Contemporary Mathematics

The Shift Incidence matrix and Hurwitz space components

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Contents

Abstract. Hurwitz spaces are parameter spaces for families of (nonsingular Riemann surface) covers, $\varphi : X \to \mathbb{P}^1_z$, of the Riemann sphere uniformized by a variable z. Two types – differing by covering equivalances – play a role in most investigations: *absolute* and *inner*. *Reduced* equivalence figures in applications connecting to classical moduli spaces. Covers have a monodromy group, G_φ, constant on a connected component of a Hurwitz space. Another constant on such a connected component is the set of conjugacy classes $\mathbf{C}$ – cardinality $r_{\mathbf{C}}$ – that appear when one identifies an element of a Nielsen class (defined by $(G, \mathbf{C})$) with those covers having a fixed (unordered) set of branch points.

When a Hurwitz space has more than one component – several *braid orbits* on Nielsen classes, $r \ge 4$ – that is significant. For one, for finding the *moduli definition field* of those components. Two tools have successfully applied to distinguishing component geometry.

1. A *lift invariant* when G has a nontrivial *Schur multiplier*.
2. The **sh**-incidence matrix of this paper's title.

#2 is our main topic (but relating to #1): an algorithm for computing components based on *cusp orbits*. The case $r = 4$ compares *reduced* Hurwitz spaces with the special case of modular curves. We show the algorithm on two distinct series of Nielsen classes, relating components and their cusps to many Hurwitz space issues.

Primary 11F32, 11G18, 11R58; Secondary 20B05, 20C25, 20D25, 20E18, 20F34 Moduli of covers, j-line covers, Hurwitz monodromy group, Spin and Frattini Scovers, Lift invariants

1. Braid action and the objects of applications

[**BFr02**, §2.10] introduced the **sh**-incidence matrix. That paper used it to display Hurwitz space components that lay above – in a *modular tower* structure – the reduced Hurwitz space $\mathcal{H}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$ corresponding to Galois covers with group A_5 having branch cycles given by 4 repetitions of the 3-cycle conjugacy class. It proved Main **MT** conj. 1.16 for this case, using genus computing tool Thm. 1.9 on reduced Hurwitz spaces of 4 branch point covers. The literature now contains a fuller enhancement of cusp types, and an expansion of what towers of Hurwitz spaces should be included in **MT**s. Using that, we show the general value of the **sh**-incidence matrix.

§1.1 lists the problems for which the **sh**-incidence algorithm identification of components of Hurwitz spaces is valuable. Our two big example sections illustrate the extra structure on cusps that arises on Hurwitz spaces, and how the graphical display helps visualize components. We did all the results on the examples with standard proofs. Still, we hope this will guide an expert on, say, **GAP** (or **Maple** or **Magma**) to produce a user interface for the

sh-incidence computation to help test those unsolved problems we describe here and in [**Fr20b**].

§1.2 reminds of Nielsen classes, and the braid group action on them. This includes practical definitions of Hurwitz spaces that suffice for this paper. §1.3 separates out material on reduced Hurwitz spaces. These are closer to classical moduli spaces than the non-reduced spaces. We note this while doing our examples. Finally, §1.4 does an exposition on ℓ-Frattini lattice quotients, from which we get the canonical towers of reduced Hurwitz spaces and the applications to *Modular Towers* (**MT**s) whose applications (for the *regular Inverse Galois Problem* and generalizing Serre's Open Image Theorem) are our the ultimate objects of study.

[**FrV91**] and [**FrV92**] and [**BFr02**], greatly expanded applications benefitting from analysis of Hurwitz spaces and *moduli definition fields* of their components. Especially, we focus on recognizing the differences between the covers in distinct components, of a given Hurwitz space.

1.1. Goals of the paper. The name **sh**-incidence (shift incidence) matrix derives from its use of the *shift*, from the *braid group*, on elements of a *Nielsen class* defined by a finite group, G, and some generating conjugacy classes, $\mathbf{C}$, of G. §1.1.1 lists the most elementary goals – about cusps and components - of the **sh**-incidence algorithm. §1.1.2 lists refined data goals on components and *moduli definition fields*: That is, identifying cusp types and definition fields involved in applying Hurwitz spaces to solutions coming from their moduli.

When combined with §1.4, these are the generalization to Hurwitz spaces of results one would expect for modular curves. Example: detecting if in appropriate towers of Hurwitz spaces that are j-line covers if the genus of high levels goes to ∞. We compute those goals on our examples.

Finally, §1.4.4 gives a result based on using the *lift invariant* (Def. 2.8) which has a simple conclusion – under a stable homotopy hypothesis – that points to Hurwitz space components with moduli definition field $\mathbb{Q}$ that have already played a serious role in the regular inverse Galois problem. Since the **sh**-incidence matrix applies to all Hurwitz spaces, there is no technical reason for restricting to spaces of covers with only $r = 4$ branch points. Though our examples don't satisfy stable homotopy, a lift invariant is still relevant to organizing properties of their components.

1.1.1. *Conjugacy classes.* Suppose $\mathbf{C}'$ is a generating collection of G conjugacy classes. Denote the gcd of the order of elements in $\mathbf{C}$ by $N_{\mathbf{C}}$.

DEFINITION 1.1. Refer to a conjugacy class collection $\mathbf{C}$ as a *rational untion* if for each $k \mod N_{\mathbf{C}}$, $\mathbf{C}^k \stackrel{\text{def}}{=} \{g^k \mid g \in \mathbf{C}\} = \mathbf{C}$.

We will always assume $\mathbf{C}'$ is a rational union, giving the collection

$$\mathbb{M}_{\mathbf{C}'} \stackrel{\text{def}}{=} \{\mathbf{C} \text{ is a rational union supported exactly on classes of } \mathbf{C}'\}. \tag{1.1}$$

For $\mathbf{C} \in \mathbb{M}_{\mathbf{C}'}$, denote the multiplicity of $\mathrm{C}' \in \mathbf{C}'$ in $\mathbf{C}$ by $r_{\mathrm{C}',\mathbf{C}}$. Consider

$$\mathbb{M}_{\mathbf{C}',\geq r'} = \{\mathbf{C} \in \mathbb{M}_{\mathbf{C}'} \mid r_{\mathrm{C}',\mathbf{C}} \geq r' \text{ for all } \mathrm{C}' \in \mathbf{C}'\}. \tag{1.2}$$

When the number of classes, $r \stackrel{\text{def}}{=} r_{\mathbf{C}}$ is 3, the group action (1.11) is through S_3. Except for induction purposes, we assume $r \geq 4$. The **sh**-incidence matrix automatically reveals properties of Hurwitz spaces associated to $(G, \mathbf{C})$ by idenitfying cusps with *cusp group* orbits in the Nielsen class. We recall the basic definitions, especially how braids act on the Nielsen classes (§1.2), and the combinatorial cusps (§1.3). Then, §2 produces the algorithm at the heart of the paper.

§3 and §4 are examples of using the algorithm. Each section is a series of related Nielsen classes, related to other papers featuring three themes.

(1.3a) Why cusp orbit geometry often catches the the effect of a *lift invariant* (Def. 2.8).

(1.3b) What orbits are easily recognized as having trivial lift invariant. In this paper these are Harbater-Mumford orbits.[1]

(1.3c) When more than one orbit has trivial lift invariant there are unsolved problems.

Here is the distinction between the two methods.

(1.4a) The **sh**-incidence matrix precisely catches all braid orbits (Hurwitz space components).

(1.4b) Cusps with distinct lift invariants are in different components, but several components may have the same lift invariant.

REMARK 1.2. A complete treatment of Riemann's Existence Theorem, as in Thm. 1.4, starting from [**Ahl79**], is at [**Fr90**]. A practical exposition, using Davenport's Problem takes up the first sections of [**Fr12**].

[**Vo96**] primarily aims at group theorists who are into the regular Inverse Galois Problem. It complements [**Se92**] ([**Fr94**] ties them together). These are less group theoretic, and more arithmetic/geometric than [**MM99**].

[1] These have a natural generalization, including those given by g-ℓ' cusp type of (2.7).

1.1.2. *Cusps and moduli definition fields.* To simplify we state results when the cover equivalence class is *inner* (1.10):[2] about Galois covers where the definition field includes coordinates of automorphisms of the cover.

The **BCL** gives a cyclotomic field, $\mathbb{Q}_{G,\mathbf{C}}$ with this relation to a definition field K of any cover $\varphi : W \to \mathbb{P}^1_z$ (and its automorphisms) corresponding to a point $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})$ on a Hurwitz space corresponding to $(G, \mathbf{C})$.

(1.5a) K contains $\mathbb{Q}_{G,\mathbf{C}}(\boldsymbol{p})$ (adjoin the coordinates of $\boldsymbol{p}$ to $\mathbb{Q}_{G,\mathbf{C}}$).

(1.5b) If fine moduli holds, then φ over $\mathbb{Q}_{G,\mathbf{C}}(\boldsymbol{p})$ exists.

An example result [**FrV91**], the one we use most often, is that $\mathbb{Q}_{G,\mathbf{C}} = \mathbb{Q}$ if and only if $\mathbf{C}$ is a rational union (Def. 1.1).

For inner Hurwitz spaces where fine moduli means G has no center, (1.5b) then produces a regular realization of G with branch cycles in $\mathbf{C}$ over this field (and no smaller field). Our examples uniformly take the case $\mathbb{Q}^{\text{in}}_{G,\mathbf{C}} = \mathbb{Q}$. In practice the case where the cover equivalence is absolute (Thm. 1.4) is also of significance. For that there is a corresponding field $\mathbb{Q}^{\text{abs}}_{G,\mathbf{C}}$, contained in $\mathbb{Q}^{\text{in}}_{G,\mathbf{C}}$.

It would be super if, for any Hurwitz space component, say $\mathcal{H}'$, we could decorate the braid orbit, $\mathcal{O}'$, corresponding to that component (as gleaned from (1.3a) and (1.3b)) to get an analog, $\mathbb{Q}_{\mathcal{O}'}$, of $\mathbb{Q}_{G,\mathbf{C}}$ for the inner and absolute cases. See Prob. 1.19 and Rem. 1.20. That there are cases where we can't yet do this – as appears in our examples – is the point of (1.3c).

The **sh**-incidence matrices are graphic devices to display Hurwitz space components. Princ. 1.7 serves as a Hurwitz space definition corresponding to the equivalences between sphere covers that we use. The groups of our example sections are accessible to anyone with a 1st year graduate algebra course: A_n, $n \geq 5$ in §3, and $(\mathbb{Z}/\ell^{k+1})2 \times^s \mathbb{Z}/3$, $\ell > 3$ an odd prime in §4.

The first series extends [**LO08**] which considered only absolute Hurwitz spaces, It thereby concluded a simply stated result; one, however, that didn't reveal the complexity of the inner Hurwitz spaces. In this example we precisely give the analog of $\mathbb{Q}_{G,\mathbf{C}}$ for all components.

The second series is natural as a test/illustration of the conjectures formulated in [**Fr20b**, §5] for generalizing Serre's Open Image Theorem (**OIT**; as begun in [**Se68**]). The in progress book [**Fr21**] aims to complete that. In these examples, though, the Hurwitz spaces have several components. Indeed, we special attention to those that have more than one *Harbater-Mumford* – with trivial lift invariant – components. We explain what it would take to give the analog of $\mathbb{Q}_{G,\mathbf{C}}$ for these components.

[2] There are versions when the covers are absolute equivalence classes.

Both series are level 0 of a natural tower of Hurwitz spaces, akin to modular curve towers, and for which **MT**s (§1.4) are a practical analog for modular curves of explicitly showing the rise of the genus of components in these towers. We explain this, and apply this to our examples.

The **sh**-incidence matrix is based on using *cusp group orbits*, the equivalent of *cusps* on the geometric space. The labeling of cusps is what gives structure to the **sh**-incidence matrix and a handle on information on reduced Hurwitz spaces. Our examples are for $r = 4$ – where the spaces have dimension 1. §1.3 gives the definitions for what we will compute including components and their cusps, genuses and whether we have fine moduli.

1.2. Braid action on Nielsen Classes. With notation as used in Thm. 1.4, unless otherwise said, we make these two assumptions.

(1.6a) Conjugacy classes, $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$, in G are *generating*.

(1.6b) G is given as a transitive subgroup of S_n.

Meanings: (1.6a) $\implies$ the full collection of elements in $\mathbf{C}$ generates G; and (1.6b) $\implies$ the cover associated to *branch cycles* $\boldsymbol{g}$ is connected. Even with (1.6a), it may be easy to decide if there exists $\boldsymbol{g} \in G^r \cap \mathbf{C}$ that generates.

Denote projective r-space by $\mathbb{P}^r$ and its discriminant locus by D_r. The Riemann sphere, $\mathbb{P}^1_z \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$, is a copy of $\mathbb{P}^1$ with z – as in 1-complex dimensional analysis – as a uniformizing variable. Then, D_r is the image of the *fat diagonal* locus Δ_r on $(\mathbb{P}^1_z)^r$ of two or more equal coordinates, by modding out by the action of the symmetric group, S_r.

Thm. 1.4 and Thm. 1.6 combinatorially describe, respectively, absolute and inner equivalence classes of Riemann surface covers with invariants $(G, \mathbf{C})$ with specific

$$\text{unordered branch points } \boldsymbol{z} \in \mathbb{P}^r \setminus D_r \stackrel{\text{def}}{=} U_r. \tag{1.7}$$

1.2.1. *Explaining Nielsen classes,* $\mathrm{Ni}(G, \mathbf{C})^\dagger$*.* This is based on choosing a collection of classical generators, $\mathcal{P} = \{P_1, \dots, P_r\}$, for the fundamental group of the r-punctured sphere $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ as documented in Rem. 1.2.

The cover comes from the mapping $P_i \mapsto g_i$, $i = 1, \dots, r$, producing a permutation representation $\pi(U_{\boldsymbol{z}}, z_0) \to G \le S_n$. How this uses the theory of the fundamental group to give a degree n cover $f^0 : W^0 \to U_{\boldsymbol{z}}$ is explained in many places. It is convenient to quote [**Fr20b**, §2.2], as we will use this reference to connect to more detailed work on Nielsen classes.

Completing the converse to, say, Thm. 1.4 or is not immediate. You must fill holes over $\boldsymbol{z}$ in f_0 to get the desired $f : W \to \mathbb{P}^1_z$, as documented a'la Rem. 1.2 in [**Fr80**, Chap. 4].

DEFINITION 1.3. Two covers ${}_if : {}_iW \to \mathbb{P}^1_z, i = 1,2$ are *absolutely equivalent* if there exists a continuous $\varphi : {}_1W \to {}_2W$ such that ${}_2f \circ \varphi = {}_1f$.

Denote the subgroup of the normalizer, $N_{S_n}(G)$, of G in S_n that permutes a given collection, $\mathbf{C}$, of conjugacy classes, by $N_{S_n}(G, \mathbf{C})$.

THEOREM 1.4. *Assume* $\boldsymbol{z} \stackrel{\text{def}}{=} z_1, \dots, z_r \in \mathbb{P}^1_z$ *distinct. Then, some degree* n *cover* $f : W \to \mathbb{P}^1_z$ *with branch points* $\boldsymbol{z}$*, and* $G = G_f \le S_n$ *produces classes* $\mathbf{C}$ *in* G*, if and only if there is* $\boldsymbol{g} \in G^r \cap \mathbf{C}$ *with these properties:*

(1.8a) $\langle \boldsymbol{g} \rangle = G$ *(generation); and*
(1.8b) $\prod_{i=1}^r g_i = 1$ *(product-one).*

Two such ${}_1\boldsymbol{g}, {}_2\boldsymbol{g}$ *represent absolutely equivalent covers if, for* $h \in N_{S_n}(G, \mathbf{C})$, $h_2\boldsymbol{g}h^{-1} = {}_1\boldsymbol{g}$. *Indeed,* r*-tuples satisfying* (1.8) *give all possible Riemann surface covers – up to absolute equivalence – with these properties.*

The index of an element $g \in S_n$ is n minus the # of disjoint cycles in g. Refer to one of those covers attached to $\boldsymbol{g}$ as $f_{\boldsymbol{g}} : W_{\boldsymbol{g}} \to \mathbb{P}^1_z$.

(1.9) The genus $\mathbf{g}_{\boldsymbol{g}}$ of $W_{\boldsymbol{g}}$ appears in $2(\deg(f) + \mathbf{g}_{\boldsymbol{g}} - 1) = \sum_{i=1}^r \text{ind}(g_i)$.

DEFINITION 1.5. Given $(G, \mathbf{C})$, the collection of $\boldsymbol{g}$ satisfying (1.8) is the *Nielsen class* $\text{Ni}(G, \mathbf{C})$, with $\text{Ni}(G, \mathbf{C})^\dagger$, $\dagger = \text{abs}$ referencing absolute equivalence from the representation $T : G \to S_n$.

Similarly for *inner* Nielsen classes, whose elements correspond to Galois closures, $\hat{f} : \hat{W} \to \mathbb{P}^1_z$, of the covers labeled $f_{\boldsymbol{g}}$ above, with an explicit isomophism $\psi : \text{Aut}(\hat{f}/\mathbb{P}^1_z) \to G$.

THEOREM 1.6. *Suppose given two such* $(\hat{f}_1, \psi_1), (\hat{f}_2, \psi_2)$ *and a continuous* $\hat{\varphi} : \hat{W}_1 \to \hat{W}_2$ *for which* $\hat{f}_2 \circ \hat{\varphi} = \hat{f}_1$. *This induces*

$$\varphi^* : \text{Aut}(\hat{W}_2/\mathbb{P}^1_z) \to \text{Aut}(\hat{W}_1/\mathbb{P}^1_z) \text{ by}$$
$$a \in \text{Aut}(\hat{W}_2) \mapsto (\hat{\varphi})^{-1} \circ a \circ \hat{\varphi} \in \text{Aut}(\hat{W}_1).$$

Then, φ *is an inner equivalence if*

(1.10) $\psi_2 \circ \hat{\varphi}^* \circ \psi_1^{-1}$ *is an inner automorphism of* G.

Equivalence classes of such pairs $(\hat{f}, \psi)$ *correspond to elements of* $\text{Ni}(G, \mathbf{C})^\dagger$, *with* $\dagger = \text{in}$. *Or, denote them* $\text{Ni}(G, \mathbf{C})/G$ *with the* $g \in G$ *action mapping* $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C})$ *to conjugation by* g *distributed on all entries of* $\boldsymbol{g}$.

