilde{\mathcal{O}}_r / N_r$;
- $\mathcal{W}_r = \tilde{\mathcal{O}}_r / \langle Z(B_r), N_r \rangle$;
- $\mathcal{V}_r = \tilde{\mathcal{U}}_r / Z(H_r)$.

We obtain the following commutative diagram:

$$\begin{array}{ccccc} \tilde{\mathcal{O}}_r & \xrightarrow{N_r} & \hat{\mathcal{O}}_r & \xrightarrow{\text{inj}} & \tilde{\mathcal{U}}_r \\ {\scriptstyle \mathbb{Z}}\downarrow & & {\scriptstyle \mathbb{Z}/2}\downarrow & & \downarrow{\scriptstyle \mathbb{Z}/2} \\ \check{\mathcal{O}}_r & \longrightarrow & \mathcal{W}_r & \xrightarrow{\text{inj}} & \mathcal{V}_r \\ & & {\scriptstyle M_r}\downarrow & & \downarrow{\scriptstyle M_r} \\ & & \mathcal{O}_r & \xrightarrow{\text{inj}} & \mathcal{U}_r \end{array}$$

The fiber of $\mathcal{V}_r \to \mathcal{U}_r$ over $\underline{\boldsymbol{x}}$ may be identified with $\Lambda/\mathrm{Inn}(G_r)$, upon which the mapping class group M_r acts regularly.

2. Hurwitz Spaces

2.1. Deformation Spaces.

2.1.1. *Deformation Equivalence.* Let Y be a compact orientable surface of genus g, and let $\mathcal{R}(Y)$ be the set of all surjective continuous maps $\varphi : Y \to \mathbb{P}^1$ such that Y admits a complex structure which makes φ analytic. Then there is a bijective correspondence between $\mathcal{R}(Y)$ and the set of equivalence classes of ramified covers with covering space of genus g. This set is a topological space when endowed with the compact open topology. A path in $\mathcal{R}(Y)$ is a *deformation*.

Let $\mathcal{R}(Y, r)$ denote the subspace of $\mathcal{R}(Y)$ consisting of those covers with exactly r branch points. We say that two ramified covers are *deformation equivalent* if they lie in the same path component of $\mathcal{R}(Y, r)$, for some r. From this point of view, the space $\mathcal{R}(Y, r)$ is difficult to analyze. We use braid groups and Nielsen tuples to decipher it.

2.1.2. *Nielsen Sets.* Let $G \leq S_n$ be a transitive group and let r be a positive integer. Let G^r denote the cartesian product of G with itself r times. The *total Nielsen set* of G of rank r is

$$\mathrm{Ni}(G, r)^{\mathrm{to}} = \{\boldsymbol{g} \in G^r \mid \langle \boldsymbol{g} \rangle = G \text{ and } \Pi \boldsymbol{g} = 1\}.$$

Thus $\mathrm{Ni}(G, r)^{\mathrm{to}}$ is the collection of rank r Nielsen tuples in G. The Artin braid group acts on $\mathrm{Ni}(G, r)^{\mathrm{to}}$ via the formula

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_r)Q_i = (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_r).$$

Let $\mathrm{Abs}(G)$ denote the subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms which preserve the conjugacy class of a one point stabilizer in G; then $\mathrm{Abs}(G) \cong N_{S_n}(G)/C_{S_n}(G)$. Any subgroup of $\mathrm{Abs}(G)$ acts on $\mathrm{Ni}(G, r)^{\mathrm{to}}$ coordinatewise in a manner which commutes with the braid action.

The *inner Nielsen set* of G of rank r is

$$\mathrm{Ni}(G,r)^{\mathrm{in}} = \mathrm{Ni}(G,r)^{\mathrm{to}}/\mathrm{Inn}(G);$$

we call its elements *inner tuples.*

The *absolute Nielsen set* of G of rank r is

$$\mathrm{Ni}(G,r)^{\mathrm{ab}} = \mathrm{Ni}(G,r)^{\mathrm{to}}/\mathrm{Abs}(G);$$

we call its elements *absolute tuples.*

Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be a tuple of distinct points in $\mathbb{P}^1$, and select a basepoint $x_0 \in \mathbb{P}^1$ not among them. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be a bouquet with respect to $(\boldsymbol{x}, x_0)$. These data define correspondences between

- **(a)** the set $\mathrm{Ni}(G,r)^{\mathrm{in}}$ and the set of equivalence classes of static ramified covers of $\mathbb{P}^1$ with branch points in $\boldsymbol{x}$ which are the normal closures of covers whose monodromy group is G;
- **(b)** the set $\mathrm{Ni}(G,r)^{\mathrm{ab}}$ and the set of equivalence classes of ramified covers of $\mathbb{P}^1$ with branch points in $\boldsymbol{x}$ and monodromy group G.

Typically, the Hurwitz monodromy group does not act on $\mathrm{Ni}(G,r)^{\mathrm{to}}$, since the Hurwitz relation **(B3)** does not act trivially. However, the action B_r on $\mathrm{Ni}(G,r)^{\mathrm{to}}$ descends to a well-defined actions of H_r on $\mathrm{Ni}(G,r)^{\mathrm{in}}$ and $\mathrm{Ni}(G,r)^{\mathrm{ab}}$. These actions correspond to the continuous deformation of covers described by the elements of the Nielsen sets along a loop in $\mathcal{U}_r$; an equivalent condition for two covers to be deformation equivalent is that the corresponding Nielsen tuples lie in the same orbit under this action.

2.1.3. *Deformation Spaces.* The action of the Hurwitz monodromy group on the sets $\mathrm{Ni}(G,r)^{\mathrm{in}}$ and $\mathrm{Ni}(G,r)^{\mathrm{ab}}$ canonically produces topological covers of the configuration space $\mathcal{U}_r$ in which every point in the covering space corresponds to a static ramified cover, as we now as we now describe.

Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover with branch points $\boldsymbol{x}$ and branch cycle description $\boldsymbol{g} \in \mathrm{Ni}(G,r)^{\mathrm{to}}$ with respect to some bouquet. The set of equivalence classes of covers with the same branch points to which φ can be continuously deformed are described by the orbit of $\boldsymbol{g}$ under the action of H_r on $\mathrm{Ni}(G,r)^{\mathrm{ab}}$. The stabilizer S of $\boldsymbol{g}$ is a subgroup of H_r which canonically produces to a cover $\mathcal{H}(G,\boldsymbol{g})^{\mathrm{ab}} \to \mathcal{U}_r$. Recall that the space $\mathcal{H}(G,\boldsymbol{g})^{\mathrm{ab}}$ may be defined as the set of all paths in $\mathcal{U}_r$ emanating from $\underline{\boldsymbol{x}}$ modulo homotopy and concatenation in S. Thus each point $\mathfrak{p} \in \mathcal{H}(G,\boldsymbol{g})^{\mathrm{ab}}$ corresponds to the cover obtained by continuously deforming φ along a path in $\mathcal{U}_r$ representing $\mathfrak{p}$. Each orbit of H_r on the Nielsen class produces such a space.

Let $\hat{\varphi} : \hat{Y} \to \mathbb{P}^1$ be the normal closure of φ, together with an isomorphism $\tau : G \to \mathrm{Aut}(\hat{\varphi})$; two such static covers are equivalent if their branch cycle descriptions differ by an inner automorphism, with continuous deformation also deforming the map τ. The action of H_r on $\mathrm{Ni}(G,r)^{\mathrm{in}}$ canonically produces a cover $\mathcal{H}(G,\boldsymbol{g})^{\mathrm{in}} \to \mathcal{U}_r$, whose points correspond to static covers.

The *inner deformation space* of (G, r) is the union of the components

$$\mathcal{H}(G,r)^{\mathrm{in}} = \bigcup_{\boldsymbol{g}\in \mathrm{Ni}(G,r)^{\mathrm{in}}} \mathcal{H}(G,\boldsymbol{g})^{\mathrm{in}}.$$

The *absolute deformation space* of (G, r) is

$$\mathcal{H}(G,r)^{\mathrm{ab}} = \bigcup_{\boldsymbol{g}\in \mathrm{Ni}(G,r)^{\mathrm{ab}}} \mathcal{H}(G,\boldsymbol{g})^{\mathrm{ab}}.$$

As part of the definition, each of these spaces is equipped with an assignment of an isomorphism class of covers to each point; such an assignment is determined by a single appropriate choice for one point on each component. Since the stabilizer in H_r of an inner tuple is necessarily contained in the stabilizer of an absolute tuple, we obtain this sequence of covers:

$$\Psi : \mathcal{H}(G,r)^{\mathrm{in}} \xrightarrow{\Xi} \mathcal{H}(G,r)^{\mathrm{ab}} \xrightarrow{\Phi} \mathcal{U}_r.$$

These maps are understood as follows:

(a) $\Psi : [\psi, \tau] \mapsto \mathrm{Bpt}(\psi)$;

(b) $\Phi : [\varphi] \mapsto \mathrm{Bpt}(\varphi)$;

(c) $\Xi : [\hat{\varphi}, \tau] \mapsto [\varphi]$.

Since these deformation spaces cover $\mathcal{U}_r$, which is an algebraic variety, the theorem of Grauert and Remmert says that they themselves have an algebraic structure. Much more can be said; the following is essentially part of [**FV91**], Theorem 1.

THEOREM 2 (Fried-Volklein Theorem). *Let $G \leq S_n$ be a transitive group generated by $r-1$ elements. Then $\mathcal{H}^{\mathrm{in}} = \mathcal{H}(G,r)^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{ab}} = \mathcal{H}(G,r)^{\mathrm{ab}}$ have a unique structure as algebraic varieties defined over $\mathbb{Q}$ so that the maps*

$$\Psi : \mathcal{H}^{\mathrm{in}} \xrightarrow{\Xi} \mathcal{H}^{\mathrm{ab}} \xrightarrow{\Phi} \mathcal{U}_r$$

are defined over $\mathbb{Q}$. Let K be an algebraically closed subfield of $\mathbb{C}$ and let $\beta \in \mathrm{Aut}(K)$ so that β acts on the K points of $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{ab}}$. Then

(a) *if $\mathfrak{q} = [\psi, \tau] \in \mathcal{H}^{\mathrm{in}}$ with $\mathrm{Bpt}(\psi) \subset K$, we have $\beta(\mathfrak{q}) = [\psi^\beta, \tau^\beta]$;*

(b) *if $\mathfrak{p} = [\varphi] \in \mathcal{H}^{\mathrm{ab}}$ with $\mathrm{Bpt}(\varphi) \subset K$, we have $\beta(\mathfrak{p}) = [\varphi^\beta]$.*

The second part of this theorem implies that the minimum field of definition of a point on a Hurwitz space is equal to the field of moduli of the corresponding cover.

The components of these deformation spaces are defined over $\bar{\mathbb{Q}}$. The absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the components, and the disjoint union of the components in an orbit is defined over $\mathbb{Q}$. We would like to find conditions relating to the group G which allow us to pick out these orbits.

2.2. Hurwitz Spaces.

2.2.1. *Conjugacy Class Tuples.* Let $G \leq S_n$ be a transitive group and let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be an r-tuple of conjugacy classes from G. We introduce some notation to facilitate considering this as a set with multiplicity. For $\boldsymbol{D} = (D_1, \ldots, D_r)$ another r-tuple of conjugacy classes from G, say that $\boldsymbol{D}$ is *similar* to $\boldsymbol{C}$, and write $\boldsymbol{D} \sim \boldsymbol{C}$, if there exists $\sigma \in S_r$ such that $D_i = C_{i\sigma}$, all i. For $\boldsymbol{g} = (g_1, \ldots, g_r) \in G^r$, say that $\boldsymbol{g}$ *satisfies* $\boldsymbol{C}$, and write $\boldsymbol{g} \vDash \boldsymbol{C}$, if there exists $\sigma \in S_r$ such that for all $i \in \mathbb{N}_n$, we have $g_i \in C_{i\sigma}$. Set $\|\boldsymbol{C}\|$ equal to the least common multiple of the orders of the elements in the conjugacy classes of $\boldsymbol{C}$.

If $n \in \mathbb{Z}$, set $\boldsymbol{C}^n = (C_1^n, \ldots, C_r^n)$; we say that $\boldsymbol{C}$ is *rational* if $\boldsymbol{C}^n \sim \boldsymbol{C}$ whenever $\gcd(n, \|\boldsymbol{C}\|) = 1$. By the branch cycle argument, if $\boldsymbol{g}$ is a Nielsen tuple corresponding to a static cover defined over $\mathbb{Q}$, can $\boldsymbol{C}$ is its tuple of conjugacy classes, then $\boldsymbol{C}$ is rational.

If $\alpha \in \mathrm{Aut}(G)$, set $\alpha(\boldsymbol{C}) = (\alpha(C_1), \ldots, \alpha(C_r))$; we say that $\boldsymbol{C}$ is *characteristic* if $\alpha(\boldsymbol{C}) \sim \boldsymbol{C}$ for every $\alpha \in \mathrm{Aut}(G)$. Let $\mathrm{Abs}(G, \boldsymbol{C}) = \{\alpha \in \mathrm{Abs}(G) \mid \alpha(\boldsymbol{C}) \sim \boldsymbol{C}\}$.

2.2.2. *Nielsen Classes.* A necessary condition for the two ramified covers of $\mathbb{P}^1$ to be deformation equivalent is that their monodromy groups and associated conjugacy classes be the same. This leads to our next series of definitions.

The *total Nielsen class* of $(G, \boldsymbol{C})$ is

$$\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}} = \{\boldsymbol{g} \in G^r \mid \Pi \boldsymbol{g} = 1,\ \langle \boldsymbol{g} \rangle = G,\ \text{and } \boldsymbol{g} \vDash \boldsymbol{C}\}.$$

This is the set of all Nielsen tuples satisfying $\boldsymbol{C}$.

The *inner Nielsen class* of $(G, \boldsymbol{C})$ is

$$\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}} = \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}} / \mathrm{Inn}(G).$$

The inner Nielsen classes partition the inner Nielsen set $\mathrm{Ni}(G, r)^{\mathrm{in}}$, so that each tuple $\boldsymbol{C}$ produces a distinct block.

The *absolute Nielsen class* of $(G, \boldsymbol{C})$ is

$$\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{ab}} = \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}} / \mathrm{Abs}(G, \boldsymbol{C}).$$

The absolute Nielsen class embeds into the absolute Nielsen set $\mathrm{Ni}(G, r)^{\mathrm{ab}}$. We note that if $\boldsymbol{C}$ is not similar to $\alpha(\boldsymbol{C})$ for some $\alpha \in \mathrm{Aut}(G)$, then $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$ and $\mathrm{Ni}(G, \alpha(\boldsymbol{C}))^{\mathrm{in}}$ form different blocks of $\mathrm{Ni}(G, r)^{\mathrm{in}}$. However, if $\alpha \in \mathrm{Abs}(G)$, then $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{ab}}$ and $\mathrm{Ni}(G, \alpha(\boldsymbol{C}))^{\mathrm{ab}}$ have the same image in $\mathrm{Ni}(G, r)^{\mathrm{ab}}$.

2.2.3. *Hurwitz Spaces.* The *inner Hurwitz space* of $(G, \boldsymbol{C})$, denoted $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$, consists of the collection of components of $\mathcal{H}(G, r)^{\mathrm{in}}$ whose points correspond to static covers whose associated conjugacy classes are given by $\boldsymbol{C}$.

The *absolute Hurwitz space* of $(G, \boldsymbol{C})$, denoted $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{ab}}$, consists of the collection of components of $\mathcal{H}(G, r)^{\mathrm{ab}}$ whose points correspond to covers whose associated conjugacy classes are given by $\boldsymbol{C}$. This is the image of $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$ under the map $\Psi : \mathcal{H}(G, r)^{\mathrm{in}} \to \mathcal{H}(G, r)^{\mathrm{ab}}$.

Even though the conjugacy classes of ramification are well-defined in the monodromy group of the cover, their whereabouts under a permutation representation depends on the enumeration of the fiber, and can get lost under absolute equivalence. In this way, distinct inner Hurwitz spaces can map to the same absolute space.

The degree of the map $\Xi : \mathcal{H}(G,r)^{\text{in}} \to \mathcal{H}(G,r)^{\text{ab}}$ is $|\text{Out}(G)|$. The degree of the restriction $\Xi : \mathcal{H}(G,\boldsymbol{C})^{\text{in}} \to \mathcal{H}(G,\boldsymbol{C})^{\text{ab}}$ is $[\text{Abs}(G,\boldsymbol{C}) : \text{Inn}(G)]$. Let $\mathcal{H}^{\text{ab}}$ be a component of the absolute space, and let $\mathcal{H}^{\text{in}}$ be its preimage in the inner space. If $\mathcal{H}^{\text{in}}$ is connected, then $\text{Aut}(\Xi \restriction_{\mathcal{H}^{\text{in}}}) \cong \text{Abs}(G,\boldsymbol{C})/\text{Inn}(G)$.

The branch cycle argument implies that a necessary condition for a Hurwitz space to be defined over $\mathbb{Q}$ is that the corresponding tuple be a rational tuple. This turns out to be sufficient, which we state here (see [**FV91**] Theorem 1).

Theorem 3. *Let $G \leq S_n$ and let $\boldsymbol{C}$ be a tuple of conjugacy classes from G. Then $\mathcal{H}(G,\boldsymbol{C})^{\text{in}}$ is defined over $\mathbb{Q}$ if and only if $\boldsymbol{C}$ is a rational tuple of conjugacy classes.*

3. Reduced Hurwitz Spaces

3.1. Reduction of Configuration Spaces.

3.1.1. *General Reduction.* Let X be topological space and let $\text{Aut}(X)$ be the group of homeomorphisms from X to itself. Let $A \leq \text{Aut}(X)$. Then A acts on $\mathcal{C}^r(X)$ on the left coordinatewise:

$$\alpha((x_i)_j) = (\alpha(x_i))_j.$$

The quotient space of this action is the *reduced* configuration space of X of rank r with respect to A, which we denote by $\mathcal{C}^r(X,A)$, with quotient map $\Pi : \mathcal{C}^r(X) \to \mathcal{C}^r(X,A)$.

The actions of S_r and A on $\mathcal{C}^r(X)$ commute, so the fiber coproduct of Σ and Π can be obtained by reducing $\mathcal{C}^r(X,A)$ by the action of S_r, or by reducing $\mathcal{C}_r(X)$ by the action of A, or by reducing $\mathcal{C}^r(X)$ by the action of $A \times S_r$. Denote the quotient space by $\mathcal{C}_r(X,A)$; this is the *reduced symmetrized* configuration space of X. This situation is summarized by the diagram:

$$\begin{array}{ccc} \mathcal{C}^r(X) & \xrightarrow{\pi} & \mathcal{C}^r(X,A) \\ \sigma\big\downarrow & & \big\downarrow\bar{\sigma} \\ \mathcal{C}_r(X) & \xrightarrow[\underline{\pi}]{} & \mathcal{C}_r(X,A) \end{array}$$

where we denote symmetrization by S_r with an underbar, and reduction by A with an overbar.

Let $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathcal{C}^r(X)$, and set

$$\underline{\boldsymbol{x}} = \sigma(\boldsymbol{x}) \leftrightarrow \{x_1, \ldots, x_r\};$$

$$\overline{\boldsymbol{x}} = \pi(\boldsymbol{x});$$

$$\underline{\overline{\boldsymbol{x}}} = \bar{\sigma}(\pi(\boldsymbol{x})) = \underline{\pi}(\sigma(\boldsymbol{x})).$$

Consider the fiber $E = \sigma^{-1}(\underline{\boldsymbol{x}})$. The setwise stabilizer $U = \text{Stb}_A\{x_1, \ldots, x_r\}$ acts on E, inducing a homomorphism $\varphi : U \to S_r$.

Let $F = \bar{\sigma}^{-1}(\underline{\bar{\boldsymbol{x}}})$; then $\pi \restriction_E : E \to F$ is surjective, and the points of F correspond to the orbits of U on E. The action of S_r on E descends to a transitive action on F, and the stabilizer in S_r of $\underline{\boldsymbol{x}} \in F$ under this action is $\varphi(U)$. Thus $|F| = [S_r : \varphi(U)]$.

3.1.2. *Sharply Transitive Reduction.* Proceed under the additional assumption that A is sharply k-transitive, where $k < r$; by a theorem of Tits (see [**DM96**] Theorem 7.6B), $k \leq 3$ if X is infinite. In this case, let us call k the *reduction rank*; here, the map $\varphi : U \to S_r$ discussed above is injective by sharpness.

Select distinct distinguished points $W = \{w_1, \ldots, w_k\} \subset X$. For every $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathcal{C}^r(X)$ there exists a unique $\alpha \in A$ such that $\alpha(\boldsymbol{x}) = (w_1, \ldots, w_k, \alpha(x_{k+1}), \alpha(x_r))$. This identifies $\mathcal{C}^r(X, A)$ with $\mathcal{C}^{r-k}(X \smallsetminus W)$. Indeed, the map $\mathcal{C}^r(X) \to A \times \mathcal{C}^{r-k}(X \smallsetminus W)$ constructed in this way is a homeomorphism, where A is endowed with the compact-open topology as a subset of $\mathrm{Aut}(X)$. From this viewpoint, reduction is merely projection onto the second factor, and in turn, the fibers are seen to be homeomorphic to A, which is homeomorphic to $\mathcal{C}^k(X)$. Thus the map π admits a section, given by $(x_{k+1}, \ldots, x_r) \mapsto (w_1, \ldots, w_k, x_{k+1}, \ldots, x_r)$.

3.1.3. *Holomorphic Reduction.* Let X be a complex manifold and let $\mathrm{Hol}(X)$ denote the subgroup of $\mathrm{Aut}(X)$ consisting of analytic self homeomorphisms. Then $\mathrm{Hol}(X)$ is a natural candidate for the reduction group. The resulting space $\mathcal{C}^r(X, \mathrm{Hol}(X))$ parameterizes holomorphism classes of subsets of X of cardinality r.

3.2. Reduction of Sphere Configuration Spaces. Consider the case where $X = \mathbb{P}^1_z$, the Riemann sphere together with a uniformizing coordinate. Choice of this coordinate determines an identification $\mathrm{Hol}(\mathbb{P}^1_z) \cong \mathrm{PSL}_2(\mathbb{C})$. It is well known that the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{P}^1_z$ is sharply three transitive.

Let $\mathcal{J}^r = \mathcal{C}^r(\mathbb{P}^1_z, \mathrm{PSL}_2(\mathbb{C}))$ and $\mathcal{J}_r = \mathcal{C}_r(\mathbb{P}^1_z, \mathrm{PSL}_2(\mathbb{C}))$. We interpret these spaces as follows:

- $\mathcal{J}^r$ is the parameter space of holomorphism classes of ordered r-tuples of points on $\mathbb{P}^1$;
- $\mathcal{J}_r$ is the parameter space of holomorphism classes of subsets of $\mathbb{P}^1$ with cardinality r.

Consider the map $\bar{\sigma} : \mathcal{J}^r \to \mathcal{J}_r$. Let $d = \max\{|\bar{\sigma}^{-1}(j)| \mid j \in \mathcal{J}_r\}$, and set $Y = \{j \in \mathcal{J}_r \mid |\bar{\sigma}^{-1}(j)| < d\}$ and $Z = \bar{\sigma}^{-1}(Y)$. Then $\bar{\sigma} \restriction_{\mathcal{J}^r \smallsetminus Z} : \mathcal{J}^r \smallsetminus Z \to \mathcal{J}_r \smallsetminus Y$ is a normal topological cover.

3.3. Reduction of Inner Hurwitz Spaces. Let $\tilde{\mathcal{U}}_r \to \mathcal{U}_r$ be the universal cover of $\mathcal{U}_r$, and view the points of $\tilde{\mathcal{U}}_r$ as homotopy classes paths in $\mathcal{U}_r$ based at some point $\underline{\boldsymbol{x}} = \{x_1, \ldots, x_r\}$. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on paths by left composition, preserving homotopy classes, and thus acts on the points of $\tilde{\mathcal{U}}_r$. Denote the quotient of this action by $\tilde{\mathcal{U}}_r^{\mathrm{rd}}$; we obtain a map $\tilde{\mathcal{U}}_r^{\mathrm{rd}} \to \mathcal{J}_r$. Any cover of $\mathcal{U}_r$ can be similarly reduced; if $\mathcal{H}$ is a component of a rank r Hurwitz space, we obtain maps

$$\tilde{\mathcal{U}}_r^{\mathrm{rd}} \to \mathcal{V}_r^{\mathrm{rd}} \to \mathcal{H}^{\mathrm{rd}} \to \mathcal{J}_r.$$

The points of $\mathcal{H}^{\mathrm{rd}}$ correspond to weak equivalence classes of ramified covers of $\mathbb{P}^1$, and the fiber of $\mathcal{V}_r^{\mathrm{rd}} \to \mathcal{J}_r$ over $\overline{\underline{\boldsymbol{x}}}$ corresponds to the set of $\mathrm{PSL}_2(\mathbb{C})$ orbits of classical generators about $(\underline{\boldsymbol{x}}, x_0)$ modulo conjugation in $\pi_1(\mathbb{P}^1 \setminus \underline{\boldsymbol{x}}, x_0)$.

Let G be a finite group and let $\boldsymbol{C}$ be a tuple of conjugacy classes from G. The *reduced inner* Hurwitz space of $(G, \boldsymbol{C})$, denoted by $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}}$, is the collection of reduced components of $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$.

We know that $d = \frac{r!}{|U|}$, where U is a setwise stabilizer of r points of minimal order. For $r = 3$, $U = S_3$ for every set of three points, and $\mathcal{J}^3 \to \mathcal{J}_3$ is a map from a single point to a single point. For $r \geq 5$, U is trivial for almost every set of 5 points, and the reduced cover has degree $r!$.

The reduction map $\mathcal{H} \to \mathcal{H}^{\mathrm{rd}}$ is continuous. In particular, $\mathcal{H}^{\mathrm{rd}}$ is connected, and covers $\mathcal{J}_r$. If the rank is $r = 3$, then $\mathcal{J}_r$ is a point, thus so is $\mathcal{H}^{\mathrm{rd}}$. A reduced rank three Hurwitz space consists of one point for each component of the unreduced space. For rank $r \geq 5$, the dimension of the cover drops but its degree remains the same. Thus we concentrate on the case $r = 4$.

4. Reduced Rank Four Hurwitz Spaces

4.1. Reduction of Rank Four Sphere Configuration Spaces. Recall that every elliptic curve is uniquely identified up to holomorphic isomorphism by its j-invariant, and that the j-line $\mathbb{P}^1_j$ parameterizes their isomorphism classes. The λ-line $\mathbb{P}^1_\lambda$ parameterizes elliptic curves together with an ordering of their involutive points, producing a natural map $\mathbb{P}^1_\lambda \to \mathbb{P}^1_j$. To motivate modular towers, and to give background for our main example, we briefly review this in chapter III. In this section we produce these spaces as reduced rank four sphere configuration spaces.

Consider the reduction of the cover $\mathcal{U}^r \to \mathcal{U}_r$ when the rank is $r = 4$. Let $W = \{0, 1, \infty\}$ be our set of preferred points, and note that $\mathcal{U}^1 = \mathbb{P}^1$. Every ordered tuple $(z_1, z_2, z_3, z_4) \in \mathcal{U}^4$ is equivalent modulo $\mathrm{PSL}_2(\mathbb{C})$ to a unique tuple of the form $(0, 1, \infty, \lambda)$. This identifies $\mathcal{J}^4$ with $\mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$. Closure and symmetrization of this space produces a ramified cover between copies of the Riemann sphere which can be expressed as a rational function, which is unique up to composition with another linear fractional transformation acting on the closure of $\mathcal{J}_4$. We use group actions and covering theory to find a satisfactory representative for this class of rational functions, which we will denote by $j : \mathbb{P}^1_\lambda \to \mathbb{P}^1_j$. Under this map, $\{0, 1, \infty\}$ comprise a single fiber.

The setwise stabilizer in $\mathrm{PSL}_2(\mathbb{C})$ of four points on $\mathbb{P}^1$ contains a Klein four group consisting of transformations which swap the points in pairs, as can be computed using the cross ratio. The cross ratio is

$$z \mapsto \frac{z - z_1}{z - z_3} : \frac{z_2 - z_1}{z_2 - z_3},$$

where $(z_1, z_2, z_3) \mapsto (0, 1, \infty)$. Each element of $\mathcal{J}_r$ is represented by a tuple of four points of the form $\boldsymbol{x} = (0, 1, \infty, \lambda)$. Then the nontrivial transformations of this Klein four group are:

$$z \mapsto \frac{\lambda}{z} \qquad \leftrightarrow \qquad \texttt{(1 3)(2 4)};$$

$$z \mapsto \frac{z-\lambda}{z-1} \qquad \leftrightarrow \qquad \texttt{(1 4)(2 3)};$$

$$z \mapsto \frac{z-1}{z-\lambda} : \frac{0-1}{0-\lambda} \qquad \leftrightarrow \qquad \texttt{(1 2)(3 4)}.$$

Denote this Klein four group in S_4 by K_4, and view S_3 in S_4. The points in the fiber over $\overline{\underline{\boldsymbol{x}}}$ correspond to the cosets of K_4 in S_4, which are represented by the elements of S_3. We determine the values of λ such that the stabilizer of $\underline{\boldsymbol{x}}$ is larger than K_4. Suppose $\alpha \in S_3$ corresponds to a linear fractional transformation f which stabilizes $\underline{\boldsymbol{x}}$. Then

(a) $\alpha = \texttt{(1 2)} \Rightarrow f(z) = 1 - z \Rightarrow \lambda = \frac{1}{2}$;

(b) $\alpha = \texttt{(1 3)} \Rightarrow f(z) = \frac{1}{z} \Rightarrow \lambda = -1$;

(c) $\alpha = \texttt{(2 3)} \Rightarrow f(z) = \frac{z}{z-1} \Rightarrow \lambda = 2$;

(d) $\alpha = \texttt{(1 2 3)} \Rightarrow f(z) = \frac{1}{1-z} \Rightarrow \lambda = \frac{1 \pm i\sqrt{3}}{2}$ (the same for $\alpha = \texttt{(1 3 2)}$).

In particular, these points are isolated, so the cover j has degree $[S_4 : K_4] = 6$.

Note that if $f(z) = 1 - z$, then $f(2) = -1$, and if $f(z) = \frac{1}{z}$, we have $f(2) = \frac{1}{2}$. Thus $\{-1, \frac{1}{2}, 2\}$ lie in the same fiber over $\mathbb{P}^1_j$, each point having ramification index two. Let $\zeta = e^{\pi i/3}$; then $\{\zeta, \zeta^{-1}\}$ forms another fiber, each point having ramification index three. So j is a normal ramified cover with three branch points, group S_3, and branch cycle description of shape $(\texttt{(3)(3)}, \texttt{(2)(2)(2)}, \texttt{(2)(2)(2)})$. This Nielsen class consists of a single element, so j is completely determined by this data, up to weak equivalence of covers.

Choose $\{\zeta, \zeta^{-1}\}$ to be the zeros of j, and $\{0, 1, \infty\}$ to be the poles. Then j is a scalar multiple of $\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$. Without further choices, it is forced upon us that $\{-1, \frac{1}{2}, 2\}$ lie in the same fiber of this rational function, and indeed it is so; each has value $\frac{27}{4}$. Divide by this quantity so that the third branch point is 1; this yields

$$j(\lambda) = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

This formula is precisely that which computes the classical j value of the elliptic curve defined by the equation $y^2 = x(x-1)(x-\lambda)$.

4.2. Reduction of Rank Four Hurwitz Spaces.

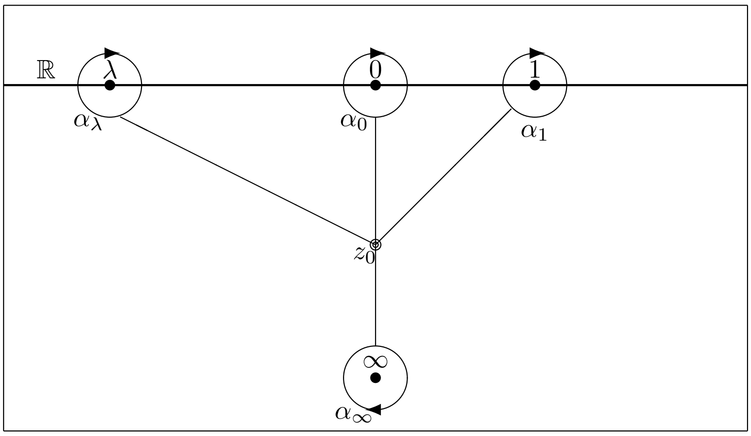
4.2.1. *Reduced Rank Four Mapping Class Group.* In this section we discuss the maximal quotient of the braid group which acts nontrivially on reduced classical tuples, expanding upon the original formulation in [**DF99**] and [**BF02**].

Select $\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in \mathcal{U}^4$ so that $\overline{\underline{\boldsymbol{z}}} \notin \{0, 1\}$, and let $X = \mathbb{P}^1_z \setminus \underline{\boldsymbol{z}}$. Let $\mathrm{Hol}(X)$ denote the group of holomorphic automorphisms of X, and view $\mathrm{Hol}(X)$ as the setwise stabilizer in $\mathrm{PSL}_2(\mathbb{C})$ of $\underline{z}$. Let $\mathrm{Aut}(X)$ denote the group of orientation preserving continuous automorphisms of X. We have an inclusion $\mathrm{Hol}(X) \to \mathrm{Aut}(X)$ which descends to a map $\mathrm{Hol}(X) \to \mathrm{Map}(X) = M_4$. Since every element of $\mathrm{Hol}(X)$ has nontrivial action on $\underline{\boldsymbol{z}}$, the latter map is injective.

Let K_4 denote the image of $\mathrm{Hol}(X)$ in M_4. Select a basepoint $z_0 \in X$ and let $\Lambda^{\mathrm{in}} = \Lambda(\boldsymbol{z})^{\mathrm{in}}$ denote the set of equivalence classes of classical tuples in $\mathbb{P}^1$ with respect to $(\boldsymbol{z}, z_0)$, modulo conjugation. Now M_r acts regularly on Λ^{in}, and two classical tuples are equivalent modulo $\mathrm{PSL}_2(\mathbb{C})$ if and only if they lie in the same K_4 orbit. Thus $\Lambda^{\mathrm{in,rd}} = \Lambda^{\mathrm{in}}/K_4$ is the set of inner reduced classical tuples.

In order to compute the action of K_4 on Λ^{in}, recall that the map $H_4 \to M_4$ discussed in subsection 1.5.3 is given by the induced action of H_4 on X. Let $\hat{K}_4$ denote the pullback of K_4 to H_4, so that $\hat{K}_4$ is the subgroup of H_4 which has trivial action $\Lambda^{\mathrm{in,rd}}$. Since the kernel $Z(H_4)$ of the map $H_4 \to M_4$ is a group of order two, $Z(H_4) \to \hat{K}_4 \to K_4$ is a central extension and $|\hat{K}_4| = 8$. We now explicitly compute $\hat{K}_4$.

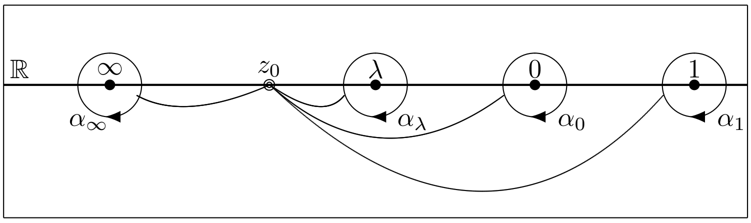
We choose an basepoint for $\mathcal{U}_4$; this choice effects our results only up to inner automorphism of H_4. Let $\boldsymbol{z} = (0, 1, \infty, \lambda)$ so that $j(\lambda) \notin \{0, 1, \infty\}$. Choose λ to be a negative real number. Let $f(z) = \frac{\lambda}{z}$, and let $z_0 = -i\sqrt{|\lambda|}$ so that $f(z_0) = z_0$, and z_0 becomes a convenient basepoint for the computation. Let α_0, α_1, α_∞, and α_λ denote the homotopy classes of paths in $\mathbb{P}^1$ which proceed in lines from z_0 towards 0, 1, ∞, and λ, in that order, go around these points in small disks, and proceed back to z_0 along the same lines, as indicated below.



Paths for braid computation of $f(z) = \frac{\lambda}{z}$.

Let $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_\infty, \alpha_\lambda)$ denote the classical tuple thus described, and compose these paths with f to obtain $f(\boldsymbol{\alpha}) = (\alpha_\infty, \alpha_\lambda, \alpha_0, \alpha_1)$. This effect is given by the square of the shift, that is, $f(\boldsymbol{\alpha}) = \boldsymbol{\alpha}(q_1 q_2 q_3)^2$.

Next we consider the linear fractional transformation $f(z) = \lambda\frac{z-1}{z-\lambda}$. The fixed points of this transformation are $\lambda \pm \sqrt{\lambda^2-\lambda}$; select $z_0 = \lambda - \sqrt{\lambda^2-\lambda}$ as a basepoint. In order to relate this computation to the previous one, we draw a line from $-i\sqrt{|\lambda|}$ to $\lambda - \sqrt{\lambda^2-\lambda}$, and concatenate it to the paths above to adjust the basepoint. This effects our computation only up to inner automorphism of the fundamental group, and so has no effect on inner equivalence classes of classical tuples. With this adjustment, paths for this calculation are drawn in the following diagram.



Paths for braid computation of $f(z) = \lambda\frac{z-1}{z-\lambda}$.

Compose these paths with f and rewrite the result in terms of the original paths to see that $f(\boldsymbol{\alpha}) = (\alpha_\infty^{-1}\alpha_1\alpha_\infty, \alpha_1^{-1}\alpha_\infty^{-1}\alpha_0\alpha_\infty\alpha_1, \alpha_\lambda, \alpha_\infty)$. Conjugate on the right by α_λ and use the product one condition to see that, up to inner equivalence, we have $f(\boldsymbol{\alpha}) = (\alpha_0\alpha_1\alpha_0^{-1}, \alpha_0, \alpha_\lambda, \alpha_\lambda^{-1}\alpha_\infty\alpha_\lambda)$. A braid which has this effect is $q_1q_3^{-1}$.

Let $a = (q_1q_2q_3)^2$ and $b = q_1q_3^{-1}$. Clearly $\hat{K}_4 = \langle a, b\rangle$, and ab has the same effect on $\boldsymbol{\alpha}$ as does $f(z) = \frac{z-\lambda}{z-1}$. Note that $a^2 = b^2 = z$, the unique involution generating the center of H_4, and in particular, a and b have order four. If s is the shift in H_4, we have seen that $q_i^s = q_{i-1}$; since $a = s^2$, we have $b^s = q_3q_1^{-1} = b^{-1}$, so a and b are noncommuting elements of order four, which tells us that $\hat{K}_4$ is isomorphic to the quaterions.

Now s commutes with a and normalizes $\langle b\rangle$. Moreover q_1 commutes with b and $a^{q_1} = q_1^{-1}q_1^{a^{-1}}a = q_1^{-1}q_3a = b^{-1}a \in \hat{K}_4$. Since q_1 and s generate H_4, this shows that $\hat{K}_4 \triangleleft H_4$. The images of a, b, and ab in M_4 are the nontrivial elements of K_4, which is normal in M_4.

The *reduced mapping class group* of rank 4 is $\bar{M}_4 = M_4/K_4 = H_4/\hat{K}_4$. It is the quotient of H_4 by the additional relation

(B4) $Q_1 = Q_3$.

Plug this relation into relations **(B2)** and **(B3)** for this simplification:

$$\bar{M}_4 = \langle Q_1, Q_2 \mid Q_1Q_2Q_1 = Q_2Q_1Q_2,\ Q_1Q_2Q_1Q_1Q_2Q_1\rangle;$$

the second relation is the Hurwitz relation. Use the first relation to rewrite the second relation as $Q_1Q_2Q_1Q_2Q_1Q_2$. Let $\gamma_0 = Q_1Q_2$ and $\gamma_1 = Q_1Q_2Q_1$ inside this group; the reason for this notation will become clear in the next subsection. We have $\gamma_1^2 = 1$ by the Hurwitz relation and $\gamma_1^3 = 1$ by its rewritten form. Also $Q_1 = \gamma_0^{-1}\gamma_1$ and $Q_2 = \gamma_1\gamma_0^{-1}$, so $\langle Q_1, Q_2\rangle = \langle\gamma_0, \gamma_1\rangle$.

We show that

$$\bar{M}_4 = \langle \gamma_0, \gamma_1 \mid \gamma_0^3,\ \gamma_1^2 \rangle$$

is an alternate presentation. Set $Q_1 = \gamma_0^{-1}\gamma_1$ and $Q_2 = \gamma_1\gamma_0^{-1}$. It suffices to derive the relations for the first presentation from those for the second. Now $Q_1Q_2 = \gamma_0^{-1}\gamma_1\gamma_1\gamma_0^{-1} = \gamma_0$, since γ_1 has order two and γ_0 has order three; also $Q_1Q_2Q_1 = \gamma_0 Q_1 = \gamma_1$. Thus $Q_1Q_2Q_1 = \gamma_1 = \gamma_1\gamma_0^{-1}\gamma_0 = Q_2\gamma_0 = Q_2Q_1Q_2$. Finally $Q_1Q_2Q_1Q_1Q_2Q_1 = \gamma_1^2 = 1$. This completes the demonstration.

4.2.2. *Reduced Rank Four Mapping Class Cover.* Let $\mathbb{H}$ denote the open upper half plane. Its group of holomorphic self homeomorphisms is $\mathrm{Hol}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$. Consider the set of lattices in $\mathbb{C}$ of the form $\mathbb{Z} \oplus \tau\mathbb{Z}$, where $\tau \in \mathbb{H}$. Then $\mathrm{PSL}_2(\mathbb{R})$ acts on this set via its action on $\mathbb{H}$. The kernel of the action is $\mathrm{PSL}_2(\mathbb{Z})$. Let R denote the set of points in $\mathbb{H}$ which have nontrivial stabilizers in $\mathrm{PSL}_2(\mathbb{Z})$; let $Y = \mathbb{H} \setminus R$ and let $X = Y/\mathrm{PSL}_2(\mathbb{Z})$. We obtain a normal topological cover $Y \to X$ with group $\mathrm{PSL}_2(\mathbb{Z})$.

Let $\Psi : \mathcal{V}_4^{\mathrm{rd}} \to \mathcal{J}_4$ be the reduced mapping class cover of rank four. As an aside, note that since the center of H_4 is contained in $\hat{K}_4$, we have $\tilde{\mathcal{U}}_4^{\mathrm{rd}} = \mathcal{V}_4^{\mathrm{rd}}$. Let $\mathcal{J}^\circ = \mathcal{J}_4 \setminus \{0,1\}$, $\mathcal{V}^\circ = \mathcal{V}_4^{\mathrm{rd}} \setminus \Psi^{-1}(\{0,1\})$, and $\Psi^\circ = \Psi\restriction_{\mathcal{V}^\circ}$. Then $\Psi^\circ : \mathcal{V}^\circ \to \mathcal{J}^\circ$ is a topological cover.

Let $j \in \mathcal{J}^\circ$; then $j = \underline{\underline{\boldsymbol{z}}}$ for some $\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in \mathcal{U}^r$. The fiber over j corresponds to classical tuples on $\mathbb{P}^1$ with respect to $\underline{\boldsymbol{z}}$ modulo inner automorphisms of $\pi_1(\mathbb{P}^1 \setminus \underline{\boldsymbol{z}})$ and modulo the action of $\mathrm{PSL}_2(\mathbb{C})$. The action of the fundamental group of $\mathcal{J}^\circ$ on this fiber, via path lifting, is the effect on the classical tuples of continuous motion in $\mathbb{P}^1$ of the points $\underline{\boldsymbol{z}}$ via the braid action, modulo reduction; it is the action of $\bar{M}_4$. Thus $\mathrm{Aut}(\Psi^\circ) = \bar{M}_4$.

It is well known that $\mathrm{PSL}_2(\mathbb{Z})$ is freely generated by an element S of order three (which stabilizes $e^{2\pi i/6} \in \mathbb{H}$) and an element T of order two (which stabilizes $i \in \mathbb{H}$). The isomorphism $\bar{M}_4 \to \mathrm{PSL}_2(\mathbb{Z})$ given by $(\gamma_0, \gamma_1) \mapsto (S, T)$ establishes an isomorphism $\mathbb{H} \to \mathcal{V}_4^{\mathrm{rd}}$.

4.2.3. *Reduced Rank Four Nielsen Classes.* Let $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$ be a rank four inner Nielsen class. The group K_4 acts on $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$ via its lift to $\hat{K}_4$. Since $\hat{K}_4 \triangleleft H_4$, its orbits create a block system for the action of H_4 on $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$. Let $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}} = \mathrm{Ni}(G, \boldsymbol{C})/\hat{K}_4$ denote the set of blocks; this is the *reduced Nielsen class.*

The action of H_4 on $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$ descends to an action of $\bar{M}_4$ on $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$. The points of $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ correspond to weak equivalence classes of ramified covers with specified ramification in $(G, \boldsymbol{C})$ over a given $\mathrm{PSL}_2(\mathbb{C})$ equivalence class of branch points.

4.2.4. *Reduced Rank Four Hurwitz Spaces.* Let $\Phi : \mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{J}_4$ be the cover given by reduction of an inner Hurwitz space of rank 4. In this case, $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ is a Riemann surface. For $j \in \mathcal{J}_4 \setminus \{0,1\}$, the points in the fiber over j correspond to the members of $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$.

Let $\mathcal{J}^\circ = \mathbb{P}^1_j \setminus \{0,1\}$ and let $\mathcal{H}^\circ = \mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}} \setminus \Phi^{-1}(\{0,1,\infty\})$. Let $\Phi^\circ = \Phi\restriction_{\mathcal{H}^\circ}$. Then $\Phi^\circ : \mathcal{H}^\circ \to \mathcal{J}^\circ$ is a topological cover of the punctured sphere, which induces a ramified cover $\Phi^\bullet : \mathcal{H}^\bullet \to \mathcal{J}^\bullet$, ramified over $j = 0, 1, \infty$.

The cover $\varphi^\bullet$ is produced by the action of $\bar{M}_4$ on the reduced Nielsen class $\mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in,rd}}$. Enumerating the set $\mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in,rd}}$ induces a permutation representation which can, in some cases, be explicitly computed. For this to completely describe the cover $\mathcal{H}(G,\boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{J}_4$, we need explicit paths in $\mathbb{P}^1_j \setminus \{0,1,\infty\}$ with respect to which a branch cycle description can be stated.

4.3. Images of Braid Generators in $\mathbb{P}^1_j \setminus \{0,1,\infty\}$.

4.3.1. *Branch Point Set Images on the j-line.* Let $\mathcal{U}_4 = \mathbb{P}^4 \setminus D_4$ and let $\boldsymbol{z} = \{z_1,z_2,z_3,z_4\} \in \mathcal{U}_4$. The reduction map $j : \mathcal{U}_4 \to \mathbb{P}^1_j \setminus \{\infty\}$ maps a set of four unordered points to the j-value of the corresponding elliptic curve. We wish to construct a formula for j as a function of $\boldsymbol{z}$. It simplifies matters if we assume $z_4 = \infty$, and this suffices for our purposes. If we map z_2 to 1 and z_3 to 0, then z_1 maps to

$$\lambda(\boldsymbol{z}) = \frac{z_1 - z_3}{z_2 - z_3}.$$

Recall that

$$j(\lambda) = \frac{4}{27}\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

Now let $a = (z_1 - z_3)$ and $b = (z_2 - z_3)$ so that $a - b = z_1 - z_2$ and $\lambda = \frac{a}{b}$. Then

$$\begin{aligned}
\frac{27}{4}j(\boldsymbol{z}) &= \frac{(\frac{a^2}{b^2} - \frac{a}{b} + 1)^3}{\frac{a^2}{b^2}(\frac{a}{b} - 1)^2} \\
&= \frac{(a^2 - ab + b^2)^3}{a^2b^2(a-b)^2} \\
&= \frac{[(z_1^2 - 2z_1z_3 + z_3^2) - (z_1z_2 - z_1z_3 - z_2z_3 + z_3^2) + (z_2^2 - 2z_2z_3 + z_3^2)]^3}{(z_1-z_3)^2(z_2-z_3)^2(z_1-z_2)^2} \\
&= \frac{(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1)^3}{(z_1-z_2)^2(z_2-z_3)^2(z_3-z_1)^2}.
\end{aligned}$$

This yields

$$j(\boldsymbol{z}) = \frac{4}{27}\frac{[(z_1+z_2+z_3)^2 - 3(z_1z_2 + z_2z_3 + z_3z_1)]^3}{(z_1-z_2)^2(z_2-z_3)^2(z_3-z_1)^2}.$$

Note that this function is symmetric in z_1, z_2, and z_3.

