

Chebyshev-derived spindown parameters for gravitational wave signals from pulsars¹

S.R. Valluri and M.D. Fried

Abstract: The master equation described by Badri Krishnan et al. (*Phys Rev. D*, **70**, 082001 (2004)) for the time-frequency pattern using the $\mathcal{F}$ -statistic is studied in the context of Chebyshev-polynomial modified spindown parameters for the case of gravitational wave pulsar signals. The Chebyshev-polynomial method enables an analytic and numeric evaluation of the Fourier transform (FT) for both the non-demodulated and $\mathcal{F}$ -statistic demodulated FT.

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Résumé : Nous étudions l'équation maîtresse décrite par Badri Krishnan et al. (*Phys Rev. D*, **70**, 082001 (2004)) pour le patron en temps-fréquence via la statistique $\mathcal{F}$ dans le contexte de paramètres modifiés de spin des polynômes de Chebyshev dans l'étude des ondes gravitationnelles (GW) générées par un pulsar. La méthode des polynômes de Chebyshev permet d'évaluer analytiquement et numériquement la transformée de Fourier (FT) à la fois pour le cas sans démodulation et avec démodulation à statistique $\mathcal{F}$ (DeFT).

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1. Introduction

The data analysis for continuous gravitational waves (GW) is an important problem for ground-based interferometric detectors that demands analytic, computational, and experimental ingenuity. In their interesting paper on the Hough transform search for continuous GWs, Badri Krishnan et al. [1] emphasize that rapidly rotating neutron stars (NS) are expected to be the primary sources, which the current generation of Earth-based GW detectors might be able to detect them. In addition, Abbott et al. [2, 3] indicate that recent data analysis of the Laser Interferometer Gravitational-Wave Observatory (LIGO) and Anglo-German GEO 600 science runs have given upper limits to the GWs emitted by the pulsar J1939+2134 and its LIGO equatorial ellipticity [3]. Data analysis of future runs [4] is expected to provide upper limits below other astrophysical constraints and eventually lead to successful detection. Abbott et al. [3] also considered the coherent integration of the detector's output for an observation time of approximately 17 days using a Bayesian time-domain method and frequentist frequency-domain approach. They targeted a single known pulsar for fixed sky location and spindown rate provided from radio observations, thereby rendering the searches computationally inexpen-

sive. Doppler frequency modulation (FM) and pulsar spindown dictate that the power in each successive Fourier transform (FT) should be taken from the frequency bins where signal peaks occur, rather than from using the same frequency bins repeatedly [1].

Future continuous GW searches involve longer data stretches of the order of months. Hierarchical searches become necessary due to a larger frequency band, wider sky search, unknown sources, more spindown parameters, and the cost of fully coherent techniques. The stack slide method, which is a Radon transform, slides the frequency bins of FT to line up the signal peaks and then sum the power. The Hough transform (HT) is a special case of the Radon transform. Instead of a sum of power, 0 and 1 are used according to whether or not the power exceeds a threshold or meets a suitable criterion [1]. The HT is a robust parameter estimator of multidimensional image patterns.

In recent works [5, 6], we have implemented the FT of the Doppler-shifted GW signal from a pulsar with the plane wave expansion in spherical harmonics (PWESH). In ref. 5, we consider a simplified spindown model using Chebyshev polynomials, which have shown their utility in the application of maximum likelihood estimates. Chebyshev-polynomial interpola-

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tion is also useful because of its minimax property. Chebyshev polynomials are used in our work in an attempt to formulate a generic analysis of pulsar spindown. It is worth mentioning that their incorporation has facilitated the representation of the spindown parameters in terms of phases along the unit circle in the complex plane.

Brady et al. [7] indicate that most of the signal-to-noise ratio (S/N) is accumulated during the final stages of spindown of pulsars. There could exist a class of pulsars that spin down mainly due to GW reaction. Assuming that the mean birth rate for such pulsars in our galaxy is τ_B^{-1} , the nearest one should be a distance $r = R\sqrt{\tau_B/\tau}$ from the Earth, where $R \simeq 10$ kpc is the radius of the galaxy, and τ is the age of the pulsar [7].

