THE $(g-1)$-SUPPORT COVER OF THE CANONICAL LOCUS

By

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One of Max Noether's achievements [1, 2] was a characterization of non-hyperelliptic compact Riemann surfaces S by the property that the vector space of holomorphic q-differentials, q an integer >1, is spanned by q-fold products of holomorphic differentials chosen from a basis $\vartheta_1, \ldots, \vartheta_g$ of the vector space of holomorphic differentials on S. Noether's proof depended on showing that for a non-hyperelliptic surface there always exists a positive divisor D of degree $g-2$ on S, such that for any $s \in S$ there is exactly one holomorphic differential ϑ (up to multiplication by a constant) whose divisor, (ϑ), contains $D+s$ in its support. An alternate proof of Noether's theorem based on the following result has been suggested: S is non-hyperelliptic if and only if there exists on S a holomorphic differential ϑ whose divisor $(\vartheta) = s_1 + \cdots + s_{2g-2}$ has the property that $s_i \neq s_j$, $i \neq j$, and if $\hat{\vartheta}$ is a non-zero holomorphic differential on S and $i_1, \ldots, i_{g-1}$ are integers with $1 \leqq i_1 < i_2 < \cdots < i_{g-1} \leqq 2g-2$ then

$$(\hat{\vartheta}) \text{ has } s_{i_1} + \cdots + s_{i_{g-1}} \text{ in its support} \tag{1}$$

if and only if $\hat{\vartheta} = c\vartheta$ for some $c \in \mathbf{C}$.

The existence of such ϑ is an immediate corollary of our main result which we now describe.

Let S be a compact Riemann surface of genus $g \geqq 2$ and let $S^{(n)}$ be the complex manifold consisting of positive divisors of degree n on S. Consider the map

$$\psi : S^{(g-1)} \times S^{(g-1)} \rightarrow S^{(2g-2)} \tag{2}$$

defined by $(D_1, D_2) \rightarrow D_1 + D_2$. Let Z be the subset of $S^{(2g-2)}$ consisting of divisors, (ϑ), of holomorphic differentials on S. For any map $\alpha : W \rightarrow V$ and U a subset of V denote the set $\{w \in W \mid \alpha(w) \in U\}$ by $\alpha^{-1}(U)$.

Main Theorem (Theorem 8). *For $g \geqq 3$, S is non-hyperelliptic if and only if $\psi^{-1}(Z)$ is an irreducible subset of $S^{(g-1)} \times S^{(g-1)}$.*

The proof of our main theorem is based on two further characterizations of hyperellipticity. Let pr_i, $i = 1,2$, denote projection onto the i-th factor of $S^{(g-1)} \times S^{(g-1)}$. For any S there is a unique irreducible component V_{g-1} of $\psi^{-1}(Z)$ for which

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$\mathrm{pr}_i : V_{g-1} \to S^{(g-1)}$ is surjective and generically of degree 1, $i = 1, 2$ (Lemma 4). We shall denote the map from S^{g-1} to $S^{(g-1)}$ that maps $(s_1, \ldots, s_{g-1})$ to $s_1 + \cdots + s_{g-1}$ by Λ_{g-1}. Let $\overline{\mathrm{pr}}_i$ denote projection onto the i-th factor of $S^{g-1} \times S^{g-1}$, $i = 1, 2$. Consider now

(3) $$\Lambda' = \Lambda_{g-1} \times \Lambda_{g-1} : S^{g-1} \times S^{g-1} \to S^{(g-1)} \times S^{(g-1)}.$$

Then $\overline{\mathrm{pr}}_i\ ((\Lambda')^{-1}(V_{g-1})) = S^{g-1}$ and the natural map $\mathrm{pr}_i \circ \Lambda' : (\Lambda')^{-1}(V_{g-1}) \to S^{(g-1)}$ determines a map

(4) $$\lambda_i : S^{g-1} \to S^{(g-1)}, \qquad i = 1, 2.$$

Furthermore $\lambda_1 = \lambda_2 \circ h$, when both sides are defined, where $h : S^{(g-1)} \to S^{(g-1)}$ is the birational map that maps a positive divisor D to D', where $D + D' \in Z$. The result then is that the branch loci of the covers λ_1 and λ_2 are identical (and the covers are equivalent as covers of $S^{(g-1)}$) if and only if S is hyperelliptic (Theorem 5). The main ingredient of the proof is the observation that $S^{(2)}$ contains a one-dimensional variety C such that the map

(5) $$m_{g-1} : S^2 \to S^{(2g-2)}$$

given by multiplication of a divisor by $g - 1$ maps C into Z if and only if S is hyperelliptic.