4.3.2. *Braid Generator Images on the j-line.* Composing the embedding of $\mathcal{O}_4$ into $\mathcal{U}_4$ with reduction $\mathcal{U}_4 \to \mathcal{J}_4$, we obtain a map $f : \mathcal{O}_4 \to \mathcal{J}_4$. Taking particular paths for Q_1, Q_2, and Q_3 as generators for $\pi_1(\mathcal{O}_4, \boldsymbol{z}_0)$, we wish to compute the images on the j-line via the map f, taking care that the images avoid the set $\{0,1,\infty\}$. We anticipate that $(f(Q_2), f(Q_1Q_2), f(Q_1Q_2Q_3))$ have the same path lifting action on $\mathcal{V}_4^{\mathrm{rd}}$ as a bouquet $\boldsymbol{\gamma}$ on $\mathbb{P}^1_j$ with respect to $((\infty,0,1),j_0)$, where $j_0 > 1$ is a positive real number; we would like to absolutely identify this bouquet. Then this bouquet, together with the action of $\boldsymbol{\gamma}$ on the reduced Nielsen class, will produce a branch cycle description for a reduced Hurwitz space cover of $\mathbb{P}^1_j$.

Before undergoing the explicit computation, let us make some observations about what we can expect. Assume the basepoint $\boldsymbol{z} \in \mathcal{U}_r$ lies on the real line, and that the nonfixed part of the Q_i's are circles in the complex plane symmetric with respect to the real line and parameterized at a constant rate by $t \in [0,1]$. Let $j_0 = j(\boldsymbol{z})$. Then

(a) $j(t) = \overline{j(1-t)}$ (where bar indicates the complex conjugate);

(b) $j(Q_i)$ intersects the real line only at $t = 0$, $\frac{1}{2}$, and 1;

(c) if $t = 0, 1$, then $j(t) = j_0 \in (1, \infty)$;

(d) if $t = \frac{1}{2}$, then $j(t)$ is in the interval $(0,1)$ or $(\infty, 0)$;

(e) $j(Q_i)$ is symmetric with respect to the real line, and is in one half plane for $t \in (0, \frac{1}{2})$ and in the other for $t \in (\frac{1}{2}, 1)$.

Since the circles are based at real numbers and are parameterized at a constant rate, the upper part of Q_i evaluated at t is the complex conjugate of the lower part of Q_i evaluated at $1-t$. Since j is an algebraic function of $\boldsymbol{z}$, we have $j(\overline{\boldsymbol{z}}) = \overline{j(\boldsymbol{z})}$. This gives **(a)**.

The preimage of $(1,\infty)$ under $j(\lambda)$ is the real part of the lambda line, and **(c)**, **(d)** are consequences of this. The other points follow.

4.3.3. *Braid Image Computation.* First we select a basepoint for $\mathcal{U}_4$ consisting of four points on the real circle, taking care that their λ value is unramified. Select $z_1 = 0$, $z_2 = 2$, $z_3 = 6$, and $z_4 = \infty$. Then $j(\boldsymbol{z}) = \frac{4}{27}\frac{(8^2-3\cdot 12)^3}{2^2\cdot 4^2\cdot 6^2} = \frac{7^3}{3^5}$, that is,

$$j(\boldsymbol{z}) = \frac{343}{243}.$$

Call this value j_0; it is the basepoint for the image paths.

Set $v(t) = e^{-\pi i t}$ for $t \in [0,1]$ and select specific paths for Q_1 and Q_2:

$$Q_1(t) = (1 - v(t), 1 + v(t), 6, \infty)$$

$$Q_2(t) = (0, 4 - 2v(t), 4 + 2v(t), \infty)$$

Compute the image of Q_1 in $\mathbb{P}^1_j \smallsetminus \{0, 1, \infty\}$ by taking its j values along the path:

$$\begin{aligned} j(Q_1) &= \frac{4}{27}\frac{(8^2 - 3((1-v^2) + (6+6v) + (6-6v)))^3}{(2v)^2(v-5)^2(v+5)^2} \\ &= \frac{1}{27}\frac{(v^2+25)^3}{v^2(v^2-25)^2}. \end{aligned}$$

The intersection of this path with the real line occurs when the first two coordinates are complex conjugate pairs, which happens when $t = \frac{1}{2}$, that is, when $v^2 = -1$. This real intersection is

$$j\,|_{v^2=-1} = \frac{-24^3}{27} \cdot 26^2 < 0.$$

For $t \in (0, \frac{1}{2})$, the path is in either the upper of lower half plane, and for $t \in (\frac{1}{2}, 1)$, it is in the opposite half plane. Thus evaluating j at $t = \frac{1}{4}$ will give the initial direction of the path. When $t = \frac{1}{4}$, $v^2 = -i$, so compute

$$j\,|_{v^2=-i} = \frac{i(25-i)^5}{(25+1)^2},$$

whose imaginary part is positive. So this path leaves j_0, moves leftward through the upper half plane, intersects the real line in the interval $(-\infty, 0)$, and proceeds back towards j_0 in the lower half plane.

Similarly,

$$\begin{aligned} j(Q_2) &= \frac{4}{27}\frac{(8^2 - 3(16-4v^2))^3}{(4+2v)^2(4v)^2(4-2v)^2} \\ &= \frac{1}{27}\frac{(4+3v^2)^3}{v^2(v^2-4)^2}. \end{aligned}$$

Now we have

$$j\,|_{v^2=-4} = \frac{8^3}{27\cdot 4\cdot 5^2},$$

which is in the interval $(0, 1)$, and

$$j\,|_{v^2=-4i} = \frac{i(1-3i)^3(1-i)^2}{27\cdot 4}.$$

Compute the angle θ of this latter quantity:

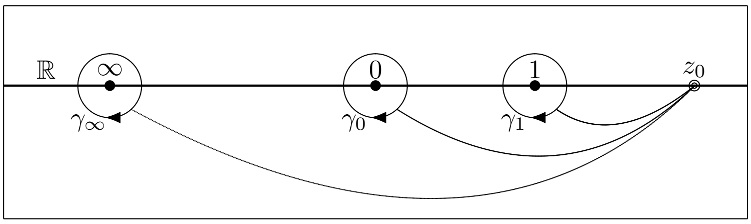
$$\theta = \frac{\pi}{2} - \frac{\pi}{4} - \arctan(3) < \frac{\pi}{4} - \arctan(\sqrt{3}) = -\frac{\pi}{12} < 0.$$

So this path starts in the lower half plane.

The above computations show that the image of the tuple $(Q_1, (Q_2Q_1)^{-1}, Q_2^{-1})$ is a bouquet for $\mathbb{P}^1_j \setminus \{0, 1, \infty\}$ with respect to $((\infty, 0, 1), j_0)$. Modulo the relations in $\bar{M}_4$, we have $Q_2^{-1} = Q_2$ and

$$(Q_1, (Q_2Q_1)^{-1}, Q_2)^{Q_1Q_2Q_1Q_2} = (Q_2, Q_1Q_2, Q_1Q_2Q_1).$$

Let γ_∞, γ_0, and γ_1 be the elements of $\pi_1(\mathbb{P}^1_j \setminus \{0, 1, \infty\}, j_0)$ which are homotopic classes of nonintersecting paths leaving j_0 and traveling in the lower half plane to $(\infty, 0, 1)$ respectively, winding clockwise around each, and returning to j_0 in the lower half plane, as indicated in this diagram.



Primary paths for cover of $\mathcal{J}_4$.

The bouquet $\boldsymbol{\gamma} = (\gamma_\infty, \gamma_0, \gamma_1)$ is homotopic to $(f(Q_1), f(Q_2Q_1)^{-1}, f(Q_2)^{-1})$. Thus the action of $\boldsymbol{\gamma}$ on the fiber of $\mathcal{V}^\bullet \to \mathcal{J}^\bullet$ over j_0, or on a reduced rank four Hurwitz space covering $\mathcal{J}^\bullet$, can be computed as the action of $(Q_2, Q_1Q_2, Q_1Q_2Q_1)$ on reduced tuples of classical generators, or on the reduced Nielsen class.

CHAPTER III

Modular Towers

1. Group Covers

1.1. Group Covers.

1.1.1. *Group Covers.* A *group cover* is a surjective homomorphism $\varphi : H \to G$ between groups. We say that cover is *finite* if H is finite. A *morphism* of group covers from $\psi : I \to G$ to $\varphi : H \to G$ is a surjective homomorphism $\xi : I \to H$ such that $\psi = \varphi \circ \xi$. This produces the category of group covers.

1.1.2. *Group Cover Factors.* Let $\varphi : H \to G$ be a group cover. A *factor* of φ is a group cover $\varphi_1 : H_2 \to H_1$ such that there exist covers $\varphi_2 : H \to H_2$ and $\varphi_0 : H_1 \to G$ with $\varphi = \varphi_0 \circ \varphi_1 \circ \varphi_2$. A factor is *trivial* if it is an isomorphism, and it is *proper* if either φ_2 or φ_0 is nontrivial. Denote the entire sequence by

$$\varphi : H \stackrel{\varphi_2}{\to} H_2 \stackrel{\varphi_1}{\to} H_1 \stackrel{\varphi_0}{\to} G,$$

and call this sequence a *factorization* of φ.

1.1.3. *Lifts of Elements.* Let $\varphi : H \to G$ be a group cover. A *lift* of $g \in G$ is an element $h \in H$ such that $\varphi(h) = g$. Let $K = \ker(\varphi)$, and suppose that K is abelian. Then G acts on K on the right by lifted conjugation, that is, we define $a^g = a^h$ for $a \in K$, where $h \in H$ is any lift of $g \in G$; this is well-defined because K is abelian, producing $G \to \mathrm{Aut}(K)$ which lifts to $H \to \mathrm{Aut}(K)$. Because of this, it makes sense to write $C_K(g)$ to mean $C_K(h)$.

We are interested in understanding the order of h from the order of g. In every case, we know that if $C_K(h) = \{1\}$, then $\mathrm{ord}(h) = \mathrm{ord}(g)$. This is because $h^{\mathrm{ord}(g)} \in C_K(h)$. If K is abelian and $\mathrm{ord}(g)$ is relatively prime to $|K|$, we can use an elementary argument to say more.

Proposition 4. *Let $\varphi : H \to G$ be a finite group cover with abelian kernel K. Let $g \in G$ with* $\gcd(\mathrm{ord}(g), |K|) = 1$. *Then there exists $h \in H$ with $\varphi(h) = g$ and* $\mathrm{ord}(h) = \mathrm{ord}(g)$. *Let $C = C_k(h)$ and let D be a complement of C in K. Then D acts regularly on h^K by conjugation, and the fiber over g is the disjoint union*

$$Kh = \bigsqcup_{a \in C} ah^D,$$

where elements of ah^D have order $\mathrm{ord}(a)\mathrm{ord}(g)$. *In particular, there are $[K : C_k(h)]$ elements of order $m = \mathrm{ord}(g)$ over g, all of which are conjugate.*

PROOF. Let $m = \mathrm{ord}(g)$ and let h be a lift of g; then $h^m \in K$. Assume $h^m = a$ is nontrivial. Since $\gcd(\mathrm{ord}(a), m) = 1$, the map $\langle a\rangle \to \langle a\rangle$ given by $x \mapsto x^m$ is an isomorphism; let $b \in \langle a\rangle$ be the preimage of a, that is, b is the unique m^{th} root of a in $\langle a\rangle$. Then $\varphi(hb^{-1}) = g$, and since h commutes with b we have $(hb^{-1})^m = h^m a^{-1} = a^n = 1$.

Thus select $h \in H$ to be a lift of g of order m. Since C is the kernel of the conjugation action of K on h^k and D acts as K/C, the action of D is faithful and transitive. It is also free, since $h^{d_1} = h^{d_2} \Rightarrow d_1 d_2^{-1} \in C \Rightarrow d_1 = d_2$. Thus the action of D on Kh breaks into $|C|$ orbits, with ah and bh in different orbits for distinct $a, b \in C$. If $a \in C$, then $\gcd(\mathrm{ord}(a), \mathrm{ord}(h)) = 1$, so $\mathrm{ord}(ah) = \mathrm{ord}(a)\mathrm{ord}(g)$. □

1.2. Group Cover Types.

1.2.1. *Elementary Covers.* An *elementary cover* is a group cover $\varphi : H \to G$ such that $\ker(\varphi)$ is an elementary abelian p-group. In this case, $M = \ker(\varphi)$ is a vector space over $\mathbb{F}_p$, and becomes a module for the group algebra $\mathbb{F}_p[G]$. The submodules of M are exactly the subgroups of M which are normal in G, so they describe the factors of the cover. This is the realm of modular representation theory, which we use only indirectly. See [**Fr95**], [**BF02**], and [**Se02**] for discussions of how modular representation theory impacts the theory of Modular Towers.

Let $g \in G$ with $m = \mathrm{ord}(g)$, $p^s = |C_M(g)|$, and $p^r = |K|$. Proposition 4 tells us that if $\gcd(m, p) = 1$, then $\varphi^{-1}(g)$ consists of p^{r-s} elements of order m, all of which are conjugate, and $p^r - p^s$ elements of order mp. What remains to be known is the order of lifts of elements which centralize an element of the kernel and whose order is divisible by p. This depends on the cover, but presently we will discover the answer in an interesting case.

1.2.2. *Central Covers.* A *central cover* is a group cover $\varphi : H \to G$ such that $\ker(\varphi) \leq Z(H)$. Note that a group cover with abelian kernel is central if and only if the action of G on $\ker(\varphi)$ is trivial.

Let G be a finitely presented group, where F is a free group of rank r and $R \triangleleft F$ with $G = F/R$. The *Schur multiplier* of G is

$$M(G) = \frac{[F,F] \cap R}{[F,R]}.$$

Up to isomorphism, this is independent of the presentation (see [**Ro93**] Section 11.4).

We would like to view $M(G)$ as the kernel of a group cover. One way to do this is to set

$$S(G) = \frac{[F,F]R}{[F,R]}.$$

Then the image of R in $S(G)$ is central, and the canonical map $\varphi : S(G) \to G$ has kernel $M(G)$. The image of φ in G is the commutator subgroup of G. If G is a perfect group, then φ is surjective, and is known as the *universal central extension* of G.

Actually one can define a cover $\varphi : S \to G$ with $\ker(\varphi) \cong M(G)$ and $\ker(\varphi) \leq Z(S) \cap [S,S]$ whenever $\gcd(|G/[G,G]|, |M(G)|) = 1$, which is unique up to isomorphism (see [**Ro93**] Exercise 11.4.15). For our purposes it is easier to work with Frattini covers.

1.2.3. *Frattini Covers.* A *Frattini cover* is a group cover $\varphi : H \to G$ with the property that no proper subgroup of H maps onto G. A group homomorphism $\varphi : H \to G$ if a Frattini cover if and only if any set of generators for G lift to generators for G. A Frattini cover is *totally nonsplit*, in the sense that no nontrivial factor of it splits. The study of a Frattini cover of a finite group produces information intrinsic to the group, yet previously hidden from view.

Let H_1, H_2, and G be finite groups, and let $\varphi_1 : H_1 \to G$ and $\varphi_2 : H_2 \to G$ be Frattini covers. Let $\varphi : H_1 \times_G H_2 \to G$ denote the fiber product. Select a minimal subgroup $H \leq H_1 \times_G H_2$ which maps surjectively to G, so that $\varphi \restriction_H : H \to G$ is a Frattini cover with φ_1 and φ_2 as factors. This construction tempts one to form a projective system of Frattini covers of G. We wish to obtain universal objects for covers of finite groups. In order to do this, we must pass to a larger category.

1.3. Universal Frattini Covers.

1.3.1. *Profinite Groups.* A *profinite group* is the projective limit, in the category of topological groups, of a system of finite groups endowed with the discrete topology. Such a limit always exists, and can be explicitly constructed (see [**FJ86**] Section 1.2). We obtain a compact topological group which is Hausdorff; in such a group, a subgroup is open if and only if it is a closed subgroup of finite index. An abstract compact topological group may be recognized as profinite if it has a basis of open subgroups whose intersection is trivial; thus a closed subgroup of a profinite group is profinite. A *morphism* of profinite groups is a continuous group homomorphism whose kernel is closed; this gives the category of profinite groups.

Let $\mathcal{C}$ be a subcategory of the category of finite groups. Then $\mathcal{C}$ induces the subcategory of pro-$\mathcal{C}$ groups as those profinite groups whose finite quotients are in $\mathcal{C}$. This gives the meaning of pro-p, pronilpotent, procyclic, and so forth. A maximal pro-p subgroup of a profinite group is called a p-Sylow subgroup. These exist by Zorn's Lemma. We say that a profinite group G is a *pro-*$\mathcal{C}$ *profree* (respectively *pro-*$\mathcal{C}$ *projective*) group if it is free (respectively projective) in the category of pro-$\mathcal{C}$ groups. Profree groups are projective.

THEOREM 5. *Profinite groups have these properties:*

- **(a)** *An epimorphism from a finitely generated profinite group to itself is an automorphism.*
- **(b)** *An open subgroup of a profree profinite group is profree.*
- **(c)** *A closed subgroup of a projective profinite group is projective.*
- **(d)** *The p-Sylows of a profinite group are closed and conjugate to each other.*
- **(e)** *All p-Sylows of a pronilpotent group are normal.*
- **(f)** *A pro-P group is projective if and only if it is profree pro-p.*

Proof. All proofs are in [**FJ86**], as follows: **(a)** is Proposition 15.3, **(b)** is Proposition 15.27, **(c)** is Corollary 20.14, **(d)** is Proposition 20.43, **(e)** is Proposition 20.44, and **(f)** is Proposition 20.37. □

1.3.2. *Frattini Subgroups.* The *Frattini subgroup* of a profinite group G is the intersection of all open maximal subgroups of G, and is denoted $\Phi(G)$. This is the set of nongenerators of G, in the sense that any set of generators will still generate after any elements from the Frattini subgroup are removed. Moreover, the Frattini subgroup is pronilpotent; that is, all of its maximal pro-p subgroups are normal, so it is isomorphic to the direct product of its p-Sylow subgroups.

A homomorphism $\varphi : H \to G$ between profinite groups is a Frattini cover if and only if $\ker(\varphi) \leq \Phi(H)$; hence the name.

A *p-Frattini cover* is a Frattini cover whose kernel is a pro-p group. Since the kernel of a Frattini cover is nilpotent, such a cover is the fiber product of p-Frattini covers.

1.3.3. *Universal Frattini Cover.* The *universal Frattini cover* of a finite group G is a Frattini cover $\psi : \tilde{G} \to G$ which is versally repelling in the category of Frattini covers of G. View this as the projective limit of all finite Frattini covers of G. Such an object always exists, and is unique up to isomorphism, although it admits nontrivial automorphisms.

To see that the universal Frattini cover always exists, let r be the rank of G, that is, r is the minimal number of generators for G. Let $\tilde{F}_r$ be the free profinite group on r generators. Map the generators for $\tilde{F}_r$ to generators for G. Select a minimal subgroup $\tilde{G}$ of $\tilde{F}_r$ which maps surjectively onto G. Since $\tilde{F}_r$ is free, $\tilde{G}$ is projective (in fact, we may characterize the universal Frattini cover as the minimal projective cover; see [**FJ86**] Proposition 20.33). If we select different generators in G, we obtain another cover, say by group $\tilde{G}^*$. Use the projective property plus the Frattini property to obtain surjective maps $\tilde{G} \to \tilde{G}^*$ and $\tilde{G}^* \to \tilde{G}$. Since these groups are profinite, these maps must be isomorphisms.

1.3.4. *Universal p-Frattini Cover.* Let G be a finite group and let $\psi : \tilde{G} \to G$ be the universal Frattini cover of G. Then $\Phi(\tilde{G}) = \psi^{-1}(\Phi(G))$. Since $\ker(\psi)$ is a subgroup of a pronilpotent group, it is also pronilpotent, and is the product of its pro-Sylow subgroups. The primes p which divide the order of G are exactly those that contribute nontrivial portions $\ker(\psi)$.

Let p be prime and let ${}_p\tilde{G}$ denote the quotient of $\tilde{G}$ by the product of the Sylow q-subgroups of $\ker(\psi)$, where q is prime to p. Then ψ factors through $\varphi : {}_p\tilde{G} \to G$; this is the *universal p-Frattini cover* of G. Let $K = \ker(\varphi)$; this is a pro-p group. If p does not divide the order of G, then K is trivial; assume p divides the order of G.

Let P be a p-Sylow subgroup of G. Then $\tilde{P} = \varphi^{-1}(P)$ is a p-Sylow subgroup of $\tilde{G}$, and $[\tilde{P} : K] = |P|$. In particular, K is a closed subgroup of finite index, and since $\tilde{G}$ is projective, K is a profree pro-p group.

Set $\ker_0 = K$, and inductively define

$$\ker_{i+1} = \ker_i^p[\ker_i, \ker_i],$$

where by convention we take the closed normal subgroup generated by these elements. Set ${}_p^0\tilde{G} = G$, and define ${}_p^i\tilde{G} = {}_p\tilde{G}/\ker_i$. This gives a sequence of finite groups

$$\cdots \to {}_p^{i+1}\tilde{G} \to {}_p^i\tilde{G} \to \cdots \to G$$

such that the kernel between successive steps is an elementary abelian p-group. The universal p-Frattini cover of ${}_p^i\tilde{G}$ is ${}_p\tilde{G}$ for each i in this sequence.

1.3.5. *Universal Elementary p-Frattini Cover.* It is often convenient to change notation and set $G_k = {}_p^k\tilde{G}$. Consider the group cover $G_{k+1} \to G_k$, and label its kernel M_k, so that $M_k = \ker_k/\ker_{k+1}$. This cover is universal for Frattini covers of G_k with elementary abelian p-group kernel, and is referred to as the *universal elementary p-Frattini cover* of G_k. View M_k as an $\mathbb{F}_p$ vector space with a G_k action given by lifted conjugation, that is, M_k is an $\mathbb{F}_p[G_k]$ module, which we refer to as the *universal elementary p-Frattini module* of G_k.

1.3.6. *Subgroup Frattini Principle.* Let G be a finite group with universal p-Frattini cover $\varphi : {}_p\tilde{G} \to G$. Let $H \leq G$; then $\varphi^{-1}(H)$ is a closed subgroup of ${}_p\tilde{G}$, and so it is projective; the map $\varphi\restriction_{\varphi^{-1}(H)}: \varphi^{-1}(H) \to H$ lifts to a map $\psi : \varphi^{-1}(H) \to {}_p\tilde{H}$, which is necessarily surjective and maps $\ker_0(G)$ onto $\ker_0(H)$. This induces a surjective homomorphism $M_0(G) \to M_0(H)$, which is an isomorphism of $\mathbb{F}_p[H]$ modules if and only if ψ is an isomorphism of profinite groups (see [**FK97**] Subgroup Frattini Principle 2.3).

Let $\varphi : {}_p^1\tilde{G} \to G$ be the universal elementary p-Frattini cover. The considerations above induce a surjective homomorphism $\psi : \varphi^{-1}(H) \to {}_p^1\tilde{H}$ such that $\psi^{-1}(M_0(H)) = M_0(G)$. Use this is find the order of lifts of elements in G; compare the following with Proposition 4.

Proposition 6. *Let G be a finite group and let $\varphi : {}_p^1\tilde{G} \to G$ be its universal elementary p-Frattini cover. Let $g \in G$ be of order pm and let $\tilde{g} \in {}_p^1\tilde{G}$ with $\varphi(\tilde{g}) = g$. Then* $\operatorname{ord}(\tilde{g}) = p^2m$.

Proof. Without loss of generality, we may assume that $m = 1$, so that $H = \langle g \rangle$ is cyclic of order p. Clearly $\operatorname{ord}(\tilde{g}) \leq p^2$. The universal elementary p-Frattini cover of H is cyclic of order p^2, say ${}_p^1\tilde{H} = \langle x \rangle$, with kernel $\langle x^2 \rangle$. Now $\varphi^{-1}(H)$ maps surjectively onto this with $\tilde{g}$ not in $\langle x^2 \rangle$. Thus p^2 divides the order of $\tilde{g}$, and hence equals it. □

1.3.7. *Split Groups.* Let $G = K \rtimes H$, with $\gcd(|K|, |H|) = 1$. Then the universal Frattini cover of G is $\tilde{G} \cong \tilde{K} \rtimes \tilde{H}$. This remains true for the universal p-Frattini covers; that is ${}_p\tilde{G} \cong {}_p\tilde{K} \rtimes {}_p\tilde{H}$ (see [**Ri85**] Theorem 3.2).

Let p be a prime dividing the order of G and let $P \leq G$ be a Sylow p-subgroup of G. Suppose that $P \lhd G$. Then $G = P \rtimes H$, where $H \cong G/P$ has order relatively prime to p, and ${}_p\tilde{H} \cong H$. The universal p-Frattini cover of a p-group is ${}_p\tilde{F}_t$, where t is the rank of the group. Thus ${}_p\tilde{G} \cong {}_p\tilde{F}_t \rtimes H$.

The profinite Nielsen-Scheier formula reveals that the kernel of the map ${}_p\tilde{F}_t \to P$ is a profree pro-p group on $1 + (t-1)|P|$ generators (see [**FJ86**] Proposition 15.27). Thus the universal elementary p-Frattini module of P is a vector space over $\mathbb{F}_p$ of this dimension.

1.4. Central Frattini Covers.

1.4.1. *Universal Central Frattini Cover.* Let G be a finite group and let $\tilde{\varphi} : \tilde{G} \to G$ be its universal Frattini cover, with $\ker(\varphi) = K$. Set

$$\hat{G} = \frac{\tilde{G}}{[\tilde{G}, K]}.$$

The induced map $\hat{\varphi} : \hat{G} \to G$ is called the *universal central Frattini cover.* Its kernel $\ker(\hat{\varphi}) = K/[\tilde{G}, K]$ is called the kernel *universal central Frattini kernel.* These properties are clear from the construction:

- **(a)** $\hat{\varphi}$ is a Frattini cover;
- **(b)** $\ker(\hat{\varphi}) \leq Z(\hat{G})$;
- **(c)** $\hat{\varphi}$ is versally repelling for Frattini covers of G with central kernels.

Let $\tilde{\xi} : \tilde{G} \to \hat{G}$ be the canonical homomorphism. Let X be a minimal set of generators for $\tilde{G}$; note X is finite. Let F be the free group on X, and let $\iota : F \to \tilde{G}$ be the induced homomorphism. Let $R = \ker(\tilde{\varphi} \circ \iota)$. Then $\ker(\tilde{\xi} \circ \iota) = [F, R]$, and since $\hat{G}$ is finite, we have $\hat{G} \cong F/[F, R]$. The image of $([F, F] \cap R)/[F, R]$ in $\hat{G}$ is $([\tilde{G}, \tilde{G}] \cap K)/[\tilde{G}, K]$; this is the part of the universal central Frattini kernel which comes for the Schur multiplier. If G is perfect, the universal central Frattini kernel is the Schur multiplier, and the universal central extension is the universal central Frattini cover.

1.4.2. *Universal Central Elementary p-Frattini Cover.* Let $\tilde{\varphi} : {}_p\tilde{G} \to G$ be the universal p-Frattini cover of G, and let $K = \ker(\tilde{\varphi})$. Set

$${}_p\hat{G} = \frac{{}_p\tilde{G}}{K^p[{}_p\tilde{G}, K]},$$

and let $\hat{\varphi} : {}_p\hat{G} \to G$ be the induced map. Call this the *universal central elementary p-Frattini cover* of G. Its kernel is called the *universal central elementary p-Frattini kernel* of G. Then

- **(a)** $\hat{\varphi}$ is a Frattini cover;
- **(b)** $\ker(\hat{\varphi}) \leq Z(\hat{G})$ is an elementary p-group;
- **(c)** $\hat{\varphi}$ is versally repelling for Frattini covers of G with central elementary p-group kernels.

We say that G is *p-perfect* if $\gcd([G : [G, G]], p) = 1$. If G is p-perfect, then the universal central elementary p-Frattini kernel of G is the maximal elementary p-group quotient of the Schur multiplier.

1.4.3. *Antecedent Central Elementary p-Frattini Cover.* Recall $\ker_{k+1} = \ker_k^p[\ker_k, \ker_k]$. Set $\ker_k^* = \ker_k^p[\tilde{G}, \ker_k]$, and $\ker_k' = (\ker^*)^p[\ker_k, \ker_k^*]$. Obtain a map $\ker_k/\ker_k^* \to \ker_{k+1}/\ker_{k+1}'$ by $x \mapsto x^p$. This is injective (see [**BF02**] Proposition 9.6 and [**FK97**] Schur Multipliers Result 3.3);

denote the pullback of the image to ${}_p\tilde{G}$ by $\ker''_{k+1}$. Then

$$\ker^*_{k+1} \leq \ker'_{k+1} \leq \ker''_{k+1} \leq \ker_{k+1} \leq \ker^*_k \leq \ker_k.$$

We call $\ker''_{k+1}/\ker_{k+1}$ the *antecedent* of the universal central elementary p-Frattini kernel at level k. When G is p-perfect, this is the part of the elementary p-group quotient of Schur multiplier which is induced from the previous level.

With $\ker''_0 = \ker^*_0$, set

$$ {}^k_p\hat{G}^* = \frac{{}_p\tilde{G}}{\ker''_k}; $$

we call ${}^k_p\hat{G}^* \to {}^k_p\tilde{G}$ the *antecedent central elementary p-Frattini cover* at level k.

Let $G_k = {}^k_p\tilde{G}$. Let $\hat{G}_k \to G_k$ be the universal central elementary p-Frattini cover of G_k. Let $M_k = \ker(G_{k+1} \to G_k)$ and $V_k = \ker(G_{k+1} \to \hat{G}_k)$. Then the antecedent $\hat{G}^*_{k+1} \to G_{k+1}$ is characterized as the central Frattini cover of G_{k+1} with the property that the elements of M_k which lift in $\hat{G}^*_{k+1}$ to have order p are exactly those in V_k.

2. Factored Covers

2.1. Factored Topological Covers.

2.1.1. *Factored Topological Covers.* Let $\psi : Z \to X$ and $\varphi : Y \to X$ be topological covers. A *factored topological cover* is a strong morphism from ψ to φ; that is, it is a map $\xi : Z \to Y$ such that $\psi = \varphi \circ \xi$, in which case ξ is necessarily a topological cover. Given φ and ξ, we construct ψ by composition, and given ψ and ξ, we construct φ via $\varphi = \psi \circ \xi^{-1}$, which is well-defined. However, given ψ and φ, $\mathrm{Aut}(\varphi)$ acts regularly on the set of possible ξ's which satisfy $\psi = \varphi \circ \xi$; yet all such ξ's are equivalent as covers. Use notation

$$\psi : Z \xrightarrow{\xi} Y \xrightarrow{\varphi} X,$$

or (ψ, ξ), to describe the factored cover.

2.1.2. *Automorphism Group Homomorphisms.* Let $\psi : Z \to X$ be a normal cover. Each subgroup H of $\pi_1(X, x_0)$ containing $K = \psi_*(\pi_1(Z, z_0))$ acts discretely on Z to produce covers $\xi : Z \to Y = \bar{Z}$ and $\varphi : Y \to X$, with $\psi = \varphi \circ \xi$. Then ξ is a normal cover with $\mathrm{Aut}(\xi) \cong H/K$, and φ equivalent to the cover produced by H as above. We may view $\mathrm{Aut}(\xi)$ as the subgroup of $\mathrm{Aut}(\psi)$. Then φ is a normal cover if and only if $\mathrm{Aut}(\xi) \triangleleft \mathrm{Aut}(\psi)$, in which case the map $\xi_* : \mathrm{Aut}(\psi) \to \mathrm{Aut}(\varphi)$ given by $\alpha \mapsto \xi \circ \alpha \circ \xi^{-1}$ is well defined with kernel $\mathrm{Aut}(\xi)$, and $\mathrm{Aut}(\varphi) \cong \mathrm{Aut}(\psi)/\mathrm{Aut}(\xi)$. Otherwise, the conjugates of $\mathrm{Aut}(\xi)$ in $\mathrm{Aut}(\psi)$ produce equivalent covers. We have an order reversing bijection between conjugacy classes of subgroups of $\pi_1(X, x_0)$ containing $\psi_*(\pi_1(Z, z_0))$ and equivalence classes of covers of X through which ψ factors. These conjugacy classes of subgroups correspond to conjugacy classes of subgroups of the automorphism group.

2.1.3. *Monodromy Group Homomorphisms.* If $\psi : Z \to X$ is not normal, the correspondence is between conjugacy classes of subgroups of $\mathrm{Mon}(\psi)$ and equivalence classes of covers of X through which the normal closure $\hat{\psi} : \hat{Z} \to X$ factors. A cover of X is a factor of ψ if and only if the corresponding subgroup of $\mathrm{Mon}(\psi)$ is contained in $\mathrm{Stb}(\psi)$ (the stabilizer).

Consider a factored cover $\psi : Z \xrightarrow{\xi} Y \xrightarrow{\varphi} X$. Let $n = \deg(\psi)$, $m = \deg(\varphi)$, and $d = \deg(\xi)$, with $n = md$. Select basepoints $x_0 \in X$, $y_0 \in \varphi^{-1}(x_0)$, and $z_0 \in \xi^{-1}(y_0)$. The core of $\varphi_*(\pi_1(Y, y_0))$ in $\pi_1(X, x_0)$ contains the core of $\psi_*(\pi_1(Z, z_0))$ in $\pi_1(X, x_0)$, inducing a canonical homomorphism $\mathrm{Mon}(\psi) \to \mathrm{Mon}(\varphi)$. For computation, we view these monodromy groups as embedded in permutation groups.

Let $Y_{x_0} = \varphi^{-1}(x_0)$ and $Z_{x_0} = \psi^{-1}(x_0)$ be the fibers over the basepoint. Enumerate these sets so that z_0 and y_0 correspond to 1. This produces a function $e : \mathbb{N}_n \to \mathbb{N}_m$ induced by restriction of ξ to the fibers. It also produces monodromy representations $T_\psi : \pi_1(X, x_0) \to S_n$ and $T_\varphi : \pi_1(X, x_0) \to S_m$. Set $H = T_\psi(\pi_1(X, x_0))$, $V = T_\psi(\psi_*(\pi_1(Z, z_0)))$, $G = T_\varphi(\pi_1(X, x_0))$, and $U = T_\varphi(\varphi_*(\pi_1(Y, y_0)))$. Thus $V = \mathrm{Stb}_H(1)$ and $U = \mathrm{Stb}_G(1)$.

The function e induces a homomorphism $f : H \to G$ which produces a morphism of group actions; that is, with H and G acting on the right of $\mathbb{N}_n$ and $\mathbb{N}_m$ respectively, we have $e(ih) = e(i)f(h)$. This satisfies $f(V) \leq U$. Let $K = \ker(g)$; since V is coreless in H, we have $V \cap K = \{1\}$, so $f \restriction_V : V \to U$ is injective. This implies that $|H|/n \leq |G|/m$, and that $|K| \leq d$.

Let $T = f^{-1}(U)$ so that $K = K_H(T)$. Note that T is the setwise stabilizer in H of $e^{-1}(1)$. If $\psi_*(\pi_1(Z), z_0) \leq \ker(T_\varphi)$, then T corresponds to the cover φ, and K corresponds to its normal closure. The monodromy group of ξ is isomorphic to $T/K_T(V)$, and its action is given by the action of T on $e^{-1}(1)$, or equivalently, on the right cosets of V in T.

2.2. Factored Ramified Covers.

2.2.1. *Factored Ramified Covers.* A *factored ramified cover* is a sequence $\psi : Z \xrightarrow{\xi} Y \xrightarrow{\xi} X$ of nonconstant analytic maps between compact connected Riemann surfaces. If $\mathrm{Bpt}(\psi)$ is the set of branch points of ψ, remove them from X and their preimages from Y and Z to obtain a factored topological cover $\psi^\circ : Z^\circ \xrightarrow{\xi^\circ} Y^\circ \xrightarrow{\varphi^\circ} X^\circ$.

Clearly we have a containment of the branch points $\mathrm{Bpt}(\varphi) \subset \mathrm{Bpt}(\psi)$. If $\mathrm{Bpt}(\varphi) = \mathrm{Bpt}(\psi)$, then call (ψ, ξ) *conservative*, because the branch point set is conserved. This is the primary situation in this dissertation. The opposite condition is $\mathrm{Bpt}(\xi) \cap \varphi^{-1}(\mathrm{Bpt}(\varphi)) = \varnothing$, to which we give the moniker *liberal*. These definitions are of most interest in the case the Y (and therefore X) is of genus zero.

2.2.2. *Branch Cycle Descriptions of Factored Covers.* Let $\psi : Z \xrightarrow{\xi} Y \xrightarrow{\varphi} X$ be a factored ramified cover such that Y is of genus zero. Then $\xi : Z \to Y$ has a branch cycle description. Given any two of ψ, φ, and ξ, the equivalence class of the third is completely determined. Thus, its branch cycle description with respect to a given bouquet is also determined, up to equivalence.

The monodromy group of ξ can be computed as specified in the previous section. However, this process does not find appropriate generators for the monodromy group which will lead to a branch cycle description for ξ.

PROBLEM 7. *Given branch cycle descriptions for any two of ψ, φ, and ξ, find a branch cycle description of the third.*

Here, the phrase branch cycle description includes the classical generators and the branch cycles, so that the cover is determined. Thus, this problem incorporates the following problem.

PROBLEM 8. *Let $\boldsymbol{x}$ be a tuple of points in $\mathbb{P}^1$ and let $\boldsymbol{\lambda}$ be classical tuple about $\boldsymbol{x}$. Let $\varphi : Y \to X$ be a ramified cover such that the genus of Y is zero, and let $\boldsymbol{g}$ be the branch cycle description for φ with respect to $\boldsymbol{\lambda}$. Find generators for $\varphi_*(\pi_1(Y, y_0))$, written in terms of $\boldsymbol{\lambda}$, which lift to a classical tuple on Y.*

We will discuss these problems further in chapter V.

2.2.3. *Branch Cycle Descriptions from Monodromy Homomorphisms.* Let $\psi : Z \to \mathbb{P}^1$ be a ramified cover of degree n whose branch cycle description is $\boldsymbol{h} = (h_1, \ldots, h_r)$ with respect to classical generators $\boldsymbol{\lambda}$. Let $H = \langle \boldsymbol{h} \rangle \leq S_n$, and let $V = \mathrm{Stb}_H(1)$. Let $f : H \to G$ be a group homomorphism, where $G \leq S_m$, $U = \mathrm{Stb}_G(1)$, and $f(V) \leq U$. Then there exists a function $e : \mathbb{N}_n \to \mathbb{N}_m$ such that f is induced by e; define $e(j) = i$ if $1 \cdot h = j$ and $1 \cdot f(h) = i$.

The pullback of U through f and then back to the fundamental group produces a cover $\varphi : Y \to \mathbb{P}^1$ whose branch cycle description, with respect to $\boldsymbol{\lambda}$, is $(f(h_1), \ldots, f(h_r))$, such that there exists an analytic map $\xi : Z \to Y$ with $\psi = \varphi \circ \xi$.

Suppose we are given ψ as above and $\xi : Z \to Y$. Then ξ induces a block system for the action of $\mathrm{Mon}(\psi)$, which in turn produces the functions e and f, and a branch cycle description for a cover $\varphi : Y \to \mathbb{P}^1$ with $\psi = \varphi \circ \xi$. This is the easy case of the Problem 7.

3. Moduli of Elliptic Curves

3.1. Elliptic Curves.

3.1.1. *Elliptic Curves.* An *elliptic curve* (E, e) is a topological torus E endowed with a complex structure, together with a specified basepoint e. Topologically, the universal cover of E is a plane, and the cover induces a uniquely determined complex structure on the plane; this complex structure determines a coordinate system for the plane which is unique up to the action of

$$\mathrm{Hol}(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} \mid f(z) = az + b \text{ for some } a \in \mathbb{C}^*,\, b \in \mathbb{C}\}.$$

Select a coordinate z so that the universal cover $\xi : \mathbb{C}_z \to E$ satisfies $\xi(0) = e$. Any other such choice differs from this one by multiplication by some $a \in \mathbb{C}^*$.

Let $y \in E$ and let $z_1, z_2 \in \xi^{-1}(y)$. Since ξ is a normal cover, there exists a unique automorphism $\mu \in \text{Aut}(\xi)$ such that $\mu(z_1) = z_2$; since $\xi \circ \mu = \xi$, μ is necessarily an holomorphic isomorphism, so $\mu(z) = az + b$ for some $a, b \in \mathbb{C}$. If $a \neq 1$, then $\frac{b}{1-a}$ is a fixed point of μ, and any automorphism of a cover with a fixed point is the identity; take $a = 1$. Thus $z_2 = \mu(z_1) = z_1 + b$, so $z_2 - z_1 = b$.

Let $L = \xi^{-1}(e)$; then $\mu(0) = b$, so $b \in L$. Thus L is a discrete additive subgroup of $\mathbb{C}$, and the fibers of ξ are the cosets of L in $\mathbb{C}$. This canonically produces an abelian group structure on E such that ξ is a group homomorphism. Moreover, $L \cong \text{Aut}(\xi) \cong \pi_1(E, e) \cong \mathbb{Z} \times \mathbb{Z}$.

A *lattice* in $\mathbb{R}^n$ is the free abelian subgroup of the additive group $\mathbb{R}^n$ generated by a basis for $\mathbb{R}^n$. Thus a lattice in $\mathbb{C}$ is any discrete free abelian group of rank two, and $L = \xi^{-1}(e)$ is a lattice in $\mathbb{C}$. Conversely, given a lattice L in $\mathbb{C}$, we see that $\mathbb{C}/L$ is a topological torus with an holomorphic group structure, and this structure is precisely that which would be given by the above process.

3.1.2. *Isogenies.* An *isogeny* between elliptic curves is a nonconstant holomorphic map $\varphi : E_2 \to E_1$ which sends the origin to the origin. This is necessarily surjective, and the Riemann Hurwitz formula dictates that it is unramified.

A universal cover $\xi : \mathbb{C} \to E_2$ composes with φ to give a universal cover $\psi : \mathbb{C} \to E_1$; if L_2 and L_1 are the lattices thus produced, it is clear that $L_1 \leq L_2$ as a subgroup, and that φ may be viewed as the canonical homomorphism $\mathbb{C}/L_2 \to \mathbb{C}/L_1$. If ω_1 and ω_2 are generators for L_1, then there exist $m, n \in \mathbb{Z}$ such that $m\omega_1$ and $n\omega_2$ are generators for L_2. Thus φ is a normal cover of degree mn with group $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Isogenies become the obvious morphisms for a category of elliptic curves. From the construction, one sees that two lattices produce isomorphic elliptic curves if and only if one lattice is a complex scalar multiple of the other.

3.1.3. *Automorphisms of Elliptic Curves.* Let E be an elliptic curve given by a lattice L. An automorphism of E descends from an automorphism of $\mathbb{C}$ given by scalar multiplication by some $a \in \mathbb{C}$ which preserves L, so we may view the automorphism group as a subgroup of $\mathbb{C}^*$. Since $\text{Aut}(E)$ permutes the points of L which have minimal distance to the origin, $\text{Aut}(E)$ is finite, and so is a finite subgroup of the unit circle $\mathbb{U}$, and thus cyclic. Since multiplication by -1 is an automorphism, the automorphism group has even order. Let $a = e^{\pi i/n}$ be a generator for $\text{Aut}(E)$.

Without loss of generality, suppose that L is generated by $\{1, \tau\}$, where 1 is the minimal distance to the origin among nontrivial points in L. Then $\langle a \rangle \subset L$, and $\text{Aut}(E) = \langle a \rangle = L \cap \mathbb{U}$.

Typically, $|\tau| > 1$, so that $a = -1$ and $|\text{Aut}(E)| = 2$. Otherwise we may take $\tau = a$. Since $|e^{\pi i/n} - 1| < 1$ for $n \geq 4$, either $n = 2$ or $n = 3$. In these special cases, we respectively have $\tau = i$ and $|\text{Aut}(E)| = 4$, or $\tau = \frac{1+i\sqrt{3}}{2}$ and $|\text{Aut}(E)| = 6$.

3.2. Moduli of Elliptic Curves.

3.2.1. *Isomorphism Classes of Elliptic Curves.* Let $\mathcal{E}$ denote the set of all isomorphism classes of elliptic curves, $\mathcal{L}$ the set of all lattices in $\mathbb{C}$, and $\mathcal{G}$ the set of all unordered pairs $\{\omega_1, \omega_2\}$ of generators for lattices in $\mathbb{C}$. We have a sequence of maps $\mathcal{G} \to \mathcal{L} \to \mathcal{E}$. Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{G}}$ denote these sets modulo the action of $\mathbb{C}^*$, producing a well-defined sequence $\overline{\mathcal{G}} \to \overline{\mathcal{L}} \to \mathcal{E}$, where the latter map is bijective.

Let $\mathbb{H}$ denote the set of complex numbers with positive imaginary parts. For $\{\omega_1, \omega_2\} \in \mathcal{G}$, the ratio ω_2/ω_1 is nonreal, and τ is in the upper half plane if and only if τ^{-1} is in the lower half plane. Thus we identify $\mathcal{G}$ with the set of ordered pairs (ω_1, ω_2) such that ω_2/ω_1 is in $\mathbb{H}$, giving an injective map $\mathcal{G} \to \mathbb{C}^2$. The action of $\mathbb{C}^*$ projectivizes this, producing $\mathcal{G} \to \mathbb{H}_\tau \hookrightarrow \mathbb{P}^1_\tau$ given by $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2 = \tau$. Each element of $\overline{\mathcal{G}}$ may be written uniquely in the form $[1, \tau]$; thus $\overline{\mathcal{G}} \to \mathbb{H}_\tau$ is a bijection, which places a complex structure on $\overline{\mathcal{G}}$.

The group of holomorphic self homeomorphisms of the upper half plane is $\mathrm{Hol}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$, via the action of $\mathrm{PSL}_2(\mathbb{R})$ on projective points of the form $[\omega_1, \omega_2]$. This group acts transitively on $\mathbb{H}$, and this descends to an action on $\overline{\mathcal{L}}$. Two bases for $\mathbb{R}^2$ generate the same lattice if and only if they are related by an invertible matrix with integer coefficients, so the kernel of this action is $\mathrm{PSL}_2(\mathbb{Z})$. Thus $\overline{\mathcal{G}}/\mathrm{PSL}_2(\mathbb{Z})$ is identified with $\overline{\mathcal{L}}$, and the category of elliptic curves is parameterized by the upper half plane modulo the action of $\mathrm{PSL}_2(\mathbb{Z})$.

3.2.2. *The λ-line.* Let $\tau \in \mathbb{H}$, L the lattice generated by $\{1, \tau\}$, and $E = \mathbb{C}/L$, with $\xi : \mathbb{C} \to \mathbb{C}/L$ the natural homomorphism. The elements of order two in E are $\xi(1/2)$, $\xi(\tau/2)$, and $\xi((1+\tau)/2)$, generating a Klein four group $K \leq E$. The map $\iota : E \to E$ given by $y \mapsto -y$ is an holomorphic group automorphism, and the action of ι on $E \setminus K$ is discrete. Thus the quotient of this action is a Riemann surface punctured at four points, which the Riemann Hurwitz formula dictates to be of genus 0. We obtain a ramified cover $\varphi : E \to \mathbb{P}^1$ of degree two with four branch points.

Let $\psi : \mathbb{C} \to \mathbb{P}^1$ be given by $\psi = \varphi \circ \xi$. Choose a coordinate x for $\mathbb{P}^1$ so that φ, and thus ψ, are holomorphic; any other choice for x differs by an element of $\mathrm{Hol}(\mathbb{P}^1) = \mathrm{PSL}_2(\mathbb{C})$. Since $\mathrm{PSL}_2(\mathbb{C})$ is sharply three transitive, we may adjust x, as it is traditional to do, so that $\psi(0) = \infty$, $\psi(1/2) = 1$, and $\psi(\tau/2) = 0$. Denote the image of $\psi((1+\tau)/2)$ by λ. This produces a well-defined surjective map $\lambda(\tau) : \mathbb{H}_\tau \to \mathbb{C} \setminus \{0, 1\}$, which is holomorphic. We refer the closure of the image as the *λ-line*, denoted by $\mathbb{P}^1_\lambda$.

Let $f(x) = x(x-1)(x-\lambda)$ and consider $V = \{(x, w) \in \mathbb{C}^2 \mid w^2 - f(x)\}$. Project V onto $\mathbb{P}^1_x$ and compactify to obtain a ramified cover $V^\bullet \to \mathbb{P}^1_x$. This ramified cover has the same branch cycle description as φ, and so it is equivalent to φ; in particular, $E \cong V^\bullet$, which induces the structure of an algebraic variety on E.