The paper is organized as follows. Section 2 describes the GW forms from an isolated spinning NS and the techniques used in hierarchical searches. The HT is bound to play a pivotal role in GW searches. Its implementation in nonmodulated short FTs, and demodulated input data using the $\mathcal{F}$ -statistic, is elucidated in Krishnan et al. [1]. The works presented in refs. 4, 8, and 3 describe the FT and the $\mathcal{F}$ -statistic, and the expected time-frequency pattern when the search statistic is used. Section 2 gives the master equation in the ideal noiseless case and the more general case for the demodulated FT that maximizes the $\mathcal{F}$ -statistic. The Chebyshev-polynomial parameterization for spindown is then described. We use the $\mathcal{F}$ -statistic to derive a modified master equation, which is expressed in terms of total observation time (T_{obs}), coherent integration time (T_{coh}), and spindown parameters. Section 3 gives the conclusions.

2. Spindown analysis of the gravitational wave signal

The detector signal output is a linear combination of GW polarizations, $h_+(t)$ and $h_\times(t)$, that appear in the metric tensor perturbations away from flat space

$$h(t) = F_+(\mathbf{n}, \psi)h_+(t) + F_\times(\mathbf{n}, \psi)h_\times(t) \quad (1)$$

where $F_{+,\times}$ are the antenna pattern functions of the detector and depend on the direction $\mathbf{n}$ to the star and the polarization angle ψ . The wave forms for the two polarizations are

$$h_+(t) = A_+ \cos(\Phi(t)), \quad h_\times(t) = A_\times \sin(\Phi(t)) \quad (2)$$

The amplitudes $A_{+,\times}$ are determined by pulsar parameters such as its axis orientation, its ellipticity, and distance from Earth, etc. The phase of the GW signal is given by $\Phi(t)$.

The instantaneous frequency $f(t)$ of the GW as observed by the detector is approximated by the nonrelativistic Doppler formula

$$f(t) = \hat{f}(t) \left(1 + \frac{\mathbf{v}(t) \cdot \mathbf{n}}{c} \right) \quad (3)$$

$$\hat{f}(t) = f_{(0)} + \sum_{k=1}^s \frac{f_{(k)}}{k!} \left(t - t_0 + \frac{\Delta \mathbf{r}(t) \cdot \mathbf{n}}{c} \right)^k \quad (4)$$

where t_0 is the fiducial detector time at the start of the observation. The spindown parameters, $f_{(k)}$, the detector velocity,

$\mathbf{v}(t)$, and the position of the detector at time t , $\mathbf{r}(t)$, are all measured in the Solar System Barycenter (SSB) frame. Also, $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is the direction to the pulsar source and $\Delta \mathbf{r}(t) := \mathbf{r}(t) - \mathbf{r}(t_0)$. Here, c is the speed of light and s is the number of spindown search parameters.

The parameterized model for frequency and phase evolution of the pulsar frequency with spindown corrections was given by Brady and Creighton [7]. The spindown age in years is given by $\tau_{\text{min}} = f/\dot{f}$, and $|f_{(k)}| \leq \tau_{\text{min}}^{-k}$. Spindown requires many spindown parameters in the domain $[-\tau_{\text{min}}^k, \tau_{\text{min}}^k]$. For the extreme case of the GW frequency of 10^3 Hz, $\tau_{\text{min}} = 40$ years for pulsars in our galaxy, and $\tau_{\text{min}} \geq 10^7$ years for millisecond pulsars. The GW phase Φ depends on $\xi \equiv (f_{(0)}, \{f_{(k)}\}, \mathbf{n}) \equiv (f_{(0)}, \boldsymbol{\lambda})$, the relevant parameters for the Keplerian orbit, the Jovian and lunar perturbations, and the co-latitude of the detector [6]. It is assumed that the NS is moving with uniform speed relative to the Sun and is so far away that there are no observable proper-motion effects. Longer coherent stages are required for better sensitivity. One must use demodulated data, i.e., remove the effects due to Earth's spin and orbital motion, and the pulsar spindown. This demodulation must be done separately for different regions of the sky and spin-down parameter space. However, it also introduces other parameters, such as the polarization angle due to amplitude modulation. These extra parameters, which are not part of the HT search space, can be eliminated by requiring the coherent stage to produce the $\mathcal{F}$ -statistic described in refs. 4 and 8. The $\mathcal{F}$ -statistic is used in ref. 3 for analyzing the first science runs of the LIGO and GEO detectors. The HT with demodulated data, based on the $\mathcal{F}$ -statistic, removes the frequency drifts caused by the Doppler modulation and spindown. The limitation on T_{coh} is due to computational resources [1]. The demodulated parameters are denoted by $\lambda_d \equiv (f_{d(k)}, \mathbf{n}_d)$.