Ordering $\boldsymbol{z}$ would destroy most applications number theorists care about. This makes sense of saying *a cover is in the Nielsen class* $\text{Ni}(G, \mathbf{C})^\dagger$.

1.2.2. *Braid action on* $\mathrm{Ni}(G,\mathbf{C})^\dagger$. The rubric of [**Fr20b**, §2.1.1] called *dragging a cover by its branch points* gives a sense of how the braid group enters. Here we give basic facts and how two generators of the Hurwitz monodromy group, H_r, act on Nielsen classes.

(1.11a) H_r is the fundamental group $\pi_1(U_r,\boldsymbol{z}^0)$ with $\boldsymbol{z}^0\in U_r$, a basepoint.

(1.11b) Two elements generate H_r:

$$\begin{aligned} q_i:\ & \boldsymbol{g}\stackrel{\text{def}}{=}(g_1,\dots,g_r)\mapsto(g_1,\dots,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,g_{i+2},\dots,g_r);\\ \mathbf{sh}:\ & \boldsymbol{g}\mapsto(g_2,g_3,\dots,g_r,g_1) \text{ and } H_r\stackrel{\text{def}}{=}\langle q_2,\mathbf{sh}\rangle \text{ with}\\ & \mathbf{sh}\,q_i\,\mathbf{sh}^{-1}=q_{i+1},\ i=1,\dots,r{-}1.\end{aligned}$$

(1.11c) Braids, B_r, on r strings give $H_r=B_r/\langle q_1\cdots q_{r-1}q_{r-1}\cdots q_1\rangle$.

The case $r=4$ in (1.11b) is so important in examples, that in *reduced* Nielsen classes (Def. 1.8), we refer to q_2 as the *middle twist*. As usual, in notation for free groups modulo relations, (1.11c) means to mod out by the normal subgroup generated by the relation $q_1\cdots q_{r-1}q_{r-1}\cdots q_1=R_H$.

Principle 1.7. *From* (1.11)*, we get a permutation represention of* H_r *on* $\mathrm{Ni}(G,\mathbf{C})^\dagger$. *Given* $\dagger$*, that gives a cover* $\Phi\stackrel{\text{def}}{=}\Phi^\dagger:\mathcal{H}(G,\mathbf{C})^\dagger\to U_r$*: The Hurwitz space of* $\dagger$*-equivalences of covers.*

The elements in $\langle q_1\cdots q_{r-1}q_{r-1}\cdots q_1\rangle$ *have the affect*

$$\begin{aligned}&\boldsymbol{g}\in\mathrm{Ni}(G,\mathbf{C})\mapsto g\boldsymbol{g}g^{-1} \textit{ for some } g\in G. \textit{ Indeed, for}\\ &\boldsymbol{g}\in\mathrm{Ni}(G,\mathbf{C}),\{(\boldsymbol{g})q^{-1}R_Hq\mid q\in B_r\}=\{g^{-1}\boldsymbol{g}g\mid g\in G\}.\end{aligned}\tag{1.12}$$

Circumstances dictate when to identify covers ${}_if:{}_iW\to\mathbb{P}^1_z$, $i=1,2$, branched at ${}_0\boldsymbol{z}$, obtained from any one cover using the dragging-branch-points principle. One might regard *inner* (resp. *absolute*) equivalence as *minimal* (resp. *maximal*). Act by H_r on either equivalence $\mathrm{Ni}(G,\mathbf{C})^\dagger$.

1.3. Reduced Nielsen Classes. Möbius transformations $\mathrm{PGL}_2(\mathbb{C})$ act on $\mathbb{P}^1_z$, so on $(\mathbb{P}^1_z)^n$, and therefore equivariantly on $(\mathbb{P}^1_z)^n/S_n$.

Definition 1.8 (Reduced action). A cover $f:W\to\mathbb{P}^1_z$ is *reduced* equivalent to $\alpha\circ f:W\to\mathbb{P}^1_z$ for $\alpha\in\mathrm{PGL}_2(\mathbb{C})$.

Also, α acts on $\boldsymbol{z}\in U_r$ by acting on each entry. That extends to an action on any cover $\Phi^\dagger:\mathcal{H}(G,\mathbf{C})^\dagger\to U_r$, giving a reduced Hurwitz space cover:

$$\Phi^{\dagger,\mathrm{rd}}:\mathcal{H}(G,\mathbf{C})^{\dagger,\mathrm{rd}}\to U_r/\mathrm{PGL}_2(\mathbb{C})\stackrel{\text{def}}{=}J_r.\tag{1.13}$$

Somewhat abusing language, we refer to a point of J_r over which $U_r\to J_r$ is not smooth as an elliptic point.

We concentrate in this section on reduced spaces when $r=4$: §1.3.1 has the genus formula and §1.3.2 has the formula for fine moduli. Both are ingredients we expect to be able to compute based mostly on the **sh**-incidence matrix.

1.3.1. *Genus formula for $r = 4$.* When $r = 4$, $U_r/\mathrm{PGL}_2(\mathbb{C})$ identifies with $\mathbb{P}^1_j \setminus \{\infty\}$. A reduced Hurwitz space of 4 branch point covers is a natural j-line cover. Significantly it is an upper half-plane, $\mathbb{H}$, quotient by a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ [**BFr02**, §2.10]. We don't directly use that finite index subgroup. It is never a congruence subgroup except when G is closely related to a dihedral group.[3]

The cover completes to $\overline{\mathcal{H}}(G, \mathbf{C})^{\dagger,\mathrm{rd}} \to \mathbb{P}^1_j$ ramified over $0, 1, \infty$.

(1.14) Denote the group $\langle q_1 q_3^{-1}, \mathbf{sh}^2\rangle$ by $\mathcal{Q}''$.

[**BFr02**, §4.2] contains the formula whose statement in Thm. 1.9 uses

(1.15) *reduced Nielsen classes* $\mathrm{Ni}(G, \mathbf{C})^{\dagger,\mathrm{rd}} \stackrel{\mathrm{def}}{=} \mathrm{Ni}(G, \mathbf{C})^{\dagger}/\mathcal{Q}''$.

THEOREM 1.9. *Suppose a component, $\overline{\mathcal{H}'}$, of $\overline{\mathcal{H}}(G, \mathbf{C})^{\dagger,\mathrm{rd}}$ is given by a braid orbit, O, on the corresponding Nielsen classes $\mathrm{Ni}(G, \mathbf{C})^{\dagger,\mathrm{rd}}$. Then, the ramification, respectively over $0, 1, \infty$, of $\overline{\mathcal{H}'} \to \mathbb{P}^1_j$ is given by the disjoint cycles of $\gamma_0 = q_1 q_2$, $\gamma_1 = q_1 q_2 q_1$, $\gamma_\infty = q_2$ acting on O.*

The genus, $g_{\overline{\mathcal{H}'}}$, of $\overline{\mathcal{H}'}$, a la Riemann-Hurwitz, appears from

$$2(|O| + g_{\bar{\mathcal{H}}'} - 1) = \mathrm{ind}(\gamma_0)+\mathrm{ind}(\gamma_1)+\mathrm{ind}(\gamma_\infty).$$

Using reduced spaces allows us to compare Hurwitz spaces with classical moduli spaces, starting with the case G is a dihedral group, D_ℓ, ℓ a prime (which we here take to be odd) and $\mathbf{C} = \mathbf{C}_{2^4}$ is 4 repetitions of the involution conjugacy class, or more generally in the dihedral case with hyperelliptic jacobians, as in [**Fr20b**, §1.3.1 et. al.].

Being able to compute cusps and genus of components when $r = 4$, as our examples illustrate, makes very specific the value of using reduced classes of covers, and the Nielsen class version (1.15) of this. We determine a great deal from the Nielsen class, and the braid action on it.

While $\Phi^\dagger$ in Princ. 1.7 is an unramified cover of manifolds, (1.16) records geometric complications when considering $\Phi^{\dagger,\mathrm{rd}}$.

(1.16a) For $r = 4$: $\Phi^{\dagger,\mathrm{rd}}$ is finite and flat but the cover is ramified – as recorded precisely in Thm. 1.9.

(1.16b) For $r \geq 5$: Depending on the braid orbit, $\Phi^{\dagger,\mathrm{rd}}$ can have singular fibers over the image of those $\boldsymbol{z}$ in J_r fixed by some $\alpha \in \mathrm{PGL}_2(\mathbb{C})$.

Fine moduli for a reduced Hurwitz space requires an addition from fine moduli for the Hurwitz space. We expect computing that to be part of

[3] Comparing with Serre's **OIT**, [**Fr21**, Chap. 6 §3] develops this point carefully.

the **sh**-incidence computations, as explained for $r = 4$ in §1.3.2. The **sh**-incidence matrix works for $r \geq 4$, and that calls for comparing **sh**-incidence results for different values of r as in §5.

1.3.2. *Fine moduli.* This is a summary from [**BFr02**, Prop. 4.7]. In absolute equivalence, with $G \leq S_n$, denote the subgroup stabilizing 1 in the representation for absolute equivalence by $G(1)$. In a check for fine moduli we assume fine moduli for $\mathcal{H}(G, \mathbf{C})^\dagger$:

(1.17a) Absolute equivalence: $G(1)$ is self-normalizing.[4]

(1.17b) Inner equivalence: G has no center.

A standard change of variables on the j-line puts $j = 0$ and 1 as the elliptic points on U_j. Prop. 1.10 then adds the additional condition to (1.17) to get fine moduli for $r = 4$. Rem. 5.2 explains the Nielsen class element check for fine moduli when the image points are elliptic for general r.

Consider a component $\mathcal{H}_*^\dagger$ of $\mathcal{H}(G, \mathbf{C})^\dagger$ for which: the moduli definition field, $\mathbb{Q}_{\mathcal{H}_*^\dagger}$ is known. We refer, as previously to the braid orbit on the Nielsen class attached to $\mathcal{H}_*^\dagger$.

PROPOSITION 1.10 (Birational fine moduli). *Assume also $\boldsymbol{p}^{\rm rd} \in \mathcal{H}_*^{\dagger,{\rm rd}}$ has image in $U_j \setminus \{0, 1\}$. Then, there is a cover in the reduced equivalence class defined over $\mathbb{Q}_{\mathcal{H}_*^{\dagger,{\rm rd}}}(\boldsymbol{p})$ (and no smaller field) if $\mathcal{Q}''$ acts as the Klein 4-group on the braid orbit attached to $\mathcal{H}_*^\dagger$.*

For $\boldsymbol{p}^{\rm rd}$ over $j = 0$ (resp. 1), the conclusion still holds if γ_0 (resp. γ_1) in Thm. 1.9 does not fix the reduced Nielsen class element associated to $\boldsymbol{p}^{\rm rd}$.

The same conclusion holds in (1.16b) *(here there is no group like $\mathcal{Q}''$), if the image of $\boldsymbol{p}^{\rm rd} \in \mathcal{H}_*^{\dagger,{\rm rd}}$ is not an elliptic fixed point. But there is another issue: is the equivalence class of the cover isomorphic to $\mathbb{P}^1_z$ or to a quadric over the resulting field, Rem. 5.1.RETURNM*

Comments on J_r for $r \geq 5$ (below), including Rem. 5.2, describes the conclusion here also holds if the extension of α' to an action on branch cycles, does not fix for the cover corresponding to $\boldsymbol{p}$.

Both results in Prop. 1.10 − excluding Rem. 5.1 − depend only on the Nielsen class braid orbit, though the classification of the precise statement for (1.16b) is in print only in special cases in [**FrG12**].

We consider including fine moduli data as part of the computations implemented here for $r = 4$, based on the success of Thm. 1.9.

1.4. Modular Towers. For a prime ℓ, here is the definition that makes the constructions of this section canonical.

[4] For $g \in G$, if $gG(1)g^{-1} = G(1)$, then $g \in G(1)$.

DEFINITION 1.11. A cover $\psi_H : H \to G$ of profinite groups is *Frattini* (resp. ℓ-Frattini) if for any subgroup $H' \le H$ with $\psi_H(H') = G$, then $H' = H$ (resp. in addition $\ker(\psi_H)$ is an ℓ-group. For C a conjugacy class of G, consisting of elements of order d, $(d, \ell) = 1$, there is a unique conjugacy class of H – refer to it as C, too – of elements of order d lying above C.[5]

§1.4.1 gives the definitions of ℓ-Frattini lattice quotients, and then of **MT**s. §1.4.2 gives the main conjectures, noting the relevance of our tools to make computations toward them. §1.4.3 gives examples of ℓ-Frattini lattice quotients, thereby including as special cases all of our examples.

The overlapping results of [**?**] and [**?**] on the case $r = 4$, give explicit problems that call, on one hand for extending Falting's Theorem, and on the other differentiating between **MT**s (with levels that are upper half-plan quotient) that resemble modular curve towers (though they aren't) and those that don't.

1.4.1. **MT** *definition.* In our examples, a given Hurwitz space, $\mathcal{H}_0$, is level 0 of a canonical Hurwitz space tower,

$$\mathbb{H}_{G,\mathbf{C},\ell,L} \stackrel{\text{def}}{=} \{\mathcal{H}_k\}_{k=0}^{\infty} \text{ as given in (1.23).}$$

(1.18) For simplificity, and as in our examples, assume $(\ell, N_{\mathbf{C}}) = 1$.[6]

Now we explain from where, with G an ℓ-perfect group, these Ls arise. [**Fr20b**, §1.3.3] documents/surveys existence and properties for the universal ℓ-Frattini cover ${}_\ell\tilde\psi : {}_\ell\tilde G \to G$: the maximal ℓ-Frattini cover of G.[7] Modding out by the commutator $[\ker({}_\ell\tilde\psi), \ker({}_\ell\tilde\psi)]$ subgroup, produces:

$$\text{the Universal } \textit{abelianized } \ell\text{-Frattini cover of } G\text{:} \quad L_{G,\ell} \stackrel{\text{def}}{=} (\mathbb{Z}_\ell)^{\ell m_G} \to {}_\ell\tilde G_{\text{ab}} \xrightarrow{{}_\ell\tilde\psi_{\text{ab}}} G. \tag{1.19}$$

DEFINITION 1.12. The characteristic $\mathbb{Z}/\ell[G]$ ℓ-Frattini module is

$$\ker({}_\ell\tilde\psi_{\text{ab}})/\ell\ker({}_\ell\tilde\psi_{\text{ab}})) \stackrel{\text{def}}{=} {}_\ell M_G. \text{ Its } \mathbb{Z}/\ell \text{ dimension is } {}_\ell m_G.$$

The universal exponent ℓ Frattini cover of G is ${}_\ell^1\psi : {}_\ell^1G \to G$. Consider any (non-trivial) $\mathbb{Z}/\ell[G]$ quotient M' of ${}_\ell M_G$, with kernel $K_{M'}$. It is elementary that any quotient of ${}_\ell\tilde\psi_{\text{ab}}$ mapping through G is a Frattini cover, therefore giving ${}_\ell^1\psi_{M'} : {}_\ell^1G/K_{M'} \stackrel{\text{def}}{=} {}_\ell^1G_{M'} \to G$ is an ℓ-Frattini cover. We say it is *unique* if the following $\mathbb{Z}/\ell$ module has dimension 1 (see Rem. 1.15):

$$H^2(G, M') = \text{Ext}^2_{\mathbb{Z}/\ell[G]}(\mathbf{1}, M') \text{ [**Be91**, p. 70].} \tag{1.20}$$

[5] This is the most trivial case of Schur-Zassenhaus.

[6] A condition like this is necessary for the tower to be canonical. See Lem. **??**. [**Fr21**] extends it, but all our examples here use this particular condition.

[7] The treatment is a variant on that of [**FrJ86**, Chap. 22]$_2$.

Proposition 1.13. *Notation as above, there is a short exact sequence*

$$L_{M'} \to {}_\ell\tilde{G}_{M',\mathrm{ab}} \xrightarrow{{}_\ell\tilde{\psi}_{M'}} G \text{ with these properties.}^{8} \tag{1.21}$$

(1.22a) ${}_\ell\tilde{\psi}_{M'}$ *factors through* ${}^1_\ell\psi_{M'}$

(1.22b) $\ell^k L_{M'}/\ell^{k+1}L_{M'} \cong M'$ *as a* $\mathbb{Z}/\ell[G]$ *module,* $k \ge 0$.

If M' *is an indecomposable* $\mathbb{Z}/\ell[G]$ *module, then* $L_{M'}$ *is an indecomposable* $\mathbb{Z}_\ell[G]$ *module* [**Be91**, Thm. 1.9.4].

Proof. For (1.21), inductively form ${}^k_\ell\psi_{M'} : {}^k_\ell G_{M'} \to G$ as a quotient of ${}^k_\ell\psi_{\mathrm{ab}} : {}^k_\ell G_{\mathrm{ab}} \to G$.

Use the universal property of ${}^{k+1}_\ell\psi_{\mathrm{ab}} : {}^{k+1}_\ell G_{\mathrm{ab}} \to G$ for ℓ-Frattini covers of G with exponent ℓ^{k+1} kernel: This factors through ${}^k_\ell\psi_{M'}$ as

$$\psi'' : {}^{k+1}_\ell\psi_{\mathrm{ab}} : {}^{k+1}_\ell G_{\mathrm{ab}} \to {}^k_\ell G_{M'}.$$

To continue inductively, assume we have formed ${}^kK_{M'}$ the kernel of ${}^k_\ell G_{M'} \to {}^{k-1}_\ell G_{M'}$. Then, on ${}^{k+1}_\ell G_{\mathrm{ab}}$, mod out by $\ell\psi''^{-1}({}^kK_{M'}) \stackrel{\mathrm{def}}{=} {}^{k+1}K_{M'}$ to form ${}^{k+1}_\ell G_{M'} \to G$.