3.2.3. *The j-line.* Let E be an elliptic curve given uniquely by an equivalence class of lattices $\overline{L} \in \overline{\mathcal{L}}$. It is possible to select representatives of $\overline{L}$ given by generators $(1, \tau)$ such that any of the three points of order two in E is the image of any of the values $1/2$, $\tau/2$, $(1+\tau)/2$. For all but two exceptional elliptic curves, there are six possible λ values, corresponding to the action of S_3 on the elements of order two in E, This gives an S_3 action on $\mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$. The quotient space is a punctured Riemann sphere whose points correspond to equivalence classes of elliptic curves, and the map to the quotient space is branched over the two exceptions. Placing the exception with an extra order three automorphism at $j = 0$ and the exception with an extra order two automorphism at $j = 1$ completely determines coordinates for the quotient, whose closure we call the *j-line*, denoted by $\mathbb{P}^1_j$. We have a normal ramified cover $j(\lambda) : \mathbb{P}^1_\lambda \to \mathbb{P}^1_j$ with group S_3, produced as a rational function in subsection II.4.1. Composition of this with $\lambda(\tau)$ yields the function $j(\tau) : \mathbb{H} \to \mathbb{P}^1_j$ given by $j(\tau) = j(\lambda(\tau))$.

The *j-invariant* of E is $j(\tau)$. Each isomorphism class of elliptic curves is uniquely identified by its j-invariant, and the j-line is the moduli space of elliptic curves. Specifically, an elliptic curve has a minimal field of definition, which is given by the minimal field of definition of its j-invariant.

3.3. Moduli of Isogenies. The kernel of an isogeny factors into cyclic groups; thus the isogeny itself factors into isogenies with cyclic kernels. Focusing on this case, consider objects (E, N) where E is an elliptic curve and N is a subgroup of E of order n. Define

$$\Gamma_0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{n} \right\}.$$

Given two pairs (E_1, N_1) and (E_2, N_1), there is an isomorphism $E_1 \to E_2$ sending N_1 to N_2 if and only if defining τ's for E_1 and E_2 are in the same orbit of the action of $\Gamma_0(n)$ on the upper half plane. Let $Y_0(n)$ denote the upper half plane modded out by the action of $\Gamma_0(n)$; then $Y_0(n)$ forms a parameter space for equivalence classes of such pairs (E, N). The space $Y_0(n)$ is called an *open modular curve*. We obtain a map $Y_0(n) \to \mathbb{P}^1_j$ by sending the equivalence class of (E, N) to the equivalence class of E; this map dictates a compactification of $Y_0(n)$, which is denoted by $X_0(n)$, by filling in the points over $j = \infty$. Moreover, there is a natural map $X_0(n) \to X_0(m)$ whenever m divides n.

A cyclic group factors into cyclic groups of prime power order, so we might as well take $n = p^r$ for some r. This produces a sequence of open modular curves

$$\cdots \to Y_0(p^{r+1}) \to Y_0(p^r) \to \cdots \to Y_0(p) \to \mathbb{P}^1_j.$$

Let $\xi : E_2 \to E_1$ be an isogeny with $\ker(\xi)$ a cyclic group of order p^r, viewed as an unramified cover between Riemann surfaces. There is exists a cover $\varphi : E_1 \to \mathbb{P}^1$ ramified over four points, and these four points are determined up to the action of $\mathrm{PSL}_2(\mathbb{C})$. Let $\psi = \varphi \circ \xi$; this is a normal cover with normal factors, whose monodromy group is a nonabelian extension of $\mathbb{Z}/p^r$ by $\mathbb{Z}/2$; thus the monodromy group is D_{p^r}. The cover is ramified over four points with order two ramification.

Let G be D_p in the regular representation and let C be the conjugacy class in G of an involution. The branch cycle description of ψ is in the Nielsen class $\mathrm{Ni}(G, C^4)^{\mathrm{to}}$. This gives a map $Y_0(p^r) \to \mathcal{H}(G, C^4)^{\mathrm{ab,rd}}$, which is a holomorphic isomorphism which commutes with the map to $\mathbb{P}^1_j$. In this way, reduced Hurwitz spaces generalize open modular curves.

The group homomorphism $D_{p^{r+1}} \to D_{p^r}$ has a p-group kernel and the property that any lift of the involutions generating D_{p^r} also generate $D_{p^{r+1}}$. The map $Y_0(p^{r+1}) \to Y_0(p^r)$ is identified with

$\mathcal{H}(D_{p^{r+1}}, C^4)^{\text{ab,rd}} \to \mathcal{H}(D_{p^r}, C^4)^{\text{ab,rd}}$, and this latter map may be viewed as coming from the corresponding group homomorphism. The Modular Towers construction generalizes this situation, with any group G replacing D_p, and any conjugacy classes which generate G replacing the involutions.

4. Modular Towers

4.1. Hurwitz Maps.

4.1.1. *Nielsen Maps.* A *Nielsen map* is a function $\delta : \text{Ni}_1 \to \text{Ni}_2$, where Ni_1 and Ni_2 are rank r inner or absolute Nielsen classes, such that for every $\boldsymbol{g} \in \text{Ni}_1$ and every $Q \in H_r$ we have $\delta(\boldsymbol{g}Q) = \delta(\boldsymbol{g})Q$. Thus a Nielsen map is a morphism of H_r actions.

4.1.2. *Hurwitz Maps.* A *Hurwitz map* is a function $\Delta : \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are rank r inner or absolute Hurwitz spaces, which commutes with the maps to $\mathcal{U}_r$. Thus a Hurwitz map is a morphism of topological covers.

It is clear that Nielsen maps produce Hurwitz maps, and vice versa. We may also use this terminology for reduced Nielsen classes and Hurwitz spaces.

The Hurwitz maps of primary interest are those that are induced from morphisms of the ramified covers corresponding to the points on the Hurwitz space. As we have seen, such morphisms come from group covers.

4.2. Hurwitz Covers.

4.2.1. *Inner Hurwitz Covers.* Let $f : H \to G$ be a surjective homomorphism between finite groups. Let $\boldsymbol{D}$ be a collection of conjugacy classes from H and let $\boldsymbol{C} = f(\boldsymbol{D})$. This produces a function between the total Nielsen classes which descends to $\delta : \text{Ni}(H, \boldsymbol{D})^{\text{in}} \to \text{Ni}(G, \boldsymbol{C})^{\text{in}}$. Since f is a homomorphism, this map commutes with the braid action of H_r, so it is a Nielsen map, which induces a Hurwitz map $\Delta : \mathcal{H}(H, \boldsymbol{D})^{\text{in}} \to \mathcal{H}(G, \boldsymbol{C})^{\text{in}}$. Now Δ has the property that $\Delta([\psi]) = [\varphi]$, where $\varphi : Y \to \mathbb{P}^1$ is the static cover induced from $\psi : Z \to \mathbb{P}^1$ by the monodromy homomorphism f. We refer to such a Δ as an *inner Hurwitz cover.*

4.2.2. *Absolute Hurwitz Covers.* Let $H \le S_n$ and $G \le S_m$ be transitive groups and let $V = \text{Stb}_H(1)$ and $U = \text{Stb}_G(1)$. Let $f : H \to G$ be a surjective homomorphism of f such that $f(V) \le U$ and $K = \ker(f)$ is stabilized by $\text{Abs}(H)$. The latter condition is necessary for f to produce an absolute Nielsen map, which in turn produces a Hurwitz map $\Delta : \mathcal{H}(H, \boldsymbol{D})^{\text{ab}} \to \mathcal{H}(G, \boldsymbol{C})^{\text{ab}}$ which again reflects the monodromy homomorphism; call this an *absolute Hurwitz cover.* Here's a picture.

$$\begin{array}{ccc} \mathcal{H}(H, \boldsymbol{D})^{\text{in}} & \longrightarrow & \mathcal{H}(H, \boldsymbol{D})^{\text{ab}} \\ \downarrow & & \downarrow \\ \mathcal{H}(G, \boldsymbol{C})^{\text{in}} & \longrightarrow & \mathcal{H}(G, \boldsymbol{C})^{\text{ab}} \end{array}$$

4.2.3. *Restraining Conditions.* We address the issue of which group covers $f : H \to G$ and conjugacy classes are resonant for analysis of Hurwitz covers. In practice, the technique for analyzing this situation will certainly be to start with knowledge of the Hurwitz space of G, and attempt to lift it to knowledge of the Hurwitz space for H.

If f is a Frattini cover, any lift of $\boldsymbol{g} \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}}$ to $\boldsymbol{h}$ will generate H, although the product may or may not be 1; focus on this case. Every Frattini cover has a nilpotent kernel, and factors into covers with elementary p-group kernels. Any p-Frattini cover of G is a quotient of ${}_p^k\tilde{G}$ for some k. Will will take these are the primary examples, but the first few lemmas can be stated for the case of abelian kernels.

If C is a conjugacy class, let $\mathrm{ord}(C)$ denote the order of any element in it. If $\boldsymbol{C}$ is a tuple of conjugacy classes, let $\mathrm{ord}(\boldsymbol{C})$ be the least common multiple of the orders of the elements in the conjugacy classes.

Let $f : H \to G$ be a group cover with abelian kernel K. Choice of conjugacy classes breaks into two distinct cases: $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$, and $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) > 1$. In this dissertation, we assume the first case. The next proposition is essentially [**Fr95**] Lemma 3.7, where the proof uses the Schur-Zassenhaus Theorem.

PROPOSITION 9. *Let $f : H \to G$ be a cover of finite groups whose kernel K is abelian, and let C be a conjugacy class in G. If $\gcd(\mathrm{ord}(C), |K|) = 1$, then there exists a unique conjugacy class $D \subset H$ such that $\mathrm{ord}(D) = \mathrm{ord}(C)$ and $f(D) = C$.*

PROOF. By Proposition 4, all elements in H which lift $g \in G$ and have the same order as g are conjugate. Since a lift of a conjugate is a conjugate of a lift, the rest follows. □

Thus in the situation of the above proposition, we use the notation $\boldsymbol{C}$ to denote the conjugacy classes in H as well as in G.

4.2.4. *Lifts of Nielsen Tuples.* The size of the fiber over a given Nielsen tuple may be bounded in terms of the sizes of the kernel centralizers of its entries.

PROPOSITION 10. *Let $f : H \to G$ be Frattini cover with abelian kernel K and let $\boldsymbol{C}$ be a rank r tuple of conjugacy classes of G with $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$. Let $c_i = [K : C_K(g_i)]$, where $g_i \in C_i$. Let $\boldsymbol{g} \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$, and let $X = \{\boldsymbol{h} \in \mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in}} \mid f(\boldsymbol{h}) = \boldsymbol{g}\}$. Then*

$$|X| \leq \frac{|Z(H)| \prod_{i=1}^{r-1} c_i}{|K||Z(G)|}.$$

PROOF. Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ and assume that there exists $\boldsymbol{h} = (h_1, \ldots, h_r) \in \mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}$ with $f(\boldsymbol{h}) = \boldsymbol{g}$. Let V_i be a complement of $C_K(g_i)$ in K. Let $\boldsymbol{v} = (v_1, \ldots, v_{r-1}) \in V_1 \times \ldots \times V_{r-1}$ and set $\boldsymbol{h}^{\boldsymbol{v}} = (h_1^{v_1}, \ldots, h_{r-1}^{v_{r-1}}, (h_1^{v_1} \cdots h_{r-1}^{v_{r-1}})^{-1})$. Note that the last entry lies over g_r; it is in the same conjugacy class as h_r if and only if it has the same order as h_r. The last entry is forced upon us by the product one condition, so by Proposition 4, all tuples in $\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}$ over $\boldsymbol{g}$ are of this form, and the tuples of this form are distinct. Thus there are $\prod_{i=1}^{r-1} c_i$ preimages of $\boldsymbol{g}$ with product one and the correct conjugacy classes in the first $r-1$ slots. Adjust by the number of inner automorphisms to obtain the result. □

4.2.5. *Lifts of Nielsen Classes.* Let $f : H \to G$ be a Frattini cover with abelian kernel K. Let $\boldsymbol{C}$ be a tuple of conjugacy classes in G such that $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$. Set

$$\mathrm{Ni}_f(G, \boldsymbol{C})^{\mathrm{to}} = \{\boldsymbol{g} \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}} \mid \exists \boldsymbol{h} \in \mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}} \text{ such that } f(\boldsymbol{h}) = \boldsymbol{g}\}.$$

We now generalize the argument of [**BF02**] Lemma 7.9 to count the size of a lifted Nielsen class, under certain conditions.

PROPOSITION 11. *Let $f : H \to G$ be Frattini cover with abelian kernel K and let $\boldsymbol{C}$ be a rank r tuple of conjugacy classes of G with* $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$. *If $K \leq Z(H)$, then*

$$|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in}}| = |\mathrm{Ni}_f(G, \boldsymbol{C})^{\mathrm{in}}|.$$

PROOF. By Proposition 4, if $g \in \cup\boldsymbol{C}$, there exists a unique element $h \in H$ with $\mathrm{ord}(h) = \mathrm{ord}(g)$. Thus there is only one choice for a lift of a given Nielsen tuple, and this choice is in $\mathrm{Ni}_f(G, \boldsymbol{C})$ if the product is one. Thus $|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}| = |\mathrm{Ni}_f(G, \boldsymbol{C})^{\mathrm{to}}|$. Since the kernel is central, $|\mathrm{Inn}(H)| = |\mathrm{Inn}(G)|$, and the result follows. □

PROPOSITION 12. *Let $f : H \to G$ be a Frattini cover with abelian kernel K. Let $\boldsymbol{C}$ be a tuple of conjugacy classes from G with* $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$. *Suppose that for every $g \in \cup\boldsymbol{C}$, we have $C_K(g) = \{1\}$. Then*

- **(a)** *for every $\boldsymbol{g} \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}}$ there exists $\boldsymbol{h} \in \mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}$ such that $f(\boldsymbol{h}) = \boldsymbol{g}$;*
- **(b)** $|\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}}| = |\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}}||K|^{r-1}$*;*
- **(c)** $|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in}}| = \frac{|\mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in}}||K|^{r-2}|Z(H)|}{|Z(G)|} = \frac{|\mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in}}||K|^{r-2}}{[Z(G):f(Z(H))]}$.

PROOF. Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ and let $(g_1, \ldots, g_r) \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}}$. By Proposition 4 and the hypothesis, the fiber over g_i consists entirely of elements of the same order as g_i, and are all conjugate. Let $h_i \in f^{-1}(g_i)$ for $i = 1, \ldots, r-1$, and let $h_r = (\prod_{i=1}^{r-1} h_i)^{-1}$. Then $f(h_r) = g_r$, so h_r has the same order as g_r, and $\boldsymbol{h} = (h_1, \ldots, h_r) \in \mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}$ with $f(\boldsymbol{h}) = \boldsymbol{g}$. There are $|K|^{r-1}$ choices for $h_1, \ldots, h_{r-1}$, so $|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}| = |\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}}||K|^{r-1}$, giving **(b)**. Divide both sides by the number of inner automorphisms to obtain **(c)**. The second equal sign of **(c)** results from the fact that the hypothesis implies that $Z(H)$ injects into $Z(G)$. □

Let $f : H \to G$ be a group homomorphism with abelian kernel K, and let $\boldsymbol{C}$ be a tuple of conjugacy classes from G with $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$. We say that $\boldsymbol{C}$ has a *common centralizer complement* with respect to f if there exists $V \leq K$ with $V \lhd H$ such that V is a complement in K of $C_H(C_i)$ for $i = 1, \ldots, r$.

Proposition 13. *Let $f : H \to G$ be Frattini cover with abelian kernel K and let $\boldsymbol{C}$ be a rank r tuple of conjugacy classes of G with $\gcd(\mathrm{ord}(\boldsymbol{C}), |K|) = 1$ and a common centralizer complement V with respect to f. Then*

$$|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in}}| = \frac{|\mathrm{Ni}_f(G, \boldsymbol{C})^{\mathrm{in}}||V|^{r-1}|Z(H)|}{|K||Z(G)|}.$$

Proof. Let $\bar{H} = H/V$. Since Nielsen tuples generate the group, the kernel of $\bar{f} : \bar{H} \to G$ is central, so Proposition 11 implies that $|\mathrm{Ni}(\bar{H}, \boldsymbol{C})^{\mathrm{to}}| = |\mathrm{Ni}_{\bar{f}}(G, \boldsymbol{C})^{\mathrm{to}}|$. The map $H \to \bar{H}$ with kernel V satisfies the hypothesis of Proposition 12, so $|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}| = |\mathrm{Ni}(\bar{H}, \boldsymbol{C})^{\mathrm{to}}||V|^{r-1}$. Moreover $\mathrm{Ni}_{\bar{f}}(G, \boldsymbol{C})^{\mathrm{to}} = \mathrm{Ni}_f(G, \boldsymbol{C})^{\mathrm{to}}$, and we have $|\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{to}}| = |\mathrm{Ni}_f(G, \boldsymbol{C})^{\mathrm{to}}||V|^{r-1}$. Divide both sides by the number of inner automorphisms to obtain the result. □

4.3. Modular Towers.

4.3.1. *Modular Towers.* Let G be a finite group whose order is divisible by p, and let $\boldsymbol{C}$ be a tuple of conjugacy classes from G with $\gcd(\mathrm{ord}(\boldsymbol{C}), p) = 1$. An *inner Modular Tower* is the sequence of Hurwitz spaces

$$\cdots \to \mathcal{H}({}_p^{k+1}\tilde{G}, \boldsymbol{C})^{\mathrm{in}} \to \mathcal{H}({}_p^{k}\tilde{G}, \boldsymbol{C})^{\mathrm{in}} \to \cdots \to \mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$$

induced from the universal elementary p-Frattini covers. We call $\mathcal{H}({}_p^{k}\tilde{G}, \boldsymbol{C})^{\mathrm{in}}$ the k^{th} *level* of the Modular Tower.

For each $k \geq 0$, select a coreless subgroup ${}^kU \leq {}_p^k\tilde{G}$ such that ${}^{k+1}U$ maps into ${}^kU$. Embed ${}_p^k\tilde{G}$ in S_{n_k}, where $k = [{}_p^k\tilde{G} : {}^kU]$, via its coset representation. An *absolute Modular Tower* is the resulting sequence of absolute Hurwitz spaces. Apply the action of $\mathrm{PSL}_2(\mathbb{C})$ on either the inner or absolute Modular Tower to obtain a *reduced Modular Tower.* In general, denote a Modular Tower by $\mathbf{MT}_p(G, \boldsymbol{C})$, with extra decoration if we wish to concentrate on inner, absolute, or reduced versions.

Recall that the points on a Hurwitz space are defined over the field of moduli of a corresponding cover, and that if either G is centerless in the inner case, or G is self-normalizing in the absolute case, a cover exists in each equivalence class which is defined over its field of moduli. Thus we would like to know when these conditions lift through the Modular Tower. See [**Fr95**] Definition 3.5 and Problem 3.8 for a discussion of this. We report the following.

Theorem 14. *If G is perfect and centerless, then ${}_p^k\tilde{G}$ is perfect and centerless, for $k \geq 0$.*

Proof. [**Fr95**] Lemma 3.6. □

4.3.2. *Modular Tower Sublevels.* Let $\mathbf{MT}_p(G, \boldsymbol{C})^{\mathrm{in}}$ be an inner Modular Tower. Let $G_k = {}_p^k\tilde{G}$ and $M_k = \ker(G_{k+1} \to G_k)$. Suppose that $K \leq M_k$ is normal in G_{k+1}, and let $H = G_{k+1}/K$. The sequence $G_{k+1} \to H \to G_k$ of Frattini covers induces a sequence of Hurwitz spaces,

$$\mathcal{H}(G_{k+1}, \boldsymbol{C})^{\mathrm{in}} \to \mathcal{H}(H, \boldsymbol{C})^{\mathrm{in}} \to \mathcal{H}(G_k, \boldsymbol{C})^{\mathrm{in}};$$

we call $\mathcal{H}(H, \boldsymbol{C})^{\text{in}}$ a *sublevel* of level $k+1$ of the inner Modular Tower. Analogously define this for absolute Modular Towers and their reduced versions.

Information regarding sublevels of a Modular Tower can help push knowledge of level k to knowledge of level $k+1$, as is the technique in chapter VII.

4.3.3. *Obstruction.* Let $f : H \to G$ be a Frattini cover with abelian kernel K, and let $\boldsymbol{C}$ be a tuple of conjugacy classes in G such that $\gcd(\text{ord}(\boldsymbol{C}), |K|) = 1$. Let $\boldsymbol{g} \in \text{Ni}(G, \boldsymbol{C})^{\text{to}}$, and select $\boldsymbol{h} \in H^r$ which lifts $\boldsymbol{g}$ to a tuple of elements of the same order(s). Let $a = \Pi \boldsymbol{h} \in K$; then $\boldsymbol{h} \in \text{Ni}(H, \boldsymbol{C})^{\text{to}}$ if and only if $a = 1$. Note that a is unaffected by the action of braiding, and the conjugacy class of a is invariant for inner tuple classes. Set

$$\nu_f(\boldsymbol{g}) = \{a \in K \mid \Pi \boldsymbol{h} = a \text{ for some } \boldsymbol{h} \text{ with } f(\boldsymbol{h}) = \boldsymbol{g} \text{ and } \boldsymbol{h} \vDash \boldsymbol{C}\}.$$

This is the *lifting invariant* of $\boldsymbol{g}$ with respect to f; it is a union of orbits under the action of G on K, which are conjugacy classes in H. If O is an orbit for the action of H_r on $\text{Ni}(G, \boldsymbol{C})^{\text{in}}$, then ν_f is constant on O; that is, it is a braid invariant, and we can set $\nu_f(O) = \nu_f(\boldsymbol{g})$ for any $\boldsymbol{g} \in O$.

Let $\mathcal{H}_O$ be the component of $\mathcal{H}(G, \boldsymbol{C})^{\text{in}}$ corresponding to O. The preimage of $\mathcal{H}_O$ in $\mathcal{H}(H, \boldsymbol{C})^{\text{in}}$ is a collection of components. If $1 \notin \nu_f(\boldsymbol{g})$, then this collection is empty, and we say that $\mathcal{H}_O$ is *obstructed* by f.

Consider the case where $f : G_{k+1} \to G_k$ as in the previous subsection. If a component of $\mathcal{H}(G_k, \boldsymbol{C})^{\text{in}}$ is obstructed by f, we say it is *obstructed at level* $k+1$. There is a precise group theoretical necessary condition for this.

THEOREM 15. *Let* $\mathbf{MT}_p(G, \boldsymbol{C})^{\text{in}}$ *be an inner Modular Tower of a group* G. *If* $\mathbf{MT}_p(G, \boldsymbol{C})^{\text{in}}$ *is obstructed at level* $k+1$, *then the universal elementary* p*-Frattini cover* $f : G_{k+1} \to G_k$ *factors as* $G_{k+1} \to H_2 \to H_1 \to G_k$ *such that* $\ker(H_1 \to H_2) = C_p \leq Z(H_1)$, *with* C_p *cyclic of order* p.

PROOF. [**FK97**] Obstruction Lemma 3.2. □

The conclusion above is equivalent to saying that $G_{k+1} \to G_k$ has a central elementary p-Frattini factor. This implies that elements in G_k relatively prime to p have nontrivial centralizers in M_k, so this result includes Proposition 12 **(a)**.

CHAPTER IV

Real Points

1. Kappa Operators

1.1. Real Covers.

1.1.1. *Complex Conjugation.* Let $\eta : \mathbb{P}^1 \to \mathbb{P}^1$ denote complex conjugation. Then η is the unique nontrivial field automorphism of $\mathbb{C}$ which is continuous, and we can use this to our advantage to detect real points on Hurwitz spaces. The ideas of this section have their roots in [**DF90**] and [**DF94**], who in turn cite [**Hu91**] and [**KN71**].

Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover. View Y as an embedded projective variety in $\mathbb{P}^n$, and let $\hat{\eta}$ denote the action of complex conjugation on $\mathbb{P}^n$. Set $Y^\eta = \hat{\eta}(Y)$ and define the cover $\varphi^\eta : Y^\eta \to \mathbb{P}^1$ by $\varphi^\eta = \eta \circ \varphi \circ \hat{\eta}^{-1}$. This induces an function $\eta_* : \mathrm{Aut}(\varphi) \to \mathrm{Aut}(\varphi^\eta)$ given by $\alpha \mapsto \hat{\eta} \circ \alpha \circ \hat{\eta}^{-1}$, which is a group isomorphism.

1.1.2. *Real Covers.* We say that $\varphi : Y \to \mathbb{P}^1$ is a *real cover* if it is defined over $\mathbb{R}$. This is the case exactly if $\varphi = \varphi^\eta$; suppose this is so. Then $Y = Y^\eta$, which identifies $\mathrm{Aut}(\varphi)$ with $\mathrm{Aut}(\varphi^\eta)$, so that η_* becomes an automorphism of $\mathrm{Aut}(\varphi)$. An automorphism $\alpha \in \mathrm{Aut}(\varphi)$ is defined over $\mathbb{R}$ if and only if $\alpha = \eta_*(\alpha)$. The subgroup of $\mathrm{Aut}(\varphi)$ of automorphisms defined over $\mathbb{R}$ is the set of points fixed by $\eta_* \in \mathrm{Aut}(\mathrm{Aut}(\varphi))$. View $\hat{\eta} \in \mathrm{Sym}(Y)$ and $\mathrm{Aut}(\varphi) \leq \mathrm{Sym}(Y)$. The subgroup of $\mathrm{Aut}(\varphi)$ consisting of automorphisms defined over $\mathbb{R}$ is $C_{\mathrm{Aut}(\varphi)}(\hat{\eta}) \leq \mathrm{Sym}(Y)$.

We say that φ is a *real Galois cover* if φ is a normal cover defined over $\mathbb{R}$ such that every automorphism of φ is defined over $\mathbb{R}$. This occurs exactly when $\eta_* \in \mathrm{Aut}(\mathrm{Aut}(\varphi))$ is the identity, so $C_{\mathrm{Aut}(\varphi)}(\hat{\eta}) = \mathrm{Aut}(\varphi)$. This implies that the function field extension of φ over $\mathbb{C}$ descends to a Galois extension over $\mathbb{R}(x)$.

Let $(\varphi : Y \to \mathbb{P}^1, \tau : G \to \mathrm{Aut}(\varphi))$ be a static cover, and set $\tau^\eta = \eta_* \circ \tau$. We say that (φ, τ) is a *real static cover* if φ is a real cover and $\tau = \tau^\eta$, that is, $\eta_* \in \mathrm{Aut}(\mathrm{Aut}(\varphi))$ is the identity. This happens if and only if φ is a real Galois cover; it is independent of τ.

1.1.3. *Pseudoreal Covers.* We say that φ is a *pseudoreal cover* if φ is equivalent to φ^η. This implies that the branch points of φ are an algebraic set over $\mathbb{R}$, i.e., the nonreal points among the branch points come in complex conjugate pairs.

All real covers are pseudoreal. It may or may not be the case that a pseudoreal cover is equivalent to a cover which is defined over $\mathbb{R}$. If φ is pseudoreal, then the field of moduli of φ is contained in

$\mathbb{R}$. If $\mathrm{Aut}(\varphi)$ is trivial or φ is normal, then φ can be defined over its field of moduli (see [**FV91**] Section 1.5 and [**DF94**] Sections 2.4 and 3.4), so in this case, φ is equivalent to a real cover.

We say that (φ, τ) is a *pseudoreal static cover* if (φ, τ) is equivalent, as a static cover, to $(\varphi^\eta, \tau^\eta)$. Here, φ is normal, and if additionally $\mathrm{Aut}(\varphi)$ is centerless, then (φ, τ) can be defined over its field of moduli.

Pseudoreal covers arise from the fact that our method of specifying covers by branch cycle descriptions identifies them only up to equivalence. Without extra conditions, we can only hope to detect the field of moduli. Static covers arise from our interest in identifying fields of definition of the automorphisms, using the combinatorics supplied by branch cycle descriptions.

1.2. General Kappa Operators.

1.2.1. *Conjugation of Branch Cycle Descriptions.* Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover whose branch points $\boldsymbol{x} = (x_1, \ldots, x_r)$ form and algebraic set over $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a basepoint for $X = \mathbb{P}^1 \setminus \underline{\boldsymbol{x}}$. Let $Y_{x_0} = \varphi^{-1}(x_0)$ be the fiber over x_0, and let $\epsilon : Y_{x_0} \to \mathbb{N}_n$ be an enumeration of Y_{x_0}, which induces a monodromy representation $T : \pi_1(X, x_0) \to S_n$ whose image is G.

Let λ be a loop in X based at x_0, and let $\bar{\lambda} = \eta \circ \lambda$. Since η is continuous and $\eta(x_0) = x_0$, $\bar{\lambda}$ is also a loop at x_0, which induces an automorphism $\eta_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ given by $[\lambda] \mapsto [\bar{\lambda}]$.

We may compose a lift of λ with $\hat{\eta}$ to obtain a lift of $\bar{\lambda}$. If the lift of λ to $y_1 \in Y_{x_0}$ ends at $y_2 \in Y_{x_0}$, then the lift of $\bar{\lambda}$ to $\hat{\eta}(y_1)$ ends at $\hat{\eta}(y_2)$.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be a classical tuple with respect to $(\boldsymbol{x}, x_0)$, and let $\bar{\boldsymbol{\lambda}} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_r)$. Also let $\bar{\epsilon} : \hat{\eta}(Y_{x_0}) \to \mathbb{N}_n$ be given by $\bar{\epsilon}(\hat{\eta}(y)) = \epsilon(y)$. Let $g_i = T(\lambda_i)$ so that $\boldsymbol{g} = (g_1, \ldots, g_r)$ is a branch cycle description for φ with respect to $\boldsymbol{\lambda}$ and ϵ. Composing paths in $\mathbb{P}^1$ with η and their lifts to Y with $\hat{\eta}$ shows that $T(\boldsymbol{\lambda}) = \boldsymbol{g}$ is the branch cycle description for φ^η with respect to $\bar{\boldsymbol{\lambda}}$ and $\bar{\epsilon}$. Thus $T(\bar{\boldsymbol{\lambda}})$ is the branch cycle description for φ^η with respect to $\bar{\bar{\boldsymbol{\lambda}}} = \boldsymbol{\lambda}$ and $\bar{\epsilon}$.

Let $G \hookrightarrow S_n$ and let $\boldsymbol{g} \in \mathrm{Ni}(G, r)^{\mathrm{to}}$, and let $T_{\boldsymbol{g}} : \pi_1(X, x_0) \to G$ be given by $\lambda_i \mapsto g_i$. Let $\kappa_{\boldsymbol{\lambda}} : \mathrm{Ni}(G, r)^{\mathrm{to}} \to \mathrm{Ni}(G, r)^{\mathrm{to}}$ be defined by $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = T_{\boldsymbol{g}}(\bar{\boldsymbol{\lambda}})$. Then $\kappa_{\boldsymbol{\lambda}}$ is an involutive permutation of the Nielsen class which detects the effect of complex conjugation of covers. We call $\kappa_{\boldsymbol{\lambda}}$ the *kappa operator* with respect to $\boldsymbol{\lambda}$. We call any bouquet isotopic to $\boldsymbol{\lambda}$ *admissible* for $\kappa_{\boldsymbol{\lambda}}$.

PROPOSITION 16. *Let $\varphi : Y \to \mathbb{R}$ be a ramified cover whose branch points are an algebraic set over $\mathbb{R}$, and let $\hat{\varphi} : Y \to \mathbb{R}$ be the normal closure of φ. Let $\tau : G \to \mathrm{Aut}(\hat{\varphi})$ be an isomorphism. Let $\boldsymbol{\lambda}$ be a bouquet with respect to the branch points of φ and a real basepoint, and let $\boldsymbol{g}$ be a branch cycle description of φ with respect to $\boldsymbol{\lambda}$, and $G = \langle \boldsymbol{g} \rangle$. Then*

- **(a)** *φ is a pseudoreal cover if and only if $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) \equiv \boldsymbol{g} \pmod{\mathrm{Abs}(G)}$;*
- **(b)** *$(\hat{\varphi}, \tau)$ is a pseudoreal static cover if and only if $\kappa_{\boldsymbol{g}}(\boldsymbol{g}) \equiv \boldsymbol{g} \pmod{\mathrm{Inn}(G)}$.*

1.2.2. *Complex Conjugators.* Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover defined over $\mathbb{R}$, and continue notation from above. Then the fiber Y_{x_0} is an algebraic set over $\mathbb{R}$, so $\hat{\eta}(Y_{x_0}) = Y_{x_0}$, and $\hat{\eta}$ acts on

Y_{x_0}. The enumeration ϵ of Y_{x_0} produces an element $c \in S_n$ describing this action, where $c = \bar{\epsilon} \circ \epsilon^{-1}$, as well as an antihomomorphism $\mathrm{Aut}(\varphi) \to S_n$ whose image we denote by A. Clearly c is an element of order two, unless the entire fiber consists of real points, in which case c is trivial. We call c the *complex conjugator* of φ with respect to the enumeration of the fiber.

Let λ be a loop in X based at x_0. The continuity of η and $\hat{\eta}$ leads to the conclusion that $T([\bar{\lambda}]) = cT([\lambda])c$. In particular, $c \in N_{S_n}(G)$.

Consider the significance of this when φ is a normal cover, in which case G is in its regular representation. The automorphism group of φ is $A = C_{S_n}(G)$, and as we have seen, $G = C_{S_n}(A)$. Thus if φ and all of its automorphisms are defined over $\mathbb{R}$, then $c \in G$.

A necessary condition for a cover $\varphi : Y \to \mathbb{P}^1$ to be able to be defined over $\mathbb{R}$ is

$$\exists c \in N_{S_n}(G) \text{ such that } c^2 = 1 \text{ and } \kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = \boldsymbol{g}^c.$$

This is sufficient when $\mathrm{Aut}(\varphi)$ is trivial or φ is normal, because in these cases, φ can be defined over its field of moduli. Thus under these conditions, a cover with branch cycle description $\boldsymbol{g}$ with respect to $\boldsymbol{\lambda}$ can be defined over $\mathbb{R}$ if and only if $\boldsymbol{g}$ is a fixed point under the action of $\kappa_{\boldsymbol{\lambda}}$ on $\mathrm{Ni}(G, r)^{\mathrm{ab}}$.

We explain further. Suppose $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = \boldsymbol{g}^a$ for some $a \in N_{S_n}(G)$. Since $\boldsymbol{g}$ generates G and a has involutive action on $\boldsymbol{g}$, $\mathrm{Inn}(G)$ contains a unique involution whose action is that of a, and we have $a^2 \in C_{S_n}(g)$. When $C_{S_n}(G) = \mathrm{Aut}(\varphi)$ is trivial, we automatically have $a^2 = 1$, and a is uniquely determined to be c. When G is in its regular representation, $\mathrm{Aut}(G)$ embeds in $N_{S_n}(G)$, so there exists $b \in N_{S_n}(G)$ with $b^2 = 1$ and $\boldsymbol{g}^a = \boldsymbol{g}^b$. But here we cannot find c just by looking at the group.

A necessary condition for a normal cover $\varphi : Y \to \mathbb{P}^1$ to be able to be defined over $\mathbb{R}$ together with its automorphisms is

$$\exists c \in G \text{ such that } c^2 = 1 \text{ and } \kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = \boldsymbol{g}^c.$$

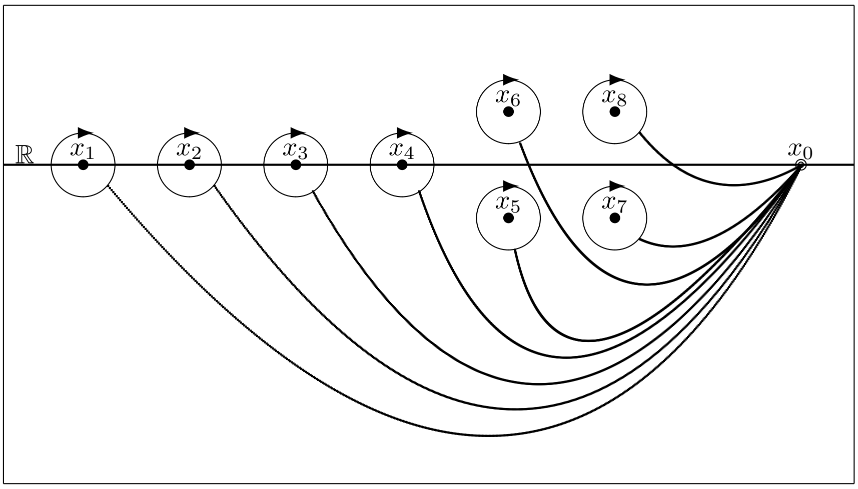
This is also sufficient (see [**DF94**] Section 3.4). When $\mathrm{Mon}(\varphi)$ has a trivial center, c is uniquely determined by its action. Under the condition of a trivial center, an static cover can be defined over $\mathbb{R}$ if and only if its branch cycle description $\boldsymbol{g}$ with respect to $\boldsymbol{\lambda}$ is a fixed point under the action of $\kappa_{\boldsymbol{\lambda}}$ on $\mathrm{Ni}(G, r)^{\mathrm{in}}$.

1.3. Specific Kappa Operators.

1.3.1. *Debes-Fried Kappa Operators.* In order to compute the operator $\kappa_{\boldsymbol{\lambda}}$, one selects specific paths for $\boldsymbol{\lambda}$, reflects them across the real axis, and rewrites the result in terms of the original paths. We review the paths used in [**DF90**] and the resulting formulae, which were then again applied in [**DF94**]. Our paths are morally the same, although we have taken the liberty to write them with counterclockwise loops, as is standard in this dissertation.

If $x \in \mathbb{C}$, let $\Re(x)$ and $\Im(x)$ respectively denote its real and imaginary parts. Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be a tuple of branch points in $\mathbb{P}^1$, defined as a set over $\mathbb{R}$. Suppose that s of these points are real; we call this an (r, s) *branch point configuration.* Order the points so that the real points are first,

x_{s+2t+1} is conjugate to x_{s+2t+2} with $\Im(x_{s+2t+1}) < 0$, and otherwise so that $\Re(x_i) \le \Re(x_{i+1})$, where $\infty \le x$ for any real x. Select $x_0 \in \mathbb{R}$ so that $x_0 > \Re(x_i)$ for all i. Draw the simplest paths which proceed from x_0 to the points in the given order, as indicated below with four real branchpoints and two pairs of complex conjugates:



Representative paths for the Debes-Fried Kappa Operator.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be the resulting bouquet, which we call a *Debes-Fried bouquet*. Reflect these paths across the real axis to compute $\bar{\boldsymbol{\lambda}}$. Let $\rho_i = \prod_{j=i}^{r} \lambda_j$. Then

$$\bar{\lambda}_i = \begin{cases} \rho_{i+1}^{-1}\lambda_i^{-1}\rho_{i+1} & \text{if } i \le s; \\ \rho_{i+2}^{-1}\lambda_{i+1}^{-1}\rho_{i+2} & \text{if } i = s+2t+1; \\ \rho_{i+1}^{-1}\lambda_{i-1}^{-1}\rho_{i+1} & \text{if } i = s+2t+2. \end{cases}$$

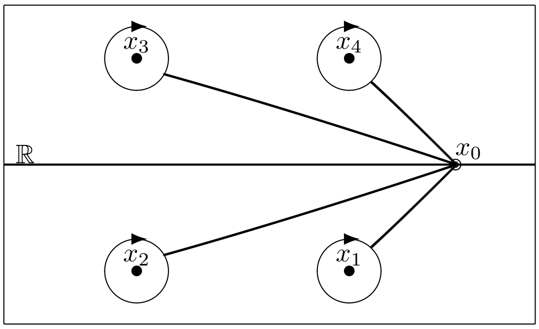
Substitute g_i for λ_i to obtain the effect of $\kappa_{\boldsymbol{\lambda}}$ on the Nielsen tuple $\boldsymbol{g} = (g_1, \ldots, g_r)$. Operators on Nielsen sets given by this formula, with r branch points of which s are real, we will call *Debes-Fried kappa operators* of type (r, s), denoted by $\kappa_{(r,s)}$. To give the flavor of the results one can expect from these considerations, we review an application from [**DF94**].

PROPOSITION 17. *Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover which is Galois over $\mathbb{R}$, all of whose branch points are real, and let $G = \mathrm{Mon}(\varphi)$. Then G is generated by involutions.*

PROOF. We paraphrase [**DF94**]. Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be the branch cycle description of the cover with respect to a Debes-Fried bouquet and some enumeration of the fiber over a basepoint x_0, with $G = \langle \boldsymbol{g} \rangle \le S_n$. Let $c \in G$ be the complex conjugator, and let $\kappa_{(r,r)}(\boldsymbol{g}) = (\bar{g}_1, \ldots, \bar{g}_r) = c\boldsymbol{g}c$. Set $a_i = \prod_{j=i+1}^{r} g_j$ for $i = 1, \ldots, r-1$, so that $ca_ic = \prod_{j=i+1}^{r} \bar{g}_i$, and compute that this latter product is a_i^{-1}. Thus ca_i has order two, and $G = \langle c, ca_1, \ldots, ca_{r-1} \rangle$. □

1.3.2. *Reflection Kappa Operators.* We introduce a simplified formula for the case of complex conjugate pairs. The bouquet we use can be constructed for any cover without real branch points. In this case, select $x_0 \in \mathbb{R}$ such that x_0 is larger than the maximum real part of one of the branch points, and so that lines in $\mathbb{C}$ passing through x_0 intersect at most one branch point. Enumerate the branch points in decreasing order of the slopes of these lines. Draw paths from the basepoint along these lines toward the branch points, around and back. Call the resulting bouquet $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$.

Assume that the branch points are an algebraic set over $\mathbb{R}$; in this case, r is even. Then the set of lines described above is invariant under complex conjugation.



Representative paths for the Reflection Kappa Operator.

The action of conjugation on the bouquet results in the formula

$$\kappa_{\boldsymbol{\omega}}(g_1, \ldots, g_r) = (g_r^{-1}, \ldots, g_1^{-1}).$$

Because of the shapes of the paths, we call $\kappa_{\boldsymbol{\omega}}$ the *reflection kappa operator.*

1.4. Harbater-Mumford Covers.

1.4.1. *Harbater-Mumford Tuples.* A *Harbater-Mumford tuple* is a Nielsen tuple $\boldsymbol{g} = (g_1, \ldots, g_r)$ of even rank such that $g_{2t+1} = g_{2t}^{-1}$; this definition is from [**Fr95**], and is used extensively in [**BF02**]. A *Harbater-Mumford cover* is a ramified cover whose branch points are complex conjugate pairs and whose branch cycle description with respect to a Debes-Fried bouquet is a Harbater-Mumford tuple. We note that having a given tuple as a branch cycle description with respect to some bouquet is a braid invariant; thus a *Harbater-Mumford component* of a Hurwitz space is a component which corresponds to the orbit of a Harbater-Mumford tuple under the braid action. See [**Fr95**] Section III.F for an interpretation of these covers in terms of coalescence of the branch points.

This dissertation makes use of the easy combinatorics provided by the shape of the branch cycle description, and we offer a different geometric interpretation which reflects our usage.

1.4.2. *Superreal Covers.* Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover. Let $A \subseteq (\mathbb{P}^1 \setminus \mathrm{Bpt}(\varphi))$, $Y_A = \varphi^{-1}(A)$, and $\varphi_A = \varphi \restriction_A$. Then $\varphi_A : Y_A \to A$ is a topological cover, perhaps with disconnected covering space. Let $\iota : A \to \mathbb{P}^1$ denote inclusion. This induces an injective homomorphism $\iota^* : \mathrm{Aut}(\varphi) \to \mathrm{Aut}(\varphi_A)$.

Consider the case where A is homeomorphic to a circle; for example, perhaps A represents a classical generator for the cover. Let Z_d denote the cyclic group of order d. Let $d_1, \ldots, d_t$ be the distinct degrees of the components of Y_A over A, and let n_{d_i} be the number of components with degree d_i. Then $\mathrm{Aut}(\varphi_A) \cong \oplus_{i=1}^t Z_{d_i} \wr S_{n_{d_i}}$.

Assume that φ has no real branch points, and specify that $A = \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. Let $Y_{\mathbb{R}} = \varphi^{-1}(\mathbb{P}^1(\mathbb{R}))$ and let $\varphi_{\mathbb{R}} = \varphi \restriction_{Y_{\mathbb{R}}}$. Let $\iota : Y_{\mathbb{R}} \to Y$ denote the inclusion map. In this case, the invariants d_i and n_{d_i} can easily be determined from the branch cycle description of φ with respect to the simple bouquet $\boldsymbol{\omega}$. The real circle is homotopic to the product of the last $r/2$ paths; the disjoint cycle decomposition of this product indicates the effect of lifting the loop given by $\mathbb{P}^1(\mathbb{R})$. Thus the number of components of $Y_{\mathbb{R}}$ is the number of disjoint cycles, and the degrees are the lengths of the cycles. If no two components of $Y_{\mathbb{R}}$ have the same degree, then $\mathrm{Aut}(\varphi)$ is abelian. On the other hand, if φ is normal, then all these degrees are the same, and $\mathrm{Aut}(\varphi_{\mathbb{R}}) \cong Z_d \wr S_{n/d}$.

If φ is real, then the map $Y \mapsto Y^{\eta}$ induces an orientation reversing self homeomorphism of Y, denoted by $\hat{\eta}$, which is an automorphism of $\varphi_{\mathbb{R}}$. A *superreal cover* is a real cover $\varphi : Y \to \mathbb{P}^1$ without real branch points such that $\hat{\eta} \in \iota^*(\mathrm{Aut}(\varphi))$; conjugation of φ produces an automorphism of $\varphi_{\mathbb{R}}$.

Proposition 18. *Let $\varphi : Y \to \mathbb{P}^1$ be a ramified cover whose branch points are pairs of complex conjugates. Let $\boldsymbol{\lambda}$ be a Debes-Fried bouquet for complex conjugate pairs, and let $\boldsymbol{g}$ be the branch cycle description of φ with respect to $\boldsymbol{\lambda}$. The following are equivalent:*

- **(a)** $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = \boldsymbol{g}$*;*
- **(b)** $\boldsymbol{g}$ *is a Harbater-Mumford tuple;*
- **(c)** φ *is a Harbater-Mumford cover.*

If additionally φ is defined over $\mathbb{R}$, these are equivalent to

- **(d)** φ *is a superreal cover.*

If additionally φ is normal and $\mathrm{Aut}(\varphi)$ *is centerless, or* $\mathrm{Aut}(\varphi)$ *is trivial, these are equivalent to*

- **(e)** *every point in $Y_{\mathbb{R}}$ is real.*

Proof. That **(a)** implies **(b)** is an inductive calculation, and that **(b)** implies **(a)** is substitution. Also **(b)** $\Leftrightarrow$ **(c)** by definition. Note **(a)** strongly implies that φ is pseudoreal.

Let $\langle \boldsymbol{g} \rangle = G \leq S_n$ be the monodromy group of φ, and let $c \in N_{S_n}(G)$ be the complex conjugator of φ. Now **(a)** implies that $c \in C_{S_n}(G)$, which is identified with $\mathrm{Aut}(\varphi)$. Since c determines an automorphism of $\varphi_{\mathbb{R}}$, we see that φ is superreal if it is real.

The additional conditions for **(e)** ensure that c is trivial. □

1.5. Summary of Formulae. Recall the paths $\boldsymbol{\gamma} = (\gamma_\infty, \gamma_0, \gamma_1)$ which were drawn in chapter II subsection 4.3. These paths are admissible for the Debes-Fried operator for three real branch points. Set $\kappa_s = \kappa_{(4,s)}$. In the case of two pairs of complex conjugate branch points, these determine a circle in $\mathbb{C}$, and we select the basepoint x_0 on this circle. Using the product one relation, we have

$$\begin{aligned}
\kappa_{\boldsymbol{\gamma}}(g_1, g_2, g_3) &= (g_1^{-1}, (g_2^{-1})^{g_3}, g_3^{-1});\\
\kappa_4(g_1, g_2, g_3, g_4) &= (g_1^{-1}, (g_2^{-1})^{g_1^{-1}}, (g_3^{-1})^{g_4}, g_4);\\
\kappa_2(g_1, g_2, g_3, g_4) &= (g_1^{-1}, (g_2^{-1})^{g_1^{-1}}, g_4^{-1}, g_3^{-1});\\
\kappa_0(g_1, g_2, g_3, g_4) &= ((g_1^{-1})^{g_2^{-1}g_1^{-1}}, (g_2^{-1})^{g_1^{-1}}, g_4^{-1}, g_3^{-1});\\
\kappa_{\boldsymbol{\omega}}(g_1, g_2, g_3, g_4) &= (g_4^{-1}, g_3^{-1}, g_2^{-1}, g_1^{-1}).
\end{aligned}$$

2. Beta Operators

2.1. Abstract Kappa Operators.

2.1.1. *Abstract Kappa Operators.* We begin this section by generalizing the idea behind the kappa operators that have been developed. If we replace complex conjugation with any self homeomorphism of $\mathbb{P}^1$ which preserves a set of points, we can again rewrite image paths in terms of the original paths to obtain operators on Nielsen classes. Behind this is an automorphism of the fundamental group of $\mathbb{P}^1$ minus the branchpoints, which is induced by the homeomorphism. Thus we may work more generally with such automorphisms.