For the sake of completeness we include a quick proof of Noether's theorem as a corollary of Theorem 8. The Galois theoretic observations around (3) and (4) have further consequences that will appear in later investigations.

§1. Further notation and terminology

The arguments used in this paper fall under the heading of "generic point" arguments. Such arguments deal exclusively with projective varieties and depend on field theoretic manipulations. Since the reader may not be sufficiently familiar with such arguments we give here a brief review of some of the key points. In what follows $\mathbf{P}^n$ denotes projective n-space over $\mathbf{C}$.

Let $V \subseteq \mathbf{P}^n$ be a projective variety (i.e., V is defined by a finite set of homogeneous polynomials, and V is irreducible). Denote the field of meromorphic functions on V by $\mathbf{C}(V)$. Similarly, if V_0 is an affine open (nonempty) subset of V, then $\mathbf{C}[V_0]$ denotes the ring of holomorphic functions on V_0 and the quotient field of $\mathbf{C}[V_0]$ is $\mathbf{C}(V)$. Following [5] we assume that all fields that appear here are contained in one algebraically closed field T_{un} of infinite transcendence dimension over $\mathbf{C}$: the *universal domain*. We produce a generic point for the variety V through the following process.

Let V be of dimension k. Then there exists a finite morphism $\varphi : V \to \mathbf{P}^k$ (the Noether normalization lemma [4, p. 253]). First we produce a generic point $\boldsymbol{x}^{\mathrm{gen}}$ for

$\mathbf{P}^k$. By definition $\boldsymbol{x}^{\mathrm{gen}} \in \mathbf{P}^k(T_{\mathrm{un}})$ (points with coordinates in T_{un}) and if $\boldsymbol{x}^{\mathrm{gen}} = (x_0^{\mathrm{gen}}, \ldots, x_k^{\mathrm{gen}})$ then $\mathbf{C}(\mathbf{P}^k) \cong \mathbf{C}(x_i^{\mathrm{gen}}/x_j^{\mathrm{gen}}, x_j^{\mathrm{gen}} \neq 0, 0 \leqq i, j \leqq k)$. Indeed, for any point $\boldsymbol{x} \in \mathbf{P}^n(T_{\mathrm{un}})$ denote the field with the allowable ratios of its coordinates adjoined by $\mathbf{C}(\boldsymbol{x})$. Then there is a unique subvariety $V \subseteq \mathbf{P}^n$ with $\boldsymbol{x} \in V(T_{\mathrm{un}})$ such that $\mathbf{C}(V) \cong \mathbf{C}(\boldsymbol{x})$ and $\boldsymbol{x}$ is called a generic point for V. Let $t_1, \ldots, t_k$ be elements of T_{un} that are algebraically independent over $\mathbf{C}$, take $\boldsymbol{x}^{\mathrm{gen}} = (1, t_1, \ldots, t_k)$ and let $\boldsymbol{v}^{\mathrm{gen}}$ be *any* point of $V(T_{\mathrm{un}})$ such that $\varphi(\boldsymbol{v}^{\mathrm{gen}}) = \boldsymbol{x}^{\mathrm{gen}}$. Such a $\boldsymbol{v}^{\mathrm{gen}}$ exists because T_{un} is algebraically closed. Then $\boldsymbol{v}^{\mathrm{gen}}$ is a generic point of V and $\mathbf{C}(\boldsymbol{v}^{\mathrm{gen}}) \cong \mathbf{C}(V)$.

We give a crucial word of warning. A given variety V of dimension $k \geqq 1$ has many generic points. Indeed, any $\boldsymbol{x} \in V(T_{\mathrm{un}})$, with the property that $\mathbf{C}(\boldsymbol{x})$ has transcendence dimension k over $\mathbf{C}$, is a generic point of V. Furthermore, if x_1^{gen} and x_2^{gen} are two generic points of V, there is a field isomorphism $\lambda : \mathbf{C}(x_1^{\mathrm{gen}}) \to \mathbf{C}(x_2^{\mathrm{gen}})$, which, when applied to the coordinates of x_1^{gen}, actually maps x_1^{gen} to x_2^{gen}. In short, the fields $\mathbf{C}(x_1^{\mathrm{gen}})$ and $\mathbf{C}(x_2^{\mathrm{gen}})$ are isomorphic by mapping x_1^{gen} to x_2^{gen}. We must make a distinction between this situation and the rarer event that the coordinates of x_2^{gen} are actually rational functions in the coordinates of x_1^{gen} and rather than just an isomorphism of fields there is an equality $\mathbf{C}(x_1^{\mathrm{gen}}) = \mathbf{C}(x_2^{\mathrm{gen}})$. In this case the automorphism of $\mathbf{C}(x_1^{\mathrm{gen}})$ that induces the map $x_1^{\mathrm{gen}} \to x_2^{\mathrm{gen}}$ is called a *birational automorphism* (it may induce an automorphism only on a Zariski open subset of V). A crucial example for this paper arises in considering a hyperelliptic surface S with canonical involution τ. If $s^{\mathrm{gen}} \in S(T_{\mathrm{un}})$ is a generic point for S then $\tau(s^{\mathrm{gen}})$ is also.