For the final profinite group cover given by ${}_\ell\tilde{\psi}_{M'}$, take the projective limit of these group covers of G. □

Refer to ${}_\ell\tilde{\psi}_{M'}$, or by abuse $L_{M'}$, as an *ℓ-Frattini lattice quotient* (of ${}_\ell\tilde{\psi}_{\mathrm{ab}}$). Now we seriously use (1.18): $(N_{\mathbf{C}}, \ell) = 1$ to equate the conjugacy classes $\mathbf{C}$ in ${}^k_\ell G_{M'}$ with those in $G = {}^0_\ell G_{M'}$ following Def. 1.11. From Prop. 1.13, using (1.13), form the canonical Hurwitz space sequence

$$\mathbb{H}_{G,\mathbf{C},\ell,L} \stackrel{\mathrm{def}}{=} \{\mathcal{H}({}^k_\ell G_{M'}, \mathbf{C})^{\dagger,\mathrm{rd}} \stackrel{\mathrm{def}}{=} \mathcal{H}_k\}_{k=0}^\infty. \tag{1.23}$$

Definition 1.14 (**MT**). A *modular tower* on $\mathbb{H}_{G,\mathbf{C},\ell,L}$ is a projective sequence of absolutely irreducible components $\mathbb{H}' = \{\mathcal{H}'_k\}_{k=0}^\infty$ on $\mathbb{H}_{G,\mathbf{C},\ell,L}$. That is, $\mathcal{H}'_{k+1}$, a component of $\mathcal{H}_{k+1}$, maps to $\mathcal{H}'_k$ by the natural map, Or, its a projective sequence of braid orbits on $\mathbb{N}_{G,\mathbf{C},\ell,L} \stackrel{\mathrm{def}}{=} \{\mathrm{Ni}({}^k_\ell G_{M'}, \mathbf{C})^\dagger\}_{k=0}^\infty$.

Remark 1.15. The uniqueness definition in (1.20) extends to consider all related coefficients ($\mathbb{Z}/\ell^k$ and $\mathbb{Z}_\ell$). We think that ${}^1_\ell\psi_{M'}$ in Prop. 1.13 is unique for Frattini covers of G with kernel M' if and only if (1.21) is unique with properties (1.22a) and (1.22b). The only if part is easy, but at this time we don't have a complete proof of the other direction.

[8]Referencing (1.21) just by M' is a simplification of notation, since M' can appear, in cases, as a quotient of ${}_\ell M_G$ several ways. Usually this won't be a problem.

1.4.2. *Main conjectures.* As previously, denote the moduli definition field for $\mathcal{H}'_k$ by $\mathbb{Q}_{\mathbb{H}'_k}$. Relevant to (1.26b), there are two possibilities for $\mathbb{H}'$.

(1.24) The ascending sequence $\{\mathbb{Q}_{\mathcal{H}'_k}\}_{k=0}^{\infty}$ either stabilizes at $K_{\mathbb{H}'}$ ($[K_{\mathbb{H}'}:\mathbb{Q}]<\infty$) or there is no degree bound over $\mathbb{Q}$.

Since we are using *reduced* Hurwitz spaces, when $r=4$ all components of the $\mathcal{H}_k$ s are upper half-plane quotients and natural $\mathbb{P}^1_j\setminus\{\infty\}$ with j the well-known j-line parameter [**BFr02**, §2.3]. Using the **sh**-incidence algorithm, Thm. 1.9 can efficiently compute the genus of the natural projective completion of the Hurwitz space, presented as a ramified cover of the j-line. Conj. 1.16 has been proven, but the interest has these points based on the notion of ℓ-cusp (2.7). Assume $\mathbb{H}'$ is a **MT** on $\mathbb{H}_{G,\mathbf{C},\ell,L}$.

(1.25a) If for some value of k, a component has at least 2 ℓ-cusps, then Thm. 1.9 shows the genus goes up rapidly.

(1.25b) Unless (1.25a) holds, at this time we have no way to estimate when the genus of $\mathcal{H}'_k$ has surpassed 1.

Our basic assumption (1.1) on the moduli definition field implies all tower levels have moduli definition field $\mathbb{Q}$. They have, however, in our examples several components, and some of those will have moduli definition field larger than $\mathbb{Q}$.

Though condition $r=4$ is not necessary, that case poses unsolved problems (geometric and diophantine) for the solved case, $r=4$, of Conj. 1.16. Problems for which **sh**-incidence data is relevant.

CONJECTURE 1.16. Let K be a number field. High tower levels satisfy:

(1.26a) All components have general type; and

(1.26b) there are no K points.

Statements (1.27) on ${}_\ell M_G$ – using considerable modular representation theory – give some sense of tools at our disposal for using Prop. 1.13. They are respectively [**?**, Chap. 3, Prop. 1.26] (or [**Fr95**, Proj. Indecomp. Lem. 2.3] with help from [**?**]) and [**?**, Chap. 3, Prop. 1.27] (or [**Fr95**, Prop. 2.7]). [**?**] collects these tools under four *ℓ-Frattini principles*. That material also has detailed explanations of the cohomology involved.

(1.27a) It is indecomposable (if not then its summands would be obvious examples of $M^\star$) and $\dim_{\mathbb{Z}/\ell}(H^2(G,M_G))=1$ (see Lem. **??**).

(1.27b) Describing it requires having explicitly only the projective indecomposables belonging to the *principal* block representations.[9]

[9] $\mathbb{Z}/\ell[G]$ decomposes as a sum of indecomposable 2-sided ideals (blocks) corresponding to writing 1 as a direct sum of primitive central idempotents. The block "containing" the identity representation is the principle block [**Be91**, §6.1].

Given ${}^1_\ell G \to G$, its kernel is ${}_\ell M$, the characteristic ℓ-Frattini $\mathbb{Z}/\ell[G]$. We call $\tilde{\psi}^\circ : \tilde{G}^\circ \to G$ an ℓ-Frattini lattice quotient if it is an ℓ-Frattini cover, with kernel a $\mathbb{Z}_\ell[G]$ module, free as a $\mathbb{Z}_\ell$ module.

1.4.3. *Examples of ℓ-Frattini lattice quotients.*

EXAMPLE 1.17 (Frattini Central extensions). Consider the smallest possible non-trivial ℓ-Frattini central extension $\psi_H : H \to G$: $\ker(\psi_H) = \mathbb{Z}/\ell$ is in the center of H. We also assume G is centerless. Now consider another quotient of ${}^1_\ell G$ given by $\psi^\circ : G^\circ \to G$. First consider what happens if ψ° factors through ψ_H. RETURNM △

1.4.4. *A model for results on components.* Since the **sh**-incidence matrix works for all r, a separate §5 comments on reduced Hurwitz spaces for $r \geq 5$. The topic isn't yet fully developed. One issue is that for a collection of distinct conjugacy classes, $\mathbf{C}'$, by increasing the multiplicity of the classes from $\mathbf{C}'$ appearing in $\mathbf{C}$, we get sequences of spaces that ought to be related, and for which we can see relations between their respective **sh**-incidence matrices.

The following *stable homotopy* result puts our definitions together where they apply for all r. Use the notation of (1.2).

PROPOSITION 1.18. *There is an r' for which, if $\mathbf{C} \in \mathbb{M}_{\mathbf{C},\geq r'}$ there is one component of the Hurwitz space $\mathcal{H}(G, \mathbf{C})$ consisting of covers with trivial lift invariant. Further, the moduli definition field of that component is $\mathbb{Q}$.*[10]

PROOF. This is an extension of [**FrV91**], which only considered the case when the lift invariant is trivial, but stated a version of this in [], which says for r' large each value of the lift invariant is achieved by exactly one component. With our assumption that $\mathbb{Q}_{G,\mathbf{C}} = \mathbb{Q}$, components are mapped among each other by the $G_\mathbb{Q}$ action. Thus, it suffices to see the component attached to the trivial lift invariant is stable under $G_\mathbb{Q}$.

Showing this amounts to showing that any cover, $\varphi : W \to \mathbb{P}^1_z$, defined over $\bar{\mathbb{Q}}$ in the component with trivial lift invariant is mapped to another with trivial lift invariant. The characterization of having trivial lift invariant considers a representation cover $\psi_\star : G^\star \to G$ (central Frattini extension)

[10]We know for certain as in Rem. 4.1, usually the moduli definition field would not be $\mathbb{Q}$ for the other components.

and the natural map $\mathrm{Ni}(G^\star, \mathbf{C}) \to \mathrm{Ni}(G, \mathbf{C})$.[11] Trivial lift invariant means:

$$\prod_{i=1}^r g_i^\star = 1. \text{ That is, there is an unramified cover } \varphi_\star : W_\star \to \mathbb{P}^1_z \text{ factoring through } \varphi. \tag{1.28}$$

Using Thm. **??**, that gives a Galois cover $\varphi_\star : W^\star \to \mathbb{P}^1_z$. From RET, this cover is defined over $\bar{\mathbb{Q}}$. This characterization doesn't depend on the choice of $\boldsymbol{g}$ representing φ. From [**Fr77**, §5], there is a connected Zariski open family

$$\mathcal{T}_\star \xrightarrow{\Psi_\star} \mathcal{H}_0 \times \mathbb{P}^1_z \text{ that includes a cover equivalent to } \varphi_\star.$$

For $\sigma \in G_{\mathbb{Q}}$, apply σ to all these spaces and covers. That gives

$$\text{a new family } \mathcal{T}_\star^\sigma \xrightarrow{\Psi_\star^\sigma} \mathcal{H}_0 \times \mathbb{P}^1_z \text{ containing } \varphi_\star^\sigma.$$

A component with trivial lift invariant corresponds to a braid orbit represented by $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ that is the image of $\boldsymbol{g}^\star \in \mathrm{Ni}(G^\star, \mathbf{C})$. Our assumption is there is precisely one braid orbit of such. Therefore, the assumptions say that the unique component consisting of covers with trivial lift invariant has moduli definition field $\mathbb{Q}$.[12] □

Contrary to Prop. 1.18, when $r = 4$ the lift invariant may not separate components. In most of our examples it does not. Therefore, the moduli definition field of the components of covers with trivial lift invariant requires more work. The foremost arithmetic geometry problem on Hurwitz space components.

PROBLEM 1.19. Find the *moduli definition field*, (1.5) of each component when there is more than one component of a Hurwitz space (see Rem. 1.20).

REMARK 1.20. The standout unsolved case of Prob. 1.19 is where, among several Hurwitz space components, more than one has **HM** type.[13] One possibility here is when the $G_{\mathbb{Q}}$ orbit of one component contains them all. The other extreme is when the moduli definition field of all components is $\mathbb{Q}$. That leaves the problem of finding an explanation (geometric?) for why there are distinct components.

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[11] Again, for simplicity compatible with our examples, we assume $(|\ker(\psi_\star)|, N_| bfC) = 1$, so that we may lift any element of $g \in \mathbf{C}$ to a unique same order element $g^\star$ in $G^\star$.

[12] With a little extra notation, if $\mathbb{Q}_{G,\mathbf{C}}$ is any field, the same argument would show the result is the component with trivial lift invariant has the same moduli definition field.

[13] This is a natural subcase of several components having trivial lift invariant, but this one case has arisen several times.

2. sh-incidence Algorithm

The **sh**-incidence matrix entwines the interaction of the braids and the group G, showing up in invariants of components of Hurwitz spaces through a labeling on cusps. All spaces are normal varieties. In this paper a † superscript corresponds to inner or absolute equivalence. Reduced Hurwitz spaces, denoted $\mathcal{H}^{\dagger,\mathrm{rd}}$ have dimension $r_{\mathbf{C}} - 3$. Their components, though, correspond one-one with the braid components of the Hurwitz spaces $\mathcal{H}^{\dagger}$. Here is the difference between them.

(2.1a) For $r = 4$, $\mathcal{H}^{\dagger,\mathrm{rd}}$ (resp. $\overline{\mathcal{H}}^{\dagger,\mathrm{rd}}$) is a nonsingular (finite, flat) cover of U_j (resp. $\mathbb{P}^1_j$).[14]

(2.1b) For $r \geq 5$, $\boldsymbol{p} \in \mathcal{H}^{\dagger,\mathrm{rd}}$, is a singular point if RETURNM

2.1. Twist orbits.

The **sh**-incidence matrix is given in Def. 2.4. Its rows (and columns), listed in Lem. 2.2, are labeled by cusp orbits. Def. 2.1 gives the most easily identified Nielsen class elements defining a very common type of cusp orbit. It also turns out to be problematic, though rare, when more than two braid orbits contain these types of cusps.

The name combines the phenomena of [**Ha84**] and [**Mu72**] that appears on the boundary of a Hurwitz space component containing an **HM** rep. The significance was that under mild explicit conditions this appearance allowed showing the moduli definition field of such a component would be $\mathbb{Q}$, even if there was more than one component, as in [**Fr95**, Thm. 3.21].

Definition 2.1. An element $\boldsymbol{g} \in \mathrm{Ni}(G,\mathbf{C})$ is a *Harbater-Mumford* (**HM**) representative if it has the form $(g_1, g_1^{-1}, \dots, g_s, g_s^{-1})$ (so $2s = r$). A braid orbit O is said to be **HM** if the orbit contains an **HM** rep.

There is a superficial similarity between the Nielsen class $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})$, with one braid orbit, and $\mathrm{Ni}(A_4, \mathbf{C}_{3^4})$ with two braid orbits. Both have braid orbits with **HM** reps. Yet, both are one case of a natural series of Nielsen classes, very different in their behavior.

Lem. 2.2, for all r, lists q_i orbits. Note though, one index i suffices for the **sh**-incidence matrix. For historical reasons we choose $i = 2$.

Lemma 2.2. *Let $\boldsymbol{g} \in \mathrm{Ni}(G,\mathbf{C})$ be a Nielsen class representative. With $\mu = g_i g_{i+1}$, the orbit of Q_i on $\boldsymbol{g}$ is the collection*

$$(\boldsymbol{g})q_i^k = \begin{cases} (g_1, \dots, g_{i-1}, \mu^l g_i \mu^{-l}, \mu^l g_{i+1} \mu^{-l}, g_{i+2}, \dots) & \text{for } k{=}2l \\ (g_1, \dots, g_{i-1}, \mu^l g_i g_{i+1} g_i^{-1} \mu^{-l}, \mu^l g_i \mu^{-l}, g_{i+2}, \dots) & \text{for } k{=}1{+}2l. \end{cases}$$

[14]Cover here means *ramified* cover with, with flat meaning, having fibers of locally constant dimentions.

If $r=4$, and $\boldsymbol{g}\in \mathrm{Ni}(G,\mathbf{C})^\dagger$ is an **HM** *rep.* $(g_1,g_1^{-1},g_2,g_2^{-1})$*, then so are all elements in the $\mathcal{Q}''$, of* (1.14)*, orbit of $\boldsymbol{g}$.*

Then, the $\mathcal{Q}''$ orbit length on $O_{\boldsymbol{g}}$ is $4/(m+1)$ with m the count of conjugacies, in † equivalence, given by

$$hg_1h^{-1}=g_2 \text{ or } g_1=hg_2^{-1}h^{-1} \text{ with } h^2=1; \text{ or } h(g_1,g_2)h^{-1}=(g_1^{-1},g_2^{-1}).$$

If G (resp. $N_{S_n}(G,\mathbf{C})$) has no center (resp. element that centralizes G) for $\dagger=\mathrm{in}$ *(resp.* abs*) then h is determined by these conditions.*

Proof. The formula of the 1st paragraph comes from the definition of q_i. Similarly, if you apply $q_1q_3^{-1}$ or $\mathbf{sh}^2$ to an **HM** rep, then it is immediate the result is another **HM** rep.

Now consider the $\mathcal{Q}''$ orbit length, which is 4 divided by the number of elements $q\in\mathcal{Q}''$ for which $(\boldsymbol{g})q=h\boldsymbol{g}h^{-1}$ for some $h\in G$ (resp. $h\in N_{S_n}(G,\mathbf{C})$ if $\dagger=\mathrm{in}$ (resp. $\dagger=\mathrm{abs}$). The cases are similar, so we will just do the 1st.

If $h^2=1$ for which $g_1=hg_2h^{-1}$, then $g_1^{-1}=hg_2^{-1}h^{-1}$ and

$$h((\boldsymbol{g})\mathbf{sh}^2)h^{-1}=h(g_2,g_2^{-1},g_1,g_1^{-1})h^{-1}=\boldsymbol{g}.$$

If h_1,h_2 both satisfied these conditions then $h_1h_2^{-1}$ would commute with $\boldsymbol{g}$, contrary to the centralizing assumption. Either one or all three of those conditions are satisfied. Add that number to 1 (for the trivial element in $\mathcal{Q}''$) to get m. □

Remark 2.3. General expectation from Lem. 2.2 is that Q_2 orbits would have length $2\cdot\mathrm{ord}(g_ig_{i+1})\stackrel{\mathrm{def}}{=}2o$. There is an important exception –Prop. **??** – for which the orbit length (even without concerns about centralizers) is half that expectation. The first condition is that o is odd. It applies to the cusp labeled ${}_{\mathrm{c}}O_{1,3}^3$ in the Ni_0^+ block of the case $\mathrm{Ni}(A_4,\mathbf{C}_{3^4})$ §3.1.

2.2. Listing cusps for the *sh*-incidence matrix. Now we give the algorithm using Def. 2.4, the **sh**-incidence matrix, for computing braid orbits on specific reduced Nielsen classes.

For, S, a set of representatives in $\mathrm{Ni}(G,\mathbf{C})$ and any equivalence relation $\bullet$ on the Nielsen class, denote by $S^{q_2,\bullet}$ (resp. $S^{\mathbf{sh},\bullet}$) the collection of $\bullet$ equivalence classes of q_2 (resp. **sh**) orbits.

We use O for braid orbits on a Nielsen class. For q_2 (cusp) orbits the notation will be ${}_{\mathrm{c}}O$, with the understanding $\bullet$-equivalence holds.

If $r=4$ and $\bullet$ is one of the reduced equivalences, then **sh** has order 2, thereby producing a symmetric matrix.

DEFINITION 2.4. List $\bullet$-equivalence classes, ${}_{\rm c}O_1, \dots, {}_{\rm c}O_u$, of q_2 (*cusp*) orbits. The **sh**-incidence matrix $A(G, \mathbf{C})$ has (i, j) term $|(O_i)\mathbf{sh} \cap O_j|$.

Denote the transpose of an $n \times n$ matrix by ${}^{\rm tr}T$. Equivalence $n \times n$ matrices A and $TA^{\rm tr}T$ running over permutation matrices T associated to elements of S_n. Refer to a matrix A as in *block form* if there are matrices $B_1, \dots, B_u$, for which A has the form of a $u \times u$ diagonal matrix with diagonal entries $B_1, \dots, B_u$. The proof of Lem. 2.5 is the algorithm. Rem. 2.10 and Rem. 2.12 add extra comments.

LEMMA 2.5. *For some T, $A(G, \mathbf{C})$ is in block form with the block rows (and columns) labeled by cusp orbits whose union of elements consists of a single braid orbit on* $\mathrm{Ni}(G, \mathbf{C})$ *under $\bullet$-equivalence.*

PROOF. Start with any q_2 orbit and label it ${}_{\rm c}O_{1,1}$. Then form the sequence

$$(2.2) \qquad {}_{\rm c}O_{1,1} \to ({}_{\rm c}O_{1,1}^{\mathbf{sh},\bullet})^{q_2,\bullet} \to ((({}_{\rm c}O_{1,1}^{\mathbf{sh},\bullet})^{q_2,\bullet})^{\mathbf{sh},\bullet})^{q_2,\bullet} \dots$$

until the sequence stops. The result will be a union of distinct q_2 orbits under $\bullet$-equivalence.