2.1.2. *Fundamental Automorphisms.* Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be a tuple of points from $\mathbb{P}^1$. Set $X = \mathbb{P}^1 \setminus \underline{\boldsymbol{x}}$ and let $x_0 \in X$. Let $G_r = \pi_1(X, x_0)$ and let G be a group which can be generated by $r-1$ elements. Let $\mathrm{Epi}(G_r, G)$ denote the set of all epimorphisms from G_r to G. Choose a classical tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ with respect to $(\boldsymbol{x}, x_0)$. This choice induces a function

$$\Omega_{\boldsymbol{\lambda}} : \mathrm{Epi}(G_r, G) \to \mathrm{Ni}(G, r)^{\mathrm{in}} \quad \text{given by} \quad f \mapsto f(\boldsymbol{\lambda}).$$

The fibers of $\Omega_{\boldsymbol{\lambda}}$ consist of epimorphisms which differ by conjugation in G_r. Now $\mathrm{Aut}(G_r)$ acts on the right of $\mathrm{Epi}(G_r, G)$ via composition, which induces an action on $\mathrm{Ni}(G, r)^{\mathrm{in}}$, explicitly given by

$$f(\boldsymbol{\lambda})\alpha = f(\alpha(\boldsymbol{\lambda})).$$

Let $\alpha_{\boldsymbol{\lambda}}$ denote the right operator on $\mathrm{Ni}(G, r)^{\mathrm{in}}$ induced by α in this way.

Let κ denote the automorphism of G_r induced by complex conjugation, where $\underline{\boldsymbol{x}}$ is an appropriate set of points. Then $\kappa_{\boldsymbol{\lambda}}$, as previously defined, is equal to the operator so labeled from this new point of view.

To see how this depends on the choice of $\boldsymbol{\lambda}$, recall that any other choice of classical generators is of the form $\boldsymbol{\lambda}Q$ for some $Q \in B_r$. Let $\boldsymbol{g} = f(\boldsymbol{\lambda})$. Since (by definition) braid action commutes with any homomorphism $f \in \mathrm{Epi}(G_R, G)$, we have

$$(\boldsymbol{g}Q)\alpha_{\boldsymbol{\lambda}Q} = f(\boldsymbol{\lambda}Q)\alpha_{\boldsymbol{\lambda}Q} = (f \circ \alpha)(\boldsymbol{\lambda}Q) = f(\boldsymbol{\lambda})\alpha_{\boldsymbol{\lambda}}Q = \boldsymbol{g}\alpha_{\boldsymbol{\lambda}}Q.$$

Thus $\alpha_{\boldsymbol{\lambda Q}} = Q^{-1}\alpha_{\boldsymbol{\lambda}}Q$.

2.2. General Beta Operators.

2.2.1. *Hurwitz Kernel Generators.* Define the following elements of B_r:

$$
\begin{aligned}
R_1 &= Q_1 \cdots Q_{r-2} Q_{r-1}^2 Q_{r-2} \cdots Q_1; && \\
R_2 &= Q_2 \cdots Q_{r-2} Q_{r-1}^2 Q_{r-2} \cdots Q_1^2 && = Q_1^{-1} R_1 Q_1; \\
R_3 &= Q_3 \cdots Q_{r-2} Q_{r-1}^2 Q_{r-2} \cdots Q_1^2 Q_2 && = Q_2^{-1} R_2 Q_2; \\
&\vdots && \\
R_r &= Q_{r-1} Q_{r-2} \cdots Q_2 Q_1^2 Q_2 \cdots Q_{r-1} && = Q_{r-1}^{-1} R_{r-1} Q_{r-1}.
\end{aligned}
$$

Let $\boldsymbol{R} = (R_1, \ldots, R_r)$. This may be viewed as a "universal Nielsen tuple", as we now discuss.

Proposition 19. *The elements $R_1, \ldots, R_r$ generate $N_r = \ker(B_r \to H_r)$. The braid action of B_r on G_r and selection of a classical tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ induces a surjective homomorphism*

$$\psi_{\boldsymbol{\lambda}} : N_r \to G_r \quad \text{given by} \quad R_i \mapsto \lambda_i,$$

where the braid action of R_i on $\boldsymbol{\lambda}$ equals the left conjugation action of λ_i. The kernel of $\psi_{\boldsymbol{\lambda}}$ is cyclic, generated by $\prod_{i=1}^r R_i = (Q_1 \cdots Q_{r-1})^{\pm 2r}$, and $N_r / \langle \Pi \boldsymbol{R} \rangle \cong G_r$.

Proof. Recall the shift $S = Q_1 \cdots Q_{r-1}$, and the central element $Z = S^r$. The Hurwitz relation is R_1, and N_r is its normal closure in B_r. Since $\{Q_1, Q_1Q_2, \ldots, S\}$ generate B_r, and $\{R_1, \ldots, R_r\}$ is the orbit of R_1 under conjugation by these generators, these elements generate a normal subgroup, which is N_r. Compute that

$$\boldsymbol{\lambda} R_i = \lambda_i \boldsymbol{\lambda} \lambda_i^{-1} \ (\mathrm{mod}\ \Pi \boldsymbol{\lambda} = 1).$$

Thus the braid action of N_r on G_r induces a surjective homomorphism $N_r \to \mathrm{Inn}(G_r)$ given by mapping R_i to left conjugation by λ_i. This is the opposite map of the restriction to N_r of the map $B_r \to \mathrm{Aut}(G_r)$ we previously discussed. Compose this with the inverse of the isomorphism $G_r \to \mathrm{Inn}(G_r)$, given by that fact that G_r is centerless, to obtain $\psi_{\boldsymbol{\lambda}}$.

Let F_r be the free group generated by $\hat{\lambda}_1, \ldots, \hat{\lambda}_r$, with map $F_r \to G_r$ given by $\hat{\lambda}_i \mapsto \lambda_i$. The kernel is cyclic, generated by $\prod_{i=1}^r \hat{\lambda}_i$. This factors through $\psi_{\boldsymbol{\lambda}}$, showing that $\ker(\psi_{\boldsymbol{\lambda}}) = \langle \prod_{i=1}^r R_i \rangle$. In particular, $N_r / \langle \Pi \boldsymbol{R} \rangle \cong G_r$.

This kernel is necessarily a subgroup of $Z(B_r) = \langle Z \rangle = \ker(B_r \to \mathrm{Aut}(G_r))$, generated by the lowest power of Z or Z^{-1} which is in N_r. From subsection 1.4.4, this element is Z^2 or Z^{-2}. □

Proposition 20. *Let Q_i be a standard generator for B_r. Then*

$$Q_i R_j Q_i^{-1} = \begin{cases} R_i R_{i+1} R_i^{-1} & \text{if} \quad j = i; \\ R_i & \text{if} \quad j = i+1; \\ R_j & \text{otherwise.} \end{cases}$$

Proof. By construction, $R_i^{Q_i} = R_{i+1}$, so $Q_i R_{i+1} Q_i^{-1} = R_i$. Now suppose that $j \notin \{i, i+1\}$. Assume $j < i$; the other case is similar. Compute

$$\begin{aligned}
R_j Q_i &= Q_j \cdots Q_{r-1}^2 \cdots Q_1^2 \cdots Q_{j-1} Q_i \\
&= Q_j \cdots Q_{r-1}^2 \cdots Q_i Q_{i-1} Q_i \cdots Q_1^2 \cdots Q_{j-1} && \text{relation } \mathbf{(B1)} \\
&= Q_j \cdots Q_{r-1}^2 \cdots Q_{i-1} Q_i Q_{i-1} \cdots Q_1^2 \cdots Q_{j-1} && \text{relation } \mathbf{(B2)} \\
&= Q_j \cdots Q_{i-1} Q_i Q_{i-1} \cdots Q_{r-1}^2 \cdots Q_1^2 \cdots Q_{j-1} && \text{relation } \mathbf{(B1)} \\
&= Q_j \cdots Q_i Q_{i-1} Q_i \cdots Q_{r-1}^2 \cdots Q_1^2 \cdots Q_{j-1} && \text{relation } \mathbf{(B2)} \\
&= Q_i R_j.
\end{aligned}$$

Finally, since Q_i commutes with R_j unless $j \in \{i, i+1\}$, we have

$$\begin{aligned}
R_i R_{i+1} R_i^{-1} &= Q_i Q_{i+1} \cdots Q_{r-1}^2 \cdots Q_1^2 \cdots Q_{i-1} R_{i+1} Q_{i-1}^{-1} \cdots Q_1^{-2} \cdots Q_{r-1}^{-2} \cdots Q_i^{-1} \\
&= Q_i \cdots Q_{r-1}^2 \cdots Q_i R_{i+1} Q_i^{-1} \cdots Q_{r-1}^{-2} \cdots Q_i^{-1} \\
&= Q_i \cdots Q_{r-1}^2 \cdots Q_i (Q_i^{-1} R_i Q_i) Q_i^{-1} \cdots Q_{r-1}^{-2} \cdots Q_i^{-1} \\
&= Q_i R_i Q_i^{-1}.
\end{aligned}$$

□

2.2.2. *Beta Operators.* Let G be a group generated by $r-1$ elements and select $\boldsymbol{g} \in \mathrm{Ni}(G,r)^{\mathrm{to}}$. Let $f_{(\boldsymbol{\lambda},\boldsymbol{g})} : G_r \to G$ be given by $\boldsymbol{\lambda} \mapsto \boldsymbol{g}$. Let $\psi_{\boldsymbol{g}} : N_r \to G$ be given by $\psi_{\boldsymbol{g}} = f_{(\boldsymbol{\lambda},\boldsymbol{g})} \circ \psi_{\boldsymbol{\lambda}}$; that is, by $R_i \mapsto g_i$. We note that the dependence on $\boldsymbol{\lambda}$ is now extraneous, since we have seen that $N_r/\langle \Pi \boldsymbol{R} \rangle \cong G_r$, and its necessity as a connection to braiding disappears if G is centerless; in that case, $\psi_{\boldsymbol{g}}(R_i) = g_i$ is the unique element of G whose conjugation action equals the braiding action of R_i.

Let $\beta \in \mathrm{Aut}(N_r)^{\mathrm{opp}}$, and define the right action of β on $\mathrm{Ni}(G,r)^{\mathrm{to}}$ by

$$\boldsymbol{g}\beta = \psi_{\boldsymbol{g}}(R_1^{\beta}, \ldots, R_r^{\beta}).$$

By Proposition 20, if β is left conjugation by $Q \in B_r$ on N_r, then the above action gives $\boldsymbol{g}\beta = \boldsymbol{g}Q$. Thus this naturally extends the braid action. If we take β to be an inner automorphism of N_r, given as left conjugation by R, then the effect on tuples is that of conjugation by $\psi_{\boldsymbol{g}}(R)$.

Let $\beta \in \mathrm{Aut}(N_r)$. Also denote by β the induced map

$$\beta : \mathrm{Ni}(G,r)^{\mathrm{in}} \to \mathrm{Ni}(G,r)^{\mathrm{in}};$$

this is what we refer to as a *beta operator*.

Let α be a self homeomorphism of $\mathbb{P}^1$ which stabilizes a set $\{x_1, \ldots, x_r\}$ and fixes ∞. Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ and use $\underline{\boldsymbol{x}}$ as a basepoint for $\mathcal{U}_r$. Then α induces an self homeomorphism of $\mathcal{U}_r$, which in turn induces an automorphism of $H_r = \pi_1(\mathcal{U}_r, \underline{\boldsymbol{x}})$ which lifts to an automorphism $\beta \in \mathrm{Aut}(B_r)$. This β stabilizes N_r, and is a candidate for a beta operator.

2.2.3. *Conjugation Beta Operators.* Let $x_1, \dots, x_r \in \mathbb{R}$ with $x_1 < \dots < x_r$. Let $Q_1, \dots, Q_{r-1}$ denote the standard generators for the braid group, as outlined in chapter II. The image of Q_i is a circle whose center is on the real line. Then complex conjugation induces an automorphism of $B_r = \pi_1(\mathcal{O}_r, \boldsymbol{x})$ given by this effect on the generators:

$$\beta : (Q_1, \dots, Q_{r-1}) \mapsto (Q_1^{-1}, \dots, Q_{r-1}^{-1}).$$

2.3. Specific Beta Operators.

2.3.1. *Focus on $r = 4$.* We intend to compute the complex conjugation operator given in the above manner for the case $r = 4$. In this case, our generators for N_r are

$$R_1 = Q_1Q_2Q_3Q_3Q_2Q_1;$$
$$R_2 = Q_2Q_3Q_3Q_2Q_1Q_1;$$
$$R_3 = Q_3Q_3Q_2Q_1Q_1Q_2;$$
$$R_4 = Q_3Q_2Q_1Q_1Q_2Q_3.$$

2.3.2. *Conjugation Beta Operator.* Let $x_1, x_2, x_3, x_4 \in \mathbb{R}$ with $x_1 < x_2 < x_3 < x_4$. Let Q_1, Q_2, and Q_3 denote the standard generators for the braid group, as outlined in chapter II.

Proposition 21. *Let $\beta : \mathrm{Ni}(G, r)^{\mathrm{in}} \to \mathrm{Ni}(G, r)^{\mathrm{in}}$ denote the beta operator induced by*

$$(Q_1, Q_2, Q_3) \mapsto (Q_1^{-1}, Q_2^{-1}, Q_3^{-1}).$$

Let $\kappa_4 : \mathrm{Ni}(G, r)^{\mathrm{in}} \to \mathrm{Ni}(G, r)^{\mathrm{in}}$ be the Debes-Fried kappa operator for four real branch points. Then

$$\boldsymbol{g}\beta = \boldsymbol{g}\kappa_4 \quad \textit{for every } \boldsymbol{g} \in \mathrm{Ni}(G, r)^{\mathrm{in}}.$$

Proof. It suffices to check this on $\boldsymbol{R}$. We have $\boldsymbol{R}\kappa_4 = (R_1^{-1}, (R_2^{-1})^{R_1^{-1}}, (R_3^{-1})^{R_4}, R_4^{-1})$. Clearly $R_1^\beta = R_1^{-1}$ and $R_4^\beta = R_4^{-1}$. Proposition 20 implies that

$$R_2^\beta = Q_2^{-1}Q_3^{-1}Q_3^{-1}Q_2^{-1}Q_1^{-1}Q_1^{-1} = Q_1^2R_2^{-1}Q_1^{-2} = Q_1R_1^{-1}Q_1^{-1} = R_1R_2^{-1}R_1^{-1};$$
$$R_3^\beta = Q_3^{-1}Q_3^{-1}Q_2^{-1}Q_1^{-1}Q_1^{-1}Q_2^{-1} = Q_3^{-2}R_3^{-1}Q_3^2 = Q_3^{-1}R_4^{-1}Q_3 = R_4^{-1}R_3^{-1}R_4.$$

□

3. Real Points on Hurwitz Spaces

3.1. Real Components on Hurwitz Spaces.

3.1.1. *Real Components of the Configuration Space.* Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^n$, and let $\mathbb{T}^n = \times_{i=1}^n \mathbb{S}^1$ be the n dimensional torus. These are smooth manifolds. Identify $\mathbb{S}^1$ with real projective one space, $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

The set of real points in $\mathcal{U}^r$, denoted $\mathcal{U}^r(\mathbb{R})$, may be viewed as the configuration space $\mathcal{C}^r(\mathbb{S}^1)$, which is a connected space homeomorphic to $\mathbb{T}^r \setminus \Delta^r(\mathbb{S}^1)$.

The set of real points of $\mathcal{U}_r$, denoted $\mathcal{U}_r(\mathbb{R})$, consists of one component for each possible configuration of the branch points. The number of components is $\frac{r}{2} + 1$ if r is even, and $\frac{r+1}{2}$ if r is odd.

Let $\mathcal{R}_{(r,s)}$ denote the component of $\mathcal{U}_r(\mathbb{R})$ whose points correspond to subsets of $\mathbb{P}^1$ containing r points, of which s are real. Then $\mathcal{R}_{(r,s)}$ is homeomorphic to $\mathcal{C}_s(\mathbb{S}^1) \times \mathcal{C}_{(r-s)/2}(\mathbb{H})$.

Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be a tuple of points from $\mathbb{P}^1$ such that $\underline{\boldsymbol{x}} \in \mathcal{R}_{(r,s)}$. The inclusion $\mathcal{R}_{(r,s)} \hookrightarrow \mathcal{U}_r$ induces a group homomorphism $\pi_1(\mathcal{R}_{(r,s)}, \underline{\boldsymbol{x}}) \to \pi_1(\mathcal{U}_r, \underline{\boldsymbol{x}})$, where the range is the Hurwitz monodromy group H_r; let $H_{(r,s)}$ denote the image. The components of the preimage of $\mathcal{R}_{(r,s)}$ in a Hurwitz space $\mathcal{H}(G, \boldsymbol{C})^{\text{in}}$ correspond to the orbits of $H_{(r,s)}$ on $\text{Ni}(G, \boldsymbol{C})^{\text{in}}$. With $r = 4$, appropriate choices produce $H_{(4,4)} = \langle Q_1 Q_2 Q_3 \rangle$ and $H_{(4,0)} = \langle Q_1 Q_3^{-1} \rangle$.

3.1.2. *Real Components of the Hurwitz Space.* Let $\mathcal{H} = \mathcal{H}(G, \boldsymbol{C})^{\text{in}}$ be an inner Hurwitz space, with branch point map $\Phi : \mathcal{H} \to \mathcal{U}_r$ defined over $\mathbb{R}$. Complex conjugation acts on this cover via an embedding of $\mathcal{H}$ into projective space such that $[\varphi] \mapsto [\varphi^\eta]$, where $[\varphi]$ denotes the point on $\mathcal{H}$ corresponding to the cover φ. Thus $[\varphi]$ is real point on $\mathcal{H}$ if and only if φ is equivalent to φ^η.

Let $\mathcal{U}_\mathbb{R} = \mathcal{U}_r(\mathbb{R})$, $\mathcal{H}_\mathbb{R} = \Phi^{-1}(\mathcal{U}_\mathbb{R})$, and $\Phi_\mathbb{R} = \Phi \restriction_{\mathcal{H}_\mathbb{R}}$. The κ operator acts locally to ensure that each component of $\mathcal{H}_\mathbb{R}$ is of one of three types:

- **(a)** the component is defined over $\mathbb{R}$, all points in the component are defined over $\mathbb{R}$;
- **(b)** the component is defined over $\mathbb{R}$, but no point in the component is defined over $\mathbb{R}$;
- **(c)** the component is a complex conjugate of another component.

Consider case **(a)**. Our production of the complex conjugator c depended not only on $\boldsymbol{\lambda}$ but also on an enumeration of the fiber. Since we can continue an fiber enumeration along a path in $\mathcal{U}_\mathbb{R}$, we see that we can choose c to be constant on any component defined over $\mathbb{R}$. If G is centerless, c is uniquely determined from its action on a given $\boldsymbol{g}$. We consider how c depends on a representative. In a manner similar to braiding, the κ operator commutes with conjugation inside G. If $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = \boldsymbol{g}^c$, then $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}^x) = \kappa_{\boldsymbol{\lambda}}(\boldsymbol{g})^x = \boldsymbol{g}^{cx} = \boldsymbol{g}^{xc^x}$. Thus the conjugacy class of c is well-defined, and it becomes an invariant of the real component.

3.1.3. *Real Tuples of Conjugacy Classes.* Let $G \leq S_n$ and let $\boldsymbol{C} = (C_1, \ldots, C_n)$ be a tuple of conjugacy classes from G. Set $\boldsymbol{C}^{-1} = (C_1^{-1}, \ldots, C_n^{-1})$. We call $\boldsymbol{C}$ a *real tuple* of conjugacy classes if $\boldsymbol{C}^{-1} \sim \boldsymbol{C}$.

Complex conjugation of a loop causes its winding number around a real point to be negated. If λ is a classical loop around $x \in \mathbb{R}$, $\bar{\lambda}^{-1}$ is also. Thus the action of a kappa operator on the Nielsen set $\text{Ni}(G, r)^{\text{to}}$ restricts to an action on $\text{Ni}(G, \boldsymbol{C})^{\text{to}}$ if and only if $\boldsymbol{C}$ is a real tuple. The following is implied by [**Fr95**] Lemma C.1.

THEOREM 22. *Let $G \leq S_n$ and let $\boldsymbol{C}$ be a tuple of conjugacy classes from G. Then $\mathcal{H}(G, \boldsymbol{C})^{\text{in}}$ is defined over $\mathbb{R}$ if and only if $\boldsymbol{C}$ is a real tuple of conjugacy classes.*

3.2. Real Points on Reduced Hurwitz Spaces.

3.2.1. *Cover Points and Brauer Points.* Let G be a centerless group. Then, every point on an inner Hurwitz space for G produces a static cover defined over the minimum field of definition for the point. This is no longer the case for reduced inner Hurwitz spaces.

Let $\mathfrak{p} \in \mathcal{H}(G,r)^{\mathrm{in,rd}}$, and let K be its minimum field of definition. We say that $\mathfrak{p}$ is a *K-cover point* if $\mathfrak{p}$ is represented by a cover $\varphi : Y \to \mathbb{P}^1$ which is Galois over K. Otherwise, $\mathfrak{p}$ is a *K-Brauer point.* See [**BF02**] section 4.4 for an in-depth discussion of this.

Consider the case $K = \mathbb{R}$ and $r = 4$. The action of a κ operator on an inner Nielsen class is well-defined modulo reduction. This is because if $\alpha(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation, then $\bar{\alpha}(z) = \frac{\bar{a}z+\bar{b}}{\bar{c}z+\bar{d}}$ is also. Thus if φ and ψ are weakly equivalent covers with $\alpha\varphi = \psi$, then $\bar{\psi} = \overline{\alpha\varphi} = \bar{\alpha}\bar{\varphi}$. Moreover, the setwise stabilizer in $\mathrm{PSL}_2(\mathbb{C})$ of a set of four points defined over $\mathbb{R}$ is actually in $\mathrm{PSL}_2(\mathbb{R})$, so if one cover with these branch points is defined over $\mathbb{R}$, then so are its reduced equivalent covers with the same branch points.

The point $\mathfrak{p}$ corresponds to a reduced inner Nielsen tuple $\boldsymbol{g}$, which is a set of inner Nielsen tuples. The κ operator on $\mathrm{Ni}(G,r)^{\mathrm{in}}$ may permute the points within the set without fixed points, while leaving the set fixed. If this is the case, then $\mathfrak{p}$ is defined over $\mathbb{R}$, and so it is a $\mathbb{R}$-Brauer point. It is the action of κ on reduced inner Nielsen classes that discovers the real points on the reduced Hurwitz space. For such a point, it the action of κ on inner tuples inside a reduced inner tuple which detects whether it is a cover or a Brauer point; it either acts trivially, or it acts without fixed points.

3.2.2. *Cover Intervals.* Each cover with four branch points produces a point on $j \in \mathcal{J}_4$ which is the $\mathrm{PSL}_2(\mathbb{C})$ equivalence class of the branch points. If the cover is defined over $\mathbb{R}$, the configuration of the branch points tells us something about the j value. The following is [**BF02**] Lemma 6.5.

PROPOSITION 23. *Let $\varphi : X \to \mathbb{P}^1_z$ is a four branch point cover over $\mathbb{R}$ with either 0 or 4 real branch points. Then, the corresponding j value is in the interval $(1, +\infty)$ along the real line.*

If φ has, instead, two complex conjugate and two real branch points, then the corresponding j value is in the interval $(-\infty, 1)$.

PROOF. Recall the cross ratio of distinct points $z_1, \dots, z_4$: $\lambda_{\boldsymbol{z}} = \frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}$ (see [**Ah79**] page 79). Four points in complex conjugate pairs (or on the real line) lie on a circle and the cross ratio is real. The cross-ratio is invariant under a transform of the points $\boldsymbol{z}$ by $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$. Since there is an $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ that takes two complex conjugate pairs of points to four real points, with no loss assume $\boldsymbol{z}$ has either two or four real points in its support. For these cases apply $\beta \in \mathrm{PSL}_2(\mathbb{R})$ to assume $0 = z_1$ and $\infty = z_2$. Then, $\lambda_{\boldsymbol{z}} = \frac{z_4}{z_3}$.

In the former case $\lambda_{\boldsymbol{z}}$ runs over the unit circle (excluding 1) and in the latter case over all real numbers (excluding 0, 1 and ∞). The $j_{\boldsymbol{z}}$ value corresponding to $\lambda_{\boldsymbol{z}}$ is $j(\lambda) = \frac{4}{27}\frac{(1-\lambda_{\boldsymbol{z}}+\lambda_{\boldsymbol{z}}^2)^3}{\lambda_{\boldsymbol{z}}^2(1-\lambda_{\boldsymbol{z}})^2)}$.

For $\lambda_{\boldsymbol{z}} \in \mathbb{R} \setminus \{0,1\}$ the connected range of $j_{\boldsymbol{z}}$ includes large positive values and is bounded away from 0. So the range of $j_{\boldsymbol{z}}$ for real $\lambda_{\boldsymbol{z}}$ is $(1,\infty)$. For $\lambda_{\boldsymbol{z}} = e^{2\pi i\theta} = \lambda(\theta)$ in the unit circle (minus 1), the range of $j_{\boldsymbol{z}}$ includes both sides of 0. Also, for θ close to 1, the numerator of $j_{\boldsymbol{z}}$ is positive and bounded, while the denominator is approximately $(i\theta)^2$. Therefore the range is the interval $(-\infty, 1)$. □

3.2.3. *Reduction of κ Operators.* Let G be a centerless group and let $\boldsymbol{C}$ be a real tuple of conjugacy classes of G. We relate the effects of various kappa operators on $\mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in}}$, to see their effect on $\mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in,rd}}$. It convenient and harmless to assume that the kappa operators act from the right.

Proposition 24. *Let $\boldsymbol{g} \in \mathrm{Ni}(G,\boldsymbol{C})^{\mathrm{in}}$. Let $\alpha = (Q_1Q_2Q_3)^2$ and $\beta = Q_1Q_3^{-1}$ be generators for the reduction kernel. Then $\boldsymbol{g}\kappa_{\boldsymbol{\omega}}\alpha\beta = \boldsymbol{g}\kappa_4$, so $\kappa_{\boldsymbol{\omega}} = \kappa_4$ on reduced Nielsen classes.*

Proof. Compute:

$$\begin{aligned}
\boldsymbol{g}\kappa_{\boldsymbol{\omega}}\alpha\beta &= (g_1,g_2,g_3,g_4)\kappa_{\boldsymbol{\omega}}(Q_1Q_2Q_3)^2(Q_1Q_3^{-1})\\
&= (g_4^{-1},g_3^{-1},g_2^{-1},g_1^{-1})(Q_1Q_2Q_3)^2(Q_1Q_3^{-1})\\
&= ((g_3^{-1})^{g_4},(g_2^{-1})^{g_4},(g_1^{-1})^{g_4},(g_4^{-1})^{g_4})(Q_1Q_2Q_3)(Q_1Q_3^{-1})\\
&= ((g_2^{-1})^{g_3g_4},(g_1^{-1})^{g_3g_4},(g_4^{-1})^{g_3g_4},(g_3^{-1})^{g_3g_4})(Q_1Q_3^{-1})\\
&= ((g_1^{-1})^{g_2g_3g_4},(g_2^{-1})^{g_3g_4},(g_3^{-1})^{g_3g_4},(g_4^{-1})^{g_3^{-1}g_3g_4})\\
&= ((g_1^{-1}),(g_2^{-1})^{g_3g_4},(g_3^{-1})^{g_4},(g_4^{-1}))\\
&= (g_1,g_2,g_3,g_4)\kappa_4\\
&= \boldsymbol{g}\kappa_4.
\end{aligned}$$

Thus $\kappa_{\boldsymbol{\omega}}$ and κ_4 are equal on the reduced Nielsen class. □

The above computation was inspired by a geometric picture. Let $\boldsymbol{z} = (z_1,z_2,z_3,z_4)$ arranged clockwise along a circle, with z_1 conjugate to z_4, z_2 conjugate to z_3, and z_1, z_2 in the lower half plane, as in the picture describing κ_ω. Let $\boldsymbol{x} = (x_1,x_2,x_3,x_4)$ as in the picture of κ_4. Let $\alpha \in \mathrm{PSL}(\mathbb{C})$ be such that $\alpha(\boldsymbol{z}) = \boldsymbol{x} \subseteq \mathbb{R}$ preserving order, so the $\alpha(z_i) = x_i$ for $i = 1,2,3,4$. Then α maps the circle inscribed by $\boldsymbol{z}$ to the real line. Since α does not fix the real line (setwise), it is not defined over $\mathbb{R}$, so if φ is a ramified over $\boldsymbol{z}$ and defined over $\mathbb{R}$, then $\alpha(\varphi)$ is not defined over $\mathbb{R}$. However, the paths chosen for the operator $\kappa_{\boldsymbol{\omega}}$ map to paths admissible for the operator κ_4, which shows geometrically why these operators are equal on the reduced Nielsen class.

Let $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a bouquet admissible for κ_0. Rewrite the paths in the bouquet $\boldsymbol{\omega}$ in terms of the paths in $\boldsymbol{\lambda}$; one sees that up to homotopy we have

$$(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (\omega_1\omega_2\omega_1^{-1},\omega_1\omega_3\omega_1^{-1},\omega_1,\omega_4).$$

Find a braid that takes one bouquet to the other:

$$\boldsymbol{\omega} Q_1 Q_2 = (\omega_1\omega_2\omega_1^{-1}, \omega_1\omega_3\omega_1^{-1}, \omega_1, \omega_4) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \boldsymbol{\lambda}.$$

Therefore,

$$\boldsymbol{g}\kappa_0 = \boldsymbol{g} Q_2^{-1} Q_1^{-1} \kappa_{\boldsymbol{\omega}} Q_1 Q_2.$$

We now consider the practical implications of these considerations.

3.2.4. *Computational Implications.* Enumerate the elements of $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$, and compute the action of $\bar{M}_4$ on this set to obtain the monodromy group $M \leq S_n$ (where $n = |\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}|$) of the cover $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{J}_4$ with respect to some basepoint $j_0 \in (1, \infty)$. According to Proposition 23, the real points in the fiber over j_0 are fixed points of the action of a kappa operator for configurations of either 4 real branchpoints or 2 pairs of complex conjugate branchpoints.

These kappa operators act directly on the Nielsen class to produce elements in $N_{S_n}(M)$, and they reflect the action of complex conjugation on the cover $\mathcal{H} \to \mathcal{J}_4$, which is defined over $\mathbb{R}$ by Theorem 22. Therefore, there exists $c \in N_{S_n}(M)$ which reflects the action of complex conjugation on the fiber over the basepoint. Let $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_\infty)$ denote the images of these paths in M. Then

$$\boldsymbol{\gamma}^c = (\gamma_\infty^{-1}, (\gamma_0^{-1})^{\gamma_1}, \gamma_1).$$

Such a c with this effect is unique up to multiplication by an element of $C_{S_n}(M)$; note that every outer automorphism of G produces such an element.

Assume that $K_4 = \ker(M_4 \to \overline{M_4})$ acts faithfully on $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$. Every point in the fiber over j_0 is represented by two inner classes of covers with four real branchpoints (related by $(Q_1Q_2Q_3)^2$) and two inner classes of covers with no real branchpoints (related by $(Q_1Q_3^{-1})$). These points correspond to integers in the enumeration only by attaching the same bouquet to every element in the inner Nielsen class (for some configuration of points mapping to j_0), and taking the corresponding cover, and reducing. Each bouquet produces a different κ operator, which we also view as elements of $N_{S_n}(M)$. We ask which bouquet $\boldsymbol{\lambda}$ produces a kappa operation $\kappa = \kappa_{\boldsymbol{\lambda}}$ such that $\boldsymbol{\gamma}^\kappa = \boldsymbol{\gamma}^c$.

PROPOSITION 25. *Let c be the image of κ_4 in $N_{S_n}(M)$. Then*

$$\boldsymbol{\gamma}^c = (\gamma_\infty^{-1}, (\gamma_0^{-1})^{\gamma_1}, \gamma_1).$$

PROOF. It suffices to show that if β is the beta operator induced by complex conjugation as in Proposition 21, then β has the desired effect. Let q_i and γ_j also denote their images in M. Then

$$\begin{aligned}
\boldsymbol{\gamma}^\beta &= (q_2^\beta, (q_1q_2)^\beta \gamma_1^{-2}, (q_1q_2q_1)^\beta) \\
&= (q_2^{-1}, q_1^{-1}q_2^{-1}q_1^{-1}(q_2^{-1}q_1^{-1})q_1^{-1}q_2^{-1}q_1^{-1}, q_1^{-1}q_2^{-1}q_1^{-1}) \\
&= (\gamma_\infty^{-1}, (\gamma_0^{-1})^{\gamma_1}, \gamma_1).
\end{aligned}$$

□

Thus the complex conjugator equivalent to $\kappa_{\boldsymbol{\gamma}}$ is given by the action of κ_4 on the Nielsen class. However, our preferred operator for the detection of real points will be κ_0, because of its relation to Harbater-Mumford tuples. These are now related.

Let $q = q_1 q_2 = \gamma_0$ be the image of this braid in M. Let κ_4, κ_0, and $\kappa_{\boldsymbol{\omega}}$ denote the images in M. Then $\kappa_4 = \kappa_{\boldsymbol{\omega}} = \kappa_0^q$. In particular, let F_κ be the set of integers fixed by an operator κ. Then $F_{\kappa_0}^q = F_{\kappa_4}$.

3.2.5. *Following.* Let $\mathcal{H}$ be the closure of a component of a reduced inner Hurwitz space, with ramified cover $\mathcal{H} \to \mathbb{P}^1_j$. Select a basepoint $j_0 \in (1, \infty)$, and let $y \in \mathcal{H}$ be a point over j_0. For $x = 1$ or $x = \infty$, let $\delta_{x,y}$ denote the cycle of γ_x which involves y.

PROPOSITION 26. *Let y be a real point over j_0. If $\mathrm{ord}(\delta_{x,y}) = 2n$, then $y\gamma_x^n$ is also real.*

PROOF. Let c be the complex conjugator for $\mathcal{H} \to \mathbb{P}^1_j$. Let $\mathrm{ord}(\delta_x^y) = 2n$. Since y is real, it is a fixed point of c, so $\delta_{x,y}^c = \delta_{x,y}^{-1}$, and the unique other point involved in $\delta_{x,y}$ which is fixed by c is $y\delta_{x,y}^n$. □

There is a geometric interpretation of this. Starting at y, move along the preimage of the closed interval $[1, \infty]$ in $\mathcal{H}$ towards the ramification point over 1. If this point is ramified, continue through the shift node and back towards the fiber over ∞. Alternately and repeatedly apply γ_1 and γ^∞ to the appropriate orders, until either one of the nodes does not have even order, or y is again achieved. If all nodes have even order, this produces a real component of kH.

On the other hand, if one of the nodes of γ_1 or γ_∞ involving a real point does not have even order, this process discovers real points over the interval $(-\infty, 1)$.

4. Harbater-Mumford Fibers

4.1. The Case $r = 4$ and $p = 2$.

4.1.1. *The Case $r = 4$.* We focus for the rest of the paper on the case $r = 4$. Here, the reduced Hurwitz spaces are quotients of the upper half plane covering the j-line.

4.1.2. *The Case $p = 2$.* The case $p = 2$ presents a special situation for a Modular Tower, given by the following.

PROPOSITION 27. *Let $\mathbf{MT}_p(G, \boldsymbol{C})^{\mathrm{in}}$ be an inner Modular Tower with centerless groups of even order and $p = 2$. Let $\Phi_k : \mathcal{H}_{k+1} \to \mathcal{H}_k$ denote the map of the inner Hurwitz spaces between the indicated levels. Let $[\psi] \in \mathcal{H}_{k+1}(\mathbb{R})$. Then $\mathrm{Bpt}(\psi) \not\subseteq \mathbb{R}$. If $\mathrm{Bpt}(\psi) \cap \mathbb{R} = \varnothing$, then $\Phi_k([\psi])$ is a Harbater-Mumford cover.*

PROOF. Since G is centerless and ψ is a static cover, it is defined over $\mathbb{R}$. All involutions of G_{k+1} are in the Frattini subgroup, so the branch points of ψ cannot be contained in $\mathbb{R}$ by Proposition 17.

Let $f_k : G_{k+1} \to G_k$ be the universal elementary 2-Frattini cover. Let $\boldsymbol{h}$ be a branch cycle description for ψ with respect to a bouquet $\boldsymbol{\lambda}$ which is admissible for $\kappa_{(r,0)}$. Then $\boldsymbol{g} = f_k(\boldsymbol{h})$

is a branch cycle description for φ with respect to $\boldsymbol{\lambda}$, where $[\varphi] = \Phi_k([\psi])$. Then there exists $c \in G_{k+1}$ such that $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{h}) = \boldsymbol{h}^c$. Since G_{k+1} is centerless, c is an involution, so $c \in \ker(f_k)$. Clearly $f_k(\kappa_{\boldsymbol{\lambda}}(\boldsymbol{h})) = \kappa_{\boldsymbol{\lambda}}(f_k(\boldsymbol{h}))$; therefore $\kappa_{\boldsymbol{\lambda}}(\boldsymbol{g}) = \boldsymbol{g}$. □

This shows that the real points on level $k+1$ of an inner Modular Tower with $p = 2$ lie over points on level k given by Harbater-Mumford covers. We call points given by Harbater-Mumford covers *Harbater-Mumford points*. Since Harbater-Mumford tuples always lift to the next level, projective systems of real points on a Modular Tower with $p = 2$ are exactly those given by Harbater-Mumford points.

4.2. Duals and Perturbations.

4.2.1. *Setup.* Let $f : H \to G$ be a Frattini cover between centerless groups with characteristic elementary 2-group kernel K. Let $\boldsymbol{C}$ be a tuple of conjugacy classes in G whose elements have odd order. Our goal is to understand the cover $\mathcal{H}(H, \boldsymbol{C})^{\text{in,rd}} \to \mathcal{H}(G, \boldsymbol{C})^{\text{in,rd}}$. To do this, we first analyze the fiber over a Harbater-Mumford tuple. It clarifies notation in what follows if we sometimes denote the identity of K by e.

4.2.2. *Complements.* Application of Proposition 4 requires finding a complement for the centralizer in K of an element $h \in H$. We can do this explicitly, as follows.

Proposition 28. *Let H be a finite group with a normal abelian subgroup K. Let $h \in H$ and set $V = \{a^{-1}a^h \mid a \in K\} \cup \{1\}$. Then*

(a) $V = [K, h] \leq K$;

(b) $K = C_M(h) \oplus V$.

Proof. Let $a \in K$; then $a^h \in K$, so $V \subseteq K$. The elements of V are commutators: $[a, h] = a^{-1}a^h$. Clearly $1 \in V$. Let $a_1, a_2 \in A$. Since K is abelian, $a_1^{-1}a_1^h a_2^{-1}a_2^h = (a_1a_2)^{-1}(a_1a_2)^h \in V$, so this is indeed a subgroup of K. Moreover, this shows that the map $K \to V$ given by $a \mapsto a^{-1}a^h$ is a group homomorphism. The kernel is exactly $C_M(h)$, producing the splitting in **(b)**. □

4.2.3. *Duals.* Let $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1}) \in \text{Ni}(G, \boldsymbol{C})^{\text{to}}$ be a Harbater-Mumford tuple, and let $\boldsymbol{h} = (h_1, h_1^{-1}, h_2, h_2^{-1}) \in \text{Ni}(H, \boldsymbol{C})^{\text{to}}$ be a Harbater-Mumford tuple over $\boldsymbol{g}$, that is, $f(\boldsymbol{h}) = \boldsymbol{g}$. Let $w \in K$. The *dual* of $\boldsymbol{h}$ with respect to w is

$$\boldsymbol{h}^{[e|w]} = (h_1, h_1^{-1}, h_2^w, (h_2^{-1})^w).$$

This is another Harbater-Mumford tuple in the fiber of $\boldsymbol{g}$. If $w \in C_K(h_2)$, then $\boldsymbol{h}^{[e|w]} = \boldsymbol{h}$.

Suppose that w_1 and w_2 are in different cosets of $C_K(h_2)$ in K, but that $\boldsymbol{h}^{[e|w_1]} = \boldsymbol{h}^{[e|w_2]}$ in the inner Nielsen class. Then there exists $c \in C_K(g_1)$ such that $w_2 = w_1c$ and $w_1w_2c \in C_K(g_2)$. Since H is centerless, generated by h_1 and h_2, we have $C_K(g_1) \cap C_k(g_2) = \{1\}$. If W is a complement for $C_K(g_1) \oplus C_K(g_2)$ in K, then $\boldsymbol{h}^{[e|W]} = \{\boldsymbol{h}^{[e|w]} \mid w \in W\}$ is the complete set of duals of $\boldsymbol{h}$. We have $|\boldsymbol{h}^{[e|W]}| = \frac{|K|}{|C_K(g_1)||C_K(g_2)|}$.

4.2.4. *Perturbations.* Let $a \in K$. Then $h_2^{-1}h_1(h_1^{-1})^a$ lies over g_2, so there exists $b \in K$ such that $h_1(h_1^{-1})^a h_2^b h_2^{-1} = 1$. The *perturbation* of $\boldsymbol{h}$ with respect to a is

$$h^{[a|e]} = (h_1, (h_1^{-1})^a, h_2^b, h_2^{-1}).$$

Say that b *fulfills* a. Clearly we can restrict a to a complement V of $C_K(g_1)$ in K, in which case b is determined up to an element of $C_V(g_2)$. Then $\boldsymbol{h}^{[V|e]} = \{\boldsymbol{h}^{[a|e]} \mid a \in V\}$ is the complete set of perturbations of $\boldsymbol{h}$, with $|\boldsymbol{h}|^{[V|e]} = [K : C_K(g_1)]$.

The perturbation with respect to a is *homogeneous* if a fulfills a. This occurs exactly when a centralizes the middle product:

$$h_1(h_1^{-1})^a h_2^a h_2^{-1} = 1 \Leftrightarrow (h_1^{-1}h_2)^a = h_1^{-1}h_2.$$

4.2.5. *Description of the Fiber.* Let $a, w \in K$, and set

$$\boldsymbol{h}^{[a|w]} = (h_1, (h_1^{-1})^a, h_2^{bw}, (h_2^{-1})^w),$$

where b fulfills a. Let W be a complement of $C_K(g_1) \oplus C_K(g_2)$ in K, and let V be a complement of $C_K(g_1)$ in K.

Proposition 29. *The fiber over* $\boldsymbol{g}$ *in* $\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in}}$ *is*

$$\boldsymbol{h}^{[V|W]} = \{\boldsymbol{h}^{[a|w]} \mid a \in V, w \in W\}.$$

Proof. First note that the perturbations of distinct Harbater-Mumford tuples are nonoverlapping, so the perturbations of the duals are all distinct members of $\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in}}$. Thus

$$|\boldsymbol{h}^{[V|W]}| = \frac{|K|^2}{|C_K(g_1)|^2 |C_K(g_2)|}.$$

By Proposition 10, this is largest possible size of the entire fiber. □

4.2.6. *Real Points in the Fiber.* Henceforth, we take the real points on the reduced Hurwitz space to be those corresponding to Nielsen tuples which are fixed by the κ_0 operator.

Proposition 30. *Let* $\boldsymbol{h}^{[a|e]} = (h_1, (h_1^{-1})^a, h_2^b, h_2^{-1})$ *be a perturbation of a Harbater-Mumford tuple. Then* $\boldsymbol{h}^{[a|e]}$ *produces a real point on the inner Hurwitz space if and only if*

$$\exists c \in C_K(g_1) \mid cab^{h_2^{-1}} \in C_K(g_2).$$

Proof. Note that $h_2^b(h_2^{-1}) = bb^{h_2^{-1}}$. Compute modulo inner equivalence

$$\begin{aligned} \boldsymbol{h}^{[a|e]}\kappa_0 &= (h_1^{abb^{h_2^{-1}}}, (h_1^{-1})^{bb^{h_2^{-1}}}, h_2, (h_2^{-1})^b) \\ &= (h_1, (h_1^{-1})^a, h_2^{abb^{h_2^{-1}}}, (h_2^{-1})^{ab^{h_2^{-1}}}). \end{aligned}$$

The result follows. □

CHAPTER V

Nielsen Graphs

1. Twist Graphs

1.1. Motivation. Let $\psi^\bullet : Z^\bullet \xrightarrow{\xi^\bullet} \mathbb{P}^1_y \xrightarrow{\varphi^\bullet} \mathbb{P}^1_x$ be a factored ramified cover of compact Riemann surfaces. Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be the branch points of $\psi^\bullet$; the branch points of $\varphi^\bullet$ are among these. Remove the branch points and the fibers over them to obtain a factored topological cover $\psi : Z \xrightarrow{\xi} Y \xrightarrow{\varphi} X$. Let $x_0 \in X$ and let $\boldsymbol{\lambda}$ be a bouquet on X based at x_0 with respect to $\boldsymbol{x}$.

In the case that the genus of Y is zero, we would like to compute a branch cycle description for ξ, given branch cycle descriptions for ψ and φ with respect to $\boldsymbol{\lambda}$. We need a tuple of classical generators on Y with respect to which this branch cycle description will be given. Computing this will not yet depend on ψ, but only on φ. Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be a branch cycle description for φ with respect to $\boldsymbol{\lambda}$.

Let $y_1, \ldots, y_m$ be the fiber over x_0. Select a basepoint, which we may as well call y_1, over x_0. For each point $y \in (\varphi^\bullet)^{-1}(\boldsymbol{x})$, we obtain a loop in Y based at y_1 around y as follows. The point y is in the fiber over x_i for some i, and corresponds to one of the disjoint cycles of g_i. Let d be the order of this cycle, and select an integer j in the support of this cycle. Then λ_i^d lifts to a loop at y_j. Since $\pi_1(X, x_0)$ acts transitively on the fiber over x_0, we may find an element $\alpha_j \in \pi_1(X, x_0)$, written as a product of λ's, such that α_j lifted to y_1 ends at y_j. Then $\beta_j = \alpha_j \lambda_i^d \alpha_j^{-1}$ is an element of $\pi_1(X, x_0)$ which stabilizes y_1, so it lifts to a loop at y_1 which proceeds towards y, goes around it, and returns to y_1.

Lifting these β's to Y gives a candidate for a bouquet on Y. Indeed, one may use these β's to find the shape of the branch cycle description for the cover ξ. However, there is no guarantee that the paths in Y do not cross, so we do not yet have a bouquet or a legitimate branch cycle description for ξ. To alleviate the situation, we need to further control the paths α_j so that we are certain the corresponding β_j's proceed from y_1 in the correct order. To do this, use a special type of graph which tracks the relative order of the edges at a vertex.

We begin by introducing much terminology with respect to graphs. We have tried to make most of this standard, as put forth in [**Tr93**], although there are some minor adjustments made for our purposes. "Twist graph" and "Nielsen graph" are our terms; a similar concepts of "fat graphs" was later found in [**MP93**]. Our usage of Nielsen graphs at level zero is similar to "dessins", as we found described in [**CG95**], which points to further literature.

1.2. Graphs. A *graph* (V, E) consists of a finite set V together with a subset of the power set of V, $E \subset \mathcal{P}(V)$, such that $e \in E \Rightarrow |e| = 2$. The elements of V are called *vertices* and the elements of E are called *edges*. For $v \in V$, let $V(v) = \{w \in V \mid \{v, w\} \in E\}$ and $E(v) = \{e \in V \mid v \in e\}$. There is an obvious bijective correspondence between $V(v)$ and $E(v)$. If $w \in V(v)$, we say that v is *adjacent* is w. If $v \in e$, we say that e *involves* v. The *degree* of a vertex v, denoted $d(v)$, is $|V(v)|$.

Let (V_1, E_1) and (V_2, E_2) be graphs. A *morphism* from (V_1, E_1) to (V_2, E_2) is a function

$$f : V_1 \to V_2 \quad \text{such that} \quad \{v, w\} \in E_1 \Rightarrow \{f(v), f(w)\} \in E_2.$$

This produces the category of graphs, and defines the notion of equivalence of graphs as isomorphism in this category.

A *subgraph* (V_0, E_0) of a graph (V, E) consists of a subsets $V_0 \subset V$ and $E_0 \subset E$ such that $\{v, w\} \in E_0 \Rightarrow v, w \in V_0$. In this case we write $(V_0, E_0) \leq (V, E)$. If (V_1, E_1) and (V_2, E_2) are subgraphs of (V, E), say that $(V_1, E_1) \leq (V_2, E_2)$ if $V_1 \subset V_2$ and $E_1 \subset E_2$; that is, if (V_2, E_2) is a subgraph of (V_1, E_1). This places a partial ordering on the collection of subgraphs of a given graph.