The Fourier transform of the data $x(t)$ denoted by $\tilde{X}(f)$ is defined below. The quantity $\ln \Lambda$, where Λ is the likelihood function, is essentially the matched filter, and is given by the following form:

$$\ln \Lambda = (x(t)|h(t)) - \frac{1}{2}(h(t)|h(t)) \quad (5)$$

where $x(t)$ is the calibrated detector output and $h(t)$ is the GW form being searched for [1].

Apart from the shape parameters $\xi(f_{(0)}, f_{(n)}, \mathbf{n})$, $\ln \Lambda$ also depends on the orientation of the pulsar and the polarization angle of the wave, etc. The $\mathcal{F}$ -statistic eliminates these additional variables and allows the search for only ξ .

In an earlier work [5], we extended the Brady–Creighton model to include pulsar spindown corrections of frequency, and phase evolution [7]. We now generalize it further using a linear combination of Chebyshev polynomials $T_k(z_j)$,

$$\sum_{j=1}^s \sum_{k=1}^{\infty} \frac{f_k^{(j)}}{k!} x^k; f_k^{(j)} = (k-1)! \frac{T_k(z_j)}{\tau_{\text{min}}^k} \quad (6a)$$

where

$$-1 \leq z_j \leq 1, \quad -1 \leq T_k(z_j) \leq 1, \\ z_j = \cos \phi_j = \omega_j + \frac{1}{\omega_j}; \quad \omega_j = e^{i\phi_j} \quad (6b)$$

The Chebyshev-polynomial parameterization allows for the summation of the double series in terms of k and j , via the following identity [9]:

$$\sum_{j=1}^s \sum_{k=1}^{\infty} \frac{T_k(z_j)}{k} \left(\frac{x}{\tau_{\min}} \right)^k = -\frac{1}{2} \sum_{j=1}^s \log \left[1 - 2 \frac{x}{\tau_{\min}} z_j + \left(\frac{x}{\tau_{\min}} \right)^2 \right] \quad (7)$$

This parameterization allows the analytic development of the FT of the pulsar signal as well as the demodulated FT (DeFT). The DeFT will not be given here due to space constraints but we refer the interested reader to some of the simpler analysis done in earlier work [5, 6, 10].

We assume that the $\mathcal{F}$ -statistic has been computed using the demodulation parameters, λ_d , and the detector output consists of a signal with parameters $(f_{(0)}, \lambda)$. We denote the mismatched parameters as $\Delta\lambda \equiv \lambda - \lambda_d$. Consequently, $\Delta\mathbf{n} \equiv \mathbf{n} - \mathbf{n}_d$, $\Delta f_{(k)} \equiv f_{(k)} - f_{d(k)}$, and the residual frequency shift $\Delta f \equiv f - f_{(0)}$. The $\mathcal{F}$ -statistic is maximized when $\Delta\lambda = 0$ and $f = f_{(0)}$. Maximizing the $\mathcal{F}$ -statistic is equivalent to maximizing $|\tilde{X}(f)|^2$ where $\tilde{X}(f)$ is the DeFT defined as [1]

$$\tilde{X}(f) = \int x(t) e^{-i\Phi(t; f, \lambda_d)} dt \quad (8)$$

The Master Equation given in ref. 1 is of the form:

$$f(t) - F_0(t) = \boldsymbol{\zeta} \cdot (\mathbf{n} - \mathbf{n}_d) \quad (9a)$$

where

$$F_0(t) = f_{(0)} + \sum_{k=1}^s \frac{\Delta f_{(k)}}{k!} (\Delta t)^k \quad (9b)$$