Denote this collection by ${}_{\rm c}\mathcal{O}_1 = \{{}_{\rm c}O_{1,1}, \dots, {}_{\rm c}O_{1,b_1}\}$. Since $H_r = \langle q_2, \mathbf{sh}\rangle$, together ${}_{\rm c}\mathcal{O}_1$ contains all elements – modulo $\bullet$-equivalence – in the Nielsen class that are in the H_r orbit of any element of ${}_{\rm c}O_{1,1}$.

Label the rows and columns of the first block of your matrix, B_1, by the elements of ${}_{\rm c}\mathcal{O}_1$. In step 1 of (2.2) you iterate applications of **sh** on ${}_{\rm c}O_{1,1}$ and check for all new q_2 orbits. The (i, j)-entry is $|({}_{\rm c}O_{1,i})\mathbf{sh} \cap {}_{\rm c}O_{1,j}|$. Call ${}_{\rm c}O_i$ and ${}_{\rm c}O_j$ *neighbors* if $|({}_{\rm c}O_{1,i})\mathbf{sh}^t \cap {}_{\rm c}O_{1,j}|$ is nonzero for some t. If the process stops after one step, then all q_2 orbits are neighbors of ${}_{\rm c}O_{1,1}$. In step 2 you do the same thing to any of the new q_2 orbits, etc., until you stop getting new q_2 orbits. The resulting **sh**-incidence matrix is obtained from a maximal sequence of neighbors.

Therefore, this gives exactly one block, B_1 as described in the opening paragraph of the lemma. If further q_2 orbits in the Nielsen class haven't been used, then start again until you have used them all. Eventually you get blocks $B_1, \dots, B_u$ corresponding to unions of q_2 orbits ${}_{\rm c}\mathcal{O}_1, \dots, {}_{\rm c}\mathcal{O}_u$. $\square$

2.2.1. *Comparing inner and absolute classes.*

2.2.2. *The Lift invariant.* We here define the lift invariant, in a restricted situation, that suffices for the paper. Defining it is one thing, computing it another. Our examples take advantage of computation formulas.

The fundamental group of the *orthogonal group*,$O_n(\mathbb{R})$, is naturally written as the *spin cover* $\psi_n : \mathrm{Spin}_n \to O_n(\mathbb{R})$. Regard $\ker(\psi_n)$ as $\{\pm 1\}$. The natural permutation embedding of A_n in O_n induces the

Frattini cover $\psi_n : \mathrm{Spin}_n \to A_n$, abusing notation a little.

A braid orbit O of $\boldsymbol{g} = (g_1, \dots, g_r) \in \mathrm{Ni}(A_n, \mathbf{C})$, with $\mathbf{C}$ conjugacy classes consisting of odd-order elements, passes the (spin) lift invariant test if the natural (one-one) map $\mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}) \to \mathrm{Ni}(A_n, \mathbf{C})$ has image containing $\boldsymbol{g}$. Each g_i lifts to a same-order element $\tilde{g}_i \in \mathrm{Spin}_n$.

Definition 2.6 (Lift invariant). Then, $s_{\mathrm{Spin}_n/A_n}(O) \stackrel{\text{def}}{=} \prod_{i=1}^r \tilde{g}_i \in \ker(\psi_n)$. Generally, for ℓ-perfect G, $\psi_H : H \to G$ a central ℓ-Frattini cover, and ℓ' conjugacy classes $\mathbf{C}$:[15] for $s_{H/G}$ on a braid orbit O on $\mathrm{Ni}(G, \mathbf{C})$ substitute $A_n \to G$ and $\mathrm{Spin}_n \to H$ in $s_{\mathrm{Spin}_n/A_n}(O)$ using same-order lifts to H.

One result was that if the genus attached to $\mathrm{Ni}(A_n, \mathbf{C})$ is 0, then the lift invariant depends only on the Nielsen class and not on O, and there is an explicit computation for it.

Example 2.7. For $n = 4$, there are two classes of 3-cycles, $\mathrm{C}_{\pm 3}$, but just one for $n \geq 5$. For $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})$, $n \geq 5$, $n-1$ repetitions of C_3, $s_{\mathrm{Spin}_n/A_n}(\boldsymbol{g}) = (-1)^{n-1}$. For $n = 4$, the only genus 0 Nielsen classes of 3-cycles are $\mathrm{Ni}(A_4, \mathrm{C}_{+3^3})$ and $\mathrm{Ni}(A_4, \mathrm{C}_{-3^3})$, and the lift invariant is -1.

The short proof of [**Fr10**, Cor. 2.3] is akin to the original statements I made to Serre for [**?**]. △

Definition 2.8.

2.3. $r = 4$ and finishing the computation.

Lemma 2.9. *Now assume $r = 4$, and $\bullet$-equivalence is one of the reduced equivalences, so q_2 orbits are γ_∞ orbits. Then the sh-incidence matrix is symmetric.*

Replacing **sh** *by either γ_0 or γ_1 acting on γ_∞ orbits gives the same blocks in the resulting matrix. Then, fixed points of γ_j, $j = 0$ or 1, on any $\bar{M}_4$ orbit give nonzero entries along the diagonal of the corresponding block. Then,*

$$|({}_cO_{1,t})\mathbf{sh} \cap {}_cO_{1,t'}| = |({}_cO_{1,t})\mathbf{sh}^2 \cap ({}_cO_{1,t'})\mathbf{sh}| = |{}_cO_{1,t} \cap ({}_cO_{1,t'})\mathbf{sh}|.$$

That shows the final matrix in block form, has each block symmetric.

Proof. Take $r = 4$ and for $\bullet$-equivalence one of the reduced equivalences, where $\mathbf{sh}^2$ is the identity on braid orbits.

[15] You can drop both assumptions, as in [**FrV91**, App.], but the definition is trickier.

Use that on reduced classes $q_1 = q_3$, with relation (**??**), $q_1q_2q_1 = q_2q_1q_2$. Now consider what happens if we replace **sh** by γ_1 represented by q_1q_2. Since we start with a q_2 orbit, say ${}_{\rm c}O$, the collection with **sh** applied is given by

$$({}_{\rm c}O)\mathbf{sh} = ({}_{\rm c}O)q_1q_2q_1 = ({}_{\rm c}O)q_2q_1q_2 = ({}_{\rm c}O)q_1q_2.$$

That shows the matrix is the same with γ_0 replacing **sh**. Of course, γ_1 is **sh**. That finishes the proof of the lemma. □

REMARK 2.10 (The algorithm-Part 1). Regard the sequence of expression (2.2) as iterated steps – it shows two steps – in computing one braid orbit on the •-equivalence classes on $\mathrm{Ni}(G, \mathbf{C})$. Our examples often have one step.

We have used the example Nielsen class $\mathrm{Ni}(A_4, \mathbf{C}_{+3^2-3^2})$ several times to illustrate increasing numbers of quantities you would like to compute about them: For example, [**Fr20b**, §2.3]. Especially properties of the *two* components in each case of reduced inner and absolute spaces as j-line covers (and upper-half plane quotients) in light of their not being modular curves. We don't do this one again, but note that it is the case $\ell = 2$ of the series of §**??** which does the cases of prime $\ell > 3$. Each component corresponds to the braid orbit of two types of cusps: an **HM** rep. Def. 2.1, and a **D**(ouble)**I**(dentity) rep. These give their Hurwitz space components corresponding monikers.

It makes sense to list q_2 orbits, referring to the blocks, as

$${}_{\rm c}O_{1,1}^{w_{1,1}}, \dots, {}_{\rm c}O_{1,b_1}^{w_{1,b_1}}, {}_{\rm c}O_{2,1}^{w_{2,1}}, \dots, {}_{\rm c}O_{2,b_1}^{w_{2,b_2}}, \dots, {}_{\rm c}O_{u,1}^{w_{u,1}}, \dots {}_{\rm c}O_{u,b_u}^{w_{u,b_u}},$$

with the superscript $w_{i,j}$ the cusp width (cusp orbit length). Still, should there be more than one step, labeling within any one block probably should correspond to the step in which it appears. Especially if the seed orbit has been chosen well.

LEMMA 2.11. *For $r = 4$, in the ith block of the* **sh**-*incidence matrix, we can read off the degree of the component $\bar{\mathcal{H}}_i$ over $\mathbb{P}^1_j$ as $\sum_{j=1}^{b_i} w_{i,j}$. Further,*

$$w_{i,j} = \sum_{k=1}^{b_i} |({}_{c}O_{i,j})\mathbf{sh} \cap \mathcal{O}_{i,k}|, j = 1, \dots, b_i.$$

PROOF. This follows by recognizing the cusps as the q_2 orbits on the ith braid orbit of the Nielsen classes under reduced equivalence. This is a piece of the proof of Thm. **??**. The rest is from the combinatorics of the **sh**-incidence matrix. □

REMARK 2.12 (The algorithm-Part 2). Fixed points of γ_0 or γ_1 on a reduced orbit imply that the reduced Hurwitz space component does not have fine moduli. We can almost read that data off directly from the **sh**-incidence matrix. If there are no nonzero diagonal elements corresponding to that block in Lem. 2.5, then for certain γ_0 or γ_1 have no fixed points. Then the only test necessary for fine moduli is that $\mathcal{Q}''$ acts on that orbit as a Klein 4-group.

Yet, if their diagonal elements aren't all 0, there may, or not, be γ_0 or γ_1 fixed points. Both cases occur in the $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})$ §3.1.

2.4. Cusp Principles. Princ. 2.13, a version of [**BFr02**, Prop. 2.17], makes transparent the width of most cusps. Princ. 2.14 smooths the way between $r = 3$ and $r = 4$ for pure-cycle Nielsen classes. It is a version of [**LO08**, §4], the hardest part of their paper ($r = 4$). Our simplification results using cusps improves the efficiency in computing braid orbits.

For g_1, g_2 in a group, denote the centralizer of $\langle g_1, g_2\rangle$ by $Z(g_1, g_2)$.

Let $g_1g_2 = g_3$, and $g_2g_1 = g_3'$. Let $o(g_1, g_2) = o$ (resp. $o'(g_1, g_2) = o'$ be the length of the orbit of γ^2 (resp. γ) on (g_1, g_2). If $g_1 = g_2$, then $o = o' = 1$.

PRINCIPLE 2.13. *Assume $g_1 \ne g_2$. The orbit of γ^2 containing (g_1, g_2) is $(g_3^j g_1 g_3^{-j}, g_3^j g_2 g_3^{-j})$, $j = 0, \dots, \mathrm{ord}(g_3){-}1$. So,*

$$o = \mathrm{ord}(g_3)/|\langle g_3\rangle \cap Z(g_1, g_2)| \stackrel{\mathrm{def}}{=} o(g_1, g_2).$$

Then, $o' = 2\cdot o$, unless o is odd, and with $x = (g_3)^{(o-1)/2}$ and $y = (g_3')^{(o-1)/2}$

$$\text{(so } g_1y = xg_1 \text{ and } yg_2 = g_2x), yg_2 \text{ has order 2 and } o' = o. \tag{2.3}$$

PROOF. For t an integer,

$$(g_1, g_2)\gamma^{2t} = (g_3^t g_1 g_3^{-t}, g_3^t g_2 g_3^{-t}) \text{ and}$$
$$(g_1, g_2)\gamma^{2t+1} = (g_3^t g_1 g_2 g_1^{-1} g_3^{-t}, g_3^t g_1 g_1 g_1^{-1} g_3^{-t}).$$

The minimal t with $(g_1, g_2)\gamma^{2t} = (g_1, g_2)$ is $o(g_1, g_2)$. Further, the minimal j with $(g_1, g_2)\gamma^j = (g_1, g_2)$ divides any other integer with this property. So $j|2o(g_1, g_2)$ and if j is odd, $j|o(g_1, g_2)$.

From the above, if the orbit of γ does not have length $2o(g_1, g_2)$, it has length $o(g_1, g_2)$. Use the notation around (2.3). The expressions $g_1y = xg_1$ and $yg_2 = g_2x$ are tautologies. If o is odd, then $(g_1, g_2)q_2^o = x(g_1, g_2)q_2x^{-1}$. Assume this equals (g_1, g_2), which is true if and only if $xg_1 = g_2x = yg_2$. The expression $(g_1g_2)^o = 1$ and $xg_1yg_2 = 1$ are equivalent. Conclude $(yg_2)^2 = 1$. So long as the order of yg_2 is not 1, this shows (2.3) holds. If, however, $yg_2 = xg_1 = g_2x = g_1y = 1$, then $g_1 = g_2$, contrary to hypothesis.

This reversible argument shows the converse: $(g_1, g_2)q_2^o = (g_1, g_2)$ follows from (2.3). This concludes the proof. □

Use the notation of Princ. 2.13, in the case the elements are pure-cycle.

PRINCIPLE 2.14. *Consider the common support of* (g_2, g_3). *With no loss, unless it is empty, take it to be* $\{1, \dots, k\}$. *Then, consider the overlap,* $U(\boldsymbol{g})$, *of that with* $(\boldsymbol{g})\mathbf{mp}$. *This consists of at most two integers.*

If $|U(\boldsymbol{g})| = 1$ *(with no loss take it to be* k*) then* (g_2, g_3) *has the form*

$$((k \dots 1\,\boldsymbol{v}), (1 \dots k\,\boldsymbol{w})) \text{ with } \boldsymbol{v},\ \boldsymbol{w} \text{ and } \{1, \dots, k\} \text{ mutually disjoint.}$$

Here, $(\boldsymbol{g})\mathbf{mp} = (k\,\boldsymbol{w}\,\boldsymbol{v})$, *is an odd pure-cycle.*

If $|U(\boldsymbol{g})| = 2$, *then there is no common support in the 3-tuple* (g_4, g_1, g_4g_1). *Further,* (g_2, g_3) *has the form*

$$((k \dots i_0{+}1\,\boldsymbol{v}_1\, i_0 \dots 1\,\boldsymbol{v}_2), (1 \dots i_0\,\boldsymbol{w}_1\, i_0{+}1 \dots k\,\boldsymbol{w}_2)), \tag{2.4}$$

with the sets $\boldsymbol{v}_1$,$\boldsymbol{v}_2$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ *and* $\{1, \dots, k\}$ *pairwise disjoint and*

$$(\boldsymbol{g})\mathbf{mp} = (k\,\boldsymbol{w}_2\,\boldsymbol{v}_2)(i_0\,\boldsymbol{w}_1\,\boldsymbol{v}_1), \text{ a split-cycle.} \tag{2.5}$$

The two disjoint cycles are the inverses of g_1 *and* g_4, *giving conditions (see* (2.6)*) on the lengths (orders) of* $\boldsymbol{v}_i$ *and* $\boldsymbol{w}_i$, $i = 1, 2$, *so the* $\boldsymbol{g}$ *entries have the right orders.*

The condition $|U(\boldsymbol{g})| = 2$ *happens for some rep. in each allowed Nielsen class if and only if it is not* $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ *for some* $n \ge 4$.

PROOF. We characterize each case of (**??**). For a segment labeled $\boldsymbol{v}$ (or $\boldsymbol{w}$) in the calculations, compatible with previous notation, denote its length $o(\boldsymbol{v})$.

If $U(\boldsymbol{g})$ is empty, then g_2 and g_3 are disjoint. Otherwise, assume $k \in U(\boldsymbol{g})$. If no other letter is in $U(\boldsymbol{g})$, then consider the effect of g_2g_3 to see that by reordering $1, \dots, k$, we may assume g_3 maps $i \mapsto i{+}1$, and g_2 reverses this, for $i = 1, \dots, k{-}1$. So, these integers disappear in the support of the product, and (g_2, g_3) has the shape given in the proposition statement. The length of $(\boldsymbol{g})\mathbf{mp}$ is $1 + o(\boldsymbol{v}) + o(\boldsymbol{w})$, and $2k + o(\boldsymbol{v}) + o(\boldsymbol{w}) = d_2 + d_3$. Since d_2 and d_3 are both odd, conclude $o(\boldsymbol{v}) + o(\boldsymbol{w})$ is even, and the length of $(\boldsymbol{g})\mathbf{mp}$ is odd.

It is similar for $|U(\boldsymbol{g})| = 2$. Now consider, by cases, what happens with the complementary pair (g_4, g_1).

Suppose $|U(\boldsymbol{g})| = 2$. Then, two integers having three supports among the entries of $\boldsymbol{g}$ appear in $(\boldsymbol{g})\mathbf{mp}$. Apply the argument to $(g_4, g_1, g_4g_1 = (g_2g_3)^{-1})$ that we used on (g_2, g_3, g_2g_3). If there were further integers in the common support of (g_4, g_1, g_4g_1) and (g_4, g_1), that would give at least three integers appearing in the common support of three entries of $\boldsymbol{g}$. So, that can't happen. Similarly, if $|U(\boldsymbol{g})| = 1$, then the common support for $(g_4, g_1, (g_4g_1)^{-1})$ has also cardinality 1, different from the integer in $U(\boldsymbol{g})$.

For $\boldsymbol{g} \in \text{Ni}_{(\frac{n+1}{2})^4}$, all pairs of $\boldsymbol{g}$ entries have overlapping support. So there can be no split-cycle cusps. Given $\boldsymbol{d} \neq (\frac{n+1}{2})^4$, we now produce split-cycle cusps.

With $d_1 \le d_2 \le d_3 \le d_4$, apply a braid to produce $\boldsymbol{g}'$ with $o(g'_i) = d_{i+1}$, $i = 1, \dots, 4 \mod 4$. This assures the two smallest lengths are at positions 1 and 4. From genus 0, $\sum_{i=1}^4 d_i = n+2$ implies $d_1+d_2 \le n$. Here are equations expressing the respective segment lengths of g'_4, g'_1, g'_2, g'_3 using $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\begin{aligned} 1 + o(\boldsymbol{v}_1) + o(\boldsymbol{w}_1) = d_1, \quad 1 + o(\boldsymbol{v}_2) + o(\boldsymbol{w}_2) = d_2, \\ k + o(\boldsymbol{v}_1) + o(\boldsymbol{v}_2) = d_3, \quad k + o(\boldsymbol{w}_1) + o(\boldsymbol{w}_2) = d_4. \end{aligned} \tag{2.6}$$

Solve the equations, as in Ex. **??**, to canonically, up to absolute equivalence, produce a split-cycle cusp. For example, $d_1-1+d_2-1 = d_3-k+d_4-k$, determines k. This concludes the proposition. □

[**?**, §3.2.1] has three generic cusp types that reflect on Hurwitz space components and properties of **MT**s containing such components:

ℓ-cusps, g(roup)-ℓ' and o(nly)-ℓ'.