Let (V, E) be a graph and construct a topological space $\mathrm{CW}(V, E)$ as follows. For each edge $e \in E$, let $I_e = [0, 1]$ be a copy of the closed unit interval and let $\sigma_e : \partial I_e \to e$ be injective. Set $U = \coprod_{e \in E} I_e$ and collect the σ_e's together to produce a function $\sigma : \partial U \to V$. Then $\mathrm{CW}(V, E)$ is the fiber coproduct of σ and the inclusion $\iota : \partial U \to U$:

$$\begin{array}{ccc} \partial U & \xrightarrow{\iota} & U \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\omega_E} \\ V & \xrightarrow[\omega_V]{} & \mathrm{CW}(V, E) \end{array}$$

If $(V_0, E_0) \leq (V, E)$, then $\mathrm{CW}(V_0, E_0)$ is naturally a subspace of $\mathrm{CW}(V, E)$.

A *drawing* of a graph (V, E) on a smooth manifold X is a continuous function $f : \mathrm{CW}(V, E) \to X$ such that

- **(a)** for $v \in V$ and $x \in \mathrm{CW}(V, E)$, $f(x) = f(\omega_V(v)) \Rightarrow x = \omega_V(v)$;
- **(b)** for $t_1, t_2 \in I_e$, $f(t_1) = f(t_2) \Rightarrow t_1 = t_2$;
- **(c)** the composition $f \circ \omega_E : U \to X$ is smooth on the interior of U.

A drawing of (V, E) induces a drawing of any subgraph of (V, E). Any graph can be drawn on any Riemann surface by selecting the image of the vertex set and selecting paths according to the edges. In particular, any graph can be drawn on $\mathbb{C}$.

A drawing is an *embedding* if it is injective. A graph is *planar* if it can be embedded in $\mathbb{C}$. This is equivalent to the ability to embed the graph in $\mathbb{P}^1$. Every graph can be embedded in some compact Riemann surface; we briefly describe why. First draw the graph on $\mathbb{P}^1$. There will be a finite number of points of intersection of the edges. For each intersection, attach a tubular handle to act as a bridge and remove the point of intersection. This increases the genus by 1 for every point of intersection eliminated from the drawing.

The *genus* of a graph is the minimum genus of a compact Riemann surface in which the graph can be embedded. We point out that any embedding can be "extended" to an embedding into a manifold of arbitrarily higher genus by attaching handles along the boundaries of disks which avoid the image of the graph (see [**Tr93**] chapter 7).

An embedding of a graph into a Riemann surface X induces, for each $v \in V$, a cyclic permutation of $V(v)$ as follows. Select a smooth loop λ around a vertex which intersects every edge attached to that vertex exactly once, and intersects no vertices or other edges. Construct a cyclical ordering of the edges attached to v by following λ in a clockwise direction. This produces a transitive (cyclic) permutation of $V(v)$ which is independent of λ and dependent only on the isotopy class of the embedding.

1.3. Walks and Trees. A *walk* in a graph (V, E) is a finite sequence of vertices $(v_0, \ldots, v_n)$ with $n \geq 1$ such that $\{v_i, v_{i+1}\} \in E$ for $i \in \{0, \ldots, n-1\}$; pairs of consecutive vertices in a walk are called the edges of the walk. The number n is called the *length* of the walk. We call v_0 the *initial* vertex and v_n the *terminal* vertex of the walk. Similarly, $\{v_0, v_1\}$ and $\{v_{n-1}, v_n\}$ are the initial and terminal edges. The graph is *connected* if for every $v_1, v_2 \in V$ there exists a walk in V whose initial vertex is v_1 and whose terminal vertex is v_2.

A *subwalk* of a walk $(v_0, \ldots, v_n)$ is a walk of the form $(v_0, \ldots, v_m)$, with $m \leq n$. A *corner* is a walk of length 2. A *trail* is a walk with distinct edges. A *simple trail* is a walk with distinct vertices except possibly at the initial and terminal positions. A *circuit* is a walk whose initial vertex equals its terminal vertex. A *cycle* is a simple trail which is a circuit.

A *tree* is a connected graph which does not admit a circuit. Note that every trail in a tree is simple. Given two vertices in a tree, there is exactly one trail from one to the other. A *subtree* is a subgraph which is a tree. In the partial ordering of subgraphs, a maximal subtree is precisely a subtree which contains all vertices.

A *root* in a graph is a specified vertex, and a *rooted graph* (V, E, v) is a connected graph (V, E) together with a root v. A *rooted walk* in (V, E, v) is a walk whose initial vertex is v. In a rooted tree, there is a unique rooted trail terminating at every vertex other than the root.

A *bush* is a rooted tree (V, E, v) such that $v \in e$ for every $e \in E$. Given a rooted graph (V, E, v), obtain the corresponding rooted bush $(V, E^{(v)}, v)$ by setting

$$E^{(v)} = \{\{v, w\} \mid w \in V \smallsetminus \{v\}\}.$$

We move towards setting up an induction which converts an embedded tree into the corresponding embedded bush with special homotopy properties.

Let (V, E, v) be a rooted tree and let $w \in V \smallsetminus \{v\}$. We construct a new rooted tree $(V, E_{(w)}, v)$ whose vertex set is V such that w has a unique adjacent vertex. Let $\{w_1, w\}$ be the terminal edge of the unique trail in V from v to w. Note that since V is a tree, there are no edges in E between distinct elements of $V(w)$. Set

$$E_{(w)} = (E \smallsetminus \{\{w, w_2\} \mid w_2 \in V(w) \smallsetminus \{w_1\}\}) \cup \{\{w_1, w_2\} \mid w_2 \in V(w) \smallsetminus \{w_1\}\}.$$

Repeating this process for every vertex other than the root leads to a bush.

1.4. Twist Graphs. A *twist structure* on a graph (V, E) is a function $\delta : V \to \mathrm{Sym}(V)$ such that $\delta(v)$ is an element of order $d(v)$ which fixes every point of $V \smallsetminus V(v)$; that is, $\delta(v)$ is a cycle which acts transitively on $V(v)$. We may write δ_v instead of $\delta(v)$.

A *twist graph* (V, E, δ) is a graph (V, E) together with a twist structure δ on (V, E).

Let (V, E, δ) and (W, F, ϵ) be twist graphs. A *morphism* from (V, E, δ) to (W, F, ϵ) is a graph morphism $f : V \to W$ together with a group homomorphism $f_* : \delta(V) \to \epsilon(W)$ such that $\epsilon \circ f = f_* \circ \delta$. This produces the category of twist graphs and defines equivalence in this category.

Let $c = (v_0, v_1, v_2)$ be a corner in a twist graph. The *twist* of c, denoted $\tau(c)$, is defined to be the minimum positive integer t such that $\delta^t_{v_1}(v_0) = v_2$. Note that $\tau(v_0, v_1, v_0) = d(v_1)$.

Let c_1 and c_2 be two corners in a twist graph with the same initial edge. We say that $c_1 \leq c_2$ if $\tau(c_1) \leq \tau(c_2)$. This puts a partial ordering on the set of corners of a twist graph.

Let $W_1 = (v_0, \ldots, v_n)$ and $W_2 = (w_0, \ldots, w_m)$ be two walks in a twist graph with the same initial edge. We say that $W_1 \leq W_2$ if W_2 is a subwalk of W_1, or if $\tau(v_{j-1}, v_j, v_{j+1}) \leq \tau(w_{j-1}, w_j, w_{j+1})$, where $v_i = w_i$ for $i \leq j$ and $v_{j+1} \neq w_{j+1}$. This imposes a partial ordering on walks in a twist graph. In a graph with at least one edge, walks can always be made longer, so there are no minimal walks.

A drawing of a graph (V, E) on a Riemann surface induces a unique twist structure δ on the graph; the vertices adjacent to a given vertex v are permuted by δ_v in the order they emerge from v. Refer to this as the twist structure induced by the drawing.

A *twist drawing* of a twist graph (V, E, δ) is a drawing of the twist graph on a Riemann surface such that the twist structure is induced by the drawing. A twist drawing of any twist graph exists on any Riemann surface. A *twist embedding* of a twist graph (V, E, δ) is a twist drawing which is an embedding. Every twist graph has a twist embedding; this can be seen by taking a twist drawing and resolving the intersections as we have previously discussed. The *twist genus* of a twist graph is the minimum genus of a compact Riemann surface into which a twist embedding exists. Clearly this is greater than or equal to the genus of the underlying graph. A twist graph is *contrived* if the twist genus is greater than the genus, and a twist embedding is contrived if the genus of the image is greater than the genus of the graph.

1.5. Rooted Twist Graphs. A *root* (v, e) for a twist graph (V, E, δ) consists of a vertex $v \in V$ and an edge $e \in E$ with $v \in e$. A *rooted twist graph* (V, E, δ, v, e) is a connected twist graph (V, E, δ) together with a choice of root (v, e). We call v the *root vertex* and e the *root edge*. *Rooted subgraphs* of a rooted twist graph contain the edge e.

Let (V, E, δ, v, e) be a rooted twist graph. We extend the partial order on walks initiating at v as follows. Let $e = \{v, w\}$. Let $W_1 = (v_0, \ldots, v_n)$ and $W_2 = (w_0, \ldots, w_m)$ be walks in V with $v_0 = w_0 = v$ but $v_1 \neq w_1$. Declare $W_1 \leq W_2$ if $\tau(w, v, v_1) \leq \tau(w, v, w_1)$. In this way we obtain a linear ordering on the set of walks in V initiating at v.

Let (V, F, v) be a maximal rooted subtree of a rooted twist graph (V, E, δ, v, e), with $e = \{v, w\}$. Distinct rooted trails in (V, F, v) terminate in distinct vertices, so the linear ordering on these trails produces a linear ordering on $V \setminus \{v\}$. This in turn induces a twist structure $\delta^{(v)}$ on the rooted bush $(V, E^{(v)}, v)$; the permutation attached to v cycles the adjacent vertices according to the above linear ordering. Now w is the maximum vertex in this order, and the trails in $(V, E^{(v)}, \delta^{(v)}, v, e)$ are linearly ordered, correspond to the trails in (V, F, v) according to the terminal vertex, and this correspondence preserves the linear ordering. Call $(V, E^{(v)}, \delta^{(v)}, v, e)$ the corresponding rooted twist bush.

We now take a twist embedding of (V, E, δ, v, e) and a maximal subtree (V, F) to derive a twist embedding for the corresponding rooted twist bush. We may do this by removing extra edges from one vertex at a time. Recall that given $w \in V \setminus \{v\}$, we obtained a new rooted graph $(V, F_{(w)}, v)$. To obtain a compatible embedding of this graph, we proceed one edge at a time.

Let w_0 be the vertex that precedes w in the unique rooted trail from v to w. Select w_1 so that $\tau(w_0, w, w_1)$ is minimal. If $w_1 = w_0$, we are done, so assume that $w_0 \neq w_1$. Set

$$E_{[w]} = (E \cup \{\{w_0, w_1\}\}) \setminus \{\{w, w_1\}\},$$

and consider the graph $(V, E_{[w]})$. We find a specific embedding of this graph.

Let $f : \mathrm{CW}(V, E) \to X$ be a twist embedding of (V, E, δ, v, e) into a compact Riemann surface X. Let U be a simply connected open neighborhood of $f(\omega_V(w))$ with smooth boundary and the property that the intersection of ∂U with $f(\mathrm{CW}(V, E))$ consists of exactly one point for each edge involving w.

Consider a path which moves from w_0 along $f(\omega_E(I_{\{w_0,w\}}))$ up to its intersection with ∂U, then along ∂U in a clockwise fashion up to its intersection with $f(\omega_E(I_{\{w,w_1\}}))$, then from this intersection point to w_1. There is a slight homotopy of this path in $X \setminus \omega_V(v)$ which ends in smooth path which does not intersect the interior of $f(\omega_E(I_{\{w_0,w\}} \cup I_{\{w,w_1\}}))$; call the resulting path μ. Let $\nu : [0, 1] \to I_{\{w_0,w_1\}}$ be an appropriate parametrization, and define

$$f_{[w]} : \mathrm{CW}(V, E_{[w]}) \to X \quad \text{by} \quad f_{[w]}(x) = \begin{cases} f(x) & \text{if } x \in f(\mathrm{CW}(V, E)) \setminus \omega_E(I_{\{w,w_1\}}); \\ \mu(t) & \text{if } x = \omega_E(\nu(t)). \end{cases}$$

Now $f_{[w]}$ is an embedding of $\mathrm{CW}(V, E_{[w]})$ into X.

Inductively define $E_{[w]^{i+1}} = (E_{[w]})_{[w]}$. Repeat the above process to take an embedding of $(V, E_{[w]^i})$ and produce an embedding of $(V, E_{[w]^{i+1}})$. For $n = d(w) - 1$, we have $E_{(w)} = E_{[w]^n}$, and we obtain an embedding of $(V, E_{(w)})$ with properties inherent from the construction. Repeat this process for every vertex w other than the root to obtain an embedding of the bush $(V, E^{(v)})$.

2. Nielsen Graphs

2.1. Nielsen Graphs.

2.1.1. *Nielsen Graphs.* A *Nielsen graph* of degree m and rank r is a connected twist graph with the following properties:

- **(a)** The vertex set is partitioned in $r+1$ blocks labeled 0 through r. Blocks 1 through r are called *positive* blocks. Vertices in block 0 are called *hubs* and vertices in positive blocks are called *nodes*.
- **(b)** There are exactly m hubs, labeled 1 through m.
- **(c)** For every hub and every positive block there exists a unique edge between the hub and a node in the block. No other edges exist.
- **(d)** For every hub, the associated cycle is of the form $(u_1 \ldots u_r)$, where u_i is in block i.
- **(e)** For every hub, every minimal trail initiating at that hub is a cycle.

A *morphism* between Nielsen graphs of the same rank is a twist morphism which preserves the block numbers. This produces the category of Nielsen graphs, and defines equivalence in this category. As we will see, a Nielsen graph is an uncontrived twist graph.

2.1.2. *Nielsen Tuples produce Nielsen Graphs.* There is an equivalence of categories between Nielsen tuples and Nielsen graphs. We briefly describe this.

Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be a Nielsen tuple of degree n; thus $G = \langle \boldsymbol{g} \rangle \leq S_n$ is a transitive subgroup, and $\Pi \boldsymbol{g} = 1$. Then $\boldsymbol{g}$ produces a Nielsen graph as follows:

- the hubs are the integers $1, \ldots, n$;
- the nodes in block i are the disjoint cycles of g_i, including singletons;
- the edges are pairs $\{j, c\}$ where j is a hub and c is a cycle involving j;
- $\delta(c) = c$ for c a node.

Similarly, a Nielsen graph produces a Nielsen tuple by viewing $\delta(c)$, for c a node, as an element of S_m, and taking the product of such cycles in block i to obtain g_i. The transitivity is given by the connectedness of V and the product one condition is assured by Nielsen graph property **(e)**.

2.2. Branch Cycle Designs.

2.2.1. *Branch Cycle Designs.* A *branch cycle design* is an isotopy class of uncontrived twist embeddings of a Nielsen graph in a compact orientable manifold. These canonically produce covers of the Riemann sphere, via their correspondence with branch cycle descriptions, which we now describe.

2.2.2. *Embedded Bushes produce Bouquets.* Let (V, E, δ, v, e) be a twist bush embedded in a compact Riemann surface $X^\bullet$. Let $\{x_0, x_1, \ldots, x_r\}$ be the image in $X^\bullet$ of the vertex set, where x_0 is the image of v and $\{x_0, x_r\}$ is the image of e. Let μ_i be the path from x_0 to x_i determined by the embedding. Let Δ_i be a small disk around x_i with the property that its boundary intersects the image of $\mathrm{CW}(V, E)$ in a single point, say x_i^*. Let δ_i be a parametrization of the boundary of Δ_i in a clockwise orientation. Let μ_i^* be the path from x_0 to x_i^* along μ_i. Set $\eta_i = \mu_i^* \cdot \delta_i \cdot (\mu_i^*)^{-1}$. Then $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_r)$ is a bouquet of classical loops on $X^\bullet$ with respect to $(\boldsymbol{x}, x_0)$.

2.2.3. *Bouquets produce Embedded Bushes.* Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be a tuple of distinct points in a compact Riemann surface $X^\bullet$ and let $X = X^\bullet \smallsetminus \boldsymbol{x}$. Let $x_0 \in X$ and let $\boldsymbol{\lambda}$ be a bouquet with respect to $(\boldsymbol{x}, x_0)$, chosen so that Δ_i, δ_i, and μ_i^* all exist as above with $\lambda_i = \mu_i^* \cdot \delta_i \cdot (\mu_i^*)^{-1}$.

Set $V = \{0, \ldots, r\}$ and $E = \{\{0, i\} \mid i = 1, \ldots, r\}$. Let δ_0 be the cyclical permutation of $1, \ldots, r$ in that order; we obtain an associated twist bush (V, E, δ). Select μ_i^+ to be a smooth path in Δ_i from x_i^* to x_i such that $\mu_i = \mu_i^* \cdot \mu_i^+$ is smooth. Define $f : \mathrm{CW}(V, E) \to X^\bullet$ by $f \restriction_{\omega_E(I_{0,i})} = \mu_i$; this is a twist embedding of the associated twist bush.

2.2.4. *Ramified Covers produce Descriptions.* A ramified cover of $\mathbb{P}^1$ of degree n with r branch points produces a branch cycle description of degree n and rank r. To fix notation, we briefly recall this process. Notation will accumulate in the rest of this section.

Let $\varphi^\bullet : Y^\bullet \to \mathbb{P}^1$ be a ramified cover of degree m. Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be the branch points of the cover. Let $X = \mathbb{P}^1 \smallsetminus \boldsymbol{x}$ and $Y = Y^\bullet \smallsetminus \varphi^{-1}(\boldsymbol{x})$ and obtain a topological cover $\varphi : Y \to X$. Select a basepoint $x_0 \in X$ and let $y_1, \ldots, y_m$ be the fiber over x_0. Then $\pi_1(X, x_0)$ acts on this fiber through path lifting, creating a permutation representation $\rho : \pi_1(X, x_0) \to S_m$.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be a bouquet of classical loops with respect to $(\boldsymbol{x}, x_0)$. Let $[\lambda]$ denote the homotopy class of a loop λ. Let $g_i = \rho([\lambda_i])$ and $\boldsymbol{g} = (g_1, \ldots, g_r)$. Then $\boldsymbol{g}$ is the branch cycle description of $\varphi^\bullet$ with respect to $\boldsymbol{\lambda}$.

2.2.5. *Bouquets in the Base Space produce Twist Embeddings of Designs.* Just as selection of a bouquet puts geometric meaning to a branch cycle description in the form of a ramified cover, it simultaneously induces a twist embedding of the corresponding Nielsen graph into the covering space, thus giving a branch cycle design.

Continue notation from section 2.2.4. Set

$$V_X = \{0, \ldots, r\} \quad \text{and} \quad E_X = \{\{0, i\} \mid i = 1, \ldots, r\}.$$

Let $\delta_X : V_X \to \mathrm{Sym}(V_X)$ be given by the identity for $i > 0$ and $\delta(0) = \texttt{(1 2 ... } r\texttt{)}$. We have seen that the bouquet $\boldsymbol{\lambda}$ produces a twist embedding of the twist bush (V, E, δ) via paths $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ as constructed in subsection 2.2.3.

Let (V_Y, E_Y, δ_Y) be the branch cycle design corresponding to $\boldsymbol{g}$.

The preimage in $Y^\bullet$ of the twist embedding of the twist bush (V_X, E_X, δ_X) produces a twist embedding on $Y^\bullet$ of the branch cycle design (V_Y, E_Y, δ_Y) corresponding to $\boldsymbol{g}$. The hubs map to the preimages of the basepoint x_0 and the nodes in block i map to the preimages of the i^{th} branch point x_i. Specifically, $f_Y : \mathrm{CW}(V_Y, E_Y) \to Y^\bullet$ is constructed by defining the image of an arbitrary edge I_e; now e is an edge between a hub, say the integer j, and a node, say $c_{i,l}$ (the l^{th} cycle in the i^{th} branch permutation). Then $f_Y(\omega_E(I_e))$ equals the lift of μ_i to y_j.

2.2.6. *Twist Embeddings of Designs produce Bouquets in the Covering Space.* Let (V, E, δ) be a branch cycle design and let $f : \mathrm{CW}(V, E) \to Y^\bullet$ be a twist embedding of (V, E, δ) into a compact Riemann surface $Y^\bullet$. Let Y equal $Y^\bullet$ with the image of V removed.

Select a vertex $v \in V$ and an edge $e \in E(v)$ to use as a root. Let $(V, E^{(v)}, \delta^{(v)}, v, e^{(v)})$ be the twist bush corresponding to (V, E, δ, v, e) as described in section 1.5. In that section we selected a maximal rooted subtree of (V, E, δ, v, e) and produced a twist embedding $f^{(v)} : \mathrm{CW}(V, E^{(v)}) \to Y^\bullet$ compatible with the ordering of the vertices. Let y_0 be the image of v under this embedding. As discussed in subsection 2.2.2, this embedding produces a bouquet based at y_0.

Proposition 31. *Let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s)$ be the bouquet corresponding to the twist bush embedding $f^{(v)}$. Then $\boldsymbol{\eta}$ is a bouquet of classical loops on Y based at y_0.*

Proof. By construction the loops emanate from y_0 in the given order. Also by construction, each is a classical loop. □

2.2.7. *Classical Generators in the Covering Space.* Continue notation from subsection 2.2.5. Let $v_0 \in V_Y$ be a hub and (V_Y, E_Y^T, v_0) be a maximal rooted subtree of (V_Y, E_Y). Let $\boldsymbol{\eta}$ be the bouquet produced by the embedding $f_Y : \mathrm{CW}(V_Y, E_Y) \to Y\bullet$, with respect to these choices. We now describe the bouquet $\boldsymbol{\eta}$ on the covering space combinatorially in terms of the bouquet $\boldsymbol{\lambda}$ on the base space. This will allow us to compute information about covers of X factoring through Y using finite group theory.

Let $W = (v_0, \ldots, v_n)$ be a walk in (V_Y, E_Y); the embedding f_Y allows us to view W as a path in $Y^\bullet$ between the images of v_0 and v_n. If n is odd, then v_n is a node; associate to this walk a loop $\beta(W)$ in X based at x_0, and hence an element of $\pi_1(X, x_0)$, as follows.

Let $b(v)$ denote the block of vertex v. Set $b_j = b(v_j)$, $d_j = d(v_j)$, and $t_j = \tau(v_{j-1}, v_j, v_{j+1})$. Define

$$\alpha(W) = \prod_{\substack{0<j<n \\ j \text{ odd}}} \lambda_{b_j}^{t_j}.$$

The lift of $\alpha(W)$ to Y is a path in Y between the images of v_0 to v_{n-1}, that is, between two points in the fiber over x_0. The loop in X associated to the walk W is

$$\beta(W) = \alpha \lambda_{b_n}^{d_n} \alpha^{-1}.$$

The hub v_0 maps to some element of $\varphi^{-1}(x_0) = \{y_1, \ldots, y_m\}$, say y_0. Select $e_0 = \{0, r\}$ to use as a root edge so that the order of the adjacent vertices to v_0 corresponds to the order of the branchpoints as determined by $\boldsymbol{\lambda}$. This produces a linear order on the set of walks in (V_Y, E_Y^T) initiating at v_0. Let $W_1, \ldots, W_s$ be the trails of odd length in (V_Y, E_Y^T), with $W_i \leq W_j$ when $i \leq j$. Note that there is a unique such trail terminating at each node. Let $\beta_j = \beta(W_j)$.

Proposition 32. *Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$. Then*

(a) *η_i is homotopic in Y to the lift of β_i to y_0;*

(b) *$\langle \boldsymbol{\beta} \rangle = \varphi_*(\pi_1(Y, y_0))$;*

(c) *$\Pi \boldsymbol{\beta} = 1$.*

Proof. It suffices to demonstrate **(a)**. This follows from the construction, because the λ's proceed around x_i in a clockwise direction, and the η's were formed by avoiding the preimages of x_i's by tracing circles around them in a clockwise direction. □

We call $\boldsymbol{\beta}$ the tuple of *design generators* produced from the branch cycle design, and the selected rooted maximal tree.

3. Condensing, Crunching and Splicing

Let $\psi : Z \xrightarrow{\xi} Y \xrightarrow{\varphi} X$ be a factored cover, where $\varphi : Y \to X$ is as above. We assume that the genus of Y is zero. Let $m = \deg(\varphi)$, $d = \deg(\xi)$, and $n = \deg(\psi)$ so that $n = md$.

We use the branch cycle design produced above to construct algorithms for producing the branch cycle descriptions for ξ and ψ. *Condensing* is the process of constructing a branch cycle description for φ given ones for ψ and ξ. *Crunching* is the process of constructing a branch cycle description for ξ given ones for ψ and φ. *Splicing* is the process of constructing a branch cycle description for ψ given ones for φ and ξ.

3.1. Condensing. Let $\boldsymbol{h}$ be a branch cycle description for ψ with respect to a bouquet $\boldsymbol{\lambda}$ on X and suppose that $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ is a function which describes the map between the enumerated fibers over x_0 in $Z\bullet$ and $Y^\bullet$ given by ξ. Then the fibers of f are blocks of imprimitivity for the action of $H = \langle \boldsymbol{h} \rangle$; then $f_* : H \to S_m$ is a well defined homomorphism. Let $G = f_*(H)$ and $\boldsymbol{g} = f_*(\boldsymbol{h})$ so that $G = \langle \boldsymbol{g} \rangle$. Now $\boldsymbol{g}$ is a branch cycle description for φ with respect to $\boldsymbol{\lambda}$.

3.2. Crunching. Let $\boldsymbol{h}$ be a branch cycle description for ψ with respect to the paths $\boldsymbol{\lambda}$. We wish to find a branch cycle description for ξ with respect to $\boldsymbol{\beta}$; to do this, we need to find the action of $\boldsymbol{\beta}$ on the fiber over y_k.

The action of β_i on the fiber in Z over x_0 is given by plugging $\boldsymbol{h}$ into the description of β_i as a product of λ's; that is, we find the image of $\boldsymbol{\beta}$ in the monodromy group H of ψ via the homomorphism $\pi_1(X, x_0) \to H \leq S_n$ given by path lifting. Now these paths act on the fiber over y_1 because each stabilizes y_1. Compute the action of $\boldsymbol{\beta}$ on this fiber by restriction. This produces a branch cycle description for ξ.

3.3. Splicing. Let $\boldsymbol{u}$ be a branch cycle description for ξ with respect to the paths $\boldsymbol{\beta}$. We wish to find a branch cycle description for ψ with respect to the paths $\boldsymbol{\lambda}$; to do this, we need to find the action of $\boldsymbol{\lambda}$ on the fiber in Z over x_0. For reference in future work, we offer (without proof) an algorithm to do this, which has been implemented in [**GAP**]. This subsection may be skipped.

Let $\boldsymbol{g}$ be the branch cycle description of φ with respect to $\boldsymbol{\lambda}$, and let D be the Nielsen graph produced by $\boldsymbol{g}$. Let T be a maximal tree in D based at 1 (corresponding to $y_1 \in Y$).

First we enumerate the fiber in Z over x_0. Define a function

$$\mathrm{spl} : \mathbb{N}_m \times \mathbb{N}_d \to \mathbb{N}_n \quad \text{by} \quad \mathrm{spl}(i, j) = (i-1)d + j.$$

This function is bijective. The components of the inverse are defined as

$$\mathrm{bot} : \mathbb{N}_n \to \mathbb{N}_m \quad \text{by} \quad \mathrm{bot}(i) = [(i-1)/d] + 1,$$

where $[x]$ is largest integer less than x, and

$$\mathrm{top} : \mathbb{N}_d \to \mathbb{N}_d \quad \text{by} \quad \mathrm{top}(i) = ((i-1) \pmod{d}) + 1.$$

In this way, each integer between 1 and n has a top part and a bottom part. We enumerate the fiber in Z over x_0 so that $\xi(z_i) = y_{\mathrm{bot}(i)}$. Thus it remains to attach the top part, which is an integer between 1 and d, to each element of $\xi^{-1}(y_j)$ for $j \in \mathbb{N}_m$.

The existence of $\boldsymbol{u}$ presupposes assignment of the top part for the fiber over y_1; let $\{z'_1, \ldots, z'_d\}$ be this fiber. In order to push this enumeration to the fibers over the other y_j's, construct a path in Y from y_1 to y_j. We do this as follows. Select one of the branch points on X to be the primary branch point for this process; for simplicity, we choose the first. Then j in involved in a unique cycle c of λ_1, and there is an edge from y_j to c in D. Let W' be the unique trail in T from y_1 to c. Either y_j is the second to last vertex in W', or y_j is not in W'. In the first case, let W be the subwalk of W' which terminates at y_j. In the second, let W be W' extended by y_j; in this case, W is a walk in D but may not be in T. Either way, W produces a well-defined homotopy class of a path in Y from y_1 to y_j. Lift this path to the various points of the fiber in Z over y_1 to transfer the top enumeration to the fiber in Z over y_j. This enumerates $\psi^{-1}(x_0)$ as $\{z_1, \ldots, z_n\}$.

Now we need to compute the action of the classical generators $\boldsymbol{\lambda}$ for $\pi_1(X, x_0)$ on this fiber. The algorithm is:

(1) For $j \in \{1, ..., m\}$, apply the action of u_i to the fiber over y_j; that is, construct a permutation h_i of $\{1, \ldots, n\}$ so that $h_i^*(k) = \mathrm{spl}(\mathrm{bot}(k), \mathrm{top}(k)^{u_i})$.

(2) Construct a conjugator v so that if $h_i = (h_i^*)^v$, then $(h_1, \ldots, h_s)$ is a branch cycle description for ψ.

Construct v as follows: Let $W_{i,j}$ denote the walk described above, only this time to any node involving y_j over branch point x_i. Then $W_{1,j}$ and $W_{i,j}$ both terminate at v_j. Concatenate the second to last vertex of $W_{1,j}$ to $W_{i,j}$ and call this L_1. Similarly concatenate the second to last vertex of $W_{i,j}$ to $W_{1,j}$ and call this L_2. Let W_k denote the walk corresponding to the k^{th} entry of $\boldsymbol{u}$. Then v is the product of u_k, in the order of the walks, for every k satisfying $L_1 \leq W_k$ and $W_k \leq L_2$, when

$L_1 \le L_2$. If $L_2 < L_1$, then switch the roles of L_1 and L_2 in the above statement and take the inverse of the product thus obtained.

We need to explain how this works. Let's first give a simple case. Let $m = 5$ and $d = 8$. Suppose that the cycle of λ_1 containing 1 is `(1 4 5 2 3)`. and that $\beta_1 = \lambda_1^5$. Lifting λ_1 to y_1, and then to the fiber $\{z'_1, \ldots, z'_8\}$ over y_1, induces and enumeration of the fiber over y_4; specifically, the endpoint of the lift to z'_3 is z_{27} since $\mathrm{spl}(4,3) = 3d + 3 = 27$. Powers of λ_1 push the enumeration around the cycle until it gets back to the beginning, at λ^5, and these fiber points have already been enumerated. But λ^5 is just β_1, and we are given the action of β_1 on $\{z'_1, \ldots, z'_8\}$; it is u_1.

All of the β's are conjugates of powers of some given λ, and the above point of view carries over to conjugates (the conjugation is part of the numbering scheme). So this gives the action of λ_i with respect to enumeration by paths which are lifts of paths to a point near ramification of λ_i. The trick is to relate the enumeration induced by varying the branch point. This is done by constructing the path which proceeds to a node over branch point 1 containing an integer j and the path which proceeds to a node over branch point i containing j. Concatenating the first path with the inverse of the second creates a loop in Y. This loop is the product of some classical generators for $\pi_1(Y, y_0)$. The action of this loop on the enumeration of the fiber over y_1 is exactly the renumbering we need to accomplish.

The walks that correspond to classical generators of $\pi_1(Y, y_0)$ are either *completely inside* the loop given by $W_{1,j} * W_{i,j}^{-1}$, or intersect it by entering the loop through the point of concatenation y_j, are exactly those W_k described by $L_1 \le W_k \le L_2$.

4. Full and Final Ramification

4.1. Designs on Reduced Rank 4 Hurwitz Spaces.

4.1.1. *Set Up.* In our study of collections of ramified covers with fixed monodromy groups, our usage of branch cycle designs is not on the covers themselves but rather on the reduced rank 4 inner Hurwitz spaces which parameterize them. We outline the idea.

Let $f : H \to G$ be a p-Frattini cover of finite groups, and let $\boldsymbol{C}$ be a rank 4 tuple of conjugacy classes with $\gcd(\mathrm{ord}(\boldsymbol{C}), p) = 1$. Let $\mathcal{H}_2 \subset \mathcal{H}(H, \boldsymbol{C})^{\mathrm{in,rd}}$ be a component and let $\mathcal{H}_1 \subset \mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ be its image, yielding a factored ramified cover

$$\psi^\bullet : \mathcal{H}_2^\bullet \xrightarrow{\xi^\bullet} \mathcal{H}_1^\bullet \xrightarrow{\varphi^\bullet} \mathcal{J}_4^\bullet.$$

Computation of the braid action on the orbit $O \subset \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ corresponding to $\mathcal{H}_1$ gives a permutation representation $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_\infty) \mapsto \overline{\boldsymbol{\gamma}}$, where $\langle \overline{\boldsymbol{\gamma}} \rangle \le S_{|O|}$ is the monodromy group of $\varphi^\bullet$; use the Riemann-Hurwitz formula to compute the genus of $\mathcal{H}_1^\bullet$. If this is zero, then the branch cycle design for $\overline{\boldsymbol{\gamma}}$ produces classical generators for $\pi_1(\mathcal{H}_1^\circ, y_0)$, where $y_0 \in kH_1^\circ$.

The design generators are written in terms of γ_0, γ_1, and γ_∞, and y_0 corresponds to a cover which is given by a Nielsen tuple $\boldsymbol{g} = \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$. To understand $\mathcal{H}_2$, we may apply these generators directly to the preimage of $\boldsymbol{g}$ in $\mathrm{Ni}(H, \boldsymbol{C})^{\mathrm{in,rd}}$ via braiding, obtaining an action on the fiber over y_0 which produces the monodromy group of the cover $\mathcal{H}_2^\bullet \to \mathcal{H}_1^\bullet$.

4.1.2. *Trivial Action.* The branch cycle designs may produce complicated generators. We would like to be able to rule out as many as we can, and then if possible, simplify the remaining paths. By rule out, we mean deduce that they have trivial action. When this is the case, no ramification occurs above the node which corresponds to the generator with trivial action, and we can eliminate that generator from the design tuple.

Trivial action here is of two types: trivial action on the next Nielsen class under consideration, or trivial action for any cover by a reduced rank 4 Hurwitz space. We consider the latter case first.

4.2. Final Ramification. Let $\mathcal{H}$ be a component of an reduced rank 4 inner Hurwitz space, with associated cover $\varphi : \mathcal{H} \to \mathcal{J}_4$. Let $y \in \varphi^{-1}(0,1)$ be a node of φ. Recall the mapping class cover ${\mathcal{V}_4}^{\mathrm{rd}} \to \mathcal{J}_4$, which is universal for reduced rank 4 Hurwitz spaces. We say that y is *finally ramified* in $\mathcal{H}$ if y is not a branch point of ${\mathcal{V}_4}^{\mathrm{rd}} \to \mathcal{J}_4$.

Any node in $\mathcal{H}_1$ over $0, 1 \in \mathcal{J}_4$ which ramifies in $\mathcal{H}_1 \to \mathcal{J}_4$ will have an unramified fiber in $\mathcal{H}_2$. This is because ramification over these points is always of prime order (order 3 and order 2 respectively). Such nodes are finally ramified. If all nodes over γ_i for $i = 0, 1$ are finally ramified, we say that γ_i is finally ramified in $\mathcal{H}_1$.

4.3. Full Ramification. Let $y \in \mathcal{H}_1$ be a node (that is, $\varphi(y) \in \{0, 1, \infty\}$). We say that y is *fully ramified* with respect to $f : H \to G$ if it is not a branch point for the cover $\xi^\bullet : \mathcal{H}_2^\bullet \to \mathcal{H}_1^\bullet$.

Suppose $\varphi(y) = \infty$; let δ_y be the γ_∞ cycle corresponding to y. Let i be an integer in the support of δ_y, corresponding to a point $y_i \in \mathcal{H}_1$ in the fiber of $\mathcal{H}_1 \to \mathcal{J}$ over a basepoint in $\mathcal{J}$, and let $\boldsymbol{g} = (g_1, g_2, g_3, g_4) \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ be the reduced Nielsen tuple corresponding to y_i.

The order of δ_y is the length of the γ_∞ orbit containing $\boldsymbol{g}$. Since γ_∞ acts as Q_2, this orbit length is tied to the *middle product order* $\mathrm{mpo}(\boldsymbol{g}) = \mathrm{ord}(g_2 g_3)$. Specifically, $\mathrm{ord}(\delta_y)$ divides $2 \cdot \mathrm{mpo}(\boldsymbol{g})$.

Let $z \in \mathcal{H}_2$ be in the fiber over y, with associated Nielsen tuple $\boldsymbol{h} = (h_1, h_2, h_3, h_4)$, so that $f(\boldsymbol{h}) = \boldsymbol{g}$. If $\mathrm{ord}(\delta_z) = \mathrm{ord}(\delta_y)$, then z is not ramified over y. If $\mathrm{ord}(\delta_y) = 2 \cdot \mathrm{mpo}(\boldsymbol{g})$, then $\mathrm{ord}(\delta_z) = 2 \cdot \mathrm{mpo}(\boldsymbol{h})$, and z ramifies if and only if $\mathrm{mpo}(\boldsymbol{h}) = p \cdot \mathrm{mpo}(\boldsymbol{g})$.

4.4. Arrangement Factorization.

4.4.1. *Arrangement Covers.* Let $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$ be an inner Hurwitz space. The Hurwitz monodromy group acts on the arrangements of the conjugacy classes through a quotient of S_r, and the stabilizer of a given arrangement produces covers $\mathcal{A}(G, \boldsymbol{C})^{\mathrm{in}} \to \mathcal{U}_r$ and $\mathcal{A}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{J}_r$, through which the corresponding Hurwitz spaces factor. The map $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}} \to \mathcal{A}(G, \boldsymbol{C})^{\mathrm{in}}$ is obtained by sending a branch cycle description to the corresponding arrangement of its conjugacy classes. One may equivalence by $\mathrm{Abs}(G)$ to obtain an absolute version of this.

Let $r = 4$. In this case, arrangement spaces can give information about final ramification. Note that in this case, H_4 acts on arrangements through S_4, and the reduction kernel $\widehat{K}_4$ from H_4 to $\overline{M}_4$ acts through the normal Klein four subgroup of S_4.

Consider the case where the conjugacy classes are distinct. Then $\mathcal{A} \to \mathcal{U}_4$ is a normal cover with group S_4. The reduced cover $\mathcal{A}^{\mathrm{rd}} \to \mathcal{J}_4$ is normal with group $S_3 = S_4/K_4$, with S_3 in its regular representation. In this case, both γ_0 and γ_1 are finally ramified in $\mathcal{A}^{\mathrm{rd}}$, and so they are finally ramified in $\mathcal{H}^{\mathrm{rd}}$.

4.4.2. *Bipolar Tuples.* Let q be an odd prime and let G be a group whose elements of order q lie in two conjugacy classes, labeled C_+ and C_-, which are swapped by an outer automorphism. We call $\boldsymbol{C}_{q^2_\pm} = (C_+, C_+, C_-, C_-)$ a *bipolar tuple* of conjugacy classes. These arise in the following situation.

Consider $G = \mathrm{PSL}_2(\mathbb{F}_q)$, where q is an odd prime. Let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{F}_p$ and $ad - bc = 1$. One quickly computes that g is an element of order q and that $h^{-1}gh = g^x$ if and only if $c = 0$, $ad = 1$, and $d^2 = x$ in $\mathbb{F}_q$. So exactly half of the nontrivial elements of the cyclic subgroup generated by g are conjugate to it. All other elements of order q are conjugate to a power of g by Sylow's theorem. Yet a change of basis given by conjugation from $\mathrm{GL}_n(\mathbb{F}_q)$ will produce an outer automorphism which swaps the conjugacy classes.

Let $\boldsymbol{C} = \boldsymbol{C}_{q^2_\pm}$ be a bipolar tuple from a group G, and consider $\mathcal{A}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{J}_4$. There are three reduced arrangements of the conjugacy classes upon which S_3 acts in its standard representation; γ_0 must act as a three cycle, so it is finally ramified in $\mathcal{A}(G, \boldsymbol{C})^{\mathrm{in,rd}}$ (see chapter VII for more details).

Suppose we can find elements in $\mathrm{SL}_2(\mathbb{F}_q)$ in conjugacy classes above C_+ and C_- whose product has order 4. Conjugate this pair by an element of the centralizer of the product to obtain a generating 4 tuple with product of order 2. Its image in $\mathrm{PSL}_2(\mathbb{F}_q)$ is a Nielsen tuple which does not lift to $\mathrm{Ni}(\mathrm{SL}_2(\mathbb{F}_q), \boldsymbol{C})^{\mathrm{in}}$.

CHAPTER VI

Automorphisms and Spin Covers

1. Universal Elementary 2-Frattini Covers of A_4 and A_5

1.1. Restricted and Induced Modules. Let G be a finite group with $H \leq G$. We will work exclusively over the field $\mathbb{F}_2$. Let $\mathbf{1} = \mathbb{F}_2$, viewed as a one dimensional module with trivial action.

Let M be a $\mathbb{F}_2[G]$ module. The $\mathbb{F}_2[H]$ module given by *restriction*, denoted $\mathrm{Res}_H^G(M)$, is given by restricting the action of G on M to the action of H on M.

Let $\mathbf{1} = \mathbf{1}_H$ be the trivial $\mathbb{F}_2[H]$ module. The $\mathbb{F}_2[G]$ module *induced* by $\mathbf{1}_H$, denoted $\mathrm{Ind}_H^G(\mathbf{1})$, is viewed to be the vector space over $\mathbb{F}_2$ of dimension $n = [G : H]$ which is freely generated by the left cosets of H in G; that is, points in $\mathrm{Ind}_H^G(\mathbf{1})$ are sums of cosets. The action of G on $\mathrm{Ind}_H^G(\mathbf{1})$ is given by the action of G on the cosets by left multiplication.

The *augmentation map* $\sigma : \mathrm{Ind}_H^G(\mathbf{1}) \to \{0, \ldots, n\}$ is given by $\sum_{i=1}^n a_i g_i H \mapsto \sum_{i=1}^n a_i$, where $g_1, \ldots, g_n$ are coset representative for H in G, and $a_1, \ldots, a_n \in \{0, 1\}$. To compress notation, we typically enumerate the cosets, producing an explicit isomorphism $\mathrm{Ind}_H^G(\mathbf{1}) \to \mathbb{F}_2^n$, and write an element of $\mathrm{Ind}_H^G(\mathbf{1})$ as a tuple containing zeros and ones. The augmentation map counts these ones.

An induced module $\mathrm{Ind}_H^G(\mathbf{1})$ always contains a submodule generated by the sum of the cosets, upon which G acts trivially; denote it by $\mathrm{Trv}_H^G(\mathbf{1})$. Let $\bar{\sigma}$ denote the quotient of the augmentation map by $\mathrm{Trv}_H^G(\mathbf{1})$; its range is $\{\bar{a} \mid a = 0, \ldots, n\}$ with $\bar{a} = \{a, n-a\}$.

1.2. Universal Elementary 2-Frattini Module of A_5.

THEOREM 33. *Let $\varphi : \frac{1}{2}\tilde{A}_5 \to A_5$ be the universal elementary 2-Frattini cover, and let D_5 be the normalizer of a five cycle in A_5. Then the universal elementary 2-Frattini module of A_5 is*

$$M_0(A_5) = \mathrm{Ind}_{D_5}^{A_5}(\mathbf{1}) / \mathrm{Trv}_{D_5}^{A_5}(\mathbf{1}).$$

PROOF. [**Fr95**] Proposition 2.4. □

Since we are interested in comparing A_4 and A_5, it is convenient to slightly modify the previous notation. Unless otherwise indicated, explicitly take A_5 to be in its standard representation, with $A_4 \leq A_5$ given by $A_4 = \mathrm{Stb}_{A_5}(5)$, and $D_5 = \langle \texttt{(1 2 3 4 5)}, \texttt{(2 5)(3 4)} \rangle \leq A_5$.

Let $U_5 = \frac{1}{2}\tilde{A}_5$ and let $M = M_0(A_5)$ be the universal elementary 2-Frattini module of A_5. By Proposition 6, $M = \{a \in U_5 \mid a^2 = 1\}$. We now use this characterization of M to describe the conjugacy classes of involutions in U_5.

Proposition 34. *The conjugacy classes of involutions in* U_5 *are*

(1) $M_2' = \{a \in M \mid \bar{\sigma}(a) = \bar{2}\}$, *with* $|M_2'| = 15$*;*

(2) $M_3' = \{a \in M \mid \bar{\sigma}(a) = \bar{3}\}$, *with* $|M_3'| = 10$*;*

(3) $M_5' = \{a \in M \mid \bar{\sigma}(a) = \bar{1}\}$, *with* $|M_5'| = 6$.

Let $V = M_2 \cup \{1\}$. *Then* V *is a submodule of* M, *and* $V = [U_5, M]$.

Proof. Enumerate the cosets of D_5 in A_5 to obtain a permutation representation $\rho : A_5 \to S_6$, which induces a linear representation $A_5 \to \mathbb{F}_2^6$, thus realizing $\mathrm{Ind}_{D_5}^{A_5}(\mathbf{1})$ as an $\mathbb{F}_2[A_5]$ module. View A_5 as acting on coordinate slots via ρ. The universal Frattini module M is the result of modding out by the fixed subspace $\mathrm{Trv}_{D_5}^{A_5}(\mathbf{1}) = \{(0,0,0,0,0,0),(1,1,1,1,1,1)\}$; thus two hexatuples are equivalent if and only if they are complementary.

If two tuples are in the same orbit, they must have the same number of ones. There are $\binom{6}{k}$ tuples with k ones. Since the one slot stabilizer D_5 acts transitively on the other slots, A_5 acts doubly transitively on six slots. Thus A_5 acts transitively on sets of tuples with 0, 1, 2, 4, 5, or 6 ones. Those with k ones are equivalent to those with $6 - k$ ones; modulo equivalence, this gives orbits of sizes 1, 6, and 15; the latter two are M_5' and M_2'.

The action of A_5 on tuples with 3 ones in not transitive, because the two point stabilizer of A_5 on slots is an involution acting on four slots as a pair of transpositions. This breaks the set of tuples with 3 ones into two orbits. However, equivalent tuples lie in separate orbits because complementation commutes with the action. Modulo equivalence, this gives one orbit M_3' containing 10 tuple classes.

Since V is the union of conjugacy classes, it is a submodule if it is a subgroup. One sees that this is so because we equivalence tuples with two cosets to their complements. It is also clear that V is a minimal subgroup of U_5, so we show it contains the stated commutators. Let $a \in M$ and $g \in U_5$. Then $[g,a] = g^{-1}aga = a^g a \in M$. If $a \in M \setminus V$, then so is a^g, and since V has index two in M, we have $aa^g \in V$. $\square$

View the elements of M as equivalence classes of hexatuples, with the equivalence class of $(z_1, z_2, z_3, z_4, z_5, z_6)$ denoted by $[z_1, z_2, z_3, z_4, z_5, z_6]$, where $z_i \in \mathbb{F}_2$.

PROPOSITION 35. *Let $g \in A_5$ have order three and let $a \in M_2'$. Then $aa^g a^{g^{-1}} = 1$. If $h \in U_5$ is a lift of g of order three, then $\langle h, a\rangle \cong A_4$.*

PROOF. The element a is an equivalence class of hexatuples, with two ones and four zeros. Renumber the slots so the orbits of g are the first three slots and the last three slots, and so that either $a = [1, 1, 0, 0, 0, 0]$ or $a = [1, 0, 0, 1, 0, 0]$. In the first case,

$$aa^g a^{g^{-1}} = [1, 1, 0, 0, 0, 0] + [0, 1, 1, 0, 0, 0] + [1, 0, 1, 0, 0, 0] = [2, 2, 2, 0, 0, 0] = [0, 0, 0, 0, 0, 0].$$

In the second case,

$$aa^g a^{g^{-1}} = [1, 0, 0, 1, 0, 0] + [0, 1, 0, 0, 1, 0] + [0, 0, 1, 0, 0, 1] = [1, 1, 1, 1, 1, 1] = [0, 0, 0, 0, 0, 0].$$

Thus $\langle a, a^h\rangle$ is a Klein four subgroup of V, and $\langle a, h\rangle$ is a semidirect Thus $\langle a, a^h\rangle$ is a Klein four subgroup of V, and $\langle a, h\rangle$ is a semidirect product isomorphic to A_4. □

PROPOSITION 36. *If $q \in \{3, 5\}$, then the map $a \mapsto C_{A_5}(a)$ produces a bijective correspondence between M_q' and the normalizers of Sylow q-subgroups of A_5.*

If $a \in M_2'$, then $C_{A_5}(a)$ is a Sylow 2-subgroup of A_5. Each element $a \in M_2'$ is in a unique Klein four subgroup $K_a \leq V$ such that the map $K_a \mapsto C_{A_5}(K_a)$ produces a bijective correspondence between $\{K_a \mid a \in M_2'\}$ and the Sylow 2-subgroups of A_5.