Δt is the time interval between each data segment, and

$$\boldsymbol{\zeta}(t) = \left[F_0(t) + \sum_{k=1}^s \frac{f_{d(k)}}{k!} (\Delta t)^k \right] \frac{\mathbf{v}(t)}{c} + \left[\sum_{k=1}^s \frac{f_{d(k)}}{(k-1)!} (\Delta t)^{k-1} \right] \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{c} \quad (9c)$$

The extension to an arbitrarily large number of spindown parameters warrants further investigation as it allows for a more flexible and convenient Chebyshev-polynomial interpolation. One can derive a convergent closed-form expression for the summation over all spindown parameters by extending the finite series from $k = 0$ to ∞ , since the spindown terms are very small after a finite value of s . We account for all possible spindown and spinup parameter values in terms of Chebyshev polynomials, $T_k(z_j)$. By allowing for varying z_j and introducing $T_k(z_j)$ for a more general parameterization one obtains,

$$\Delta f_{(k)}^{(j)} = f_{(k)}^{(j)} - f_{d(k)}^{(j)} = (k-1)! \frac{T_k(z_j)}{T_{\text{obs}}^k} \delta f, \quad \delta f = \frac{1}{T_{\text{coh}}} \quad (10a)$$

$$f_{d(k)}^{(j)} = (k-1)! \frac{T_k(z_j)}{\tau_{\min}^k}, \quad j = 1, 2, \dots, s \quad (10b)$$

The Master Equation given in ref. 1,

$$F_0(t) = f_{(0)} + \sum_{k=1}^s \frac{\Delta f_{(k)}}{k!} (\Delta t)^k \quad (11)$$

becomes, after extending each summation in t ,

$$F_0(t) = f_{(0)} + \sum_{j=1}^s \sum_{k=1}^{\infty} \frac{(k-1)! T_k(z_j)}{k! T_{\text{coh}}^k} \left(\frac{\Delta t}{T_{\text{obs}}} \right)^k \quad (12)$$

where we can expand the Master Equation further by using a pair of summation expansions. Thereby, terms in the finite series of the Master Equation [1] modify

$$\sum_{k=1}^s \frac{f_{d(k)}}{(k-1)!} (\Delta t)^{k-1} \rightarrow \frac{d}{d(\Delta t)} \left\{ \sum_{j=1}^s \sum_{k=1}^{\infty} \frac{(k-1)! T_k(z_j)}{k!} \left(\frac{\Delta t}{\tau_{\min}} \right)^k \right\} \quad (13)$$

$$\sum_{k=1}^s \frac{f_{d(k)}}{k!} (\Delta t)^k \rightarrow \left\{ \sum_{j=1}^s \sum_{k=1}^{\infty} \frac{(k-1)! T_k(z_j)}{k!} \left(\frac{\Delta t}{\tau_{\min}} \right)^k \right\} \quad (14)$$

giving the Master Equation the new extended form:

$$\begin{aligned} f(t) &= F_0(t) + \boldsymbol{\zeta} \cdot (\mathbf{n} - \mathbf{n}_d) \\ &= F_0(t) + \boldsymbol{\zeta} \cdot \Delta\mathbf{n} \\ &= \left(1 + \frac{\mathbf{v}(t)}{c} \cdot \Delta\mathbf{n} \right) \left\{ f_0 + \sum_{j=1}^s \sum_{k=1}^{\infty} \frac{(k-1)! T_k(z_j)}{k! T_{\text{coh}}^k} \left(\frac{\Delta t}{T_{\text{obs}}} \right)^k \right\} \\ &\quad + \sum_{j=1}^s \sum_{k=1}^{\infty} \left\{ \frac{(k-1)! T_k(z_j)}{k!} \left(\frac{\Delta t}{\tau_{\min}} \right)^k \right\} \frac{\mathbf{v}(t)}{c} \cdot \Delta\mathbf{n} + \frac{d}{d(\Delta t)} \left\{ \sum_{j=1}^s \sum_{k=1}^{\infty} \frac{(k-1)! T_k(z_j)}{k!} \left(\frac{\Delta t}{\tau_{\min}} \right)^k \right\} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{c} \cdot \Delta\mathbf{n} \quad (15) \end{aligned}$$