Modular curve towers have only the first two types, with the g-ℓ' cusps the special kind called shifts of **HM**.

[**?**, §3.2.1] develops these cusps when $r = 4$ (as alluded to in (**????**)):

(2.7a) For $\boldsymbol{g} = (g_1, g_2, g_3, g_4)$ in the cusp orbit (**??**), the cusp is g-ℓ' if $H_{1,4} = \langle g_1, g_4 \rangle$ and $H_{2,4} = \langle g_2, g_3 \rangle$ are ℓ' groups.
(2.7b) It is o-ℓ', if $\ell \not| g_2g_3$ but the cusp is not g-ℓ'.
(2.7c) It is an ℓ cusp otherwise.

These generalize to all r. For example:

Definition 2.15 (g-ℓ' type). For $\boldsymbol{g}$ in a braid orbit O on $\text{Ni}(G, \mathbf{C})$, we say O is g-ℓ' if $\boldsymbol{g} = (g_1, \dots, g_r)$ has a partition with elements

$$P = [g_u, g_{u+1}, \dots, g_{u+u'}] \text{ (subscripts taken } \mod r)$$

and $H_P = \langle g_u, g_{u+1}, \dots, g_{u+u'} \rangle]$ an ℓ' group for each partition.

The following is from [**?**, Prin. 3.6, Frattini Princ. 2].

Theorem 2.16. *There is a full* **MT** *over the Hurwitz space component corresponding to O if O contains a g-ℓ' representative (no need to check central Frattini extensions as in Thm.* **??***).*

The approach to more precise results has been to consider a Harbater patching converse: Identify the type of a g-ℓ' cusp that supports a Witt-vector realization of ${}_\ell\tilde G$.

Typically we label a braid orbit O in $\mathrm{Ni}(G,\mathbf{C})$ by the type of cusp it contains. In actual examples, as in §4, even these generic names get refinements where we call particular o-ℓ' cusps *double identity*.

3. The Nielsen classes $\mathrm{Ni}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^\dagger$

[**LO08**] considered absolute Nielsen classes with the conjugacy classes pure-cycle, where the absolute classes were of genus 0 covers. The groups here can only be A_n or S_n. With these stipulations, we take the case $r=4$. In addition we ask that $\mathbf{C}$ consists of just one class repeated 4 timesthe cycles are pure-cycle of the same length k, denoted C_k: just one disjoint cycle of length exceeding 1. That is, from RH (Thm. 1.9) the pure-cycle has length $\frac{n+1}{2}$ and n is odd.

An example element of C_k would be

$$(1\,2)(1\,3)\cdots(1\,k), \text{a product of } \frac{k-1}{2} \text{ 2-cycles.}$$

Therefore, G is A_n if and only if $n \equiv 1 \mod 4$.[16] Here † indicates the equivalence could be either inner or absolute. As set up in §2.2.1 and done in particular in §3.1.4, we go from absolute to inner classes starting with the result lacking complications for absolute classes of [**LO08**]. The difficulty for inner classes, reverts in Prop. 3.3 to whether 2 pure-cycle elements of order $\frac{n+1}{2}$ in $A_{\frac{n+1}{2}}$ are conjugate in this group.

We can write out the **sh**-incidence matrix for all $n \equiv 1 \mod 4$ in this case. Prop. 3.2 shows the most easily stated distinction between the cases $n \equiv 1 \mod 8$ (two components) and $n \equiv 5 \mod 8$ (just one component) using the algorithm of §2 encapsulated in Table 4. S3.2 has the actual sh-incidence display. Table 3 ($n=5$) doesn't tackle all cusp matters. So, Table 7 has $n=13$, the next case for $n \equiv 5 \mod 8$, to show the concise abs-inn sh-incidence form. Prop. 3.17 has how the genus of the reduced spaces comes from these matters.

3.1. Braid orbits on $\mathrm{Ni}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\dagger,\mathrm{rd}}$.

§3.1.1 uses **HM** reps. to set up the most memorable result. §3.1.2 decorates the $x_{i,j}$ notation to precisely label the shift of an ${}_{\mathrm{r}}$cusp (Def. 3.1). §3.1.3 displays the shifted Nielsen class representatives (denoted ${}_{k,u}\boldsymbol{g}'$) that dominate the rest of the computation.

[16] The case for all $r \ge n-1$ and $k=3$ has already appeared (Rem. 3.19).

Then, §3.1.4 has the **sh**-incidence display for absolute classes, used to get that for inner classes.

3.1.1. *Shifts of elements in* **HM** *reps.* Fixing the 1st and 4th entries in pure-cycle reps. Prop. **??** says all g-2′ cusps are **sh** applied to **HM** cusps $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Also, all remaining cusps are pure-cycle.

With $x_{i,j} = (i\, i{+}1 \cdots j)$, inner **HM** class have one of two reps:

$$\mathbf{HM}_1 \stackrel{\text{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$
$$\mathbf{HM}_2 = (\mathbf{HM}_1)q_1 \stackrel{\text{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

Definition 3.1 (Splitting ${}_{\mathrm{c}}O_{k,2k}$). For $k = n{-}2\ell$, ℓ as in (**????**), let ${}_{\mathrm{c}}O'_{k;1}$ (resp. ${}_{\mathrm{c}}O'_{k;2}$) be all inner (reduced also from Lem. **??**) classes in $R'_{k,1}$ (resp. $R'_{k,2}$). We call these ${}_{\prime\prime}$cusps. The concept p-${}_{\prime\prime}$cusps (extending p-cusps) makes sense.

Proposition 3.2. *For $n \equiv 5 \mod 8$, $\mathbf{HM}_1$ and $\mathbf{HM}_2$ are not inner equivalent. So, there is one braid orbit on* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

For $n \equiv 1 \mod 8$, if $h \in S_n \setminus A_n$, there is no braid between $\boldsymbol{g}$ and $h\boldsymbol{g}h^{-1}$. So, there are two braid orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

3.1.2. *Organizing conjugation parities.* We use special notation for Prop. 3.3. For $V = \{\frac{n+m}{2}, \dots, n\}$, $m \geq 3$ denote the alternating (resp. symmetric) group acting on V by $A_{\frac{n+m}{2},n}$ (resp. $S_{\frac{n+m}{2},n}$). Also, $k{-}1{-}|2u{-}(k{-}1)| \stackrel{\text{def}}{=} m_{k,u}$ and

$$(\boldsymbol{n}k1)\mathbf{mp} = (x_{\frac{n+k}{2},\frac{n+1}{2}}\, x_{1,\frac{k-1}{2}}) \stackrel{\text{def}}{=} \alpha_{k,1} \text{ (as in Lem. ??)}.$$

The cycle $\alpha_{k,1}$ maps $\frac{n+1}{2} \mapsto 1$. Yet, sometimes we group $\frac{n+1}{2}$ with $\{1, \dots, \frac{k-1}{2}\}$. As i runs from $\frac{n+1}{2}$ toward $\frac{k-1}{2}$, use $x'_{i,j}$ to mean the segment from i to j, including interpreting $x'_{\frac{n+1}{2},j}$ $(i = \frac{n+1}{2})$ as $(\frac{n+1}{2}\; 1 \dots j)$. Example:

$$\alpha_{k,1} = (x_{\frac{n+k}{2},\frac{n+3}{2}}\, x'_{\frac{n+1}{2},\frac{k-1}{2}}).$$

Also, the end points of the list for ${}_{k,u}\boldsymbol{g}$ as u varies include cases where we mean to indicate $x_{i,j}$ is empty. Example: $x_{\frac{n+k-2u}{2},\frac{n+3}{2}}$ appears as the 2nd segment of the first entry of (3.2), with $0 \leq u \leq \frac{k-1}{2}$. In all terms except $u = \frac{k-1}{2}$, $\frac{n+k-2u}{2} \geq \frac{n+3}{2}$. So, denote it $x''_{\frac{n+k-2u}{2},\frac{n+3}{2}}$ to mean $x''_{\frac{n+1}{2},\frac{n+3}{2}}$ is empty.

Similarly, for $x''_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}$ appearing in the 2nd segment of the 2nd entry of (3.2): When $u = 0$ take it to mean $x''_{\frac{n+k}{2},\frac{n+k+2}{2}}$ is empty.

3.1.3. *Finding conjugating elements.* Use the ${}_{\mathrm{c}}O'_{k;j}$ convention of §**??**, and the splitting of ${}_{\mathrm{c}}O_{k,2k}$ into ${}_{\prime\prime}$cusps (Def. 3.1), to include these, too. We could have denoted ${}_{\mathrm{c}}O'_{k;1}$ by ${}_{\mathrm{c}}O'_{\boldsymbol{n}k1;1}$, and similarly formed ${}_{\mathrm{c}}O'_{\boldsymbol{g};1}$ for Nielsen class rep. $\boldsymbol{g}$.

Explicit asides on the sh-incidence column for the cusp of $\boldsymbol{n}31$ help follow notation. The $\boldsymbol{n}31$ row of Table 1 is $(x_{2,\frac{n+3}{2}}, (\frac{n+3}{2} \ \dots \ n\,1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$, with middle product $\alpha_{3,1} = (1\ \frac{n+3}{2}\ \frac{n+1}{2})$. Conjugate the 2nd and 3rd positions of $\boldsymbol{n}31$ by $\alpha_{3,1}^u$, then shift. Here are the 1st and 2nd positions of the result for $u = 1, 2$:

$$ {}_{3,1}\boldsymbol{h} = ((x_{n,\frac{n+5}{2}}\ \tfrac{n+1}{2}\ 1), x_{\frac{n+1}{2},1}) \qquad {}_{3,2}\boldsymbol{h} = ((x_{n,\frac{n+5}{2}}\ 1\ \tfrac{n+3}{2}), x_{\frac{n+1}{2},1}). \tag{3.1}$$

The respective products of entries of ${}_{3,1}\boldsymbol{h}$ and ${}_{3,2}\boldsymbol{h}$ are

$$(x_{n,\frac{n+5}{2}}\ x_{\frac{n+1}{2},2}) \text{ and } (x_{n,\frac{n+5}{2}}\ 1\ x_{\frac{n+1}{2},2}\ \tfrac{n+3}{2}).$$

Key to Prop. 3.3: Apply $q_1^{-1}q_3$, leaving the reduced Nielsen class unchanged. Once ℓ is given in Prop. 3.3, denote the entries of $\boldsymbol{n}\ell2$ by $(g_1'', g_2'', x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$.

PROPOSITION 3.3. *Each* ${}_{c}O'_{n;1}$, ${}_{c}O'_{n;2}$ *and* ${}_{c}O'_{n-2;2}$ *intersect once with* $({}_{c}O'_{3,1})\mathbf{sh}$ *giving all nonzero sh-incidence entries in the column (or row) of* ${}_{c}O'_{3;j}$, $j = 1, 2$ *(*Table 5*). Elements of* $({}_{c}O'_{k;1})\mathbf{sh}$ *have reps. of form*

$${}_{k,u}\boldsymbol{g} \stackrel{\text{def}}{=} (\alpha_{k,1}^u x_{n,\frac{n+1}{2}} \alpha_{k,1}^{-u}, \alpha_{k,1}^u (x'_{\frac{n+1}{2},\frac{k-1}{2}}\ x_{\frac{n+k+2}{2},n}) \alpha_{k,1}^{-u}, \bullet, x_{\frac{n+1}{2},1})$$

in ${}_{c}O'_{\ell;1} \cup {}_{c}O'_{\ell;2}$, $\ell = n - m_{k,u}$ *with* $m_{k,u} = 2u$ *(resp.* $2(k-1-u)$*) for* $0 \le u \le \frac{k-1}{2}$ *(resp.* $\frac{k+1}{2} \le u \le k-1$*). For* $0 \le u \le \frac{k-1}{2}$, ${}_{k,u}\boldsymbol{g}$ *is inner equivalent to some* ${}_{k,u}\boldsymbol{g}'$ *with 4th entry* $x_{\frac{n+1}{2},1}$ *and with respective 1st and 2nd entries*

$$(x_{n,\frac{n+k+2}{2}}\ x''_{\frac{n+k-2u}{2},\frac{n+3}{2}}\ x_{1,1+u}), (x_{1+u,\frac{k+1}{2}}\ x''_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}\ x_{\frac{n+k+2}{2},n}). \tag{3.2}$$

Conclude: ${}_{k,u}\boldsymbol{g} \in {}_{c}O'_{\ell;2}$ *(resp.* ${}_{c}O'_{\ell;1}$*) if* $\frac{\ell-1}{2}$ *is odd (resp. even), if and only if for some* $\beta' \in A_{\frac{n+3}{2},n}$ *and* j,

$$\beta' g_2' (\beta')^{-1} = \alpha_{\ell,2}^j g_2'' \alpha_{\ell,2}^{-j}, \text{ with } g_2' \text{ the 2nd entry of } {}_{k,u}\boldsymbol{g}'. \tag{3.3}$$

Here is the analog of expression (3.2) *for* $\frac{k+1}{2} \le u \le k-1$, *with* $k-1-u = u'$:

$$(x_{n,\frac{n+k+2}{2}}\ x_{1,1+u'}\ x_{\frac{n+k}{2},\frac{n+3+2u'}{2}}), (x_{\frac{n+3+2u'}{2},\frac{n+3}{2}}\ x_{\frac{n+1}{2}-(\frac{k-3-2u'}{2}),\frac{n+1}{2}}\ x_{\frac{n+k+2}{2},n}). \tag{3.4}$$

PROOF. Three distinct sets of form $\boldsymbol{g}^{\text{in,rd}}$ comprise $({}_{c}O'_{3;1})\mathbf{sh}$. From Table 2, ${}_{c}O'_{n;j}$, $j = 1, 2$, gives two. Then, $\boldsymbol{g}^* = ((x_{\frac{n+1}{2},n}\ \frac{n+3}{2}), \boldsymbol{h}^{3,1}, x_{2,\frac{n+3}{2}})$ is the 3rd, and it is in the cusp of one of ${}_{c}O'_{n-2;j}$, $j = 1, 2$. Apply $q_1^{-1}q_3$ to $\boldsymbol{g}^*$ to get the 4th entry $x_{\frac{n+1}{2},1}$. Conjugating (§**??**) by $x_{1,\frac{n+1}{2}}$ gives:

$${}_{3,1}\boldsymbol{g}' \stackrel{\text{def}}{=} ((x_{n,\frac{n+5}{2}}\ x_{1,2}), (2\ x_{\frac{n+3}{2},n}), \bullet, x_{\frac{n+1}{2},1}).$$

Now conjugate the first entry of ${}_{3,1}\boldsymbol{g}'$ to the first entry of

$$\boldsymbol{n}n{-}22 = ((x_{\frac{n+5}{2},n}\ x_{1,2}), x_{2,\frac{n+3}{2}}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}).$$

The conjugation is by $\beta'_{3,1}$: It inverts $|x_{n,\frac{n+5}{2}}|$ (parity +1 from Lem. **??**). The result $\boldsymbol{g}^\dagger$ has the same 1st and 4th entries as $\boldsymbol{n}n{-}22$. Lem. **??** now says $\text{Cusp}_{\boldsymbol{g}^*} = {}_{\text{c}}O'_{n-2;2}$. Rem. 3.4 shows $k=3$ matches the general case.

That accounts for the column in the ${}_{\prime\prime}$cusp sh-incidence (comment following Lem. **??**) for the cusp of $\boldsymbol{n}31$. Analogously, for $\boldsymbol{n}32$: one intersection with each of the cusps for $\mathbf{HM}_1$ and $\mathbf{HM}_2$; one intersection with the cusp of $\boldsymbol{n}n{-}21$.

Now we extend the pattern above for computing into which ${}_{\prime\prime}$cusp does the shift of inner, reduced elements fall in the columns of the other ${}_{\text{c}}O'_{k;1}$. (It is then automatic to deduce the same for ${}_{\text{c}}O'_{k;2}$.) With $\alpha_k \stackrel{\text{def}}{=} \alpha_{k,1}$, as in the statement, apply Princ. **??** and $q_1^{-1}q_3$ as above to get

$$_{k,u}\boldsymbol{g} \stackrel{\text{def}}{=} (\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}, \alpha_k^u(\tfrac{n+1}{2}\, x_{1,\frac{k-1}{2}}\, x_{\frac{n+k+2}{2},n})\alpha_k^{-u}, \bullet, x_{\frac{n+1}{2},1}).$$

The following steps give the sh-incidence matrix:

(3.5a) As (k,u) varies, $3 \le k \le n$ odd and $0 \le u \le k{-}1$, compute

$$\ell = \text{ord}((_{k,u}\boldsymbol{g})\mathbf{mp}) = \text{ord}(x_{\frac{n+1}{2},1}\cdot \alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}).$$

(3.5b) Conjugate $_{k,u}\boldsymbol{g}$ by some power $x_{1,\frac{n+1}{2}}$ to get $_{k,u}\boldsymbol{g}'$ with 1st entry having the same segment from $\{1,\dots,\frac{n+1}{2}\}$ as does g''_1.

(3.5c) Conjugate $_{k,u}\boldsymbol{g}'$ by $\beta_{k,u} \in S_{\frac{n+3}{2},n}$ to get $\boldsymbol{g}^\dagger$ with 1st entry g''_1.

(3.5d) Find $\beta'_{k,u}$ centralizing $\langle g''_1, x_{\frac{n+1}{2},1}\rangle$ so $\beta' g^\dagger(\beta')^{-1} \in {}_{\text{c}}O'_{\boldsymbol{n}\ell 2;1}$.

To help with notation (3.6) gives an example of "translate segment" §**??**. For $k=7$ and $0 \le u \le 6$, it has the 1st (col. 1) and 2nd (col. 2) entries of $_{k,u}\boldsymbol{g}$: That is of the shift of the ${}_{\prime\prime}$cusp ${}_{\text{c}}O_{7,1}$. Here $\alpha_7 = (\frac{n+7}{2}\,\frac{n+5}{2}\,\frac{n+3}{2}\,\frac{n+1}{2}\,1\,2\,3)$ $(k=7)$.