If $a \in M$, then $C_{U_5}(a) = \varphi^{-1}(C_{A_5}(a))$. Therefore

(a) $a \in M_2' \Rightarrow |C_{U_5}(a)| = 128$;
(b) $a \in M_3' \Rightarrow |C_{U_5}(a)| = 192$;
(c) $a \in M_5' \Rightarrow |C_{U_5}(a)| = 320$.

PROOF. The action of A_5 on M_q' is transitive on $x = |M_q'|$ points, so the one point stabilizers are of index x in A_5. If $q = 5$, then $x = 6$ and these are the D_5 subgroups which normalize a Sylow 5-subgroup. If $q = 3$, then $x = 10$ and these are the S_3 subgroups which normalize a Sylow 3-subgroup.

If $q = 2$, then $x = 15$ and the one point stabilizers are the K_4 subgroups which are the Sylow 2-subgroups of A_5. Thus let $a \in M_2'$ and let $K = C_{A_5}(a)$ be the centralizing 2-Sylow. The other two points in M_2' centralized by K are in the orbit of a three cycle in A_5 which normalizes K.

The statement that $C_{U_5}(a) = \varphi^{-1}(C_{A_5}(a))$ reiterates that the conjugation action of U_5 on M is given by lifting elements from A_5. The final statement on orders follows from these considerations. □

1.3. Universal Elementary 2-Frattini Module of A_4.

THEOREM 37. *Let* $\varphi : \frac{1}{2}\tilde{A}_5 \to A_5$ *be the universal elementary* 2*-Frattini cover, and let* $Z_2 = A_4 \cap D_5$*. Then* $\varphi\restriction_{\varphi^{-1}(A_4)} : \varphi^{-1}(A_4) \to A_4$ *is the universal elementary* 2*-Frattini cover of* A_4*, and*

$$M_0(A_4) = \mathrm{Res}_{A_4}^{A_5}(M_0(A_5)) = \mathrm{Ind}_{Z_2}^{A_4}(\mathbf{1}).$$

PROOF. [**Fr95**] Proposition 2.9, or [**BF02**] Proposition 5.6. □

Henceforth, with φ as above, set $U_4 = \varphi^{-1}(A_4) \cong \frac{1}{2}\tilde{A}_4$.

PROPOSITION 38. *Let* K_4 *denote the Sylow* 2*-subgroup of* A_4*, and let* C *be a conjugacy class of three cycles in* A_4*. Then the conjugacy classes of involutions in* U_4 *are*

(1) $J_1 = C_M(K_4) \setminus \{1\} \subset M_2'$*, with* $|J_1| = 3$*;*
(2) $J_2 = M_2' \setminus J_1$*, with* $|J_2| = 12$*;*
(3) $J_3 = \cup_{g\in C} C_M(g) \subset M_3'$*, with* $|J_3| = 4$*;*
(4) $J_4 = M_3' \setminus J_3$*, with* $|J_4| = 6$*;*
(5) $J_5 = M_5'$*, with* $|J_5| = 6$*.*

The proper nontrivial submodules of $M_0(A_4)$ *are* $V_1 = J_1 \cup \{1\}$*,* $V_3 = J_3 \cup V_1$*, and* V*. Again,* $V = [U_4, M]$*.*

PROOF. The list of conjugacy classes, and the fact that V_1 is a submodule, follow from Proposition 36. The appearance of J_1 and J_3 are obtained by collecting together the elements which are centralized by some Sylow 2-subgroup of A_4. If $a_1, a_2 \in J_3$, compute directly from the cosets that $a_1a_2 \in V_1$. Thus V_3 is a subgroup, and so is a submodule. □

PROPOSITION 39. *Let* $a \in M$*. Then*

(a) $a \in J_1 \Rightarrow |C_{U_4}(a)| = 128$*;*
(b) $a \in J_2 \Rightarrow |C_{U_4}(a)| = 32$*;*
(c) $a \in J_3 \Rightarrow |C_{U_4}(a)| = 96$*;*
(d) $a \in J_4 \Rightarrow |C_{U_4}(a)| = 64$*;*
(e) $a \in J_5 \Rightarrow |C_{U_4}(a)| = 64$*.*

PROOF. We have $C_{U_4}(a) = C_{U_5}(a) \cap U_4$. The elements of J_1 are centralized by the preimage of $K_4 \in A_4$. The elements of J_2 are centralized by elements of M; if $a \in J_2$, its full centralizer in U_5 comes from a conjugate of K_4 in A_5. For J_3, an involution in A_5 which normalizes a 3-Sylow is not in A_4, and does not lift. □

2. Automorphisms of U_4

2.1. Automorphisms of Universal Frattini Covers. Let $\varphi : H \to G$ be a group homomorphism with characteristic kernel. For $\alpha \in \mathrm{Aut}(H)$, define $\alpha_* \in \mathrm{Aut}(G)$ by $\alpha_*(g) = \varphi(\alpha(h))$, where $h \in \varphi^{-1}(g)$. This produces a well-defined homomorphism $\varphi_* : \mathrm{Aut}(H) \to \mathrm{Aut}(G)$ given by $\alpha \mapsto \alpha_*$. Let $\mathrm{Aut}(H, \varphi) = \ker(\varphi_*)$; this is the group of automorphisms of H which preserve the cosets of $\ker(\varphi)$. Also set $\mathrm{Inn}(H, \varphi) = \mathrm{Aut}(H, \varphi) \cap \mathrm{Inn}(H)$, and $\mathrm{Out}(H, \varphi) = \mathrm{Aut}(H, \varphi)/\mathrm{Inn}(H, \varphi)$. Then $\mathrm{Out}(H, \varphi)$ is the image of $\mathrm{Aut}(H, \varphi)$ in $\mathrm{Out}(H)$.

It is clear that $\varphi_*(\mathrm{Inn}(H)) \leq \mathrm{Inn}(G)$, and if φ is surjective, this is equality. If φ is a Frattini cover, then every inner automorphism of G lifts to an inner automorphism of H. In this case, $|\mathrm{Out}(H)| = [\varphi_*(\mathrm{Aut}(H)) : \mathrm{Inn}(G)]|\mathrm{Out}(H, \varphi)|$.

Proposition 40. *Let G be a finite group and let $\varphi : {}_p^1\tilde{G} \to G$ be its universal elementary p-Frattini cover, with $M = \ker(\varphi)$. Then*

- **(a)** *M is characteristic in ${}_p^1\tilde{G}$;*
- **(b)** *$\varphi_* : \mathrm{Aut}({}_p^1\tilde{G}) \to \mathrm{Aut}(G)$ is an epimorphism, with kernel $\mathrm{Aut}({}_p^1\tilde{G}, \varphi)$;*
- **(c)** *$\bar{\varphi}_* : \mathrm{Out}({}_p^1\tilde{G}) \to \mathrm{Out}(G)$ is an epimorphism, with kernel $\mathrm{Out}({}_p^1\tilde{G}, \varphi)$.*

Proof. By Proposition 6, $M = \{a \in {}_p^1\tilde{G} \mid a^p = 1\}$, so it is characteristic. Thus every automorphism of ${}_p^1\tilde{G}$ descends to an automorphism of G. If $\alpha \in \mathrm{Aut}(G)$, then $\alpha \circ \varphi$ is an elementary p-Frattini cover of G, and the universal property produces $\tilde{\alpha} \in \mathrm{Aut}({}_p^1\tilde{G})$ which lifts α. ☐

Let $\psi : {}_p\tilde{G} \to G$ be the universal p-Frattini cover of G, and let $\alpha \in \mathrm{Aut}(G)$. Since $\alpha \circ \psi$ is a p-Frattini cover, there exists an homomorphism $\beta : \tilde{G} \to \tilde{G}$ such that $\psi = \alpha \circ \psi \circ \beta$. By the Frattini property, this is surjective, and since $\tilde{G}$ is profinite, it is an automorphism. Now β^{-1} descends to an automorphism of G which lifts α if and only if $\ker_0$ is stabilized by β. As noted in [**BF02**] Lemma 3.10, if $\ker_0$ is characteristic, then so is $\ker_k$ for $k \geq 1$. If all of the p-Sylows of G intersect in $\{1\}$, for example if G is a simple group, then $\ker_0$ is characteristic. However, whether or not $\ker_0$ is characteristic, we have the following.

Proposition 41. *Let $\psi : {}_p\tilde{G} \to G$ be the universal p-Frattini cover of G, and let $\alpha \in \mathrm{Aut}(G)$. Then there exists $\tilde{\alpha} \in \mathrm{Aut}({}_p\tilde{G})$ such that $\tilde{\alpha}(\ker_0) = \ker_0$, and $\psi \circ \tilde{\alpha} = \alpha \circ \psi$.*

Proof. The universal elementary p-Frattini cover of ${}_p^k\tilde{G}$ is ${}_p^{k+1}\tilde{G}$. Iterate the lifts of α given by Proposition 40; the projective limit will be an automorphism of ${}_p\tilde{G}$ as claimed. ☐

2.2. Automorphisms of p-Groups. The Burnside Basis Theorem states that the Frattini subgroup $\Phi(P)$ of a p-group P is generated by p^{th} powers and commutators (see [**Sc87**] 7.3.10 or [**Ro93**] 5.3.2, or [**FJ86**] Lemma 20.36 for the profinite case). As a consequence, $P/\Phi(P)$ is a vector space over $\mathbb{F}_p$, whose automorphism group is $\mathrm{GL}_t(\mathbb{F}_p)$; here, t is the rank of P. Let $\varphi : P \to P/\Phi(P)$ be the canonical homomorphism and let $\mathrm{Aut}(P, \varphi)$ denote the subgroup of automorphisms which are trivial on $P/\Phi(P)$. A theorem of P. Hall concludes that $|\mathrm{Aut}(P, \varphi)|$ divides p^{st}, where $p^s = |\Phi(P)|$, and that $|\mathrm{Aut}(P)|$ divides $p^{st} \prod_{i=0}^{t-1}(p^t - p^i)$ (see [**Sc87**] 7.3.11 or [**Ro93**] 5.3.3).

We apply the method proof to the universal elementary p-Frattini cover $\varphi : P_1 \to P_0 = \mathbb{F}_p^t$. In this case, $\ker(\varphi) = \Phi(P_1)$. The universal p-Frattini cover of P_0 is ${}_p\tilde{F}_t$, the pro-free pro-p group on t generators. The kernel of ${}_p\tilde{F}_t \to P_0$ is the Frattini subgroup of ${}_p\tilde{F}_t$, so it is characteristic, and the profinite version of the Nielsen-Scheier formula implies that the rank of $\Phi({}_p\tilde{F}_t)$ (and of $\Phi(P_1)$) is $s = 1 + (t-1)p^t$ (see [**FJ86**] Proposition 15.27).

Proposition 42. *Let P_0 be an elementary p-group of rank t, and let $\varphi : P_1 \to P_0$ be its universal elementary p-Frattini cover, with kernel M. Let $\boldsymbol{x} = (x_1, \dots, x_t)$ be generators of P_1, and for $\boldsymbol{a} = (a_1, \dots, a_t) \in M^t$, set $\boldsymbol{x} \odot \boldsymbol{a} = (x_1 a_1, \dots, x_t a_t)$. Let $s = 1 + (t-1)p^t$. Then*

- **(a)** *for every $\boldsymbol{a} \in M^t$ there exists a unique automorphism $\xi_{\boldsymbol{a}}$ of P_1 such that $\xi_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{x} \odot \boldsymbol{a}$;*
- **(b)** $\mathrm{Aut}(P_1, \varphi) = \{\xi_{\boldsymbol{a}} \mid \boldsymbol{a} \in M^t\}$*;*
- **(c)** $|\mathrm{Aut}(P_1)| = |\mathrm{Aut}(P_1, \varphi)| \cdot |\mathrm{Aut}(P_0)| = p^{st} \cdot \left(\prod_{i=0}^{t-1}(p^t - p^i)\right)$.

Proof. Let $\tilde{\varphi} : {}_p\tilde{P} \to P_0$ be the universal Frattini cover of P_0, with kernel $\ker_0$. Then ${}_p\tilde{P}$ is a pro-free pro-p group on t generators, given by lifting $\boldsymbol{x}$ to $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \dots, \tilde{x}_t)$. Let $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_t)$ be a lift in ${}_p\tilde{P}$ of $\boldsymbol{a}$. Since ${}_p\tilde{P}$ is pro-free, there exists an homomorphism $\tilde{\xi}_{\tilde{\boldsymbol{a}}} : {}_p\tilde{P} \to {}_p\tilde{P}$ defined by $\tilde{\boldsymbol{x}} \mapsto \tilde{\boldsymbol{x}} \odot \tilde{\boldsymbol{a}}$. Since $\tilde{\varphi}(\tilde{\boldsymbol{x}} \odot \tilde{\boldsymbol{a}})$ generates P_0, $\tilde{\xi}_{\tilde{\boldsymbol{a}}}$ is necessarily an epimorphism by the Frattini property, and since ${}_p\tilde{P}$ is profinite, it is an automorphism. Moreover, it is clear that $\tilde{\xi}_{\tilde{\boldsymbol{a}}}$ preserves $\ker_0$, and since $\ker_1$ is characteristic in $\ker_0$, this descends to an automorphism $\xi_{\boldsymbol{a}} : P_1 \to P_1$ with the prescribed effect. Every element of $\mathrm{Aut}(G_1, \varphi)$ is necessarily of this form. Since $|M| = p^s$ and $\mathrm{Aut}(P_0) \cong \mathrm{GL}_t(\mathbb{F}_p)$, the last formula follows. □

2.3. Automorphisms of Split Groups. We plan to combine Proposition 42 with the following lemma to obtain information about $\mathrm{Aut}(U_4)$. Call a semidirect product $K \rtimes H$ *faithful* if it is given by an antihomomorphism $\rho : H \to \mathrm{Aut}(K)$ with trivial kernel. Let $A = \mathrm{Aut}(K)$ and set $C_A(H) = C_A(\rho(H))$ and $N_A(H) = N_A(\rho(H))$. For $\alpha \in N_A(H)$ and $h \in H$, set $h^{\alpha^{-1}} = \rho^{-1}(\alpha(\rho(h) \circ \alpha^{-1}))$.

PROPOSITION 43. *Let $G = K \rtimes H$ be a faithful semidirect product. Let $A = \mathrm{Aut}(K)$ and let $\alpha \in N_A(H)$. Then α extends to $\alpha_* \in \mathrm{Aut}(G)$ given by $\alpha_* : kh \to \alpha(k)h^{\alpha^{-1}}$.*

PROOF. Every element in $g \in G$ may be written uniquely as $g = kh$, so the indicated map is well-defined. For simplicity of notation, identify H with $\rho(H)$. Compute

$$\begin{aligned}\alpha_*(k_1\rho_1k_2\rho_2) &= \alpha_*(k_1\rho_1(k_2)\rho_1\rho_2)\\ &= \alpha(k_1)\alpha(\rho_1(k_2))(\rho_1\rho_2)^{\alpha^{-1}}\\ &= \alpha(k_1)\rho_1^{\alpha^{-1}}(\alpha(k_2))\rho_1^{\alpha^{-1}}\rho_2^{\alpha^{-1}}\\ &= \alpha(k_1)\rho_1^{\alpha^{-1}}\alpha(k_2)\rho_2^{\alpha^{-1}}.\end{aligned}$$

□

2.4. Automorphisms of U_4. Let $\varphi : H \to G$ be a Frattini cover, and let $h \in H$. Let $\mathrm{Aut}(H, h)$ denote the group of automorphisms of H which fix h. If the conjugacy class of h in H is characteristic, then every automorphism of H differs from an element of $\mathrm{Aut}(H, h)$ by an inner automorphism. In this case, the map $\mathrm{Aut}(H, h) \to \mathrm{Out}(H, \varphi)$ is surjective. This applies for $H = U_4$ and $G = A_4$, and allows us to find $\mathrm{Out}(U_4)$.

PROPOSITION 44. *Let h_1 and h_2 be conjugate order three generators of U_4. Let $x_1 = h_1h_1h_2$ and $x_2 = h_1h_2h_1$. Then $\mathrm{ord}(x_1) = \mathrm{ord}(x_2) = 4$. Let P_1 be the normal 2-Sylow of U_4. Then $P_1 = \langle x_1, x_2\rangle$. Let $A = \mathrm{Aut}(P_1)$, $a_1, a_2 \in M$, and $\xi = \xi_{\boldsymbol{a}}$ as in Proposition 42. Then $\xi \in C_A(h_1)$ if and only if*

$$a_1 \in V \text{ and } a_2 = a_1^{h_1}.$$

In this case, ξ extends to an automorphism of U_4 such that $\xi(h_1) = h_1$ and $\xi(h_2) = h_2a_1$.

PROOF. Let $g_1, g_2 \in A_4$ be conjugate elements of order three which lift to h_1 and h_2. Then $g_1^{-1}g_2$ has order two, and its lift x_1 has order four. Let $x_3 = h_2h_1h_1$; since $x_2 = x_1^{h_1}$ and $x_3 = x_2^{h_1}$, these elements also have order four. Also x_1 and x_2 generate P_1 since they lift generators from P_0, where P_0 is the 2-Sylow of A_4. The kernel of $P_1 \to P_0$ is M, so all automorphism of P_1 which are trivial on P_0 are of the form $\xi_{\boldsymbol{a}}$, with $a_1, a_2 \in M$. Note that $x_3 = (x_2x_1)^{-1}$. Compute

$$\xi(x_3) = (x_2a_2x_1a_1)^{-1} = a_1x_1^{-1}a_2x_2^{-1} = x_1^{-1}x_2^{-1}a_1^{x_3}a_2^{x_2^{-1}} = x_3a_1^{x_3}a_2^{x_2^{-1}}.$$

Now ξ and ρ commute in A if and only if they commute on the generators x_1 and x_2. Compute

$$\rho\xi(x_1) = \rho(x_1a_1) = x_2a_1^{h_1} \quad \text{and} \quad \xi\rho(x_1) = \xi(x_2) = x_2a_2;$$

thus $a_2 = a_1^{h_1}$. Also

$$\rho\xi(x_2) = \rho(x_2 a_2) = x_3 a_2^{h_1} \quad \text{and} \quad \xi\rho(x_2) = \xi(x_3) = x_3 a_1^{x_3} a_2^{x_2^{-1}}.$$

Thus $a_2^{h_1} = a_1^{x_3} a_2^{x_2^{-1}}$; replace a_2 with $a_1^{h_1}$ to get $a_1^{h_1^{-1}} = a_1^{h_2 h_1^{-1}} a_1^{h_2^{-1} h_1^{-1}}$. Conjugate by h_1 to arrive at $a_1 = a_1^{h_2} a_1^{h_2^{-1}}$. By Proposition 35, this condition is satisfied by every $a_1 \in V$. By Proposition 43, ξ extends to an automorphism of U_4 which fixes h_1. Finally, $\xi(h_2) = h_1\xi(x_1) = h_1 x_1 a_1 = h_2 a_1$. $\square$

PROPOSITION 45. *Let h_1 and h_2 be conjugate order three generators for U_4. For each $v \in V$ there exists a unique automorphism $\nu_v \in \mathrm{Aut}(U_4, \varphi)$ such that $\nu_v(h_1) = h_1$ and $\nu_v(h_2) = h_2^v$. Let $c_i \in M$ be the nontrivial element of $C_M(h_i)$. Let W be a complement in V for $\langle c_1 c_2 \rangle$. Then the map*

$$W \to \mathrm{Out}(U_4, \varphi) \quad \textit{given by} \quad w \mapsto \nu_w \pmod{\mathrm{Inn}(U_4, \varphi)}$$

is an isomorphism.

PROOF. The map $\xi_{\boldsymbol{a}} \in \mathrm{Aut}(U_4, \varphi)$ from Proposition 44 fixes h_1 and sends h_2 to $h_2 a_1$; this defines it. For each $a_1 \in V$ there exists $v \in V$ such that $h_2 a_1 = h_2^v$, and the map $a_1 \mapsto v$ is bijective. Thus ν_v is an automorphism, and all elements of $\mathrm{Aut}(U_4, h_1)$ are of this form.

Now $\nu_{c_1 c_2}$ is the unique nontrivial inner automorphism in $\mathrm{Aut}(U_4, h_1)$. The elements of W represent the outer automorphisms from $\mathrm{Aut}(U_4, h_1)$, so $W \to \mathrm{Out}(U_4, \varphi)$ is well-defined and injective. It is surjective because any automorphism of U_4 which is trivial on A_4 differs from an element of $\mathrm{Aut}(U_4, h_1)$ by an inner automorphism of U_4. $\square$

PROPOSITION 46. *Let $h_1, h_2 \in U_4$ be conjugate generators of order three. Then there exists a unique automorphism $\mu \in \mathrm{Aut}(U_4)$ such that $\mu(h_1) = h_1^{-1}$ and $\mu(h_2) = h_2^{-1}$.*

PROOF. Let $x_1, x_2, x_3 \in U_4$ and P_1 be as in Proposition 44. Now x_1 and x_3 generate P_1; lift x_1 and x_3 to elements $\tilde{x}_1$ and $\tilde{x}_3$ in the universal p-Frattini cover $\tilde{P}_1$. Then $(x_1, x_3) \mapsto (\tilde{x}_3^{-1}, \tilde{x}_1^{-1}$ defines an automorphism of $\tilde{P}_1$, which descends to a unique automorphism $\xi \in \mathrm{Aut}(P_1)$ such that $\xi(x_1) = x_3^{-1}$ and $\xi(x_3) = x_1^{-1}$. Since $x_2 = (x_1 x_3)^{-1}$, we have

$$\xi(x_2) = \xi(x_1 x_3)^{-1} = (x_3^{-1} x_1^{-1})^{-1} = x_2^{-1}.$$

This automorphism normalizes the action of h_1, as follows. Let $\rho \in \mathrm{Aut}(P_1)$ denote conjugation by h_1; we wish to show that $\rho^\xi = \rho^{-1}$. Compute

$$\rho\xi(x_1) = \rho(x_3^{-1}) = x_1^{-1} \quad \text{and} \quad \xi\rho^{-1}(x_1) = \xi(x_3) = x_1^{-1};$$

also

$$\rho\xi(x_3) = \rho(x_1^{-1}) = x_2^{-1} \quad \text{and} \quad \xi\rho^{-1}(x_3) = \xi(x_2) = x_2^{-1}.$$

By Proposition 43, ξ extends to an automorphism $\mu \in \mathrm{Aut}(U_4)$ such that $\mu(h_1) = h_1^{-1}$. Finally, $\mu(h_2) = \mu(h_1 x_1) = h_1^{-1} x_3^{-1} = h_2^{-1}$. $\square$

The automorphism μ is an order two lift of the nontrivial outer automorphism of A_4. We put this together with our previous proposition to obtain the following.

PROPOSITION 47. *Let h_1 and h_2 be conjugate order three generators for U_4. Let W and ν_w be as in Proposition 45 and μ as in Proposition 46. A complete list of coset representatives for* $\mathrm{Out}(U_4)$ *is $\{\nu_w, \nu_w\mu \mid w \in W\}$. In particular,* $|\mathrm{Out}(U_4)| = 16$.

The next proposition follows as a corollary, and implies that there is only one Harbater-Mumford component of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{ab}}$. In chapter VII we will see that there are two Harbater-Mumford components of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. Recall that mpo denotes the middle product order.

PROPOSITION 48. *Let $\boldsymbol{h} = (h_1, h_1^{-1}, h_2, h_2^{-1}) \in \mathrm{Ni}(U_4, \boldsymbol{C}_{3^4_\pm})^{\mathrm{in}}$ be a Harbater-Mumford tuple with* $\mathrm{mpo}(\boldsymbol{h}) = 4$*. Let c_1 and c_2 be the nontrivial elements of M which centralize h_1 and h_2, respectively. Let W be a complement in V for the subgroup generated by the c_1c_2, so that $\boldsymbol{h}^{[e|W]}$ is the set of eight duals if $\boldsymbol{h}$. For every $\boldsymbol{h}^{[e|w]} \in \boldsymbol{h}^{[e|W]}$ there exists a unique automorphism $\nu = \nu^{[e|w]} \in \mathrm{Aut}(U_4)$ such that $\nu(\boldsymbol{h}) = \boldsymbol{h}^{[e|w]}$.*

3. Spin Covers

3.1. Spin Groups.

3.1.1. *Lifting Involutions.* Let $\theta : \hat{G} \to G$ be a central extension with kernel an elementary abelian 2-group. An element $g \in A_n$ of odd order has a unique lift $\hat{g} \in \hat{A}_n$ to an element of odd (the same) order; the other lifts have order $2 \cdot \mathrm{ord}(g)$. However, if the order of g is even, the order of the lift is independent of the choice of the lift. In particular, we are interested in lifting elements of order two through factors of the universal elementary 2-Frattini cover of G. One method of computation lies in computing the spin covers of G.

3.1.2. *Standard Spin Groups.* The Lie group $\mathrm{SO}_n(\mathbb{R})$ admits a degree two universal cover, denoted by $\mathrm{Spin}_n(\mathbb{R})$, which is also a topological group so that the map $\tilde{\theta} : \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}_n(\mathbb{R})$ is a nonsplit central cover whose kernel is of order two.

Let $\sigma : A_n \to \mathrm{SO}_n(\mathbb{R})$ be the linear representation induced by the standard permutation representation of A_n. The *standard spin group* of degree n is

$$\hat{A}_n = \tilde{\theta}^{-1}(\sigma(A_n)).$$

The corresponding cover $\theta : \hat{A}_n \to A_n$ is a central Frattini cover with kernel of order two, which we call the *standard spin cover* of A_n.

3.1.3. *Clifford Algebras.* The spin groups can be defined as certain subgroups of Clifford algebras. A technique of J. P. Serre uses this Clifford algebra to obtain information about the order of lifted elements. The following proposition analyzes the orders involutions lifted from A_n to $\hat{A}_n$, and was reported in [**BF02**] Proposition 5.10; it offers an interesting change of pace, so we repeat it here.

Proposition 49. *Assume $n \geq 4$, and $g \in A_n$ of order 2 is a product of $2s$ disjoint 2-cycles. Any lift $\hat{g} \in \hat{A}_n$ of g has order 4 if s is odd and 2 if s is even.*

Proof. We review the *Clifford algebra setup* used in [**Se90**]. Let C_n be the Clifford algebra on $\mathbb{R}^n$ with generators $x_1, \dots, x_n$ subject to relations

$$x_i^2 = 1,\ 1 \leq i \leq n, \text{ and } x_i x_j = -x_j x_i \text{ if } i \neq j.$$

In the Clifford algebra, write $[i\,j] = \frac{1}{\sqrt{2}}(x_i - x_j)$. Then, $[i\,j]^2 = 1$ and $[i\,j] = -[j\,i]$. The collection of $[i\,j]$ under multiplication generate a subgroup $\hat{S}_n$. Characterization: It is the central nonsplit extension of S_n whose restriction to transpositions splits, and whose restriction to products of two disjoint transpositions is nontrivial (see [**Se92**] page 97). The map $\hat{S}_n \to S_n$ appears from $[i\,j] \mapsto (i\,j)$.

So, $\hat{A}_n = A_n \times \{\pm 1\}$ if $n \leq 3$. That $\hat{A}_n \to A_n$ is nontrivial if $n \geq 4$ shows from lifts of certain elements of order 2. Example: $(1\,2)(3\,4)$ lifts to have order 4:

$$\Big(\frac{1}{\sqrt{2}}(x_1 - x_2)\frac{1}{\sqrt{2}}(x_3 - x_4)\Big)^2 = -[1\,2]^2[3\,4]^2 = -1.$$

Of course the order of a lift is conjugacy class invariant. Similarly, with $n \geq 8$,

$$([\,12][3\,4]\dots[s{-}1\,s])^2 = (-1)^{2(s-2)}([\,12][3\,4])^2([5\,6]\dots[s{-}1\,s])^2.$$

By induction, the result is $(-1)^s$: $[1\,2][3\,4]\dots[s{-}1\,s]$ has order $2^{1+\frac{1-(-1)^s}{2}}$. □

3.2. Spin Representations.

3.2.1. *Spin Covers.* Let G be a finite group, and let $\sigma : G \to S_n$ be a faithful permutation representation. Suppose that $\sigma(G) \leq A_n$; for example, if G is generated by elements of odd order, this will always be the case. Set $\hat{G} = \theta^{-1}(\sigma(G))$, and let $\theta_\sigma : \hat{G} \to G$ be given by restriction.

A *spin representation* of G is a faithful permutation representation $\sigma : G \to A_n$ such that $\theta_\sigma : \hat{G} \to G$ does not split. We call θ_σ a *spin cover* of G.

In this case, that θ_σ does not split is equivalent to it being a Frattini cover. Thus, spin covers are quotients of the universal central elementary 2-Frattini cover of G.

3.2.2. *Computing Spin Representations.* Proposition 49 offers a serious tool for computation of the order of lifts of involutions from A_n to $\hat{A}_n$. The next proposition, which is a rewording of [**BF02**] Lemma 9.13, uses the coset representation to produce a formula for applying this tool to a group embedded in A_n.

PROPOSITION 50. *Let G be a finite group generated by elements of odd order, and let T be a coreless subgroup of G. Let $\sigma : G \to A_d$ be the coset representation given by T, where $d = [G : T]$. Let $\tau : \hat{G} \to G$ be given by pullback to $\hat{A}_d$. Let J be a conjugacy class of involutions in G, and let $a \in J$. Let $a_1, \ldots, a_m \in T \cap J$ represent the orbits of T on $T \cap J$ by conjugation. Then the number of cosets of T in G fixed by right multiplication by a is*

$$f(a) = \sum_{i=1}^{m} [C_G(a_i) : C_T(a_i)].$$

Thus $d \geq f(a)$. If $|C_T(a)|$ is constant on $T \cap J$, let $o(a) = m$, so this formula becomes

$$f(a) = o(a)[C_G(a) : C_T(a)].$$

If $\hat{a} \in \tau^{-1}(a)$, then $\mathrm{ord}(\hat{a}) = 4 \Leftrightarrow (d - f) \equiv 4 \pmod 8$.

PROOF. We have $Tga = Tg \Leftrightarrow a^{g^{-1}} \in T$, so the fixed cosets are exactly those represented by elements of G whose inverses conjugate a into T. Suppose $a^{g^{-1}} = a_i$. If $g_1 \in Tg$, then $g_1 = tg$ for some $t \in T$, and $a^{g_1^{-1}} = a_i^{t^{-1}}$, so any two members of the same fixed coset conjugate a into the same T orbit. If $g_2 \in G$ with $a^{g_2} = a_i$, then $gg_2^{-1} \in C_G(a_i)$, so the cosets with representatives conjugating a to a_i are in bijective correspondence with $C_G(a_i)/C_T(a_i)$. □

3.3. Spin Covers of U_5 and U_4.

3.3.1. *Spin Covers of A_5 and A_4.* Our example base groups have easily determined spin covers.

The spin cover of $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$ is realized as $\theta : \mathrm{SL}_2(\mathbb{F}_5) \to \mathrm{PSL}_2(\mathbb{F}_5)$.

The spin cover of $A_4 \cong \mathrm{PSL}_2(\mathbb{F}_3)$ is realized as $\theta : \mathrm{SL}_2(\mathbb{F}_3) \to \mathrm{PSL}_2(\mathbb{F}_3)$.

3.3.2. *Spin Representations of U_5.* Let $\sigma : U_5 \to A_n$ be a spin representation, and let $\theta : \hat{U}_5 \to U_5$ be the corresponding spin cover. By [**BF02**] Proposition 9.12, if $\hat{a} \in \theta^{-1}(M)$, then $\mathrm{ord}(a) = 4 \Leftrightarrow \theta(a) \in M \setminus V$ and $\mathrm{ord}(a) = 2 \Leftrightarrow \theta(a) \in V$. Thus θ is the antecedent universal elementary 2-Frattini cover of U_5 (see subsection III.1.4.3). By [**BF02**] Corollary 9.16, all spin representations of U_5 are of degree 40, 60, or 120.

3.3.3. *Spin Representations of U_4.* We aim to show that U_4 has three distinct spin covers, one of which does not come from a coset representation of U_4.

PROPOSITION 51. *If $\sigma : U_5 \to A_n$ is a spin representation, and let $\tau = \sigma \restriction_{U_4}$. Then $\tau : U_4 \to A_n$ is a non-transitive spin representation, and the corresponding spin cover $\theta : \hat{U}_4 \to U_4$ is the antecedent central elementary 2-Frattini cover of U_4.*

PROOF. Since $|U_4| = 384$, U_4 cannot act transitively on 40, 60, or 120 elements.

However, if $\theta : \hat{U}_4 \to U_4$ is the pullback cover of U_4 induced by τ, nevertheless every involution in $M \setminus V \subset U_4$ lifts to an element of order four. So this cover cannot split.

Since V is the only normal index two subgroup of $M_0(A_4)$, it is clear that $U_4/V \to U_4/M$ is the universal central elementary 2-Frattini cover of A_4, giving the last claim. □

Let $\sigma : U_5 \to A_n$ be a spin representation. Let $S = \mathrm{Stb}_{U_5}(1)$ and let $T = S \cap U_4$. This induces a transitive representation $\tau : U_4 \to A_m$, where $m = [U_4 : T]$; call this the transitive representation of U_4 induced by σ. We wish to show that τ is not a spin representation.

Proposition 52. *Let $\tau : U_4 \to A_m$ be a transitive faithful representation, with $T = \mathrm{Stb}_{U_4}(1)$. Then $|T| \leq 24$, and consequently, $m \geq 16$. Moreover, $[T : T \cap M]$ divides 6.*

Proof. It suffices to consider $|T| = 32$ and $|T| = 48$.

If $|T| = 32$, then it contains an element x of order four, and $|T \cap J_2| = 6$. The elements of J_2 do not commute with x. Select distinct element $a, a^x, b, b^x \in T \cap J_2$. Now $aa^x, bb^x \in J_1$ since each is centralized by x. If $aa^x = bb^x$, then $ab = (ab)^x \in J_1 \setminus \{aa^x\}$; in any case, T contains at least two nontrivial elements of V_1, and so it contains V_1, and T is not coreless.

If $|T| = 48$, then T cannot contain an element of order four, and again $|T \cap J_2|$ contains at least six elements. Apply the previous argument, with x an element of order three.

The last statement comes from the fact that if 4 divides $[T : T \cap M]$, then the image of T in A_4 contains K_4, and any lift of K_4 contains the entire 2-Sylow of U_4; in this case, T is not coreless. □

Proposition 53. *Let $\tau : U_4 \to A_m$ be a transitive spin representation and let $\theta : \hat{U}_4 \to U_4$ be the induced cover. Then θ is not the antecedent central p-Frattini cover of U_4, and τ is not induced by a spin representation of U_5.*

Proof. Assume that θ is the antecedent central p-Frattini cover of U_4. Then $\mathrm{ord}(\hat{a}) = 4$ if $a \in M \setminus V$, and $\mathrm{ord}(\hat{a}) = 2$ if $a \in V$. Let $T = \mathrm{Stb}_{U_4}(1)$. Let Y be the 2-Sylow subgroup of T. Since T is coreless, Y cannot contain V.

Suppose $T \cap M \subset V$. For $a \in M \setminus V$, $o(a) = 0$ so the degree of the representation is congruent to four modulo eight, which implies that $|Y| \geq 16$. Since the squares of elements of order four are in $M \setminus V$, we have $Y \subset V$, so $|Y| \leq 8$, a contradiction. Thus $|T \cap M| = 2|T \cap V|$.

Suppose that $|Y| \leq 8$. Let $d = [U_4 : T]$. Then 2^4 divides d, and 2^3 divides $[C_G(a) : C_T(a)] = 64/|C_T(a)|$ for $a \in J_4$ or J_5, whence 8 divides $d - f$, a contradiction. Thus $|Y| = 16$, and from Proposition 52, $|T| = 16$. Moreover, T contains at most one coset of elements of order four, so the size of $T \cap M$ is either 8 or 16. Thus the size of $T \cap V$ is either 4 or 8. The corresponding possible sizes for $T \cap J_2$ are 2 or 6.

Suppose $|T \cap J_2| = 2$. In this case, T contains an element of order four which swaps these elements, so $o(a) = 1$, and $f(a) = [C_G(a) : C_T(a)] = 32/8 = 4$, so $d - f = 24 - 4 = 20$, and $\mathrm{ord}(\hat{a}) = 4$.

Suppose $|T \cap J_2| = 6$. Then $T \subset M$, and since M is abelian, $o(a) = 6$. Thus $f(a) = 6(32/16) = 12$, and $d - f = 24 - 12 = 12$; in this case, $\mathrm{ord}(\hat{a}) = 4$.

Either case contradicts that $a \in V \Rightarrow \mathrm{ord}(\hat{a}) = 2$, and completes the proof. □

The *lifting pattern* of a representation $\tau : U_4 \to A_n$ is

$$\mathrm{pat}_\tau = (x_1, x_2, x_3, x_4, x_5),$$

where x_i is the order of a lift of $a_i \in J_i$ to $\hat{A}_n$.

PROPOSITION 54. *Let $x \in U_4$ be an element of order four, and let $a \in M \setminus (V \cup C_M(x))$. Let $T = \langle x, a \rangle$. Then T is coreless in U_4, $|T| = 16$, and $|T \cap M| = 8$. Let $\tau : U_4 \to A_m$ be the associated transitive faithful representation. Then τ is a spin representation, and*

(a) *if $a \in J_3$, then* $\mathrm{pat}_\tau = (2, 4, 4, 2, 2)$*;*

(b) *if $a \notin J_3$, then* $\mathrm{pat}_\tau = (2, 4, 2, 4, 4)$.

PROOF. As in the proof of Proposition 53, $|T \cap V| = 4$, $|T \cap J_2| = 2$, and if $a \in J_2$, then $\mathrm{ord}(\hat{a}) = 4$. Moreover $|T \cap J_1| = 1$, so if $a \in J_1$, then $C_T(a) = T$, $o(a) = 1$, $f(a) = 128/16 = 8$, and $d - f = 24 - 8 = 16$, so $\mathrm{ord}(\hat{a}) = 2$.

Suppose $a \in J_3$; then a^x is also in J_3. If T were to contain any more elements from J_3, it would not be coreless. Thus $o(a) = 1$, $f(a) = 96/8 = 12$, and $d - f = 24 - 12 = 12$, so $\mathrm{ord}(\hat{a}) = 4$. There are two elements in $T \cap (J_4 \cup J_5)$, one of which is x^2, and the other, which we label b, is also is centralized by x. Then $C_T(a) = T$. Now $b = x^2 a^x a = x^a x$, so b is conjugate to x^2 in U_4. Thus $o(a) = 2$, $f(a) = 2(64/16) = 8$, and $d - f = 24 - 8 = 16$, so $\mathrm{ord}(\hat{a}) = 2$.

Suppose $a \in J_4$ (the case of J_5 is identical). Then $a^x \in J_4$, and $\{1, aa^x\} \in V_1$. If T contains an element of J_3, then it must contain at least two; this cannot be the case, since $x^2 \in T$. So $T \cap J_3 = \varnothing$. Thus for $b \in J_3$, we have $o(b) = 0$, $f(b) = 0$, $d - f = 24$, and $\mathrm{ord}(\hat{b}) = 2$.

Let $c_1, c_2 \in M \setminus V$ be the elements of T in $J_4 \cup J_5$ which are centralized by x; one of them is x^2. If $c_1 \in J_4$, then $c_1 a a^x \in J_5$. For $b \in J_4$, we have $o(b) = 2$, $f(b) = (64/16) + (64/8) = 4 + 8 = 12$, $d - f = 24 - 12 = 12$, so $\mathrm{ord}(\hat{b}) = 4$. □

Each lifting pattern produces a distinct spin cover. Denote them by

(a) θ_1 comes from pat $= (2, 2, 4, 4, 4)$;

(b) θ_2 comes from pat $= (2, 4, 2, 4, 4)$;

(c) θ_3 comes from pat $= (2, 4, 4, 2, 2)$.

Computations aided by [**GAP**] indicate that each of these obstructs a different set of Nielsen tuples (see chapter IX).

CHAPTER VII

Ascent of $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$

1. The Nielsen Class $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})$

1.1. Definition of $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})$. Let A_3 denote the subgroup of S_3 generated by $g =$ `(1 2 3)`, that is, cyclic of order three. Let $C_+ = C_+(A_3)$ be the conjugacy class of g (containing only g) and $C_- = C_-(A_3)$ denote the conjugacy class of $g^2 = g^{-1}$.

Let $\boldsymbol{C}_{3^2_\pm} = (C_+, C_-, C_+, C_-)$. Our interest in the Nielsen class $(A_3, \boldsymbol{C}_{3^2_\pm})$ lies in the fact that it codifies information about braiding arrangements of conjugacy classes.

1.2. Elements of $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})$. The total Nielsen class $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{to}}$ contains exactly $\binom{4}{2} = 6$ elements corresponding to the six possible arrangements of the conjugacy classes. Since A_3 is abelian, the inner classes are the same. Now A_3 has an outer automorphism of order two which sends an arrangement to its complement. We enumerate these arrangements and their complements:

List 55 (Elements of $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$).

```
[1] +-+-        [4] -+-+
[2] ++--        [5] --++
[3] +--+        [6] -++-
```

1.3. Braid Action on $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})$. The Hurwitz monodromy group H_4 acts on $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. Using the enumeration of the arrangements above, we compute that

$$Q_1 \mapsto \texttt{(1 6)(3 4)}; \quad Q_2 \mapsto \texttt{(1 2)(4 5)}; \quad Q_3 \mapsto \texttt{(1 3)(4 6)}.$$

The reduction kernel has this effect:

$$Q_1 Q_3^{-1} \mapsto \texttt{(1 4)(3 6)}; \quad (Q_1 Q_2 Q_3)^2 \mapsto \texttt{(2 5)(3 6)}.$$

Thus the reduced classes equal the absolute classes, and are represented by `[1]`, `[2]`, and `[3]`. On the reduced classes we have

$$Q_1 \mapsto \texttt{(1 3)}; \quad Q_2 \mapsto \texttt{(1 2)}; \quad Q_3 \mapsto \texttt{(1 3)},$$

so the monodromy group of the cover $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathcal{J}_4$ is S_3. The branch cycle description of this cover is

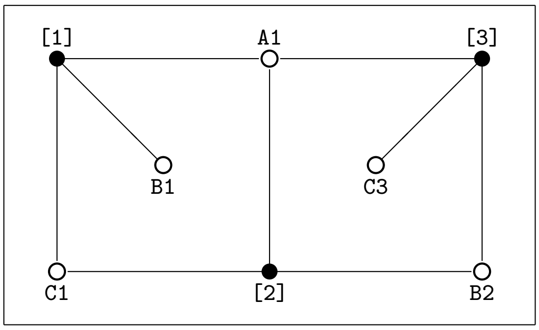
$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_\infty) \mapsto \big(\texttt{(1 3 2)}, \texttt{(2 3)}, \texttt{(1 2)}\big).$$

Apply the Riemann-Hurwitz formula to see that the genus of $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ equals zero.

1.4. Branch Cycle Design for $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}} \to \mathcal{J}_4$. Recall that the hubs of a branch cycle design are the integers of the branch cycle description, and that the nodes are the disjoint cycles, including those of length 1. In our current situation, label the hubs by `[1]`, `[2]`, and `[3]`, and nodes as follows:

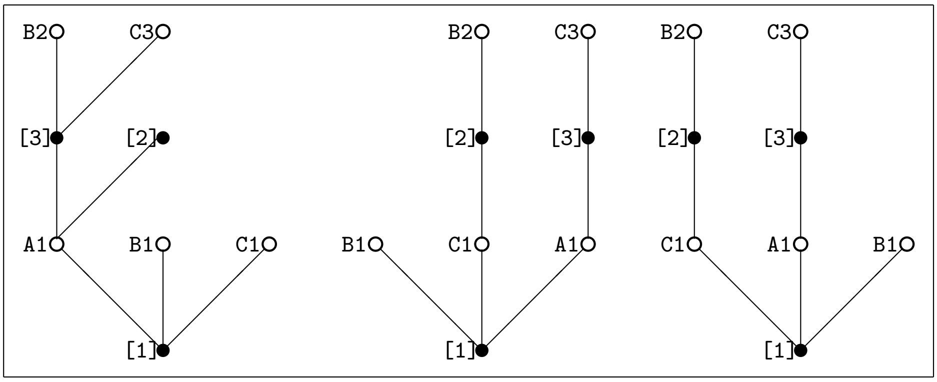
- Over 0, let `A1` = `(1 3 2)`;
- Over 1, let `B1` = `(1)` and `B2` = `(2 3)`;
- Over ∞, let `C1` = `(1 2)` and `C3` = `(3)`.

The branch cycle design for this cover is given by the following labeled planar graph. We take care that the twist sequence is accurately represented at each vertex.



Branch Cycle Design for $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}} \to \mathcal{J}_4$

Produce maximal trees initiated at specified basepoints and terminating in nodes according to the algorithm presented in chapter V. There are two parameters to this algorithm; the initial vertex and the initial edge. Select a Harbater-Mumford tuple as a basepoint. In $\text{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$, one such tuple is described by `+-+-`; that is, it is labeled `[1]`. Draw trees such that the height of a node indicates the complexity of the design generator it produces.



Recall that nodes over $0, 1 \in \mathcal{J}_4$ are finally ramified if they ramify to order three or two, respectively, and no further ramification can occur over them. The branches terminating in hubs or finally ramified nodes may be ignored.

If $\mathcal{H}$ is a reduced rank four Hurwitz space, let $\mathcal{H}^*$ denote $\mathcal{H}$ with the nodes over 0 and 1 which are not finally ramified removed. Let $\mathcal{J}_4^* = \mathcal{J}_4 \setminus \{0, 1\}$, so that the design generators are images in $\mathcal{J}^*$ of loops in $\mathcal{H}^*$. In the present case, all three trees will give the same generators for

$\pi_1(\mathcal{H}^*(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}, \texttt{[1]})$ inside $\pi_1(\mathcal{J}_4^*, j_0)$, up to order. Choose the first tree, ignore the vertices B2 and [2] and their adjacent edges, and obtain these walks and generators:

$$W_1 : \texttt{[1]} \to \texttt{A1} \to \texttt{[3]} \to \texttt{C3} \qquad \beta_1 = \gamma_0\gamma_\infty\gamma_0^{-1}$$

$$W_2 : \texttt{[1]} \to \texttt{B1} \qquad \beta_2 = \gamma_1$$

$$W_3 : \texttt{[1]} \to \texttt{C1} \qquad \beta_3 = \gamma_\infty^2$$

Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$. Any Hurwitz cover of $\mathcal{J}_4$ which factors through $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ has branch cycles over $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ given by acting on the fiber over the arrangement +-+- by $\boldsymbol{\beta}$.

2. The Nielsen Class $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$

2.1. Definition of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$. Let A_4 be the alternating group on 4 letters in its standard representation. It is generated as a subgroup of S_4 by (1 2 3) and (1 3 4). The Sylow 2-subgroup of A_4 is a normal Klein four subgroup which we denote by K. We have $A_4/K \cong A_3$, and $A_4 = K \rtimes A_3$.

There are two conjugacy classes of three cycles in A_4, each containing 4 elements. For an element of order three, its inverse is in the opposite conjugacy class. An outer automorphism (conjugation by (1 2)) swaps these conjugacy classes.

Let $C_+ = C_+(A_4)$ be the conjugacy class of (1 2 3) and $C_- = C_-(A_4)$ be the conjugacy class of (1 3 2). Let $\boldsymbol{C}_{3^2_\pm} = (C_+, C_-, C_+, C_-)$. We analyze the Nielsen class $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$.

2.2. Size of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. To compute the size of the inner Nielsen class of $(A_4, \boldsymbol{C}_{3^2_\pm})$, note that six arrangements of these conjugacy classes appear in the Nielsen class, so we may count the total number of Nielsen tuples in the ordered arrangement $\boldsymbol{C}_{3^2_\pm}$ and multiply by six, then divide by the order of A_4 to obtain the number of inner Nielsen tuples.

We begin by showing that if $g_1 \in C_+$ and $g_2 \in C_-$, then $g_2 = g_1^{-1}$ or $\mathrm{ord}(g_1g_2) = 2$. Conjugate so that $g_1 =$ (1 2 3), and consider the action of g_1 on C_-. There is one orbit of length 1 consisting of g_1^{-1} and one orbit of length 3. Suppose that g_2 is in the length 3 orbit; we show that $\mathrm{ord}(g_1g_2) = 2$. Since $g_1g_2^{g_1} = g_2g_1$ and $\mathrm{ord}(g_1g_2) = \mathrm{ord}(g_2g_1)$, it suffices to test a representative from the orbit. Such a representative is $g_2 =$ (1 2 4), and $g_1g_2 =$ (1 4)(2 3), which has order 2.