Finally, the Chebyshev-modified Master Equation now takes the form

$$f(t) = \left\{ f_0 + \frac{1}{T_{\text{coh}}} \sum_{j=1}^s -\frac{1}{2} \log \left\{ 1 - 2 \frac{\Delta t}{T_{\text{obs}}} z_j + \left(\frac{\Delta t}{T_{\text{obs}}} \right)^2 \right\} \right\} \left(1 + \frac{\mathbf{v}(t)}{c} \cdot \Delta \mathbf{n} \right) - \frac{1}{2} \log \left\{ 1 - 2 \frac{\Delta t}{\tau_{\text{min}}} z_j + \left(\frac{\Delta t}{\tau_{\text{min}}} \right)^2 \right\} \frac{\mathbf{v}(t)}{c} \cdot \Delta \mathbf{n} \\ + \frac{d}{d(\Delta t)} \left[\sum_{j=1}^s -\frac{1}{2} \log \left\{ 1 - 2 \frac{\Delta t}{\tau_{\text{min}}} z_j + \left(\frac{\Delta t}{\tau_{\text{min}}} \right)^2 \right\} \right] \left(\frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{c} \cdot \Delta \mathbf{n} \right) \quad (16)$$

The final form after simplification is

$$f(t) = f_0 - \frac{1}{2} \frac{1}{T_{\text{coh}}} \sum_{j=1}^s \left\{ \log \left(1 - \omega_j \frac{\Delta t}{T_{\text{obs}}} \right) + \log \left(1 - \frac{1}{\omega_j} \frac{\Delta t}{T_{\text{obs}}} \right) \right\} \left(1 + \frac{\mathbf{v}(t)}{c} \cdot \Delta \mathbf{n} \right) \\ - \frac{1}{2} \sum_{j=1}^s \left\{ \log \left(1 - \omega_j \frac{\Delta t}{\tau_{\text{min}}} \right) + \log \left(1 - \frac{1}{\omega_j} \frac{\Delta t}{\tau_{\text{min}}} \right) \right\} \left(\frac{\mathbf{v}(t)}{c} \cdot \Delta \mathbf{n} \right) \\ + \frac{1}{2} \sum_{j=1}^s \left\{ \left(\frac{\omega_j}{\tau_{\text{min}} - \omega_j \Delta t} \right) + \left(\frac{1}{\omega_j \tau_{\text{min}} - \Delta t} \right) \right\} \left(\frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{c} \cdot \Delta \mathbf{n} \right) \quad (17)$$

The last term in the brackets can signify the possible appearance of a pole in extreme situations. We also note that the last two terms in the expression for $f(t)$ can be written as the time derivative of a product.

The search method [1] depends on finding a signal, whose frequency evolution fits the pattern produced by the Doppler shift and the spindown for the sky locations consistent with the master equation

3. Conclusions

Each value of ξ_j , expressed in terms of phase ϕ_j and $\Delta \mathbf{n}$, determines the frequency evolution $f(t)$ as a curve in the time-frequency plane. The Chebyshev-polynomial parameterization might be useful in the application of the Hough transform for an incoherent and inexpensive gravitational-wave pulsar signal search. In both versions of the Hough-transform search, either simple non-demodulated Fourier transforms or demodulated Fourier transforms using the $\mathcal{F}$ -statistic, the Chebyshev method facilitates an analytic and numeric evaluation of the Fourier transform as well as a concise summation of the series represented by the Master Equation. An interesting question for further study is if the number of spindown parameters in gravitational-wave data and the topological invariants characteristic of the neutron star can be related in a way akin to an Atiyah–Singer index formula.

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