The value of $\ell = \text{ord}((_{k,u}\boldsymbol{g})\mathbf{mp})$ heads each row. Here ℓ is n minus two numbers: the cardinality of the subset of $\{\frac{n+k}{2},\dots,\frac{n+3}{2}\}$ (resp. of $\{1,2,\dots,\frac{n+1}{2}\}$) missing from $\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}$ (resp. moved by $\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}$ as in $x_{1,\frac{n+1}{2}}$).

$$\begin{array}{rll}
n: & x_{n,\frac{n+1}{2}} & (\frac{n+1}{2}\, x_{1,3}\, x_{\frac{n+9}{2},n}) \\
n{-}2: & (x_{n,\frac{n+9}{2}}\, x_{\frac{n+5}{2},\frac{n+1}{2}}\, 1) & (x_{1,3}\, \frac{n+7}{2}\, x_{\frac{n+9}{2},n}) \\
n{-}4: & (x_{n,\frac{n+9}{2}}\, x_{\frac{n+3}{2},\frac{n+1}{2}}\, x_{1,2}) & (x_{2,3}\, x_{\frac{n+7}{2},\frac{n+5}{2}}\, x_{\frac{n+9}{2},n}) \\
n{-}6: & (x_{n,\frac{n+9}{2}}\, \frac{n+1}{2}\, x_{1,3}) & (3\, x_{\frac{n+7}{2},\frac{n+3}{2}}\, x_{\frac{n+9}{2},n}) \\
n{-}4: & (x_{n,\frac{n+9}{2}}\, x_{1,3}\, \frac{n+7}{2}) & (x_{\frac{n+7}{2},\frac{n+1}{2}}\, x_{\frac{n+9}{2},n}) \\
n{-}2: & (x_{n,\frac{n+9}{2}}\, x_{2,3}\, x_{\frac{n+7}{2},\frac{n+5}{2}}) & (x_{\frac{n+5}{2},\frac{n+1}{2}}\, 1\, x_{\frac{n+9}{2},n}) \\
n: & (x_{n,\frac{n+9}{2}}\, 3\, x_{\frac{n+7}{2},\frac{n+3}{2}}) & (x_{\frac{n+3}{2},\frac{n+1}{2}}\, x_{1,2}\, x_{\frac{n+9}{2},n}).
\end{array} \tag{3.6}$$

The pattern is clear. For a general k the g_1 term has $x_{n,\frac{n+k+2}{2}}$ on its left side, and the g_2 term has $x_{\frac{n+k+2}{2},n}$ on its right side. For each i in the support of α_k, going in the direction $\frac{n+1}{2} \to 1$, start with $\frac{n+1}{2}$. Then, the rest of g_1 (resp. g_2) is the $\frac{k+1}{2}$ integers including i and immediately to the left (resp. right) of i in α_k. Then, $u=0 \leftrightarrow i=\frac{n+1}{2}$, $u=1 \leftrightarrow i=1$, etc. Also, for u from 0 to $\frac{k-1}{2}$ (resp. $\frac{k+1}{2}$ to $k-1$), $m_{k,u}$ is u (resp. $k-1-u$). This concludes step (3.5a).

Here is the analog for $0 \le u \le \frac{k-1}{2}$ for u (or i) of (3.6) for a general (n,k):

$$(3.7) \qquad (x_{n,\frac{n+k+2}{2}}\, x''_{\frac{n+k-2u}{2},\frac{n+3}{2}}\, x'_{\frac{n+1}{2},u}), (x'_{u,\frac{k-1}{2}}\, x''_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}\, x_{\frac{n+k+2}{2},n}).$$

Conjugating ${}_{k,u}\boldsymbol{g}$ by $x_{\frac{n+1}{2},1}$ (leaving its 4th entry unchanged) shows ${}_{k,u}\boldsymbol{g}'$, with entries expressed by (3.2), is inner equivalent to ${}_{k,u}\boldsymbol{g}$; (3.4) similarly.

Now turn to the proposition's last paragraph. The first entries of both $\boldsymbol{n}\ell 2$ and ${}_{u,k}\boldsymbol{g}'$ (as in (3.2)) each have as support the segment $|x_{1,1+u}|$, and no other integers of $\{1,\dots,\frac{n+1}{2}\}$. This concludes showing that ${}_{k,u}\boldsymbol{g}'$ satisfies (3.5a) and (3.5b). The conclusion is an interpretation of (3.5c) and (3.5d). □

Remark 3.4. Prop. 3.3 starts with $k=3$. Lem. **??** says the conclusion about $\boldsymbol{n}n{-}22$ didn't need the 2nd term in $\boldsymbol{g}^\dagger = \beta'_{3,1}({}_{3,1}\boldsymbol{g}')(\beta'_{3,1})^{-1}$. Still, a pattern emerges. Conjugating by $\langle(\boldsymbol{n}n{-}22)\mathbf{mp}\rangle \stackrel{\text{def}}{=} \langle(x_{2,\frac{n+1}{2}}\, x_{n,\frac{n+5}{2}})\rangle$ on the 2nd term $g_2^\dagger = (2\,\frac{n+3}{2}\, x_{n,\frac{n+5}{2}})$ of $\boldsymbol{g}^\dagger$ contains $x_{2,\frac{n+3}{2}}$, the second entry of $\boldsymbol{n}n{-}22$.

3.1.4. *sh-incidence for absolute Liu-Osserman spaces.* The 2nd paragraph of Prop. 3.3 lets us fill in the absolute sh-incidence table for all $n \equiv 1 \mod 4$.

(3.8a) For odd k, $1 \le k \le u$ and $1 \le u < \frac{k-1}{2}$, $({}_{\rm c}O_k, {}_{\rm c}O_{n-2u}) = 2$.

(3.8b) $({}_{\rm c}O_k, {}_{\rm c}O_{n-(k-1)}) = 1$ and, modulo symmetry, all other entries are 0.

We list cusps in descending width along the rows and columns, as in Table 4. The case $n=13$ (Table 7) shows how we build the *abs-inn* form of the inner sh-incidence matrix along a(nti)-(sub)d(iagonal)s. A less intricate version is in the absolute sh-incidence matrix: the 1-1 a-d is $\frac{n+1}{2}$ 1's, the 3-3 a-d is $\frac{n-1}{2}$ 2's, etc.

Table 1. **sh**-incidence Matrix: $r=4$ and $\mathrm{Ni}_{3^4}^{\mathrm{abs,rd}}$

Cusp orbit	${}_{\rm c}O_5$	${}_{\rm c}O_3$	${}_{\rm c}O_1$
${}_{\rm c}O_5$	2	2	1
${}_{\rm c}O_3$	2	1	0
${}_{\rm c}O_1$	1	0	0

Prop. **??** says $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ has genus $\mathbf{g}_{3^4}^{\mathrm{in,rd}} = 0$. So, its degree 9 image over $\mathbb{P}^1_j$, $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{abs,rd}}$, does too. Then, γ'_0 has at most 3 orbits of length

3, and γ_1' has at most 4 orbits of length 2. Apply RH: the maxima are necessary so that

$$2(9+\mathbf{g}_{3^4}^{\text{abs,rd}}-1)=\text{ind}(\gamma_0')+\text{ind}(\gamma_1')+\text{ind}(\gamma_\infty') \text{ gives } \mathbf{g}_{3^4}^{\text{abs,rd}}=0$$

since $\text{ind}(\gamma_\infty')=(1-1)+(3-1)+(5-1)=6$. This shows $\text{ind}(\gamma_0')=6$, and $\text{ind}(\gamma_1')=4$. So, γ_1' has one fixed point, γ_0' none.

In general, then, the sh-incidence matrix for $\text{Ni}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$ has 1's along the anti-diagonal, 2's above that, and 0's below that.

Remark 3.5. Prop. **??** shows that for $n=5$, Lem. 3.14 accounts for all the ramification from $\bar{\mathcal{H}}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$ to $\bar{\mathcal{H}}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}}$.

Remark 3.6. To save space, Prop. 3.3 left a • for the 3rd term of ${}_{k,u}\boldsymbol{g}$ and ${}_{k,u}\boldsymbol{g}'$. We fill them for Prop. 3.17: $(x_{\frac{k-1}{2},\frac{n-1}{2}}\,x_{\frac{n+3}{2},\frac{n+k}{2}})$ and $x_{\frac{k+1}{2},\frac{n+k}{2}}$ in (3.2).

3.2. Completing inner sh-incidence entries.

The absolute sh-incidence entries are in (3.8). This section produces the inner $_{\mathrm{r}}$cusp sh-incidence matrix for which we have only to find $\beta_{k,u}$ (resp. $\beta'_{k,u}$) from (3.5c) (resp. (3.5d)).

§3.2.1 does this when $u\le\frac{k-1}{2}$. §3.2.2 completes the rest and §3.2.3 combines them for the inner $_{\mathrm{r}}$cusp sh-incidence display using the §3.1.4 absolute display.

3.2.1. *sh-incidence parities — 1st half.* Start with $0<u<\frac{k-1}{2}$: Table 2 handled $u=0$ and $u=\frac{k-1}{2}$ appears in Cor. 3.8. This gives the 1st (resp. 2nd) entry of (3.2) as $g_1'\overset{\text{def}}{=}{}_{k,u}g_1'=(x_{n,\frac{n+k+2}{2}}\,x_{\frac{n+k-2u}{2},\frac{n+3}{2}}\,x_{1,1+u})$ (resp. g_2').

As in (3.5c), for such (k,u) (with $\ell=n-2u$), find $\beta_{k,u}\in S_{\frac{n+3}{2},n}$ conjugating g_1' to $(x_{\frac{n+2u+3}{2},n}\,x_{1,1+u})$, the 1st entry g_1'' of $\boldsymbol{n}\ell 2$.

The element $\beta_{k,u}=\beta_{k,u,1}\beta_{k,u,2}^u$ that works comes from these two (Lem. 3.7):

$$\beta_{k,u,2}=x_{\frac{n+3}{2},\frac{n+k}{2}} \text{ (shifts } |x_{\frac{n+k-2u}{2},\frac{n+3}{2}}| \text{ to } |x_{\frac{n+k-2(u-1)}{2},\frac{n+5}{2}}|\text{); and} \tag{3.9}$$
$$\beta_{k,u,1} \text{ inverting } |x_{\frac{n+2u+3}{2},n}| \text{ (Lem. ??).}$$

Finally, consider $u=\frac{k-1}{2}$. In the previous notation, we drop the x'' segment in $g_1'=(x_{n,\frac{n+k+2}{2}}|x_{1,\frac{k+1}{2}})$, while $g_2'=(\frac{k+1}{2}\,x_{\frac{n+k}{2},\frac{n+3}{2}}|x_{\frac{n+k+2}{2},n})$ retains its previous form. Then, the resulting $\beta_{k,\frac{k-1}{2}}$ is just $\beta_{k,\frac{k-1}{2},1}$: invert $|x_{\frac{n+k+2}{2},n}|$.

Take $a_2=(-1)^{u\cdot(\frac{k-3}{2})}$. With $u=\frac{k-1}{2}$, a_2 is $(-1)^{\frac{k-1}{2}\cdot(\frac{k-3}{2})}=1$ shows the general computation for $\beta_{k,u}$ even works for $u=\frac{k-1}{2}$.

Lemma 3.7. *If $\beta=(i_1\,i_2\,\dots\,i_k)$ (parity $(-1)^{k-1}$) and $U=|i_1\,i_2\,\dots\,i_j|$, $1\le j<k$, is in pure-cycle α, then, $\beta\alpha\beta^{-1}$ substitutes $|i_2\,i_3\dots i_{j+1}|$ (right β-shift) for U. So, $\beta_{k,u,2}^u$ has parity a_2. Also, $\beta_{k,u,1}$ has parity $(-1)^{\frac{n-2u-3}{4}}=(-1)^{\frac{1-u}{2}}$ (resp. $(-1)^{\frac{n-2u-1}{4}}=(-1)^{1-\frac{u}{2}}$) if u is odd (resp. even).*

Conclude: $\beta_{k,u}$ has parity $(-1)^{\frac{1-u}{2}+\frac{k-3}{2}}$ (resp. $(-1)^{1-\frac{u}{2}}$) if u is odd (resp. even).

With ${}_{k,u}\boldsymbol{g}^\dagger \stackrel{\text{def}}{=} \beta_{k,u}({}_{k,u}\boldsymbol{g}')\beta_{k,u}^{-1}$ (even for $u=\frac{k-1}{2}$), consider $g_2^\dagger$

$$\begin{aligned} &= \beta_{k,u}(x_{1+u,\frac{k+1}{2}}\, x_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}\, x_{\frac{n+k+2}{2},n})\beta_{k,u}^{-1} \\ (3.10)\qquad &= \beta_{k,u,1}(x_{1+u,\frac{k+1}{2}}\, x_{\frac{n+2u+1}{2},\frac{n+3}{2}}\, x_{\frac{n+k+2}{2},n})\beta_{k,u,1}^{-1} \\ &= (x_{1+u,\frac{k+1}{2}}\, x_{\frac{n+2u+1}{2},\frac{n+3}{2}}\, x_{n-\frac{k-1-2u}{2},\frac{n+2u+3}{2}}). \end{aligned}$$

Recall $\alpha_{n-2u,2}=(x_{1+u,\frac{n+1}{2}}\, x_{n,\frac{n+2u+3}{2}})$ (Lem. **??**), and $g_2''=(x_{1+u,\frac{n+2u+1}{2}})$, the 2nd entry of $\boldsymbol{n}n{-}2u2$. By inspection, $\beta_{k,u}' \stackrel{\text{def}}{=} \beta'$ that inverts $|x_{\frac{n+2u+1}{2},\frac{n+3}{2}}|$ conjugates the orbit of $\langle\alpha_{n-2u,2}\rangle$ to contain g_2''. Combine this with Lem. 3.7 for $1\le u\le \frac{k-1}{2}$.

Corollary 3.8. *For u odd (resp. even), β' has parity $(-1)^{\frac{u-1}{2}}$ (resp. $(-1)^{\frac{u}{2}}$). So, for u odd, (3.2) is in ${}_{\rm c}O'_{n-2u,2}$ if and only if $k\equiv 3 \mod 4$. Otherwise, it is in ${}_{\rm c}O'_{n-2u,1}$. For u even, (3.2) is in ${}_{\rm c}O'_{n-2u,2}$.*

Proof. Apply Lem. **??** for the 1st sentence parities. If u is odd (resp. even), then Lem. **??** says ${}_{\rm c}O'_{\boldsymbol{n}n-2u2}={}_{\rm c}O'_{n-2,2}$ (resp. ${}_{\rm c}O'_{n-2,1}$). Then, from Lem. 3.7, the parity of $\beta_{k,u}\beta'$ is $+1$ iff either u is even or u is odd and $k\equiv 3 \mod 4$. □

3.2.2. *sh-incidence parities — 2nd half.* Now consider $\frac{k+1}{2}\le u\le k{-}1$. With $u'=k{-}1{-}u$, compare 1st and 2nd entries of $\boldsymbol{n}n{-}2u'2$ with corresponding entries g_1', g_2' of (3.4). 1st conjugate (3.4) by $\beta_{k,u'}\in S_{\frac{n+3}{2},n}\times\langle x_{\frac{n+1}{2},1}\rangle$ to get ${}_{k,u'}\boldsymbol{g}^\dagger$ whose 1st and 4th entries match those of $\boldsymbol{n}n{-}2u'2$. A $\beta_{k,u'}$ inverting both segments $|x_{\frac{n+k}{2},\frac{n+3+2u'}{2}}|$ and $|x_{n,\frac{n+k+2}{2}}|$ does it. An induction computes its parity.

Lemma 3.9. *With t odd, the conjugation $\alpha_{t,j}\in S_t$ that inverts both segments $|x_{1,j}|$ and $|x_{j+1,t}|$, $1\le j\le t{-}1$ has — independent of j — parity $(-1)^{\frac{t-1}{2}}$. With t even, the analogous result is that $\alpha_{t,j}$ has parity $(-1)^{\frac{t-2j}{2}}$.*

Apply $t=\frac{n-1-2u'}{2}$ to $\beta_{k,u'}$: For u' odd (resp. even), $\beta_{k,u'}$ has parity

$(-1)^{\frac{n-3-2u'}{4}}=(-1)^{\frac{1-u'}{2}}$ *(resp. $(-1)^{1-\frac{u'}{2}}$ if $k\equiv 1 \mod 4, (-1)^{\frac{u'}{2}}$ if $k\equiv 3 \mod 4$).*

Proof. The 1st paragraph is easy. For the 2nd, we do the toughest case, when u', so t, is even. The lengths of the two segments are $\frac{n-k}{2}$ and $\frac{k-1-2u'}{2}$. These are even (resp. odd) if $k\equiv 1 \mod 4$ (resp. $k\equiv 3 \mod 4$). According to the 1st paragraph, the parity in each of these cases will be the same as for the case when the seqments have length 2 and $\frac{n-5-2u'}{2}$ (resp. 1 and $\frac{n-3-2u'}{2}$). □

We insert dividers in the relevant permutations to see the effect of a middle product translation. Denote the 2nd entry (as previously) of $\boldsymbol{n}n{-}2u'2$ by

$$g_2''=x_{1+u',\frac{n+1}{2}+u'}=(x_{1+u',\frac{n+1}{2}}|x_{\frac{n+3}{2},\frac{n+1}{2}+u'}).$$

Denote the parity for (k,u') in Lem. 3.9 by $b_{k,u',1}$. Note: $\alpha_{n-2u',2}$ translates the segment $|x_{\frac{n+1}{2}-(\frac{k-3-2u'}{2}),\frac{n+1}{2}}\, x_{n,\frac{n+k}{2}}|$ of $g_2^\dagger$ in (3.11) into $|x_{1+u',\frac{n+1}{2}}|$ of g_2''.

LEMMA 3.10. *Assume $u'>0$. So, for β' inverting $x_{\frac{n+3}{2},\frac{n+1}{2}+u'}$ and some j:*

$$\text{(3.11)}\qquad \begin{aligned} \textit{with } g_2^\dagger = &\ (x_{\frac{n+1}{2}-(\frac{k-3-2u'}{2}),\frac{n+1}{2}}\, x_{n,\frac{n+k}{2}}|x_{\frac{n+1}{2}+u',\frac{n+3}{2}}),\\ (\beta')g_2^\dagger(\beta')^{-1} = &\ \alpha^j_{n-2u',2} g_2'' \alpha^{-j}_{n-2u',2} \end{aligned}$$

Then, for u' odd (resp. even), β' has parity $b_{k,u',2}=(-1)^{\frac{u'-1}{2}}$ (resp. $(-1)^{\frac{u'}{2}}$). Conclude: (3.4) *is in* ${}_{\rm c}O'_{\boldsymbol{n}n-2u'2,1}$ *if and only if* $b_{k,u',1}b_{k,u',2}=1$. *For u' odd,* (3.4) *is in* ${}_{\rm c}O'_{n-2u',2}$. *For u' even,* (3.4) *is in* ${}_{\rm c}O'_{n-2u',2}$ *if and only if* $k\equiv 1 \mod 4$.