Moreover, if g_1 and g_2 are in the same conjugacy class and generate A_4, there product has order 3. To see this, again let $g_1 =$ (1 2 3) and select any nonidentical element in the same conjugacy class, say $g_2 =$ (1 3 4). Then $g_1g_2 =$ (1 2 4), which has order 3.

Consider the map $C_+ \times C_- \to A_4$ given by $(g_1, g_2) \mapsto g_1g_2$. The order of the product is either 1 or 2. If the order of the product is 1, then $g_2 = g_1^{-1}$. There are 4 such pairs, so there are $16 - 4 = 12$ pairs with product of order 2.

In a Nielsen tuple (g_1, g_2, g_3, g_4) of rank 4, the product of the second pair must equal the inverse of the product of the first. Call the first pair the initial pair and the second pair the terminal pair. Then if the initial pair satisfies $g_2 = g_1^{-1}$, the terminal pair satisfies $g_4 = g_3^{-1}$, and $\{g_1, g_3\}$ generates the group. The latter condition holds in our case when $g_1 \neq g_3$ (since $g_1, g_3 \in C_+$), so there are $4 \cdot 3 = 12$ tuples with trivial initial product.

Now suppose that $\mathrm{ord}(g_1 g_2) = 2$. Then $\{g_1, g_2\}$ automatically generates A_4. Let C_2 denote the conjugacy class of involutions. The 12 elements of $C_+ \times C_-$ with product of order 2 map surjectively onto C_2. Conjugation permutes the fibers over elements in C_2, so all are of the same size. If (g_1, g_2) maps to h, then $(g_1, g_2)^g$ maps to h if and only if $g \in C_{A_4}(h)$. Elements of order two are centralized by the normal Klein four subgroup of A_4, so the fiber over an element in C_2 contains four elements. Thus there are $12 \cdot 4 = 48$ tuples with initial product in C_2.

Conclude that $|\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = \frac{(12+48)\cdot 6}{12} = 30$.

2.3. Labeling elements of A_4. The elements of A_4 can be described in terms of an arbitrary pair of order three generators. Although we understand A_4 so well that this is unnecessary, the technique is presented here in this simple case to make later extensions more transparent. Notation set here will be used throughout the rest of this section.

LEMMA 56. *Let G be a group and let $g_1, g_2 \in G$ be elements of order n.*
Set $a_i = g_1^{n-i} g_2 g_1^{i-1}$ for $i = 1, \ldots, n$. Then

- **(a)** *$a_i^{g_1} = a_{i+1}$ for $i = 1, \ldots, n-1$;*
- **(b)** *$a_n^{g_1} = a_1$;*
- **(c)** *$a_i^{g_2} = a_i^{g_1}$ for $i = 1, \ldots, n$, if $\mathrm{ord}(g_1 g_2) = n = 3$.*

PROOF. Part **(a)** is a direct computation; part **(b)** follows because g_1 has order n action. If $\mathrm{ord}(g_1 g_2) = n = 3$, then $a_2^{-1} = g_2 g_1 g_2$. Compute the orbit of the action of g_2 on this. □

LEMMA 57. *Let $g_1, g_2 \in C_+(A_4)$ be distinct.*
Then $\mathrm{ord}(g_1 g_2) = 3$ and $\mathrm{ord}(g_1^{-1} g_2) = 2$. Label the following elements of A_4:

- **(1)** *e is the identity;*
- **(2)** *$a_1 = g_1 g_1 g_2 = g_1^{-1} g_2$;*
- **(3)** *$a_2 = g_1 g_2 g_1 = g_2 g_1 g_2$;*
- **(4)** *$a_3 = g_2 g_1 g_1 = g_2 g_1^{-1}$;*

Then

- **(a)** *$K = \{e, a_1, a_2, a_3\}$;*
- **(b)** *$a_1^{g_1} = a_1^{g_2} = a_2$, $a_2^{g_1} = a_2^{g_2} = a_3$, and $a_3^{g_1} = a_3^{g_2} = a_1$;*
- **(c)** *$g_1 = g_2^{a_2}$ and $g_2 = g_1^{a_2}$.*

PROOF. We have seen that $\mathrm{ord}(g_1 g_2) = 3$ and $\mathrm{ord}(g_1^{-1} g_2) = 2$. Combine this with Lemma 56 to obtain **(a)** and **(b)**. Since $\mathrm{ord}(g_1 g_2) = 3$, we have $g_2 g_1 g_2 = a_2^{-1}$, and since a_2 has order two, we have $g_2 g_1 g_2 = a_2$. Now $g_2^{a_2} = (g_1 g_2 g_1) g_2 (g_1 g_2 g_1) = (g_1 g_2)^3 g_1 = g_1$. This gives **(c)**. □

Fix $g_1 =$ `(1 2 3)` and $g_2 =$ `(1 3 4)`, and let $e =$ `()`, $a_1 =$ `(1 4)(2 3)`, $a_2 =$ `(1 2)(3 4)`, and $a_3 =$ `(1 3)(2 4)`, as in the lemma. The tuple $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$ will be the base camp upon which we mount our ascent to level one. The elements a_i will come into play in the next stage.

2.4. Elements of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$**.** To list the elements of this Nielsen class, it suffices to list five with conjugacy class arrangement (C_+, C_-, C_+, C_-), and to braid each with six braids permuting the arrangements. Up to inner equivalence, there is only one Nielsen tuple with trivial initial product in this arrangement; conjugate so that $g_1 =$ `(1 2 3)` and $g_3 =$ `(1 3 4)` to give the first entry in the list below.

Now suppose $g_2 =$ `(1 2 4)`. Then $g_1 g_2 =$ `(1 4)(2 3)` has order two, and the above argument shows that the other four inner tuples with initial product in C_2 are given by finding the second pair of entries, that is, four pairs from $C_+ \times C_-$ with product `(1 4)(2 3)`. These are given by conjugating (g_1, g_2) by the centralizer of `(1 4)(2 3)` in A_4. This yields the last four entries in this list:

List 58 (Fiber over `+-+-` in $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{rd,in}} \to \mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{rd,in}}$).

[1] ((1 2 3), (1 3 2), (1 3 4), (1 4 3))
[2] ((1 2 3), (1 2 4), (1 3 4), (2 3 4))
[3] ((1 2 3), (1 2 4), (1 4 2), (1 3 2))
[4] ((1 2 3), (1 2 4), (1 2 3), (1 2 4))
[5] ((1 2 3), (1 2 4), (2 4 3), (1 2 3))

2.5. Reduction of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$**.** The amount of reduction is a braid invariant, because we reduce by a subgroup which is normalized by braiding. Thus to compute the amount of reduction of this Nielsen class, it suffices to consider only the fiber listed above, as it will pass through every braid orbit.

Direct computation on Nielsen tuples `[1]` and `[4]` show that $(q_1 q_2 q_3)^2$ acts trivially, and $(q_1 q_3^{-1})$ maps each to a tuple to which it is absolutely equivalent. We will see that these represent all braid orbits, which shows that the amount of reduction in each braid orbit is two. Thus the discrete information which produces the reduced space (braid action on the Nielsen class) equals that which produces the absolute space (however, the reduced absolute Nielsen class is smaller).

Since the amount of reduction in $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ is also two, the degrees of the maps induced by $A_4 \to A_3$ are the same on the inner Nielsen and reduced inner Nielsen classes. Thus the list above represents the fiber over `+-+-` for the reduced inner Nielsen class as well as the inner Nielsen class.

2.6. Braid Action on $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ **via Branch Cycle Designs.** Let $\mathcal{H}^*(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ denote the reduced inner space, with points to fill in the finally ramified places over 0 and 1, and with the unramified node over 1 removed. Using branch cycle designs, we have computed that classical generators for $\mathcal{H}^*(A_3, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$ have these images in $\mathcal{J}^*$:

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_2) = (\gamma_0\gamma_\infty\gamma_0^{-1}, \gamma_1, \gamma_\infty^2).$$

Use this to compute the reduced braid action on the Nielsen class. Compute the orbits of Nielsen tuple `[1]`:

$$(\texttt{(1 2 3)}, \texttt{(1 3)}, \texttt{(1 2)}).$$

Compute the orbits of Nielsen tuple `[4]`:

$$(\texttt{(4 5)}, \texttt{(1)}, \texttt{(4 5)}).$$

Therefore

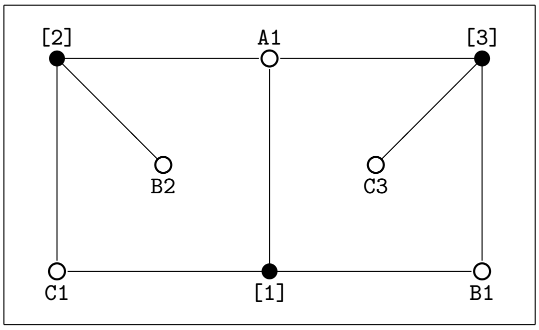
(a) H_4 has two orbits on $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\rm in}$, one of size $6 \cdot 3 = 18$ and the other of size $6 \cdot 2 = 12$;

(b) $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$ has two components, each of genus zero.

2.7. Branch Cycle Design for $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd,HM} \to \mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$. If $\mathcal{H}$ is a Hurwitz space, let $\mathcal{H}^{\rm HM}$ denote the disjoint union of its Harbater-Mumford components; in the present case, there is only one such component. The branch cycles for $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd,HM} \to \mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$ are $(\texttt{(1 2 3)}, \texttt{(1 3)}, \texttt{(1 2)})$. Label the branch points on $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$ by `A`, `B`, and `C`, and the nodes as follows:

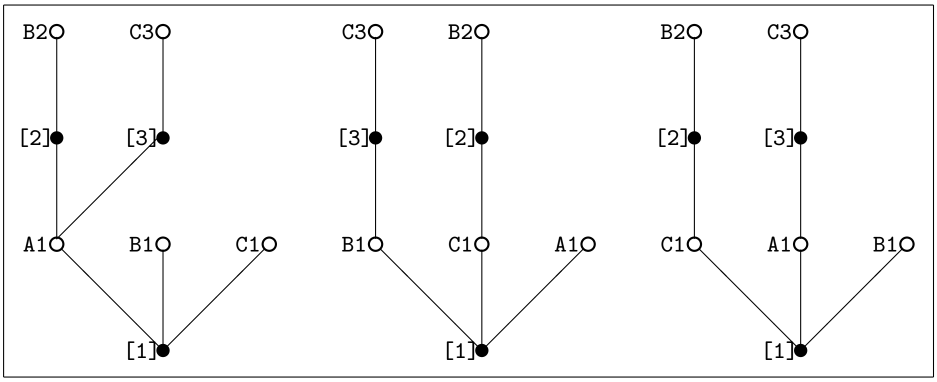
- Over `A`, let `A1` = `(1 2 3)`;
- Over `B`, let `B1` = `(1 3)` and `B2` = `(2)`;
- Over `C`, let `C1` = `(1 2)` and `C3` = `(3)`.

We obtain the following branch cycle design.



Branch Cycle Design for $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd,HM} \to \mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$

Our basepoint for covers of $\mathcal{H}^*(A_4, \boldsymbol{C}_{3^2_\pm})^{\rm in,rd}$ will be $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$, where $g_1 =$ `(1 2 3)` and $g_2 =$ `(1 3 4)`; this is the Harbater-Mumford tuple whose middle product has order two, and is enumerated `[1]` above. Draw the three possibilities for trees based at `[1]`.



Select the middle tree, and compute the maximal trails beginning at the hub `[1]` and ending at nodes within this maximal tree. Since `B` lies over $1 \in \mathcal{J}_4$, the node labeled `B1` is finally ramified, and may be omitted. We obtain these walks and generators:

$$W_1: \texttt{[1]} \to \texttt{B1} \to \texttt{[3]} \to \texttt{C3} \qquad \alpha_1 = \beta_2\beta_3\beta_2^{-1} \qquad = \gamma_1\gamma_\infty^2\gamma_1^{-1}$$

$$W_2: \texttt{[1]} \to \texttt{C1} \to \texttt{[2]} \to \texttt{B2} \qquad \alpha_2 = \beta_3\beta_2\beta_3^{-1} \qquad = \gamma_\infty^2\gamma_1\gamma_\infty^{-2}$$

$$W_3: \texttt{[1]} \to \texttt{C1} \qquad \alpha_3 = \beta_3^2 \qquad = \gamma_\infty^4$$

$$W_4: \texttt{[1]} \to \texttt{A1} \qquad \alpha_4 = \beta_1^3 \qquad = \gamma_0\gamma_\infty^3\gamma_0^{-1}$$

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Any Hurwitz cover of $\mathcal{J}$ by a reduced rank four Hurwitz space which factors through $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$ has branch cycles over $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$ given by acting on the fiber over the $\boldsymbol{g}$ by $\boldsymbol{\alpha}$.

3. The Nielsen Class $\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm})$

3.1. Definition of $\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm})$**.** Let Q_8 denote the quaternion group of order 8, generated elements i and j of order 4 with $ij = k$, and we write $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. This group has an automorphism $\sigma \in \mathrm{Aut}(Q_8)$ given by $i \mapsto j$ and $j \mapsto -k$ so that $k \mapsto -i$. This automorphism must fix the unique element of order two in Q_8. Identify σ with `(1 2 3)` and the cyclic subgroup of $\mathrm{Aut}(Q_8)$ generated by σ with A_3, and form the semidirect product $\hat{A}_4 = Q_8 \rtimes A_3$.

The center of $\hat{A}_4$ is generated by the single involution from Q_8, and $\hat{A}_4/Z(\hat{A}_4) = (Q_8/Z(Q_8)) \rtimes A_3 = A_4$. Thus $\hat{A}_4$ is a central extension of A_4 by a single involution denoted by -1. The map $\hat{A}_4 \to A_4$ is the *spin cover* of A_4; it can be identified with the cover $\mathrm{SL}_2(\mathbb{F}_3) \to \mathrm{PSL}_2(\mathbb{F}_3)$. Any generators for A_4 lift to generators for $\hat{A}_4$; it is a Frattini extension. The outer automorphism of A_4 lifts to an outer automorphism of $\hat{A}_4$.

Elements of order 2 in A_4 lift to elements of order 4 in $\hat{A}_4$, and elements of order 3 in A_4 have a unique element of order 3 in $\hat{A}_4$ above them (the other element above a three cycle has order six).

The conjugacy classes of three cycles in A_4 lift uniquely to conjugacy classes in $\hat{A}_4$, which we denote by $C_+(\hat{A}_4)$ and $C_-(\hat{A}_4)$. Let $\boldsymbol{C}_{3^2_\pm}$ denote (C_+, C_-, C_+, C_-) in either case.

The product of elements of order 3 from opposite conjugacy classes in $\hat{A}_4$ has order 1 or 4. To see this, suppose the elements are not inverses. Then the product has order 2 is A_4, and the product of the lifts is a lift of the product. Lifts of elements of order 2 have order 4.

The product of distinct elements of order 3 from the same conjugacy class in $\hat{A}_4$ has order 6. To see this, apply an automorphism to select the first element to be $\hat{g}_1 = (1, g)$ with $g =$ `(1 2 3)` in the semidirect product formulation; the conjugate element will be $\hat{g}_2 = (x, g)$ with $x \in \{i, j, k\}$. Then $\hat{g}_1\hat{g}_2 = (x^\sigma, g^2)$, whose cube is $(x^\sigma x x^{\sigma^{-1}}, 1) = (-1, 1)$.

3.2. Size of $\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$**.** With unique lifts to tuples of elements of order three, we obtain an injective map $\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm}) \to \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$ which commutes with the braid action. This map is not surjective. An element in $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$ which does not lift to an element of $\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm})$ is *obstructed* with respect to $\hat{A}_4 \to A_4$.

Let $\boldsymbol{g} = (g_1, g_2, g_3, g_4) \in \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$ and $\hat{\boldsymbol{g}} = (\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4) \in \hat{A}_4^4$, where $\hat{g}_i$ is the unique element of order 3 over g_i. Then $\boldsymbol{g}$ is obstructed if and only if $\Pi\hat{\boldsymbol{g}} = -1$. This product is a braid invariant; thus we may refer to entire orbits of the braid action as either obstructed or not.

We have seen that there are two braid orbits on $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$; these may be differentiated by this invariant. One easily sees that Harbater-Mumford tuples always lift, so their orbits are unobstructed. Now consider the tuple labeled `[4]` in List 58. It is of the form (g_1, g_2, g_1, g_2) with g_1 and g_2 in different conjugacy classes, so $\mathrm{ord}(g_1g_2) = 2$. Therefore $\mathrm{ord}(\hat{g}_1\hat{g}_2) = 4$, so $(\hat{g}_1\hat{g}_2)^2 = -1$, and this tuple is obstructed. Therefore the size of $\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ is the size of the braid orbit in $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ which contains the Harbater-Mumford tuples.

Conclude that $|\mathrm{Ni}(\hat{A}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = 18$.

4. The Nielsen Class $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})$

4.1. Definition of $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})$**.** Let Z_4 denote a cyclic group of order 4 generated by z with identity e. Let $Z_4^2 = Z_4 \times Z_4$; it is generated by (z, e) and (e, z). Then $\mathrm{Aut}(Z_4^2)$ contains an element σ of order 3 defined by $(z, e)^\sigma = (e, z)$ and $(e, z)^\sigma = (z, z^3)$. Identify `(1 2 3)` with σ and $\langle\sigma\rangle$ with A_3. Set $O_4 = Z_4^2 \rtimes A_3$. The Sylow 2-subgroup of O_4 is normal and abelian. The elements of order two in O_4 generate a normal Klein four subgroup; denote it by L. This is the Frattini subgroup of O_4. We have $O_4/L \cong A_4$.

Let $\varphi : O_4 \to A_4$ be the canonical homomorphism. This is a Frattini cover, so any lift of a set of generators of A_4 produces a set of generators for O_4. Elements of order 2 in A_4 lift to elements of order 4 in O_4. By Proposition 9, the conjugacy classes $\boldsymbol{C}_{3_\pm}$ in A_4 lift uniquely to conjugacy classes in O_4. We may consider L as an A_4 module; clearly three-cycles in A_4 act transitively on the involutions in L, and involutions in A_4 act trivially. In particular, O_4 is centerless, and the centralizer in L of an element h of order three in O_4 is trivial, so the coset of h is $hL = \{hl \mid l \in L\} = \{h^l \mid l \in L\}$.

The conjugacy classes of three cycles may be differentiated by their action on L; that is, if we enumerate the three nontrivial elements of L, an element of order 3 in O_4 acts on them as a three cycle. Those in one conjugacy class act as `(1 2 3)` and those in the other act as `(1 3 2)`. For $l \in L$, let $l^+ = l^h$ for $h \in C_+$ and $l^- = l^h$ for $h \in C_-$.

4.2. Size of $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. We have seen that $|\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = 30$.

Elements of order three in O_4 have trivial centralizers, and the map $O_4 \to A_4$ is a Frattini cover. Additionally, both groups are centerless. Thus by Proposition 12, every element of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ lifts to $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$, and

$$|\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = |L|^2 |\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = 4^2 \cdot 30 = 480,$$

with 288 of these over the Harbater-Mumford orbit of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ and 192 over the other orbit. In particular, the fiber over a point in $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ contains sixteen points.

This method will not work for computing the size of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ from the size of $\mathrm{Ni}(A_3, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ because $A_4 \to A_3$ is not a Frattini cover; the Frattini subgroup of A_4 is trivial.

4.3. Duals and Perturbations in $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. Let $\boldsymbol{h} = (h_1, h_1^{-1}, h_2, h_2^{-1})$ be a Harbater-Mumford tuple. Since $C_L(h)$ is trivial for h of order 3, the four duals and four perturbations are distinct up to inner equivalence. The fiber over $\boldsymbol{g}$ consists of the duals of $\boldsymbol{h}$ and their perturbations, for a total of 16 elements.

There are two types of Harbater-Mumford tuples in $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})$; those with middle product 4 have arrangement `+-+-` or `-+-+` and those of middle product 3 have arrangement `+--+` or `-++-`. Reduction preserves these distinctions.

Suppose that $\mathrm{mpo}(\boldsymbol{h}) = 4$. Then the middle product of $\boldsymbol{h}$ commutes with elements of L, so the perturbations are homogeneous:

$$\boldsymbol{h}^{[l|e]} = (h_1, (h_1^{-1})^l, h_2^l, h_2^{-1}).$$

4.4. Reduction of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. The action of reduction is faithful, so reduction is 4 to 1. Downstairs over A_4, reduction is 2 to 1. Thus reduction glues together pairs of fibers (via $Q_1 Q_3^{-1}$) and has a 2 to 1 action within a fiber (via $(Q_1 Q_2 Q_3)^2$).

In particular, in the unreduced inner Nielsen classes, there are 16 Harbater-Mumford tuples lying over 4 such tuples in A_4, and in the reduced classes there are 4 Harbater-Mumford tuples lying over 2. Given $\boldsymbol{h} = (h_1, h_1^{-1}, h_2, h_2^{-1})$, there is a unique nontrivial $l \in L$ such that $(h_1, h_1^{-1}, h_2^l, (h_2^{-1})^l)$ is reduction equivalent to $\boldsymbol{h}$ (via $(Q_1 Q_2 Q_3)^2$; $Q_1 Q_3^{-1}$ actually changes the arrangement of the conjugacy classes).

Let $\boldsymbol{h}^+ = (h_1, h_1^{-1}, h_2^{l^+}, (h_2^{-1})^{l^+})$ and $\boldsymbol{h}^- = (h_1, h_1^{-1}, h_2^{l^-}, (h_2^{-1})^{l^-})$; these are equivalent modulo reduction, and represent the *reduced dual* of $\boldsymbol{h}$. The perturbations remain distinct upon reduction.

4.5. Labeling Elements of O_4. Let $g_1, g_2 \in A_4$ be given by $g_1 = \texttt{(1 2 3)}$ and $g_2 = \texttt{(1 3 4)}$. Compute that

- $\operatorname{ord}(g_2 g_1) = 3$;
- $\operatorname{ord}(g_1 g_2 g_1) = 2$;
- $g_1^{g_1 g_2 g_1} = g_2$;
- $g_2^{g_1 g_2 g_1} = g_1$.

Let $h_1, h_2 \in O_4$ such that $h_1 \mapsto g_1$ and $h_2 \mapsto g_2$. Then $\operatorname{ord}(h_1 h_2) = 3$ and $\operatorname{ord}(h_1 h_2^{-1}) = 4$. Moreover, the nontrivial elements of Klein four kernel of $A_4 \to A_3$ are $g_1 g_1 g_2$, $g_1 g_2 g_1$, and $g_2 g_1 g_1$. This motivated the investigation which produced the following definitions and lemmas.

LEMMA 59. *Let $h_1, h_2 \in C_+(O_4)$ be distinct.*
Then $\operatorname{ord}(h_1 h_2) = 3$ *and* $\operatorname{ord}(h_1 h_2^{-1}) = 4$. *Label the following elements of O_4:*

(1) *e is the identity;*
(2) $a_1 = h_1 h_1 h_2 = h_1^{-1} h_2 = (h_2^{-1} h_1)^{-1}$;
(3) $a_2 = h_1 h_2 h_1 = (h_2 h_1 h_2)^{-1}$;
(4) $a_3 = h_2 h_1 h_1 = h_2 h_1^{-1} = (h_1 h_2^{-1})^{-1}$;
(5) $o_1 = a_1^2$;
(6) $o_2 = a_2^2$;
(7) $o_3 = a_3^2$.

Then

(a) $L = \{e, o_1, o_2, o_3\}$;
(b) $a_1^{h_1} = a_2$ *and* $a_2^{h_1} = a_3$, *and these elements have order* 4;
(c) $o_1^{h_1} = o_2$ *and* $o_2^{h_1} = o_3$, *and these elements have order* 2;
(d) $o_1^+ = o_2$ *and* $o_2^+ = o_3$;
(e) $h_1 = h_2^{a_2 o_1}$;
(f) $h_2 = h_1^{a_2 o_3}$.

PROOF. The conjugations in **(b)** are immediate computations, and $a_1 = h_1^{-1} h_2$ has order four since h_1^{-1} and h_2 are neither conjugates nor inverses. Part **(c)** follows by squaring, since conjugation is a homomorphism.

Now

$$\begin{aligned} o_2 &= h_1 h_2 h_1 h_1 h_2 h_1 \\ &= (h_1 h_2^{-1})(h_2^{-1} h_1)(h_1 h_2^{-1})(h_2^{-1} h_1) && [\operatorname{ord}(h_2) = 3 \Rightarrow h_2 = h_2^{-2}] \\ &= o_1 o_3 && [h_1 h_2^{-1} \text{ and } h_2^{-1} h_1 \text{ commute}]. \end{aligned}$$

Since o_1 and o_3 commute, $\{e, o_1, o_2, o_3\}$ form a Klein four group. This proves **(a)**.

Part **(d)** follows from **(c)**, recalling that all elements in C_+ act the same on the nontrivial elements of L, and all elements of C_- act in the reverse.

As for **(e)**, first note that $h_1^{o_1} = o_1 h_1 o_1 h_1^{-1} h_1 = o_1 o_1^{-} h_1 = o_2 h_1$; conjugate by o_1 and use this to see that it suffices to show that $h_2^{h_1 h_2 h_1} = o_2 h_1$. Note that since o_3 is an involution, $o_3 = o_3^{-1} = (h_1 h_2^{-1})^2$, so

$$\begin{aligned} h_2^{h_1h_2h_1} &= h_1^{-1}h_2^{-1}h_1^{-1}h_2h_1h_2h_1 \\ &= h_1(h_1h_2^{-1})(h_1^{-1}h_2)(h_1h_2^{-1})h_2^{-1}h_1 && [\mathrm{ord}(h_1)=\mathrm{ord}(h_2)=3] \\ &= h_1 o_3 (h_1^{-1}h_2)h_2^{-1}h_1 && [h_1^{-1}h_2 \text{ and } h_1h_2^{-1} \text{ commute}] \\ &= o_3^{h_1^{-1}} h_1 \\ &= o_2 h_1. \end{aligned}$$

The proof of part **(f)** is analogous, and has the following consequence. Since $h_2 = h_1^{h_1 h_2 h_1 o_1 o_2}$, we have $h_1^{h_1 h_2 h_1 o_1} = h_2^{o_2}$. □

4.6. A Fiber of $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. Continue notation from the previous subsection. For the rest of this section, we sometimes shorten notation as follows:

(1) $a = o_1$;

(2) $b = o_2$;

(3) $c = o_3$;

Consider the map $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. The fiber over $\boldsymbol{g}$ consists of perturbations of duals of $\boldsymbol{h}$. For $x, y \in O_4$, set

$$\boldsymbol{h}^{[x|y]} = (h_1, (h_1^{-1})^x, h_2^{yx}, (h_2^{-1})^y).$$

Typically $x, y \in L$, but we reserve some leeway in this regard.

LEMMA 60. $\boldsymbol{h}^{[x|y]} = \boldsymbol{h}^{[x|yb]}$ *modulo inner reduction, via* $(Q_1Q_2Q_3)^2$.

PROOF. Conjugate $\boldsymbol{h}(Q_1Q_2Q_3)^2$ by h_1h_2 to see that this tuple is equivalent to $(h_2, h_2^{-1}, h_1, h_1^{-1})$. Now conjugate by $a_2 o_1$ and apply that $h_1^{a_2 o_1} = h_2^{o_2}$. □

A complete set of representative tuples upon which design generators act can now be given.

LIST 61 (Fiber over $\boldsymbol{g}$ in $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$).

[1] $\boldsymbol{h}^{[e\|e]}$	[5] $\boldsymbol{h}^{[b\|e]}$
[2] $\boldsymbol{h}^{[e\|a]}$	[6] $\boldsymbol{h}^{[b\|a]}$
[3] $\boldsymbol{h}^{[a\|e]}$	[7] $\boldsymbol{h}^{[c\|e]}$
[4] $\boldsymbol{h}^{[a\|a]}$	[8] $\boldsymbol{h}^{[c\|a]}$

We use this characterization of the fiber over an Harbater-Mumford tuple in $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$ to further investigate the Nielsen class $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})$.

4.7. Braid Action on $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ **via Branch Cycle Designs.** Denote the components of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ which lie over $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$ by $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,Re}}$. Using branch cycle designs, we have computed that classical generators with nontrivial action for $\pi_1(\mathcal{H}^*(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}})$ inside $\pi_1(\mathcal{J}_4)$ are

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\gamma_1\gamma_\infty^2\gamma_1^{-1}, \gamma_\infty^2\gamma_1\gamma_\infty^{-2}, \gamma_\infty^4, \gamma_0\gamma_\infty^3\gamma_0^{-1}).$$

View $\boldsymbol{g} = (\texttt{(1 2 3)}, \texttt{(1 3 2)}, \texttt{(1 3 4)}, \texttt{(1 4 3)})$ as a point in $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$, and the fiber over it (as enumerated above) as points in $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,Re}}$. The action of path lifting corresponds to the braid action.

Lemma 62. *Let α be a braid which acts on perturbed duals of an arbitrary lifted $\boldsymbol{h}$ over an Harbater-Mumford tuple $\boldsymbol{g}$ induced by abelian kernel K. If $\boldsymbol{h}^{[x|e]}\alpha = \boldsymbol{h}^{[f_1(x)|f_2(x)]}$ for some functions $f_1, f_2 : K \to K$, then $\boldsymbol{h}^{[x|y]}\alpha = \boldsymbol{h}^{[f_1(x)|yf_2(x)]} = (\boldsymbol{h}^{[x|e]}\alpha)^{[e|y]}$.*

Proof. Since $\boldsymbol{h}$ is an arbitrary lift, then $\boldsymbol{h}^{[e|y]}$ is also an arbitrary lift. Replace h_2 with h_2^y throughout the computation to obtain the first equal sign. The second is explained by the fact that K is abelian. □

We now compute the action of $\boldsymbol{\alpha}$ on the fiber over $\boldsymbol{g}$ as enumerated above. All equal signs mean "modulo inner reduction". In each case, for arbitrary $x, y \in L$, compute the action on $\boldsymbol{h}^{[x|e]}$ and then insert y via the previous lemma. We compute $\boldsymbol{h}^{[x|y]}\alpha_i$, then apply this to the enumeration of the fiber over $\boldsymbol{g}$ as given by List 61 to obtain the image of α_i in under the map $\pi_1(\mathcal{H}^*(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}} \to S_8$, which is the monodromy representation of the cover $\mathcal{H}(O_4)^{\mathrm{in,rd,Re}} \to \mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$. The most difficult to compute is α_4, so we compute the first three and know that α_4 is the inverse of the product of the first three.

4.7.1. *Action of α_1.* Since $h_2^x h_2^{-1} = xx^{h_2^{-1}} = xx^- = x^+$, modulo inner reduction we have

$$\begin{aligned}\boldsymbol{h}^{[x|e]}\alpha_1 &= \big(h_1, (h_1^{-1})^x, h_2^x, (h_2^{-1})\big)\gamma_1\gamma_\infty^2\gamma_1\\ &= \big((h_1^{-1})^x, h_2^x, (h_2^{-1}), h_1\big)\gamma_\infty^2\gamma_1\\ &= \big((h_1^{-1})^x, h_2^{xx^+}, h_2^{x^+}, h_1\big)\gamma_1\\ &= \big(h_1, (h_1^{-1})^x, h_2^{xx^+}, (h_2^{-1})^{x^+}\big)\\ &= \boldsymbol{h}^{[x|x^+]}.\end{aligned}$$

Therefore

$$\alpha_1 : \boldsymbol{h}^{[x|y]} \mapsto \boldsymbol{h}^{[x|yx^+]} \qquad \Rightarrow \qquad \alpha_1 \mapsto \texttt{(1)(2)(3)(4)(5 6)(7 8)};$$

in particular, α_1 has trivial action on the first four tuples.

4.7.2. *Action of* α_2. The following computation will be used again in a more complex group; we do not use the fact that the Sylow 2-subgroup is abelian, but only that the element of the kernel (in this case L) commute with elements of this Sylow. We will use these comments:

(1) $xh = hh^{-1}xh = hx^-$ for $h \in C_-$;
(2) $(h_1h_2^{-1})^{-1} = a_3$;
(3) $x_1 = (h_1^{-1})^{x^-}h_1^{a_3} = x^+h_1^{-1}a_3^{-1}h_1a_3 = x^+a_1^{-1}a_3$;
(4) $x_2 = x^-a_2a_1^2$ so that $(h_2^{x^-})^{x_2} = h_1$;
(5) $a_3a_2 = a_1^{-1}$;
(6) $a_3x_2 = a_3x^-a_2a_1^2 = x^-a_1$;
(7) $x_1x_2 = x^+a_1^{-1}a_3x^-a_2a_1^2 = x$;
(8) $h_1^{a_3h_1^{-1}} = h_1^{a_2} = h_2^{a_3^2}$;
(9) $h_2^{a_1h_1^{-1}} = h_2^{a_3^2}$.

$$\begin{aligned}
\boldsymbol{h}^{[x|e]}\alpha_2 &= \big(h_1, (h_1^{-1})^x, h_2^x, h_2^{-1}\big)\gamma_\infty^2\gamma_1\gamma_\infty^{-2} \\
&= \big((h_1^{-1})^{xh_1^{-1}}, h_1, h_2^{-1}, h_2^{xh_2^{-1}}\big)\gamma_\infty^2\gamma_1\gamma_\infty^{-2} && [\text{via } (Q_1Q_3^{-1})] \\
&= \big((h_1^{-1})^{x^-}, h_1, h_2^{-1}, h_2^{x^-}\big)\gamma_\infty^2\gamma_1\gamma_\infty^{-2} && [\text{comment (1)}] \\
&= \big((h_1^{-1})^{x^-}, h_1^{a_3}, (h_2^{-1})^{a_3}, h_2^{x^-}\big)\gamma_1\gamma_\infty^{-2} && [\text{comment (2)}] \\
&= \big(h_2^{x^-}, (h_1^{-1})^{x^-}, h_1^{a_3}, (h_2^{-1})^{a_3}\big)\gamma_\infty^{-2} \\
&= \big(h_2^{x^-}, (h_1^{-1})^{x^-x_1}, h_1^{a_3x_1}, (h_2^{-1})^{a_3}\big) && [\text{comment (3)}] \\
&= \big(h_1, (h_1^{-1})^{x^-x_1x_2}, h_1^{a_3x_1x_2}, (h_2^{-1})^{a_3x_2}\big) && [\text{conj by } x_2] \\
&= \big(h_1, (h_1^{-1})^{x^+}, h_1^{a_3x}, (h_2^{-1})^{x^-a_1}\big) && [\text{comments (6) and (7)}] \\
&= \big(h_1, (h_1^{-1})^{x}, h_2^{a_3^2x^-}, (h_2^{-1})^{a_3^2x^+}\big) && [\text{conj by } h_1^{-1}] \\
&= \boldsymbol{h}^{[x|x^+c]}.
\end{aligned}$$

Therefore

$$\alpha_2 : \boldsymbol{h}^{[x|y]} \mapsto \boldsymbol{h}^{[x|yx^+c]} \qquad \Rightarrow \qquad \alpha_2 \mapsto \texttt{(1 2)(3 4)(5)(6)(7)(8)};$$

in particular, α_2 has trivial action on the last four tuples.

4.7.3. *Action of* α_3. The order of the middle product of a perturbation is equal to order of the middle product of the original tuple; both equal the order of h_4h_1. Now $((h_1^{-1})^x h_2^x) = ((h_1^{-1})h_2)^x = a_1$. Thus $((h_1^{-1})^x h_2^x)^{-2} = a_1^{-2} = a$, and

$$\begin{aligned} \boldsymbol{h}^{[x|e]}\alpha_3 &= (h_1, (h_1^{-1})^x, h_2^x, h_2^{-1})\gamma_\infty^4 \\ &= (h_1, (h_1^{-1})^{xa}, h_2^{xa}, h_2^{-1}) \\ &= \boldsymbol{h}^{[xa|e]}. \end{aligned}$$

Therefore

$$\alpha_3 : \boldsymbol{h}^{[x|y]} \mapsto \boldsymbol{h}^{[xa|y]} \qquad \Rightarrow \qquad \alpha_3 \mapsto \texttt{(1 3)(2 4)(5 7)(6 8)}.$$

4.7.4. *Action of* α_4. Finally, use that fact that $\Pi\boldsymbol{\alpha} = 1$ to compute

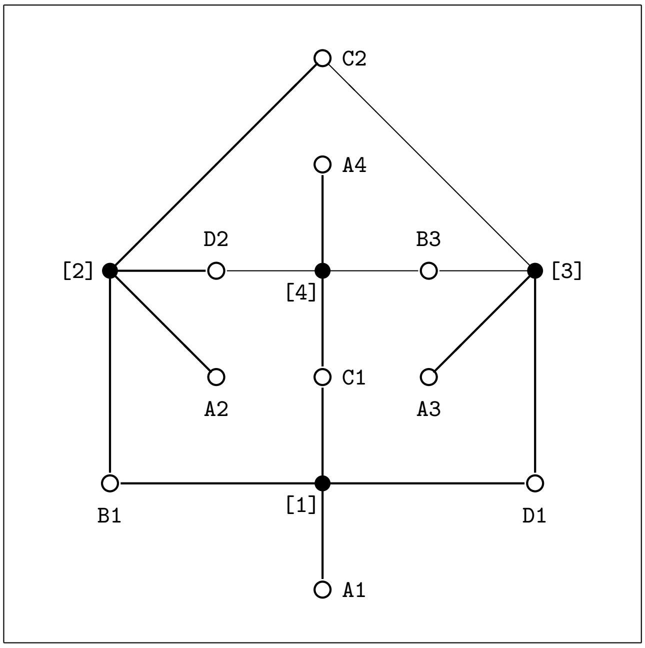
$$\begin{aligned} \alpha_4 &= (\alpha_1\alpha_2\alpha_3)^{-1} \\ &\mapsto (\texttt{(5 6)(7 8)(1 2)(3 4)(1 3)(2 4)(5 7)(6 8)})^{-1} \\ &= \texttt{(1 4)(2 3)(5 8)(6 7)}. \end{aligned}$$

4.7.5. *Conclusions regarding* $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. There are two components of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ which lie over $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$; each is a Klein four normal cover ramified over three points. Only one of these components contains Harbater-Mumford points.

4.8. Branch Cycle Design for $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}} \to \mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$. Let Y and X denote the closures of the components of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ and $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ which contain Harbater-Mumford tuples. Let $\varphi : Y \to X$ denote the canonical ramified cover. This is a three branch point normal cover whose group is Klein four. The branch cycle description for φ is given by (`(1)`, `(1 2)(3 4)`, `(1 4)(2 3)`, `(1 3)(2 4)`). We include the initial unramified point, as it may become a nontrivial branch point at a later stage. Denote the branch points by `A`, `B`, `C`, and `D`, with corresponding generators α_1, α_2, α_3, and α_4. Denote the nodes over these branch points as follows:

- `A`: `A1` = `(1)`, `A2` = `(2)`, `A3` = `(3)`, and `A4` = `(4)`;
- `B`: `B1` = `(1 2)` and `B3` = `(3 4)`;
- `C`: `C1` = `(1 4)` and `C2` = `(2 3)`;
- `D`: `D1` = `(1 3)` and `D2` = `(2 4)`.

The branch cycle design for this cover is drawn below. Since `B` lies over $1 \in \mathcal{J}_4$ and is now completely ramified, we may ignore the trails terminating at `B1` and `B3`. The other trails in the maximal tree with respect to (`[1]`, `A1`) are indicated with bold lines.



Branch Cycle Design for $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}} \to \mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$

The pertinent walks in this tree and the generators they produce are

$W_1:$ `[1]` $\to$ `A1`	$\omega_1 = \alpha_1$	$= \gamma_1\gamma_\infty^2\gamma_1^{-1}$
$W_2:$ `[1]` $\to$ `B1` $\to$ `[2]` $\to$ `C2`	$\omega_2 = \alpha_2\alpha_3^2\alpha_2^{-1}$	$= \gamma_\infty^2\gamma_1\gamma_\infty^8\gamma_1^{-1}\gamma_\infty^{-2}$
$W_3:$ `[1]` $\to$ `C1` $\to$ `[4]` $\to$ `D2`	$\omega_3 = \alpha_2\alpha_4^2\alpha_2^{-1}$	$= \gamma_\infty^2\gamma_1\gamma_\infty^{-2}\gamma_0\gamma_\infty^6\gamma_0^{-1}\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^{-2}$
$W_4:$ `[1]` $\to$ `B1` $\to$ `[2]` $\to$ `A2`	$\omega_4 = \alpha_2\alpha_1\alpha_2^{-1}$	$= \gamma_\infty^2\gamma_1\gamma_\infty^{-2}\gamma_1\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^{-2}$
$W_5:$ `[1]` $\to$ `C1` $\to$ `[4]` $\to$ `A4`	$\omega_5 = \alpha_3\alpha_1\alpha_3^{-1}$	$= \gamma_\infty^4\gamma_1\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^{-4}$
$W_6:$ `[1]` $\to$ `C1`	$\omega_6 = \alpha_3^2$	$= \gamma_\infty^8$
$W_7:$ `[1]` $\to$ `D1` $\to$ `[3]` $\to$ `A3`	$\omega_7 = \alpha_4\alpha_1\alpha_4^{-1}$	$= \gamma_0\gamma_\infty^3\gamma_0^{-1}\gamma_1\gamma_\infty^2\gamma_1^{-1}\gamma_0\gamma_\infty^{-3}\gamma_0^{-1}$
$W_8:$ `[1]` $\to$ `D1`	$\omega_8 = \alpha_4^2$	$= \gamma_0\gamma_\infty^6\gamma_0^{-1}$

5. The Nielsen Class $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})$

5.1. Definition of $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})$. Let $\hat{O}_4$ be the fiber product of $\hat{A}_4$ and O_4 over A_4, as indicated by this commutative diagram:

$$\begin{array}{ccc} \hat{O}_4 & \longrightarrow & \hat{A}_4 \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ O_4 & \xrightarrow{f_2} & A_4 \end{array}$$

Therefore $\hat{O}_4 = \{(\hat{g}, h) \in \hat{A}_4 \times O_4 \mid f_1(\hat{g}) = f_2(h)\}$. In both $\hat{A}_4$ and O_4, all elements of order two map to the identity in A_4, and centralize the normal two Sylow. Thus this remains true in $\hat{O}_4$, and the involutions are the nontrivial elements of the subgroup $\{\pm 1\} \times L$.

The kernel of the map $\hat{O}_4 \to O_4$ contains a single nontrivial element, which is an involution. This is a central Frattini cover. Every element of order 2 in O_4 lifts to two elements of order 2 in $\hat{O}_4$, so this is not a spin cover. Every element of order 3 in O_4 lifts to a unique element of order 3 in $\hat{O}_4$, and the conjugacy classes lift uniquely. Let $\boldsymbol{C}_{3^2_\pm}$ denote the conjugacy classes in $\hat{O}_4$ as well as in O_4.

5.2. Size of $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$**.** The map $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}} \to \mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ is injective. Since Harbater-Mumford tuples always admit lifts, so do their braid equivalents, so the map is surjective onto the Harbater-Mumford orbit. Also, those tuples which are obstructed with respect to $\hat{A}_4 \to A_4$ are necessarily obstructed with respect to $\hat{O}_4 \to A_4$, and there lifts to O_4 are obstructed with respect to $\hat{O}_4 \to O_4$. There remains one braid orbit in $\mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})$ to check; it suffices to show that perturbations of Harbater-Mumford tuples always lift.

Consider $\boldsymbol{h}^{[b|e]}$; this is in the non-Harbater-Mumford orbit which is not obstructed via $\hat{A}_4$. Recall $c = (h_2 h_1^{-1})^2$. Let $\hat{h}_i$ be the unique elements of order three over h_i for $i = 1, 2$. Let $\hat{c} = (\hat{h}_2 \hat{h}_1^{-1})^2$; this lifts c, has order two, and commutes with $\hat{h}_1 \hat{h}_2^{-1}$. Then $\hat{\boldsymbol{h}}^{[\hat{c}|e]}$ lies over $\boldsymbol{h}^{[c|e]}$, and $\Pi \hat{\boldsymbol{h}}^{[\hat{c}|e]} = \hat{h}_1 (\hat{h}_1^{-1} \hat{h}_2)^{\hat{c}} \hat{h}_2^{-1} = 1$. This shows that this braid orbit is unobstructed with respect to the map $\hat{O}_4 \to O_4$.

5.3. Branch Cycle Design for $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}} \to \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$**.** The components of $\mathcal{H}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})$ map isomorphically onto the unobstructed components of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})$.

Let X denote the Harbater-Mumford component of $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$, Y the Harbater-Mumford component of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$, and $\hat{Y}$ the Harbater-Mumford component of $\mathcal{H}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. Then the covers $\hat{Y} \to X$ and $Y \to X$ are isomorphic as ramified covers, and $\hat{Y} \to X$ produces the same branch cycle design as $Y \to X$. Thus we are set up for the last step in our ascent to the Harbater-Mumford components of $\mathcal{H}(\frac{1}{2}\tilde{A}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$.

6. The Nielsen Class $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})$

6.1. Definition of $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})$. Let $U_4 = \frac{1}{2}\tilde{A}_4$ be the universal exponent 2 Frattini cover of A_4. Let M be the kernel of $U_4 \to A_4$. We understand M as an A_4 module.

The map $U_4 \to A_4$ factors through the spin cover $\hat{A}_4$ of A_4; denote the kernel of $U_4 \to \hat{A}_4$ by V.

The map $U_4 \to A_4$ factors through O_4; let N be the kernel of $U_4 \to O_4$. The index of N in M is 4, so the size of N is 8.

The map $U_4 \to A_4$ factors through the map $\hat{O}_4 \to O_4$; denote the kernel of $U_4 \to \hat{O}_4$ by W. Then $W = N \cap V$, and W has index 2 in N. Fit these groups into a diagram:

$$\begin{array}{ccccccc} U_4 & \longrightarrow & \hat{O}_4 & \longrightarrow & \hat{A}_4 & & \\ & & \downarrow & & \downarrow & & \\ & & O_4 & \longrightarrow & A_4 & \longrightarrow & A_3 \end{array}$$

The conjugacy classes of three cycles in $\hat{O}_4$ lift uniquely to conjugacy classes in U_4. Again we denote these conjugacy classes by $C_+ = C_+(U_4)$ and $C_- = C_-(U_4)$, where $C_- = \{h^{-1} \mid h \in C_+\}$, and denote (C_+, C_-, C_+, C_-) by $\boldsymbol{C}_{3^2_\pm}$. Our ultimate goal is to understand $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$.

As is the case in O_4, conjugate three cycles have the same action on W; for $h \in C^+$ and $w \in W$, let $w^+ = w^h$ and $w^- = w^{h^{-1}}$.

If Y is the normal 2 Sylow of U_4, then $W = Z(Y)$.

6.2. Size of $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$. We have seen that $|\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = 288$.

Let $f : U_4 \to \hat{O}_4$ be the canonical homomorphism with kernel W. For $g \in \hat{O}_4$ of order three, the action of g on W has no nontrivial fixed points. Thus if $h \in U_4$ is over g, we have $C_W(h)$ is trivial. By Proposition 12,

$$|\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}| = \frac{|W|^2 |\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}|}{[Z(\hat{O}_4) : f(Z(U_4))]} = \frac{4^2 \cdot 288}{2} = 2304.$$

In particular, the fiber over a point in $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ contains eight points.

6.3. Labeling Elements in U_4. We label elements of U_4 in a manner analogous to our labeling in O_4, using the same names where appropriate. Here, however, the a_i's are relative to elements $h_1, h_2 \in U_4$ of order three as opposed to in O_4; they live over the identically named elements in the O_4 case.

LEMMA 63. *Let $h_1, h_2 \in C_+(U_4)$ be distinct.*
Then $\operatorname{ord}(h_1h_2) = 6$ *and* $\operatorname{ord}(h_1h_2^{-1}) = 4$. *Label the following elements of* U_4*:*

(1) *e is the identity;*
(2) $a_1 = h_1h_1h_2$;
(3) $a_2 = h_1h_2h_1$;
(4) $a_3 = h_2h_1h_1$;
(5) $o_1 = a_1^2$;
(6) $o_2 = a_2^2$;
(7) $o_3 = a_3^2$;
(8) $h_3 = (h_1h_2)^2 \in C_+$;
(9) $h_4 = (h_2h_1)^2 \in C_+$;
(10) $e_1 = (h_2h_3)^3 = [a_1, a_3^{-1}] = [a_2, a_1^{-1}] = [a_3, a_2^{-1}]$;
(11) $e_2 = (h_3h_1)^3 = [a_2, a_3] = [a_2, a_1^{-1}] = [a_3, a_1^{-1}]$;
(12) $e_3 = (h_1h_2)^3 = [a_1, a_3] = [a_1, a_2^{-1}] = [a_2^{-1}, a_3^{-1}]$;
(13) $e_4 = (h_2h_1)^3 = [a_1, a_2] = [a_2, a_3^{-1}] = [a_1^{-1}, a_3^{-1}]$;
(14) $u_1 = e_2e_3 = [a_2, o_1]$;
(15) $u_2 = e_1e_2 = [a_3, o_2]$;
(16) $u_3 = e_3e_1 = [a_1, o_3]$.