Now combine Cor. 3.8 and Lem. 3.10, with $0<u=u'<\frac{k-1}{2}$, where u (resp. u') corresponds to ${}_{k,u}\boldsymbol{g}'\in({}_{\rm c}O'_{k,1})\mathbf{sh}$ (resp. ${}_{k,u}\boldsymbol{g}^*\in({}_{\rm c}O'_{k,1})\mathbf{sh}$) in (3.2) (resp. (3.4)).

COROLLARY 3.11. *For u even: ${}_{k,u}\boldsymbol{g}'\in{}_{\rm c}O'_{n-2u,2}$; and ${}_{k,u}\boldsymbol{g}^*\in{}_{\rm c}O'_{n-2u,2}$ (resp. ${}_{\rm c}O'_{n-2u,1}$) if and only if $k\equiv 1 \mod 4$ (resp. $k\equiv 3 \mod 4$).*

For u odd: ${}_{k,u}\boldsymbol{g}^*\in \mathrm{Cusp}_{n-2u,2}$; and ${}_{k,u}\boldsymbol{g}'\in{}_{\rm c}O'_{n-2u,2}$ (resp. ${}_{\rm c}O'_{n-2u,1}$) if and only if $k\equiv 3 \mod 4$ (resp. $k\equiv 1 \mod 4$). Always: ${}_{k,\frac{k-1}{2}}\boldsymbol{g}'\in{}_{\rm c}O'_{n-k+1,2}$.

REMARK 3.12 ($u'=0$ in Lem. 3.10). The case $u'=0$ for inverting $x_{\frac{x+3}{2},1+u'}$ differs from the others: the range is descending, not ascending. The final parity value $(-1)^{u'}$ is also wrong, but Table 2 already handled that case.

3.2.3. **sh**-*incidence display.* Use Defs. **??** and 3.1 (as in Table 2), for $n\equiv 5 \mod 8$ to label columns/rows of the general ${}_{r}$cusp sh-incidence matrix as follows:

$${}_{\rm c}O'_{n;1},\ {}_{\rm c}O'_{n;2},\ {}_{\rm c}O'_{n-2;1},\ {}_{\rm c}O'_{n-2;2},\dots,{}_{\rm c}O'_{3;1},\ {}_{\rm c}O'_{3;2},\ {}_{\rm c}O_{1,2}.$$

Use the subscripts of the symbols to label its entries as $((\ell;j),(\ell';j'))$. By shortening this to $(\bar\ell,\bar\ell')$, we can throw in the (1,2) subscript as another $\bar\ell$.

§**??** lists cusps and their widths. Conclude: $2\cdot(\frac{n+1}{2})^2$ is the cover degree of

$$\bar\psi_{(\frac{n+1}{2})^4}:\bar{\mathcal{H}}^{\rm in,rd}_{(\frac{n+1}{2})^4}\stackrel{\rm def}{=}\mathcal{H}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}\to\mathbb{P}^1_j.$$

PRINCIPLE 3.13. *For $n\equiv 5 \mod 8$, the $(\bar\ell,\bar\ell')$ are all 0, 1 or 2. Also:*

(3.12a) *Symmetry:* $(\bar\ell,\bar\ell')=(\bar\ell',\bar\ell)$.

(3.12b) *Width sum: Entries in the row for* ${}_{\rm c}O'_{\ell;j}$ *(resp.* ${}_{\rm c}O_{1,2}$*) sum to* ℓ *(resp. 2).*

(3.12c) *With* $0 \le u < \frac{k-1}{2}$ *or (resp.* $u = \frac{k-1}{2}$*),* $((\ell;1),(\ell';1)) + ((\ell;1),(\ell';2)) = 2$ *(resp. 1) if* $\ell' = n-2u$*; and 0 otherwise.*

To describe the ${}_{\rm r}$cusp sh-incidence matrix of $\mathrm{Ni}^{\rm in,rd}_{(\frac{n+1}{2})^4}$ use §3.1.4 for $\mathrm{Ni}^{\rm abs,rd}_{(\frac{n+1}{2})^4}$. You get the former from the latter using symmetry and these two rules.

Rule 1: At an entry (ℓ,ℓ') (odd integers) with $3 \le \ell \le \ell' \le n$: Replace 2 (resp. 1; resp. 0) by one of these matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; (\text{ resp. } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \text{ resp. } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}).$$

In each case, you determine the replacing matrix from just one of its entries.

Rule 2: At the entry $(1,n)$ (resp. $(1,\ell)$, ℓ odd, between 3 and $n-2$; resp. $(1,1)$): Replace 1 by $(1\,1)$ (resp. $(0\,0)$; resp. 0).

Use symbols such as $\frac{2|0}{0|2}$, $\frac{0|2}{2|0}$, $\frac{1|1}{1|1}$, etc. to substitute for the 2×2 matrices, and $1|1$, $0|0$ (resp. $\frac{1}{1}$, $\frac{0}{0}$) for the 2×1 (resp. 1×2) matrices. Refer to the matrix of such substitutions as *abs-inn* form.

We already know the last three sh-incidence rows (and from Table 2 the first two), including all the data of Rule 2. The first two rows of Table 5 are from the opening paragraph of Prop. 3.3, with the last from the first line of Prop. 3.2. Table 6 renders this with even more detail in the abbreviated abs-inn form:

TABLE 2. Rows for ${}_{\rm c}O'_{3,1}$, ${}_{\rm c}O'_{3,2}$ and ${}_{\rm c}O_{1,2}$

Cusp orbit	${}_{\rm c}O'_{n;1}$	${}_{\rm c}O'_{n;2}$	${}_{\rm c}O'_{n-2;1}$	${}_{\rm c}O'_{n-2;2}$	...	${}_{\rm c}O'_{3;1}$	${}_{\rm c}O'_{3;2}$	${}_{\rm c}O_{1,2}$
${}_{\rm c}O'_{3;1}$	1	1	0	1	...	0	0	0
${}_{\rm c}O'_{3;2}$	1	1	1	0	...	0	0	0
${}_{\rm c}O_{1,2}$	1	1	0	0	...	0	0	0

TABLE 3. Abs-inn form of the rows ${}_{\rm c}O_3$ and ${}_{\rm c}O_1$

Cusp orbit	${}_{\rm c}O_n$	${}_{\rm c}O_{n-2}$	${}_{\rm c}O_{n-4}$	...	${}_{\rm c}O_3$	${}_{\rm c}O_1$
${}_{\rm c}O_3$	$\frac{1\|1}{1\|1}$	$\frac{0\|1}{1\|0}$	$\frac{0\|0}{0\|0}$	...	$\frac{0\|0}{0\|0}$	$\frac{0}{0}$
${}_{\rm c}O_1$	$1\|1$	$0\|0$	$0\|0$	...	$0\|0$	0

Use the anti-subdiagonal notation of §3.1.4. For $n = 13$, Table 7 gives the *abs-inn* form of the matrix. For example, the 1-1 a-d (read from lower left to upper right) consists of $1|1$, then five $\frac{0|1}{1|0}$'s, followed by $\frac{1}{1}$. Below the 1-1 a-d are $\frac{0|0}{0|0}$'s except at the right edge $(\frac{0}{0})$, bottom edge $(0|0)$, or lower right corner (0). Table 2 says column ${}_{\rm c}O_n$ fills down as $\frac{0|2}{2|0} \to \frac{n-3}{2}\,\frac{1|1}{1|1}$'s $\to 1|1$.

TABLE 4. Abs-inn ${}_{\eta}$cusp form for $n = 13$

Cusp orbit	${}_{\rm c}O_{13}$	${}_{\rm c}O_{11}$	${}_{\rm c}O_{9}$	${}_{\rm c}O_{7}$	${}_{\rm c}O_{5}$	${}_{\rm c}O_{3}$	${}_{\rm c}O_{1}$
${}_{\rm c}O_{13}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{1}{1}$
${}_{\rm c}O_{11}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0}{0}$
${}_{\rm c}O_{9}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2^0}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\rm c}O_{7}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1^1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\rm c}O_{5}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\rm c}O_{3}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\rm c}O_{1}$	$1\mid 1$	$0\mid 0$	$0\mid 0$	$0\mid 0$	$0\mid 0$	$0\mid 0$	0

TABLE 5. Abs-inn ${}_{\eta}$cusp form for $n = 13$

Cusp orbit	${}_{\rm c}O_{13}$	${}_{\rm c}O_{11}$	${}_{\rm c}O_{9}$	${}_{\rm c}O_{7}$	${}_{\rm c}O_{5}$	${}_{\rm c}O_{3}$	${}_{\rm c}O_{1}$
${}_{\rm c}O_{9}^{-}$	$2\mid 2$	$2\mid 2$	4	$2\mid 2$	$1\mid 1$	$0\mid 0$	0
${}_{\rm c}O_{7}^{-}$	$2\mid 2$	$2\mid 2$	$2\mid 2$	2	$0\mid 0$	$0\mid 0$	0

Superscripts 0 and 1 along the diagonal indicate fixed points for γ_0' and γ_1' a la Prop. 3.14 and Prop. 3.17.

The distinction for the sh-incidence matrix for $n = 13$ requires replacing the rows and columns respectively for ${}_{\rm c}O_7$ and ${}_{\rm c}O_9$. The process, say in the columns is to add the contributions for ${}_{\rm c}O'_{k,1}$ and ${}_{\rm c}O'_{k,2}$ across the rows, but since you also do this with rows replacing columns, that adds extra at the diagonal terms. Table 8 gives the row result, using the notation ${}_{\rm c}O_k^-$ for the collapsed form:

3.3. Elliptic fixed points. Thm. **??** outlines the absolute and inner sh-incidence matrices for the Nielsen classes $\text{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$. While related through Lem. **??**, we need more to complete finding the fixed points of γ_0' and γ_1' on Nielsen classes. We do that here to compute the absolute and reduced space genuses.

3.3.1. *An absolute γ_1' fixed point.* Lem. 3.14 locates a γ_1' fixed point that ramifies in the cover from the absolute space to the inner space.

LEMMA 3.14. *For $n \equiv 1 \mod 4$, the 1-1* a-d *diagonal term in the absolute sh-incidence corresponds to $({}_{c}O_{\frac{n+1}{2}}, {}_{c}O_{\frac{n+1}{2}})$. À la Lem.* **??**, *this arises from a fixed point $\boldsymbol{p}_1' \in \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd}$ of γ_1' (lying over $j = 1$). The only n-n* a-d *term (a 2, on the diagonal) does not correspond to points fixed by γ_v', $v = 0$ or 1.*

For $n \equiv 5 \mod 8$, the inner sh-incidence diagonal terms above the 1 in the absolute sh-incidence are both 0. This means γ_1' fixes no point of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$ over $\boldsymbol{p}_1'$. So, this and the width 1 cusp give two known points of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd}$ that ramify to $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$.

Proof. For any $n \equiv 1 \mod 4$, Prop. 3.3 says a unique absolute (reduced) class (represented by ${}_{\frac{n+1}{2},\frac{n-1}{4}}\boldsymbol{g}'$) of the cusp ${}_{\rm c}O_{\frac{n+1}{2}}$ shifts into ${}_{\rm c}O_{\frac{n+1}{2}}$. Since the sh-incidence entry is 1, Lem. **??** says this corresponds to a fixed point of γ_1'.

Now consider the n-n a-d. Table 1 explains the 2: $\boldsymbol{n}nj$, $j=1,2$ represent the two reduced classes in ${}_{\rm c}O_n \cap ({}_{\rm c}O_n)\mathbf{sh}$. We show neither γ_0 nor γ_1 fixes either. In $\bar M_4$, the shift represents γ_1, and that sends $\boldsymbol{n}n1^{\rm abs,rd}$ to $\mathbf{HM}^{\rm abs,rd}_{1,\frac{n-1}{2}}$.

Lem. **??** says this either fixes $\boldsymbol{n}n1^{\rm abs,rd}$ or it sends it to $\boldsymbol{n}n2^{\rm abs,rd}$. It is the latter: Conjugate $(\boldsymbol{n}n1)\mathbf{sh}$ by $(n\,1)(n\,2)\cdots(\frac{n+3}{2}\ \frac{n-1}{2})$ followed by $(n\ \frac{n+3}{2})(n{-}1\ \frac{n+3}{2})\cdots$. The resulting 3rd and 4th terms are $x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}$ (same as $\boldsymbol{n}n2$). Check: The resulting 1st term is also the same as that of $\boldsymbol{n}n2$.

Characterize that γ_0 fixes $\boldsymbol{n}n1$: $((\boldsymbol{n}n1)q_2^{-1})\mathbf{sh} \in \boldsymbol{n}n1^{\rm abs,rd}$. From Prin. **??**, q_2^{-1} has the effect of conjugating the 2nd and 3rd terms of $\boldsymbol{n}n1$ by $(\boldsymbol{n}n1)\mathbf{mp}^{\frac{n-1}{2}}$. The combined effect is to send $\boldsymbol{n}n1$ back to the shift of the $\mathbf{HM}$ rep. it came from in Table 1. So, it has the wrong middle product to be fixed by γ_0.

For $n \equiv 5 \mod 8$, the symbol (as in §3.2.3) for what lies above the diagonal 1-1 a-d in the inner sh-incidence matrix is $\substack{0\,1\\1\,0}$, the last sentence of Cor. 3.11. At the diagonal position, it means there are no fixed points of γ_1' (or γ_0') over $\boldsymbol{p}_1'$. ☐

3.3.2. γ_0' *fixed points.* For $n \equiv 5 \mod 8$, the degree 2 cover (Prop. **??**)

$$\bar\Psi_n^{\rm in,abs}: \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd} \to \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd}$$

maps γ_0' fixed points on the upper surface 2-1 (on)to γ_0' fixed points to the lower. Consider the reduced class of ${}_{k,u}\boldsymbol{g} = (g_1,g_2,g_3,g_4)$ (from Prop. 3.3), and denote the common support cardinality of the pair $(g_1g_2g_1^{-1}, g_4)$ by $\nu_{k,u}$. Recall (Lem. **??**): The degree of $\bar\Psi_n^{\rm abs}: \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd} \to \mathbb{P}^1_j$ is $(\frac{n+1}{2})^2$.

Proposition 3.15. *Inner sh-incidence diagonal positions correspond to (k,u) with $u = \frac{n-k}{2}$ or $\frac{3k-n-2}{2}$ (subject to the latter being ≥ 0 and $0 \le u \le k$). For such a (k,u), if the reduced absolute class of ${}_{k,u}\boldsymbol{g} = (g_1,g_2,g_3,g_4)$ is a fixed point of γ_0', $\nu_{k,u}$ equals the common support cardinality of the pair (g_1,g_4). Then:*

(3.13a) *For $0 \le u \le \frac{k-1}{2}$: $\nu_{k,u} = \frac{k-1}{2} - u$.*
(3.13b) *For $\frac{k+1}{2} \le u \le k{-}1$ and $u' = k{-}1{-}u$: $\nu_{k,u} = \frac{k-1-2u'}{2} + 1$.*

If $3|n$ (resp. $n \equiv 1 \mod 3$), then ${}_{k,\frac{n-k}{2}}\boldsymbol{g}$ (resp. ${}_{k,\frac{3k-n-2}{2}}\boldsymbol{g}$) with $3k = 2n+3$ (resp. $2n+1 = 3k$) represents the only possible absolute diagonal fixed point of γ_0'. If $n \equiv 0$ or $1 \mod 3$, then $\deg(\bar\Psi_n^{\rm abs}) \equiv 1 \mod 3$, and this is the one absolute fixed point of γ_0'. There are none if $n \equiv -1 \mod 3$.

Proof. From Prop. 3.3, among values of $u \le \frac{k-1}{2}$ one diagonal position is from (k, u_0) with $n-2u_0 = k$, or $u_0 = \frac{n-k}{2}$. A 2nd from $u_1 = u_0 + 2(\frac{k-1}{2} - u_0)$, giving the other u value in the lemma statement.

For $\boldsymbol{g} \in \text{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$, Princ. 2.14 says if there are t integers of common support in g_i and g_j, $1 \le i \ne j \le 4$, then $\text{ord}(g_i g_j) = 2(\frac{n+1}{2}) - 2t + 1$. That is, $\text{ord}(g_i g_j)$ determines the common support of g_i, g_j.

Now suppose the reduced (absolute or inner) class of ${}_{k,u}\boldsymbol{g}$ is fixed by q_1q_2 (that is by γ_0). Then, $({}_{k,u}\boldsymbol{g})\mathbf{mp} = (({}_{k,u}\boldsymbol{g})q_1q_2)\mathbf{mp}$. Applying q_2 doesn't change the middle product. So, $\text{ord}(({}_{k,u}\boldsymbol{g})q_1)\mathbf{mp}) = \text{ord}(g_1g_3) = \text{ord}(g_4g_1g_2g_1^{-1})$, computed again by a cardinality of overlap. Since the common support of (g_1, g_4) is the same as the common support of (g_2, g_3), that completes the statement before (3.13).

We have fixed g_4 to be $x_{\frac{n+1}{2},1}$. As in Prop. 3.3, denote $k-1-u$ by u'. With $\alpha_{k,1} = (x_{1,\frac{k-1}{2}}\, x_{\frac{n+k}{2},\frac{n+1}{2}})$ (Lem. **??**), for a given (k, u) we figure the cardinality, $v_{k,u}$, of the support in $\{1, \dots, \frac{n+1}{2}\}$ of $g_1g_2g_1^{-1} \overset{\text{def}}{=}$

(3.14)
$$\alpha^u_{k,1}(x_{n,\frac{n+1}{2}}(x'_{\frac{n+1}{2},\frac{k-1}{2}}\, x_{\frac{n+k+2}{2},n})x_{\frac{n+1}{2},n})\alpha^{-u}_{k,1} = \alpha^u_{k,1}(x_{\frac{n+k}{2},n}\, x_{1,\frac{k-1}{2}})\alpha^{-u}_{k,1}$$
$$= \begin{cases} (x_{\frac{n+k}{2},\frac{n+k-2u}{2}}\, x_{\frac{n+k+2}{2},n}\, x_{1+u,\frac{k-1}{2}}) \text{ for } 0 \le u < \frac{k-1}{2} \\ (x_{\frac{n+2u'+1}{2},\frac{n+3}{2}}\, x'_{\frac{n+1}{2},\frac{k-1-2u'}{2}}\, x_{\frac{n+k+2}{2},n}) \text{ for } \frac{k+1}{2} \le u \le k-1. \end{cases}$$

To see the 2nd case, substitute $\alpha^{u'+1}_{k,1}$ for $\alpha^{-u}_{k,1}$. Clearly, $v_{k,u}$ has the values in (3.13).