Then

(a) $W = \{e, u_1, u_2, u_3\}$ *and* $N = W \cup \{e_1, e_2, e_3, e_4\}$;
(b) $a_1^{h_1} = a_2$ *and* $a_2^{h_1} = a_3$, *and these elements have order* 4;
(c) $o_1^{h_1} = o_2$ *and* $o_2^{h_1} = o_3$, *and these elements have order* 2;
(d) $u_1^{h_1} = u_2$ *and* $u_2^{h_1} = u_3$, *and these elements have order* 2;
(e) $u_1^{+} = u_2$ *and* $u_2^{+} = u_3$;
(f) *e_i generates $C_N(h_i)$;*
(g) $h_1 = h_2^{a_2o_1u_2}$;
(h) $h_2 = h_1^{a_2o_3u_1}$.

PROOF. We have $\operatorname{ord}(h_1h_2) = 6$, because its image in $\hat{A}_4$ has order 6. Thus h_3 has order 3. Since h_3 is the square of a product of elements of C_+, it is also in C_+. Clearly h_3 centralizes e_3.

Let $f : U_4 \to O_4$ be the canonical homomorphism. Let $x = h_2h_3 = h_2h_1h_2h_1h_2$; since $\operatorname{ord}(f(h_1h_2)) = 3$ in O_4, $f(x) = f(h_1^{-1})$. Now the action of $f(x)$ on x^3 by lifted conjugation in independent of the lift, so x and h_1^{-1} act identically; in this case, trivially. Thus h_1 centralizes e_1. Similarly, h_2 centralizes e_2.

The images of the e_i's in $\hat{O}_4$ are nontrivial, but are trivial in O_4, so they are in $N \smallsetminus W$. Since $[N : C_N(h_i)] = 3$, e_i must generate the centralizer. In particular, the e_i's are distinct and then so are the u_i's. Moreover the u_i's must be in W, so they are the nontrivial elements of W. This proves **(a)** and **(f)**.

The conjugations in parts **(b)** and **(c)** are the same as the analogous parts in O_4. The a_i's have order 4 in U_4 because their images in $\hat{A}_4$ have order 4 and in A_4 have order 2.

Part **(e)** will follow from **(d)**, and for this part it suffices to show that $u_1^{h_1} = u_2$, which amounts to showing $e_3^{h_1} = e_1e_2e_3$. But $e_1e_2e_3 \in N \setminus W$ and is distinct from e_1, e_2, and e_3, so it must equal e_4, and one easily computes that $e_3^{h_1} = e_4$.

The first equal sign in **(10)** through **(16)** denotes definition; the others are identities. We prove only what we will use. Note that the commutators have order two, so $[a_i, a_j] = [a_j, a_i]$.

Next we show that $e_1 = [a_3, a_2^{-1}]$. Expanding the commutator gives $[a_3, a_2^{-1}] = e_3^{a_3}$. Since h_1h_2 and a_3^2 commute with e_3, we have

$$e_3^{a_3} = e_3^{a_3^{-1}} = e_3^{h_1h_2^{-1}} = e_3^{h_2} = e_3^{h_2h_1^{-1}h_1} = e_3^{a_3h_1}.$$

This shows that h_1 commutes with $e_3^{a_3}$, so $e_3^{a_3} = e_1$. The other identities of **(10)** follow by conjugating with h_1. The identities of **(12)** are obtained from these by conjugating with a_3.

To show that identity in **(16)**, compute $u_3 = e_3e_1 = [a_3, a_1][a_1, a_3^{-1}] = a_3^{-1}a_1^{-1}o_3a_1a_3^{-1}$. This element in necessarily in W, and so commutes with the element a_3^{-1} of order four. Conjugate by it to find $u_3 = [a_1, o_3]$.

Now use this identity to prove **(g)**; **(h)** is similar. Compute

$$\begin{aligned}
h_2^{a_2o_1u_2} &= u_2o_1a_2^{-1}h_2a_2o_1u_2 \\
&= u_2a_1^{-1}a_3h_2a_2o_1u_2 && [a_2^{-1} = a_1a_3 \text{ and } o_1a_1 = a_1^{-1}] \\
&= u_2a_1^{-1}o_3h_1a_2o_1u_2h_1^{-1}h_1 && [h_2 = a_3h_1] \\
&= u_2a_1^{-1}o_3a_1o_3u_1h_1 && [\text{conjugate } a_2o_1u_2 \text{ by } h_1^{-1}] \\
&= u_3[a_1, o_3]h_1 && [W \text{ centralizes the 2 Sylow}] \\
&= h_1 && [\textbf{(16)} \text{ identity}]
\end{aligned}$$

This completes the demonstration relating generators of U_4 to elements of M. □

6.4. A Fiber of $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. Continue notation from the previous subsection. For the rest of this section, we sometimes shorten notation as follows:

(1) $a = u_1$;

(2) $b = u_2$;

(3) $c = u_3$;

Consider the map $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. Let $\tilde{\boldsymbol{g}}$ denote a Harbater-Mumford tuple in $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})$ which lies over $\boldsymbol{g} \in \mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})$. The fiber over $\tilde{\boldsymbol{g}}$ consists of perturbations of duals of $\boldsymbol{h}$. For $x, y \in U_4$, set

$$\boldsymbol{h}^{[x|y]} = (h_1, (h_1^{-1})^x, h_2^{yx}, (h_2^{-1})^y).$$

Typically $x, y \in W$, but we reserve some leeway in this regard.

LEMMA 64. $\boldsymbol{h}^{[x|y]} = \boldsymbol{h}^{[x|yb]}$ *modulo inner equivalence.*

PROOF. By Lemma 63, b is the product of the generators for the centralizers in M of h_1 and h_2. Thus the claim is a particular case of Proposition 29. □

A complete set of representative tuples for the fiber of $\tilde{\boldsymbol{g}}$ can now be given.

LIST 65 (Fiber over $\tilde{\boldsymbol{g}}$ in $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}} \to \mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$).

[1] $\boldsymbol{h}^{[e|e]}$	[5] $\boldsymbol{h}^{[b|e]}$
[2] $\boldsymbol{h}^{[e|a]}$	[6] $\boldsymbol{h}^{[b|a]}$
[3] $\boldsymbol{h}^{[a|e]}$	[7] $\boldsymbol{h}^{[c|e]}$
[4] $\boldsymbol{h}^{[a|a]}$	[8] $\boldsymbol{h}^{[c|a]}$

We use this characterization of the fiber over a Harbater-Mumford tuple in $\mathrm{Ni}(\hat{O}_4, \boldsymbol{C}_{3^2_\pm})$ to further investigate the Nielsen class $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})$.

6.5. Braid Action on $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ via Branch Cycle Designs. We now determine the action of the ω's. For $x, y \in W$, let $\boldsymbol{h}^{[x|y]}$ be as before. We still have conjugacy classes consistency of the action of conjugation on elements of W; let x^+ and x^- be as before.

6.5.1. *Action of* ω_6. The node corresponding to ω_6 is fully ramified, so no further ramification can occur. Thus the action of ω_6 is trivial on the entire fiber, so

$$\omega_6 \mapsto \texttt{(1)(2)(3)(4)(5)(6)(7)(8)}.$$

6.5.2. *Action of* ω_1. The terminal product of $\boldsymbol{h}$ is one, so the shift has middle product one, and γ_∞ acts trivially on such a tuple. Thus ω_1 is trivial on a Harbater-Mumford tuple.

We have computed that $\boldsymbol{h}^{[x|y]}\alpha_1 = \boldsymbol{h}^{[x|yx^+]}$ for $\boldsymbol{h} \in \mathrm{Ni}(O_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$; this computation is equally effective for $\boldsymbol{h} \in \mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$. Now plug in to find that we have

$$\omega_1 \mapsto \texttt{(1)(2)(3)(4)(5 6)(7 8)}.$$

6.5.3. *Action of* ω_8. Consider $(h_1 h_2^x)^3$, which has order 2. Then the value of this element is independent of x, that is, $(h_1 h_2^x)^2 = (h_1 h_2)^3 = e_3$, which can be seen by pulling the x's past all the h_i's and accumulating the effects of conjugation. Use these comments:

(1) $h_1^{e_3} = h_1^c$, because e_1 centralizes h_1 and $c = e_3 e_1$;
(2) $h_2^{e_3} = h_2^a$, similarly;
(3) $e_3^{h_2^{-1}} = e_2(e_2e_3)^{h_2^{-1}} = e_2 a^- = e_1 e_2 e_3$;
(4) $h_1^{e_3^{h_2^{-1}}} = h_1^a$;
(5) $\gamma_0 = \gamma_\infty^{-1}\gamma_1^{-1}$.

Therefore

$$\begin{aligned}
\boldsymbol{h}^{[x|e]}\omega_8 &= (h_1, (h_1^{-1})^x, h_2^x, h_2^{-1})\gamma_\infty^{-1}\gamma_1^{-1}\gamma_\infty^6\gamma_1\gamma_\infty \\
&= (h_1, h_2^x, (h_1^{-1})^{xh_2^x}, h_2^{-1})\gamma_1^{-1}\gamma_\infty^6\gamma_1\gamma_\infty \\
&= (h_2^{-1}, h_1, h_2^x, (h_1^{-1})^{xh_2^x})\gamma_\infty^6\gamma_1\gamma_\infty \\
&= (h_2^{-1}, h_1^{e_3}, h_2^{xe_3}, (h_1^{-1})^{xh_2^x})\gamma_1\gamma_\infty \\
&= (h_1^{e_3}, h_2^{xe_3}, (h_1^{-1})^{xh_2^x}, h_2^{-1})\gamma_\infty \\
&= (h_1^{e_3}, (h_1^{-1})^{xh_2^x(h_2^{-1})^{xe_3}}, h_2^{xu}, h_2^{-1}) \\
&= (h_1, (h_1^{-1})^{xe_3^{h_2^{-1}}}, h_2^x, (h_2^{-1})^{e_3}) \\
&= (h_1, (h_1^{-1})^{xa}, h_2^{xaa}, (h_2^{-1})^a) \\
&= \boldsymbol{h}^{[xa|a]}.
\end{aligned}$$

Deduce that

$$\boldsymbol{h}^{[x|y]}\omega_8 = \boldsymbol{h}^{[xa|ya]} \quad \Rightarrow \quad \omega_8 \mapsto \texttt{(1 4)(2 3)(5 8)(6 7)}.$$

6.5.4. *Action of* ω_5. Consider $(h_1^x h_2^{yx})^2$ with $x, y \in W$; the value of this is independent of x and y, and equals o_1. Similar comments apply to o_2. Define these variables and derive these comments:

(1) $x_1 = (h_1(h_1^{-1})^{xo_1})$;
(2) $x_2 = ((h_1^{-1})^{xx_1}h_2^x)^2$;
(3) $x_1 = x^+o_1^{h_1^{-1}}o_1 = x^+o_3o_1$, so $\mathrm{ord}(x_1) = 2$;
(4) $o_1x_2 = a$.

Therefore

$$\begin{aligned}
\boldsymbol{h}^{[x|e]}\omega_5 &= (h_1, (h_1^{-1})^x, h_2^x, h_2^{-1})\gamma_\infty^4\gamma_1\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^{-4} \\
&= (h_1, (h_1^{-1})^{xo_1}, h_2^{xo_1}, h_2^{-1})\gamma_1\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^{-4} \\
&= (h_2^{-1}, h_1, (h_1^{-1})^{xo_1}, h_2^{xo_1})\gamma_\infty^2\gamma_1^{-1}\gamma_\infty^{-4} \\
&= (h_2^{-1}, h_1^{x_1}, (h_1^{-1})^{xo_1x_1}, h_2^{xo_1})\gamma_1^{-1}\gamma_\infty^{-4} \\
&= (h_2^{xo_1}, h_2^{-1}, h_1^{x_1}, (h_1^{-1})^{xo_1x_1})\gamma_\infty^{-4} \\
&= (h_2^{xo_1}, (h_2^{-1})^{x_2}, h_1^{x_1x_2}, (h_1^{-1})^{xo_1x_1}) \\
&= (h_1, (h_1^{-1})^{xo_1x_2}, h_2^{xo_1x_1x_2o_3o_1c}, (h_2^{-1})^{x_1o_3o_1c}) && \text{[conj by } xo_1a_2o_1b] \\
&= (h_1, (h_1^{-1})^{xa}, h_2^{xax^+c}, (h_2^{-1})^{x^+c}) \\
&= \boldsymbol{h}^{[xa|x^+c]}
\end{aligned}$$

Deduce that

$$\boldsymbol{h}^{[x|y]}\omega_5 = \boldsymbol{h}^{[xa|yx^+c]} \quad \Rightarrow \quad \omega_5 \mapsto \texttt{(1 4)(2 3)(5 7)(6 8)}.$$

6.5.5. *Conjugation by* α_2. Recall that α_2 interchanges the Harbater-Mumford tuples over $\boldsymbol{g}$, so conjugation by α_2 moves the action to the fiber over the dual. We adjust the computation for the action of α_2 slightly for the U_4 case. Thus $\boldsymbol{h}$ is in $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})$, the a_i's are relative to $\boldsymbol{h}$. Make these adjustments:

(1) $x_1 = (h_1^{-1})^{x^-} h_1^{a_3} = x^+ h_1^{-1} a_3^{-1} h_1 a_3 = x^+ a_1^{-1} a_3$;
(2) $x_2 = x^- a_2 a_1^2 u_2$ so that $(h_2^{x^-})^{x_2} = h_1$;
(3) $a_3 a_2 = a_1^{-1}$;
(4) $a_3 x_2 = a_3 x^- a_2 a_1^2 u_2 = x^- a_1 u_2$;
(5) $x_1 x_2 = x^+ a_1^{-1} a_3 x^- a_2 a_1^2 u_2 = x u_2$;
(6) $h_1^{a_3 h_1^{-1}} = h_1^{a_2} = h_2^{a_3^2}$;
(7) $h_2^{a_1 h_1^{-1}} = h_2^{a_3^2 u_1}$.

$$
\begin{aligned}
\boldsymbol{h}^{[x|e]}\alpha_2 &= \left(h_2^{x^-}, (h_1^{-1})^{x^- x_1}, h_1^{a_3 x_1}, (h_2^{-1})^{a_3}\right) \\
&= \left(h_1, (h_1^{-1})^{x^- x_1 x_2}, h_1^{a_3 x_1 x_2}, (h_2^{-1})^{a_3 x_2}\right) && \text{[conj by } x_2\text{]} \\
&= \left(h_1, (h_1^{-1})^{x^+ u_2}, h_1^{a_3 x u_2}, (h_2^{-1})^{x^- a_1 u_2}\right) && \text{[comments (6) and (7)]} \\
&= \left(h_1, (h_1^{-1})^{x u_1}, h_2^{a_3^2 u_1 x^- u_1}, (h_2^{-1})^{a_3^2 x^+ u_1}\right) && \text{[conj by } h_1^{-1}\text{]} \\
&= \boldsymbol{h}^{[x u_1 | a_3^2 x^+ u_1]}.
\end{aligned}
$$

So

$$\alpha_2 : \boldsymbol{h}^{[x|y]} \mapsto \boldsymbol{h}^{[x u_1 | y a_3^2 x^+ u_1]}.$$

It is the appearance of a_3^2 which changes the fiber.

We note that if the first position h_1 is the same in two Harbater-Mumford tuples, the u_i's written in terms of these tuples are the same. This follows from the fact that u_1 is the product of the two involutions of $N \setminus W$ which *do not* commute with h_1; then u_2 and u_3 are determined by the effect of conjugation by h_1 on u_1. Thus we can use the formula above equally as well on perturbations of duals of $\boldsymbol{h}^{[e|a_3^2]}$.

To compute the actions of the remaining generators, we conjugate by α_2. As before, it suffices to compute with $y = e$.

6.5.6. *Action of* ω_2. Since ω_6 has trivial action on the set of perturbed duals of $\boldsymbol{h}^{[e|a_3^2]}$, conjugation of it by α_2^{-1} has trivial action on the set of perturbed duals of $\boldsymbol{h}$. Therefore

$$\omega_2 \sim \omega_6^{\alpha_2} \sim \omega_6 \mapsto \texttt{(1)(2)(3)(4)(5)(6)(7)(8)}.$$

6.5.7. *Action of* ω_4. Since $\boldsymbol{h}^{[x|y]}\omega_1 = \boldsymbol{h}^{[x|yx^+]}$:

$$\begin{aligned}\boldsymbol{h}^{[x|e]}\alpha_2\omega_1\alpha_2^{-1} &= \boldsymbol{h}^{[xu_1|a_3^2x^+u_1]}\omega_1\alpha_2 \\ &= \boldsymbol{h}^{[xu_1|a_3^2x^+u_1(xu_1)^+]}\alpha_2 \\ &= \boldsymbol{h}^{[xu_1u_1|a_3^2u_3xa_3^2u_1(xu_1)^+]} \\ &= \boldsymbol{h}^{[x|x^+]}.\end{aligned}$$

Therefore

$$\omega_4 \sim \omega_1^{\alpha_2} \sim \omega_1 \mapsto \texttt{(1)(2)(3)(4)(5 6)(7 8)}.$$

6.5.8. *Action of* ω_3. Since $\boldsymbol{h}^{[x|y]}\omega_8 = \boldsymbol{h}^{[xu_1|yu_1]}$:

$$\begin{aligned}\boldsymbol{h}^{[x|e]}\alpha_2\omega_8\alpha_2^{-1} &= \boldsymbol{h}^{[xu_1|a_3^2x^+u_1]}\omega_8\alpha_2 \\ &= \boldsymbol{h}^{[xu_1u_1|a_3^2x^+u_1u_1]}\alpha_2 \\ &= \boldsymbol{h}^{[xu_1u_1u_1|a_3^2x^+u_2a_3^2x^+u_1]} \\ &= \boldsymbol{h}^{[xu_1|u_3]} \\ &= \boldsymbol{h}^{[xu_1|u_1]}.\end{aligned}$$

Therefore

$$\omega_3 \sim \omega_8^{\alpha_2} \sim \omega_8 \mapsto \texttt{(1 4)(2 3)(5 8)(6 7)}.$$

6.5.9. *Action of* ω_7. The product one condition now forces

$$\omega_7 \sim \omega_5 \mapsto \texttt{(1 4)(2 3)(5 7)(6 8)}.$$

6.5.10. *Conclusions.* The tuples of List 65 lie in three braid orbits; these orbits are $\{$`[1]`, `[4]`$\}$, $\{$`[2]`, `[3]`$\}$, and $\{$`[5]`, `[6]`, `[7]`, `[8]`$\}$. The first two orbits each contains one Harbater-Mumford tuple, and the third contains none.

There are three components in $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$ which lie over the Harbater-Mumford component of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$; two contain Harbater-Mumford points and the third does not. The Harbater-Mumford components of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$ are degree two covers of the genus zero curve $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$ ramified over four points; that is, they are elliptic curves presented in a standard way. The other component is a normal Klein four cover of $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$ ramified over six points; by the Riemann Hurwitz formula, it has genus three.

CHAPTER VIII

Analysis of $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$

1. Fields of Definition in $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$

1.1. Rationality of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$. The tuple $\boldsymbol{C}_{3^2_\pm}$ of conjugacy classes of U_4 is a rational tuple. Thus the reduced Hurwitz space $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$ is defined over $\mathbb{Q}$, and the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the components. This action fixes the space $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$, so the two components of this space are either defined over $\mathbb{Q}$, or are conjugates over $\mathbb{Q}$. Since they contain real points, their field of definition is either $\mathbb{Q}$, or it is the same real degree two extension of $\mathbb{Q}$.

We begin to explore what can be said about the common field of definition of the Harbater-Mumford components by collecting the ramification information which we have computed.

Loop	Action		Over	Cycle
γ_0	(1 3 2)			
γ_1	(1)(2 3)			
γ_∞	(1 2)(3)			
β_1	(1 2 3)	(4 5)	γ_∞	(3)
β_2	(1 3)(2)	(4)(5)	γ_1	(1)
β_3	(1 2)(3)	(4 5)	γ_∞	(1 2)
α_1	(1)(2)(3)(4)	(5 6)(7 8)	β_3	(3)
α_2	(1 2)(3 4)	(5)(6)(7)(8)	β_2	(2)
α_3	(1 3)(2 4)	(5 7)(6 8)	β_3	(1 2)
α_4	(1 4)(2 3)	(5 8)(6 7)	β_1	(1 2 3)
ω_1	(1)(2)(3)(4)	(5 6)(7 8)	α_1	(1)
ω_2	(1)(2)(3)(4)	(5)(6)(7)(8)	α_3	(2 4)
ω_3	(1 4)(2 3)	(5 8)(6 7)	α_4	(2 3)
ω_4	(1)(2)(3)(4)	(5 6)(7 8)	α_1	(2)
ω_5	(1 4)(2 3)	(5 7)(6 8)	α_1	(4)
ω_6	(1)(2)(3)(4)	(5)(6)(7)(8)	α_3	(1 3)
ω_7	(1 4)(2 3)	(5 7)(6 8)	α_1	(3)
ω_8	(1 4)(2 3)	(5 8)(6 7)	α_4	(1 4)

Summary of Design Generators

This shows that the two Harbater-Mumford components are ramified over the same four points in $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$. Thus, they have the same j-invariant; if this j-invariant is irrational, the elliptic curves cannot be defined over $\mathbb{Q}$. We intend to compute the j-invariant by finding an appropriate coordinate system for $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$. To do this, we lift coordinates through the sublevels of the Modular Tower which we have explored. This requires precise usage of the ramification at each level, as supplied by the design generators.

1.2. The j-invariant of $\mathcal{E}$. Let $\mathcal{E}$ be the closure of one of the Harbater-Mumford components of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$; then $\mathcal{E}$ is an algebraic curve of genus one, that is, $\mathcal{E}$ is an elliptic curve. We intend to find its j-invariant. Let $\mathcal{H}_3$, $\mathcal{H}_2$, $\mathcal{H}_1$, and $\mathcal{J}$ denote the closure of the image of $\mathcal{E}$ in the Hurwitz spaces $\mathcal{H}(O_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$, $\mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$, $\mathcal{H}(A_3, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}}$, and $\mathcal{J}_4$, respectively. We have a sequence of covering maps

$$\mathcal{E} \xrightarrow{\varphi_3} \mathcal{H}_3 \xrightarrow{\varphi_2} \mathcal{H}_2 \xrightarrow{\varphi_1} \mathcal{H}_1 \xrightarrow{\varphi_0} \mathcal{J},$$

where $\mathcal{H}_3$, $\mathcal{H}_2$, $\mathcal{H}_1$, and $\mathcal{J}$ have genus 0.

The map φ_3 is of degree two and ramified over four points. If we can put coordinates on $\mathcal{H}_3$ and identify these four points, then we can compute the j-invariant of $\mathcal{E}$.

Each of the maps φ_2, φ_1, and φ_0 is a rational function. We have the branch cycle descriptions of each of these maps. Indeed, each of these is a three branch point cover which belongs to a pure Nielsen class containing a single element.

Both φ_0 and φ_1 are S_3 covers with ramification of shape `((3),(2),(2))`. Any cubic polynomial with distinct roots gives this shape; set

$$f(z) = z^3 - 3z.$$

Then $f'(z) = 3z^2 - 3 = 3(z^2 - 1)$; the finite ramified points are $\{\pm 1\}$, so the branch points are $f(1) = -2$, $f(-1) = 2$, and ∞. We also have $f(-2) = -2$ and $f(2) = 2$. We can compose on the left or the right with a linear fractional transformation with rational coefficients, without changing the $\mathbb{Q}$ weak equivalence class of the cover.

The map φ_2 is a K_4 cover of shape `((2)(2),(2)(2),(2)(2))`. Let

$$g(z) = \left(\frac{z^2 - 1}{z^2 + 1}\right)^2.$$

This is a composition of $z \mapsto z^2$, then a linear fractional transformation $z \mapsto \frac{z-1}{z+1}$ which moves the branch points, followed by another $z \mapsto z^2$. Its branch points are 0, 1, and ∞, and its ramification points are $(\pm 1 \mapsto 0)$, $(0, \infty \mapsto 1)$, and $(\pm i \mapsto \infty)$.

Compose f on the left by a linear fractional transformation h_0 of $\mathcal{J}$ so that the branch points of $h_0 \circ f$ are 0, 1, and ∞; specifically, select

$$h_0 : (\infty, 2, -2) \mapsto (0, 1, \infty) \quad \text{given by} \quad h_0(z) = \frac{4}{z + 2}.$$

Compose this on the right by a linear fractional transformation h_1 of $\mathcal{H}_1$ which positions the branch points for the next step. The cover $\mathcal{H}_2 \to \mathcal{H}_1$ has shape `(2)(1)` at the unramified point over $1 \in \mathcal{J}$, shape `(2)(1)` at the ramified point over $\infty \in \mathcal{J}$, and shape `(3)` at the unramified point over infinity. Thus select

$$h_1 : (-2, 2, \infty) \mapsto (2, 1, -2) \quad \text{given by} \quad h_1(z) = \frac{-2z + 20}{z + 14}.$$

Apply f on the right; the points on the domain over which $\mathcal{H}_3$ is ramified are now labeled -2, -1, and ∞. The other point over $\infty \in \mathcal{J}$ is 2. Compose with

$$h_2 : (0,1,\infty) \mapsto (-1,\infty,-2) \quad \text{given by} \quad h_2(z) = \frac{-2z+1}{z-1}.$$

Now the points on $\mathcal{H}_2$ over which $\mathcal{H}_3$ is ramified are 0, 1, and ∞, and we are in a position to compose with g. We need to label the other point over $\infty \in \mathcal{J}$, because ramification of $\mathcal{E} \to \mathcal{H}_3$ occurs over it. Pull back 2 through h_2 and find that this point is $h_2^{-1}(2) = \frac{3}{4}$.

Let $\mathcal{H}_3$ have the coordinates so induced by

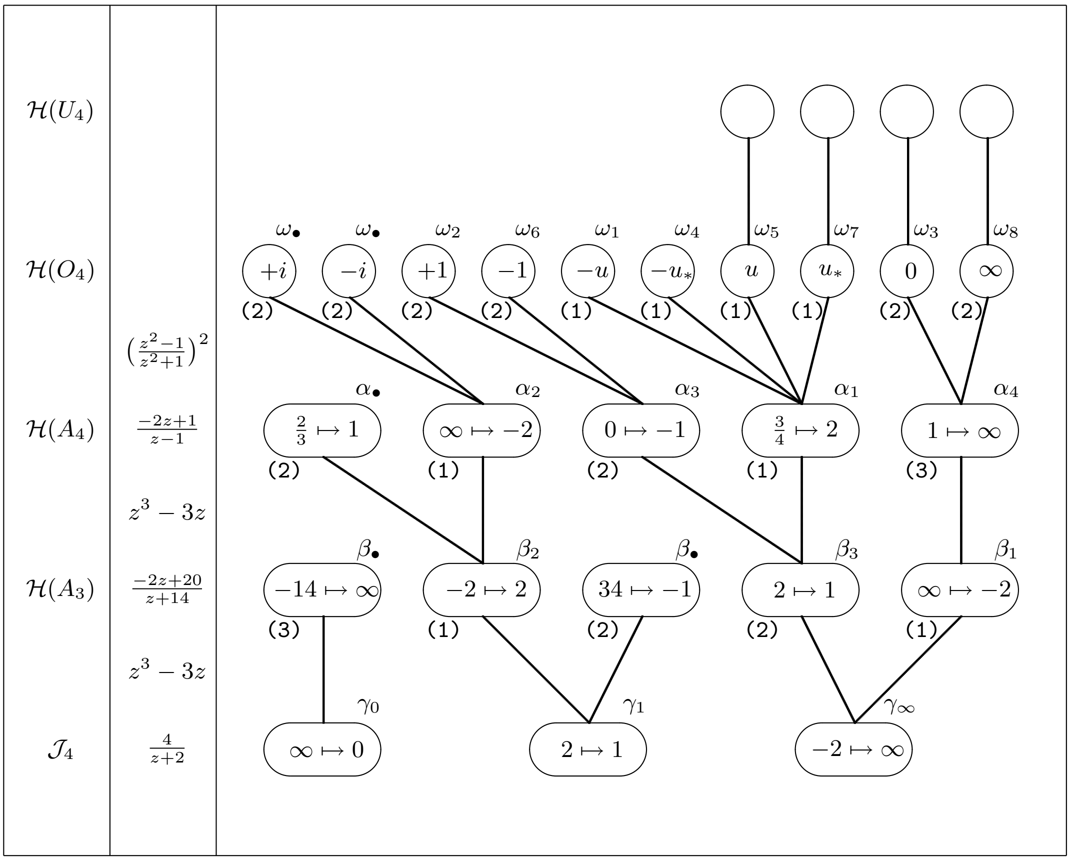
$$h : \mathcal{H}_3 \to \mathcal{J} \quad \text{given by} \quad h = h_0 \circ f \circ h_1 \circ f \circ h_2 \circ g.$$

Now $\mathcal{E} \to \mathcal{H}_3$ has two ramification points over $1 \in \mathcal{H}_2$ and two over $\frac{3}{4} \in \mathcal{H}_2$. Since $g^{-1}(1) = \{0, \infty\}$, these are two of the ramification points of $\mathcal{E} \to \mathcal{H}_3$.

Setting $g(z) = \frac{3}{4}$ shows that the fiber $g^{-1}(\frac{3}{4})$ consists of the roots of

$$z^4 - 14z + 1 = (z^2 + 4z + 1)(z^2 - 4z + 1).$$

Since both of the genus one components of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$ are ramified over the same points, this set of points must be an algebraic set over $\mathbb{Q}$; thus the other two ramification points of $\mathcal{E} \to \mathcal{H}_3$ are either $\{2 \pm \sqrt{3}\}$ or $\{-2 \pm \sqrt{3}\}$. Either choice produces the same j-invariant. Let $u = 2 + \sqrt{3}$ and $u_* = 2 - \sqrt{3}$, and assume $\{u, u_*\}$ are the ramification points. Note that $u^{-1} = u_*$.



Node Mapping Tree for $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$

Recall the formula for the j invariant when $z_4 = \infty$:

$$j(\boldsymbol{z}) = \frac{4}{27}\frac{[(z_1+z_2+z_3)^2 - 3(z_1z_2+z_2z_3+z_3z_1)]^3}{(z_1-z_2)^2(z_2-z_3)^2(z_3-z_1)^2}.$$

When $z_1 = u$, $z_2 = u_*$, and $z_3 = 0$, we have

$$j(\mathcal{E}) = \frac{4}{27}\frac{[(u+u_*)^2 - 3(uu_*)]^3}{u^2u_*^2(u-u_*)^2} = \frac{4}{27}\frac{(4^2-3)^3}{(2\sqrt{3})^2} = \frac{13^3}{3^4}.$$

This j-invariant nails down the holomorphism class of $\mathcal{E}$. It also tells us that there is some elliptic curve, defined over $\mathbb{Q}$, which is isomorphic to $\mathcal{E}$ over $\mathbb{C}$. But what does it tell us about the field of definition of $\mathcal{E}$ itself?

1.3. Equations for $\mathcal{E}$. This method fails to describe the rational points on $\mathcal{E}$, or even its field of definition. For example, consider the elliptic curve given by the equation $y^2 = f(x)$, where $f(x)$ is a cubic polynomial over $\mathbb{Q}$. Let $a \in \mathbb{C}$. Then the equation $y^2 = af(x)$ gives an elliptic curve with the same branch points, not defined over $\mathbb{Q}$ unless $a \in \mathbb{Q}$.

Even if we knew $\mathcal{E}$ were defined over $\mathbb{Q}$, this method would not decide upon the existence of rational points of $\mathcal{E}$, as we now describe.

Our choice of $\{0, 1, u, u_*\}$, as opposed to $\{0, 1, -u, -u_*\}$ as ramification points was arbitrary in the following sense: there exists a linear fractional transformation (defined over $\mathbb{Q}$) which switches these sets; it is $z \mapsto -z$. Each choice produces a different potential equation for $\mathcal{E}$. Let E_1 and E_2 be two possible elliptic curves with these branch point sets, given by equations

(1) $E_1 : y^2 = x^3 + 4x + x$;

(2) $E_2 : y^2 = x^3 - 4x + x$.

Using Cremona's computer programs **MWRANK** and **TORSION**, we find that the only rational points of E_1 are those over 0 and ∞, whereas E_2 has infinitely many.

1.4. A Moduli Problem. We now rephrase our question regarding the fields of definition of the Harbater-Mumford components.

Recall Proposition VI.48, which states that for any two inner Harbater-Mumford tuple in $\mathrm{Ni}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$ which lie over the same element of $\mathrm{Ni}(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in}}$, there is a unique outer automorphism of U_4 which sends one to the other. There are eight such automorphisms acting on the Nielsen class. Modulo reduction, half of them are trivial and half switch the two orbits. Let $\alpha \in \mathrm{Aut}(U_4)$ be an automorphism which switches the orbits.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the two components of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd,HM}}$. Define a function

$$\Phi : \mathcal{E}_1 \to \mathcal{E}_2 \quad \text{by} \quad [\varphi, \tau] \mapsto [\varphi, \tau \circ \alpha];$$

here, $[\varphi, \tau]$ denotes the reduced equivalence class of (φ, τ), where φ is a ramified cover and $\tau : G \to \mathrm{Aut}(\varphi)$ is an isomorphism. This map is holomorphic. The field of definition of $\mathcal{E}_1$ and $\mathcal{E}_2$ is $\mathbb{Q}$ if and only if this map is defined over $\mathbb{Q}$.

CHAPTER IX

GAP Results and Mysteries

Wittgenstein [**Wi21**] said

What we cannot speak about, we must pass over in silence.

I believe that it is traditional, in a Ph.D. dissertation, to ignore this advice, and attempt to say things that might be. We follow this tradition here. However, in our case, the computer language [**GAP**] can be made to speak for us, and we report on the findings we have coaxed from it.

1. GAP Programs

1.1. Groups. Most of our results have been either originally discovered or checked with the aid of the public domain computer language [**GAP**]. This is an amazingly well designed interpretive language, together with a wealth of subroutines which do group theoretical (and other algebraic) computations. We used version 3.4.

The universal exponent 2-Frattini cover of A_5 was described in [**Fr95**], and was originally taught to [**GAP**] using a package for cohomology. This cohomology package only runs under Unix, and since most of the programming was done on a DOS machine, this was the sole use we made of this package. By passing it matrices for the Frattini module, it returned generators and relations for a nonsplit extension of A_5. We found a coreless subgroup to create a permutation representation, and eventually searched for all coreless subgroups in order to both shrink the degree for faster execution, and to help understand spin representations.

Occasionally one finds that the specific situation is not amenable to the general algorithm. In our case, we wished to find automorphism groups, normalizers and centralizers, as subgroups of symmetric groups. We found [**GAP**] to be nearly interminable for our cases, even though it is extremely fast for some computations.

Since U_5 and U_4 are generated by two elements of order three, their automorphism groups were found by an exhaustive search which looked for other pairs of elements of order three to see if mapping one pair to the other produced an automorphism; these groups were returned as subgroups of $N_{S_{1920}}(U_5)$ and $N_{S_{384}}(U_4)$, acting on the elements in the regular representation. Then one may generate the full normalizer as the group generated by U_5 or U_4 and its automorphism group; this outperformed [**GAP**]'s `Normalizer` command in our case.

1.2. Covers. Our [**GAP**] programs views topological covers as given by the permutation representations on the fibers; in [**GAP**], they appear as permutation groups.

To find the automorphism group of a cover, we need the centralizer of the monodromy group in S_n. We wrote a program utilizing the explicit isomorphism $N_G(S)/S \to C_{S_n}(G)$, where S is a one point stabilizer. This outperformed [**GAP**]'s `Centralizer` command in our case, probably because the order of G was small relative to the degree.

For ramified covers of the Riemann sphere, we add an entry to the `Group` record for the branch cycle description. From this, the Riemann-Hurwitz formula can compute the genus. All of the kappa operators discussed in this paper have been implemented in [**GAP**] as functions which act on branch cycle descriptions, realized as lists of permutations.

1.3. Nielsen Classes. We implemented Nielsen classes as a [**GAP**] domain. This means that it has an operations record which instructs various general [**GAP**] commands, such as `Size`, `Elements`, and `Print`, what to do.

Our function to declare a Nielsen class takes the basic form

```
Ni := NielsenClass(<group>,<list of conjugacy classes>)
```

Here, `<group>` is a subgroup of the automorphism group of the group generated by `<list of conjugacy classes>`, and for inner classes should equal it.

The main subroutine with respect to Nielsen classes is that which finds all of its elements. Once these are found, braiding them is relatively easy, although it can be time consuming. Eventually the program creates a list of Nielsen tuples, assigning each a number, and returns the action of each braid generator Q_i as a permutation of these integers. Then the monodromy group of the Hurwitz space cover becomes the subgroup of S_n generated by these permutations, where n is `Size(Ni)`. The orbits can then be found with [**GAP**]'s `Orbits` command.

Various operators on Nielsen classes, specifically those for complex conjugation, are produced in [**GAP**] as elements of S_n.

1.4. Quotient Classes. Any block system for the braid action allows one to equivalence elements in the Nielsen class, and condense the monodromy group accordingly, along with any associated operators. We implemented absolute and reduced Nielsen classes using this idea.

Much more can be said about reduced Nielsen classes in the case of four branch points, and we have additional code for this case. In particular, the program uses the reduction kernel to find the reduced Nielsen class, and computes the genus of each component of a reduced Hurwitz space.

1.5. Branch Cycle Designs. The algorithms discussed in chapter V for finding design generators and combining branch cycle descriptions via condensing, crunching, and splicing, have all been implemented in [**GAP**].

2. GAP Results

2.1. Description of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})$.

2.1.1. *Components.* There are six components of $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$, two of genus 1, two of genus 0, and two of genus 3. The two of genus 1 contain Harbater-Mumford points, and so they are unobstructed, and the components above them at level two contain real points; label these $\mathcal{H}_{\mathtt{1A}}(U_4)$ and $\mathcal{H}_{\mathtt{1B}}(U_4)$. One of the genus three components contains real points and one does not; label these $\mathcal{H}_{\mathtt{3R}}(U_4)$ and $\mathcal{H}_{\mathtt{3I}}(U_4)$, respectively. The two genus zero components are complex conjugates; label these $\mathcal{H}_{\mathtt{0A}}(U_4)$ and $\mathcal{H}_{\mathtt{0B}}(U_4)$. The number of real points in a fiber over an appropriate branch point set is indicate has been determined through use of the kappa operators.

2.1.2. *Spin Covers.* We have described the three spin covers θ_1, θ_2, and θ_3 of U_4. Each of these obstructs a different set of components. The only unobstructed components are the Harbater-Mumford components.

2.1.3. *Automorphisms.* Outer automorphisms of U_4 swap the two genus one components and the two genus zero components. Thus the absolute spaces given by the regular representation of U_4 contains four components.

	Inner Components							Regular Components					
Comp	Deg	Red	HM	κ_4	κ_2	κ_0	Obs	Deg	Red	HM	κ_4	κ_2	κ_0
$\mathcal{H}(A_3)$	3	2	1	0	0	4		3	2	1	3	1	3
$\mathcal{H}_{\texttt{+}}(A_4)$	18	2	4	0	0	8		9	1	2	5	1	5
$\mathcal{H}_{\texttt{-}}(A_4)$	12	2	0	0	0	8	θ_0	6	2	0	4	2	4
$\mathcal{H}_{\texttt{+H}}(O_4)$	144	4	16	0	0	24		36	1	2	8	0	8
$\mathcal{H}_{\texttt{+R}}(O_4)$	144	4	0	0	0	16	θ_1	36	2	0	2	2	6
$\mathcal{H}_{\texttt{-A}}(O_4)$	96	4	0	0	0	0	θ_0	24	2	0	8	4	4
$\mathcal{H}_{\texttt{-B}}(O_4)$	96	4	0	0	0	0	θ_0						
$\mathcal{H}_{\texttt{1A}}(U_4)$	288	4	16	0	0	48		36	1	2	12	0	12
$\mathcal{H}_{\texttt{1B}}(U_4)$	288	4	16	0	0	48							
$\mathcal{H}_{\texttt{3R}}(U_4)$	576	4	0	0	0	64	θ_2, θ_3	36	2	0	4	0	12
$\mathcal{H}_{\texttt{0A}}(U_4)$	288	4	0	0	0	0	θ_1, θ_3	36	4	0	4	2	4
$\mathcal{H}_{\texttt{0B}}(U_4)$	288	4	0	0	0	0	θ_1, θ_3						
$\mathcal{H}_{\texttt{3I}}(U_4)$	576	4	0	0	0	0	θ_1, θ_2	36	4	0	0	2	4
	Reduced Inner Components							Reduced Regular Components					
Comp	Deg	Gen	HM	κ_4	κ_2	κ_0	Obs	Deg	Gen	HM	κ_4	κ_2	κ_0
$\mathcal{H}(A_3)$	3	0	1	3	1	3		3	0	1	3	1	3
$\mathcal{H}_{\texttt{+}}(A_4)$	9	0	2	5	1	5		9	0	2	5	1	5
$\mathcal{H}_{\texttt{-}}(A_4)$	6	0	0	2	0	2	θ_0	3	0	0	3	1	3
$\mathcal{H}_{\texttt{+H}}(O_4)$	36	0	4	16	0	16		18	0	2	8	0	8
$\mathcal{H}_{\texttt{+R}}(O_4)$	36	0	0	8	0	8	θ_1	9	0	0	5	1	5
$\mathcal{H}_{\texttt{-A}}(O_4)$	24	0	0	0	0	0	θ_0	12	0	0	4	2	4
$\mathcal{H}_{\texttt{-B}}(O_4)$	24	0	0	0	0	0	θ_0						
$\mathcal{H}_{\texttt{1A}}(U_4)$	72	1	4	24	0	24		36	1	2	12	0	12
$\mathcal{H}_{\texttt{1B}}(U_4)$	72	1	4	24	0	24							
$\mathcal{H}_{\texttt{3R}}(U_4)$	144	3	0	32	0	32	θ_2, θ_3	18	0	0	8	0	8
$\mathcal{H}_{\texttt{0A}}(U_4)$	72	0	0	0	0	0	θ_1, θ_3	9	0	0	5	1	5
$\mathcal{H}_{\texttt{0B}}(U_4)$	72	0	0	0	0	0	θ_1, θ_3						
$\mathcal{H}_{\texttt{3I}}(U_4)$	144	3	0	0	0	0	θ_1, θ_2	9	0	0	5	1	5

Table of Components for $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$

	θ_1	θ_2	θ_3
Lifts	`1A, 1B, 3R`	`1A, 1B, 0A, 0B`	`1A, 1B, 3C`
Obstructs	`0A, 0B, 3C`	`3R, 3C`	`0A, 0B, 3R`

Table of Obstruction for $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})$

2.2. Description of $\mathcal{H}(U_5, \boldsymbol{C}_{5^2_\pm})$. We present [**GAP**] information regarding the Hurwitz spaces relating to $\mathcal{H}(U_5, \boldsymbol{C}_{5^2_\pm})$. One notes the striking similarity with the U_4 case. Indeed, we have used [**GAP**] to find a map between the Harbater-Mumford fibers which shows that $\mathcal{H}(U_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd}} \to \mathcal{H}(A_4, \boldsymbol{C}_{3^2_\pm})^{\text{in,rd,HM}}$ and $\mathcal{H}(U_5, \boldsymbol{C}_{5^2_\pm})^{\text{in,rd}} \to \mathcal{H}(A_5, \boldsymbol{C}_{5^2_\pm})^{\text{in,rd,HM}}$ are equivalent as covers.

This is made possible by the following observation: the branch cycle design for $\mathcal{H}(A_5, \boldsymbol{C}_{5^2_\pm})^{\text{in,rd}} \to \mathcal{J}_4$, when simplified for final ramification, is identical to the branch cycle design in the A_4 case except for the occurrence of a single additional generator: $\gamma_0\gamma_\infty^6\gamma_0^{-1}$, which corresponds to the six cycle over ∞. However, one computes that this cycle acts trivially on tuples $\boldsymbol{g} = (g_1, g_2, g_3, g_4) \in \text{Ni}(A_5, \boldsymbol{C}_{5^2_\pm})^{\text{in,rd}}$ if $\text{ord}(g_1g_3) = 3$. When $\boldsymbol{g}$ is a Harbater-Mumford tuple whose middle product is four, this is the case, and the detrivialized branch cycles designs are the same.

Let θ_0 and θ_1 denote the unique spin covers of A_5 and U_5, respectively.

	Inner Components							Regular Components					
Comp	Deg	Red	HM	κ_4	κ_2	κ_0	Obs	Deg	Red	HM	κ_4	κ_2	κ_0
$\mathcal{H}_{\mathtt{+}}(A_5)$	30	2	4	14	2	6		15	1	2	7	1	7
$\mathcal{H}_{\mathtt{-}}(A_5)$	12	2	0	8	4	4	θ_0	6	2	0	4	2	4
$\mathcal{H}_{\mathtt{1A}}(U_5)$	480	4	16	0	0	48		240	2	8	0	0	40
$\mathcal{H}_{\mathtt{1B}}(U_5)$	480	4	16	0	0	48							
$\mathcal{H}_{\mathtt{3R}}(U_5)$	960	4	0	0	0	64		240	4	0	0	0	32
$\mathcal{H}_{\mathtt{0A}}(U_5)$	480	4	0	0	0	0	θ_1	240	4	0	0	0	24
$\mathcal{H}_{\mathtt{0B}}(U_5)$	480	4	0	0	0	0	θ_1						
$\mathcal{H}_{\mathtt{3I}}(U_5)$	960	4	0	0	0	0	θ_1	240	4	0	0	0	16

	Reduced Inner Components							Reduced Regular Components					
Comp	Deg	Gen	HM	κ_4	κ_2	κ_0	Obs	Deg	Gen	HM	κ_4	κ_2	κ_0
$\mathcal{H}_{\mathtt{+}}(A_5)$	15	0	2	7	1	7		15	0	2	7	1	7
$\mathcal{H}_{\mathtt{-}}(A_5)$	6	0	0	4	2	4	θ_0	3	0	0	3	1	3
$\mathcal{H}_{\mathtt{1A}}(U_5)$	120	1	4	28	0	28		120	1	4	28	0	28
$\mathcal{H}_{\mathtt{1B}}(U_5)$	120	1	4	28	0	28							
$\mathcal{H}_{\mathtt{3R}}(U_5)$	240	3	0	32	0	32		60	0	0	20	0	20
$\mathcal{H}_{\mathtt{0A}}(U_5)$	120	0	0	0	0	0	θ_1	60	0	0	12	0	12
$\mathcal{H}_{\mathtt{0B}}(U_5)$	120	0	0	0	0	0	θ_1						
$\mathcal{H}_{\mathtt{3I}}(U_5)$	240	3	0	0	0	0	θ_1	60	0	0	12	0	12

Table of Components for $\mathbf{MT}_2(A_5, \boldsymbol{C}_{5^2_\pm})$.

2.3. Description of $\mathcal{H}(U_5, \boldsymbol{C}_{3^4})$. The following table lists the same information for the case of four 3-cycles in A_5 and U_5. [**BF02**] explains is detail exactly why these things are true; in particular, there is a precise module-theoretic explanation for the two components of $\mathcal{H}(U_5, \boldsymbol{C}_{3^4})^{\mathrm{in}}$, describing exactly which tuples are obstructed by the spin cover.

	Inner Components							Regular Components					
Comp	Deg	Red	HM	κ_4	κ_2	κ_0	Obs	Deg	Red	HM	κ_4	κ_2	κ_0
$\mathcal{H}(A_5)$	18	1	2	4	2	4		9	1	1	3	3	3
$\mathcal{H}_{+}(U_5)$	1152	4	16	0	0	32		288	4	4	0	16	16
$\mathcal{H}_{-}(U_5)$	1152	4	0	0	0	0	θ_1	288	4	0	8	0	0

	Reduced Inner Components							Reduced Regular Components					
Comp	Deg	Gen	HM	κ_4	κ_2	κ_0	Obs	Deg	Gen	HM	κ_4	κ_2	κ_0
$\mathcal{H}(A_5)$	18	0	2	4	2	4		9	1	1	3	3	3
$\mathcal{H}_{+}(U_5)$	288	12	4	16	0	16		72	2	1	8	8	8
$\mathcal{H}_{-}(U_5)$	288	9	0	0	0	0	θ_1	72	2	0	8	0	8

Table of Components for $\mathbf{MT}_2(A_5, \boldsymbol{C}_{3^4})$.

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