The conclusion on the possible representatives of fixed points follows by equating k and $2(\frac{n+1}{2}) - 2\nu_{k,u} + 1$ for values of u at the diagonal positions. For (3.13a), $\nu_{k,\frac{n-k}{2}} = \frac{k-1}{2} - \frac{n-k}{2}$. For (3.13b), with $u' = (k-1) - \frac{3k-n-2}{2}$,

$$\nu_{k,\frac{3k-n-2}{2}} = 1 + \tfrac{k-1-2u'}{2} = \tfrac{2k-n+1}{2}.$$

Then, $k = 2(\frac{n+1}{2}) - (2k-n+1) + 1$ completes the condition in the statement.

$\square$

Example 3.16 (γ_0 fixed point, $n = 13$). Use Cor. 3.11 notation. Prop. 3.15 says γ_0 fixes $({}_{9,2}\boldsymbol{g}^*)^{\text{in}}$ when $n = 13$. Apply q_1q_2, and conjugate by $x^2_{7,1}$ to get

$$((x_{9,8}\, x_{2,4}\, x_{12,13}), (x_{10,11}\, x'_{7,1}\, x_{8,9}\, 13), (x_{13,12}\, x_{2,4}\, x_{11,10}), x_{7,1}).$$

Conjugate in order by $\beta_3 = x^2_{11,8}$, $\beta_2 = x_{12,13}$ and $\beta_1 = x^2_{13,10}$ to get

$$((x_{13,12}\, x_{2,4}\, x_{11,10}), (x_{8,9}\, x'_{7,1}\, x_{12,13}\, 10), \bullet, x_{7,1})$$

Then, conjugate by $\beta' = (8\,9)$, with $\beta'\beta_1\beta_2\beta_3$ of parity $+1$ to get back to ${}_{9,2}\boldsymbol{g}^*$. This shows γ_0 fixes $({}_{9,2}\boldsymbol{g}^*)^{\text{in}}$. It also fixes $(\beta'{}_{9,2}\boldsymbol{g}^*(\beta')^{-1})^{\text{in}}$. This works on γ'_0 fixed points for all $n \equiv 5 \mod 8$.

3.3.3. *Genuses of the inner and absolute spaces.* Prop. 3.17 finishes the elliptic fixed point analysis, giving absolute/inner reduced space genuses for $n \equiv 5 \mod 8$. (Also for $n \equiv 1 \mod 8$ for the absolute case; as in Lem. **??**, inner components have the same genus.) Note: For t an integer, $[(3t+m)/3]$ is t, for $m = 0$ or 1.

PROPOSITION 3.17. *Let $v = [n/5]$. Then, γ_1' has only one absolute fixed point and no inner fixed points. Exactly $2v+2$ points of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ ramify in $\bar{\Psi}_n^{\mathrm{in,abs}}$: The width 1 cusp, the point over $j = 1$ indicated in Lem. 3.14, and the $2v$ cusps ${}_cO_{7+8m}$, ${}_cO_{9+8m}$, $m = 0, \dots, v-1$.*

Conclude the following formulas for the respective genuses, $\mathbf{g}^{\mathrm{abs,rd}}$ and $\mathbf{g}^{\mathrm{in,rd}}$, of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ and $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$.

(3.15)

$$\begin{aligned} 2((\tfrac{n+1}{2})^2+\mathbf{g}^{\mathrm{abs,rd}}-1) &= ((\tfrac{n+1}{2})^2-1)/2 + 2[(\tfrac{n+1}{2})^2/3] + (\tfrac{n-1}{2})(\tfrac{n+1}{2}) \\ 2(2(\tfrac{n+1}{2})^2+\mathbf{g}^{\mathrm{in,rd}}-1) &= (\tfrac{n+1}{2})^2 + 4[(\tfrac{n+1}{2})^2/3] + 2(\tfrac{n-1}{2})(\tfrac{n+1}{2}) + 1 + v. \end{aligned}$$

PROOF. To check for fixed points of γ_1' consider first the absolute case. For each odd k between 3 and $n-2$ (1 and n are already done), and $u = \frac{n-k}{2} < \frac{k-1}{2}$, we are asking if ${}_{k,u}\boldsymbol{g}^\dagger \stackrel{\mathrm{def}}{=} (({}_{k,u}\boldsymbol{g}')\mathbf{sh})q_1^{-1}q_3$ (as in (3.2)) is conjugate by some $\beta \in S_{\frac{n+3}{2},n} \times \langle x_{\frac{n+1}{2},1}\rangle$ to ${}_{k,u}\boldsymbol{g}'$. The point of applying $q_1^{-1}q_3$ is to assure – without changing the reduced class – that they both have $x_{\frac{n+1}{2},1}$ as their 4th entry.

With $g_1^\dagger = (x_{\frac{2k-n-1}{2},\frac{k-1}{2}}\, x_{\frac{n+3}{2},\frac{n+k}{2}})$, use Rem. (3.6) to fill in ${}_{k,\frac{n-k}{2}}\boldsymbol{g}^\dagger$:

$$(g_1^\dagger, (g_1^\dagger)^{-1}(x'_{\frac{n+1}{2},\frac{2k-n-1}{2}}\, x_{\frac{n+k}{2},k+1}\, x_{\frac{n+k+2}{2},n})g_1^\dagger, \bullet, x_{\frac{n+1}{2},1}).$$

The assumption on β means it conjugates $g_1^\dagger$ to g_1'. So, the power of $x_{\frac{n+1}{2},1}$ in it would translate the segment $|x_{\frac{2k-n-1}{2},\frac{k-1}{2}}|$ to $|x_{\frac{k+1}{2},\frac{n+1}{2}}|$. That is, you add $\frac{n-k+2}{2}$ to the subscripts. That means a conjugation in the second position of the form $g''' = (x_{\frac{k+1}{2},\frac{n+1}{2}} \dots)^{-1}(x_{\frac{n-k+2}{2},\frac{k+1}{2}} \dots)(x_{\frac{k+1}{2},\frac{n+1}{2}} \dots)$ would end up as $(x'_{\frac{n+1}{2},\frac{2k-n-1}{2}} \dots)$. In each case "…" means integers in $\{\frac{n+3}{2}, \dots, n\}$.

So, it seems g''' maps $\frac{n+1}{2} \mapsto 1$ for k in the allowed range. Yet, the constituents of g''' don't even have 1 in their supports. Conclude: no appropriate β gives even an absolute fixed point.

Now we compute the genuses. Lem. **??** gives $(\frac{n+1}{2})^2$ as the degree of

$$\bar{\Psi}_n^{\mathrm{abs}} : \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}} \to \mathbb{P}_j^1$$

and the lengths of the disjoint cycles of the absolute γ_0' as all the odd integers from 1 to n with multiplicity 1. So γ_∞' has index $\frac{n+1}{2}(\frac{n+1}{2}-1)$. Write $1 + 4t = n$. According to Prop. 3.17, $\mathrm{ind}(\gamma_0')$ is $2 \cdot ((\frac{n+1}{2})^2 - 1)/3$ (resp. $2 \cdot (\frac{n+1}{2})^2/3$) if $t \equiv 0$ or $-1 \mod 3$ (resp. $1 \mod 3$). Similarly, from the above $\mathrm{ind}(\gamma_1')$ is $((\frac{n+1}{2})^2 - 1)/2$ (indicating one fixed point).

The degree doubles in going to the inner case (Lem. **??**). Above we've computed the indices from the contributions of γ'_0 and γ'_1 in the inner case, leaving the contribution of γ'_∞ (in (**??**)). The cusps $\mathcal{O}_1, \mathcal{O}_{7+m8}, \mathcal{O}_{9+m8}$ ramify of index 2 in the cover. So, if we denote the width of one of these by k, then its index is $k-1$ and the index of the cusp above it is $2k-1 = 2(k-1)+1$. For all the other absolute cusps, there there are exactly two inner cusps of the same width above it. The calculations come directly from this. □

Example 3.18 (No converse to Lem. **??**). Assume $n \equiv 1 \mod 4$; as in (3.8) or Table 4. Then, sh-incidence matrices for $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ have a 1 and $\frac{n-1}{2}$ 2's as diagonal entries. Lem. 3.14 shows the diagonal term 1 in the 1-1 a(nti)-(sub)d(iagonal) corresponds to a fixed point of γ'_1. While the 2 in the n-n a-d corresponds to no fixed points of either γ'_0 or γ'_1.

In the inner case: Table 8, for the **sh**-incidence matrix labeled Ni_0^+, has a nonzero diagonal entry, though neither γ'_0 nor γ'_1 has a fixed point.

Remark 3.19. From [**BFr02**] The kth level of the (A5, C34 , p = 2) Modular Tower passes this test for b-fine moduli, k 1 (Prop. 7.9); even for fine moduli (Ex. 8.5 and Lem. 8.1). It is not even b-fine for level k = 0.

4. Braid orbits on $\mathrm{Ni}((\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{+3^2-3^2})^{\dagger,\mathrm{rd}}$

Remark 4.1 (Moduli definition field).

5. Reduced spaces $r \geq 5$

A future implementation of the algorithm should include tests for where fine moduli at the elliptic fixed points might fail for $r > 4$.

(5.1) lists two well-known, but difficult, literary concentration points that reflect on both the prestige and and yet, still unexplored territory around the properties of J_r. We follow these by subsections of comments, respectively §5.2 and §5.1.

(5.1a) [**MuFo82**, Chap. 4, §1] refers to $\mathrm{PGL}_2(\mathbb{C})$ acting on $(\mathbb{P}^1_z)^r/S_r$ as binary quantics – not quadrics, cubics, ..., but collectively, quantics – as the simplest of reductive group actions.

(5.1b) [**Ih91**, p. 100] refers to $\mathrm{PGL}_2(\mathbb{C})$ acting on $(\mathbb{P}^1_z)^r \setminus \Delta_r$, producing the moduli Λ_r of distinct *ordered* branch points.

5.1. Comments on (5.1a).

[**Fr12**, Prop. A.8, reduction Prop.] already quotes [**MuFo82**, Thm. 1.1]: a general proposition says reduction by a reductive group on an affine scheme gives a(n affine scheme) geometric quotient without reservation. That means that reduction of a cover of affine varieties $\mathcal{H}(G,\mathbf{C})^\dagger \to U_r$ to $\mathcal{H}(G,\mathbf{C})^{\dagger,\mathrm{rd}} \to J_r$ works as well as you might expect. The proof makes use, as does most of the book, of the *Reynolds operator* [**MuFo82**, Def. 1.5]. This extends complete reducibility – as an idempotent – to the level of schemes. There are further points worth noting.

The definition of categorical and geometry quotient [**MuFo82**, p. 3-4] is compatible with the idea of a moduli space. It, the categorical quotient Y, in this case of a scheme X (over a scheme S) for an algebraic group G acting on X, is a target — by which you induce from $X \to Y$ that pullback over $Y' \to Y$ gives Y' as a categorical quotient of $X' \times_Y Y'$ by the action of G. The book aims at applying these ideas to many moduli situations, even to the moduli of abelian varieties, and through Jacobians to the moduli of curves of genus g. It doesn't seem to apply (directly) to Hurwitz spaces.

Reduced Hurwitz spaces use $U_r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\text{def}}{=} J_r$ as a target where we recognize the discriminant locus, D_r, as having these properties:[17]

(5.2) $\mathrm{PGL}_2(\mathbb{C})$ has a closed action on the invariant set U_r; and U_r is an *affine set*.

The classical theory aimed at extending this result to a maximal locus in J_r, called the *stable* points in $\mathbb{P}^r$ for which the $\mathrm{PGL}_2(\mathbb{C})$ equivalence classes would still form a quasiprojective scheme. Especially to describe it explicitly. [**MuFo82**, Chap. 4, §1] regards – expressed in 3 compressed pages – the $\mathrm{PGL}_2(\mathbb{C})$ case as the simplest of all the reductive group cases, even among the cases $G = \mathrm{PGL}_n(\mathbb{C})$.

Yet, it isn't simple. It involves most aspects of the general case. This starts (Chap. 1, §3) by extending the G on X to the vector bundle attached to an invertible sheaf $\mathcal{L}$, and thereby to the sections $H^0(X,\mathcal{L}^n)$ for all n [**MuFo82**, p. 32]. These are the *G-linearized sheaves*, denoted $\mathrm{Pic}^G(X)$.

[**MuFo82**, Chap. 1, Prop. 1.7] implies $\mathcal{L}$ (assuming it has sections with no common zeros) gives a morphism $\Psi_{\mathcal{L}} : X \to \mathbb{P}^n$, on which the G action extends, with a G-linearization of the twist sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ that induces the G-linearization on $\mathcal{L}$. Almost everything in [**MuFo82**] reverts to schemes over a characteristic 0 algebraically closed field.

[**MuFo82**, p. 33] notes that $\mathcal{O}_{\mathbb{P}^n}(1)$ admits no $\mathrm{PGL}_{n+1}(\mathbb{C})$ action. Since, however, the nth exterior power of the cotangent bundle to $\mathbb{P}^n$ does admit such an action, so does its nth exterior power (isomorphic) to $\mathcal{O}_{\mathbb{P}^n}(n{+}1)$.

[17] Meaning the image of closed sets is closed; since the spaces are separable we can just use sequences with limits to test this, where the test passes since a limit element in $\mathrm{PGL}_2(\mathbb{C})$ applied to a limit in U_r is in U_r.

Back to our case $G = \mathrm{PGL}_2(\mathbb{C})$. [**MuFo82**] seeks the *semi-stable points* which includes droppng the closed action condition in (5.2), and for which the goal is the following.

(5.3a) Find the maximal set of $\boldsymbol{z} \in \mathbb{P}^n$ for which there is an affine set U containing $\boldsymbol{z}$ that resembles the description of U_r in (5.2).

(5.3b) Replace the condition $s(\boldsymbol{z}) \neq 0$ with s the discriminant function by some G-invariant element $s \in H^0(\mathcal{L}^n)$ for some n.

(5.3c) This starts with finding that $\mathrm{PGL}_2(\mathbb{C})$-linearizable sheaf $\mathcal{L}$.

The case $n = 1$ suggests the minimal possible twisting is $n = 2$, from which [**MuFo82**, p. 77] uses the notation $(\mathbb{P}^n)^{\mathrm{ss}}/\mathrm{PGL}_2(\mathbb{C})$ as the classical quantics goal. Their statement is that:[18]

(5.4) $O_{\mathbb{P}}(2)$ admits *one* and only one $\mathrm{PGL}_2(\mathbb{C})$-linearization (cf. §1.3).

The first italicized *one* is not stated in the section for all n, just $n = 1$. It looks as if they intend this to be obvious. Here is an argument.

Use the the $\mathrm{PGL}_2(\mathbb{C})$ morphism from $(\mathbb{P}^1)^n \to \mathbb{P}^n$, as in (1.7). Then, consider an embedding of $\mathbb{P}^1 \xrightarrow{\delta_{\mathrm{t}}} \mathbb{P}^n$ as the

$$\textit{thinnest piece of the fat diagonal}\text{: } z \mapsto (z, z, \ldots, z) \in (\mathbb{P}^1)^n.$$

This embedding is equivariant for both the S_n and $\mathrm{PGL}_2(\mathbb{C})$ actions, with the former a trivial action on the range of δ_{t}.[19] This induces an isomorphism of $\mathrm{Pic}(\mathbb{P}^1/\mathbb{C})$ and $\mathrm{Pic}(\mathbb{P}^n/\mathbb{C})$ – both generated by their twist sheaves $\mathcal{O}(1)$ – that is $\mathrm{PGL}_2(\mathbb{C})$ equivariant.

[**MuFo82**, Prop. 1.5] says, in general, an invertible sheaf L on X, corresponding to $\lambda_L \in \mathrm{Pic}(X/k)$ (k a field) with a G action has a power L^n that is linearizable if and only if the corresponding point $\lambda_L^n \in \mathrm{Pic}(X/k)$ is fixed by G. Apply this to $L = \mathcal{O}_{\mathbb{P}^1}(1)$ and $n = 2$ in $\mathrm{Pic}(\mathbb{P}^1/\mathbb{C})$ to conclude $\lambda_{\mathcal{O}_{\mathbb{P}^1}(2)}$ is fixed by $\mathrm{PSL}_2(\mathbb{C})$. Therefore so is $\lambda_{\mathcal{O}_{\mathbb{P}^n}(2)}$. Therefore, $\mathcal{O}_{\mathbb{P}^n}(2)$ is $\mathrm{PGL}_2(\mathbb{C})$-linearizable.

5.2. Comments on (5.1b)**.** The spaces Λ_r, especially $r = 4$ and 5, are the locales for the profinite *Grothendieck-Teichmüller group*. That means the degree $r!$ natural map $\Lambda_r \to J_r$ does lead to some confusion, for this reason.

(5.5a) Sharp triple-transitivity of $\mathrm{PGL}_2(\mathbb{C})$ on distinct points of $\mathbb{P}^1$ identifies Λ_4 with $\mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$, and Λ_r with $(\Lambda_4)^{r-3} \setminus \Delta_{r-3}$, $r > 4$.

[18] With the subscript $\mathbb{P}$ missing the indication of which $\mathbb{P}^n$ s they refer to.

[19] The *fat diagonal*, Δ_k, on X^k is the locus where ≥ 2 coordinates are equal.

(5.5b) The upper half-plane $\mathbb{H} \subset \mathbb{C}$ modulo $\mathrm{SL}_2(\mathbb{Z})$ is unramified over $\mathbb{P}^1_j \setminus \{0, 1, \infty\}$ (U_j together with its elliptic fixed points).[20]

A convenient starting point for (5.5a) is [**Ih91**]; for (5.5b) is [**Fr95**], though both have generated series of papers. Here are some comparisons between the two, using the symbol Ih for the former, Fr for the latter. Both, for each value of $r \geq 4$, deal with towers of covers of their respective spaces Λ_r and J_r.

(5.6a)

Comments on J_r for $r \geq 5$:

REMARK 5.1.

REMARK 5.2.

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