For L any field extension of $\mathbf{C}$ contained in T_{un} and $\lambda : L \to T_{\mathrm{un}}$ any field isomorphism we may extend λ to an automorphism of T_{un}. If $L_1 \supset L$ and the degree of L_1 over L, $[L_1 : L]$, is finite, then there are $[L_1 : L]$ distinct extensions of λ to L_1. On the other hand, if $L_1 \subseteq T_{\mathrm{un}}$ and L_1 is generated over $\mathbf{C}$ by elements that are algebraically independent over L, then we can extend λ to T_{un} so that its restriction to L_1 is the identity. Finally, if $v_1^{\mathrm{gen}}, v_2^{\mathrm{gen}}, \ldots, v_l^{\mathrm{gen}}$ are generic points of V we say that they are algebraically independent if $\mathbf{C}(v_i^{\mathrm{gen}})$ is algebraically independent of the field composite of $\mathbf{C}(v_1^{\mathrm{gen}}), \ldots, \mathbf{C}(v_{i-1}^{\mathrm{gen}}), \mathbf{C}(v_{i+1}^{\mathrm{gen}}), \ldots, \mathbf{C}(v_l^{\mathrm{gen}})$, $i = 1, \ldots, l$. We conclude with some genuine Riemann surface notation. For S a compact Riemann surface of genus $g \geqq 1$, and $n \in \mathbf{Z}$, $\mathrm{Pic}(S)^{(n)}$ denotes the g-dimensional complex manifold of (not necessarily positive) divisor classes of degree n. For $n \geqq 1$ there is a natural sequence

$$S^n \xrightarrow{\Lambda_n} S^{(n)} \xrightarrow{\Psi_n} \mathrm{Pic}(S)^{(n)} \tag{6}$$

of complex analytic maps of projective complex manifolds. Here Ψ_n maps a positive divisor to its linear equivalence class. If $s_1^{\mathrm{gen}}, \ldots, s_n^{\mathrm{gen}}$ are n algebraically independent generic points of S, then $(s_1^{\mathrm{gen}}, \ldots, s_n^{\mathrm{gen}})$ is a generic point of S^n. Its image $s_1^{\mathrm{gen}} + \cdots + s_n^{\mathrm{gen}}$ in $S^{(n)}$ is a generic point of $S^{(n)}$ and $\Psi_n(s_1^{\mathrm{gen}} + \cdots + s_n^{\mathrm{gen}})$ is a generic point of the image of Ψ_n. Denote the image of $S^{(g-1)}$ in $\mathrm{Pic}(S)^{(g-1)}$ under Ψ_{g-1} by Θ, and its singular points by Θ_{sing}. Since Θ_{sing} is of codimension at least 1 in Θ, $\Psi_{g-1}(s_1^{\mathrm{gen}}+\cdots+s_{g-1}^{\mathrm{gen}})$ is not in Θ_{sing}.

Riemann's Theorem [2, pp. 291–300]. *For $x\in\Theta$, the set $\Psi_n^{-1}(x)$ consists of more than one point if and only if $x\in\Theta_{\mathrm{sing}}$.*

§2. Fundamental lemmas on Z and $\Psi^{-1}(Z)$

Throughout the rest of this paper S denotes a compact Riemann surface of genus $g\geqq 2$. We use the notation of the introduction where Z is the locus in $S^{(2g-2)}$ of canonical divisors.

Lemma 1. *The map ψ is an analytic map of complex manifolds such that for each $z\in S^{(2g-2)}$, $\psi^{-1}(z)$ consists of $\binom{2g-2}{g-1}$ points when counted with suitable multiplicity. In addition, for $z\in S^{(2g-2)}$ suitably general, for each $(D_1,D_2)\in\psi^{-1}(z)$*

(7) *D_1 is the only positive divisor linearly equivalent to D_1.*

Proof. For $s_1,\ldots,s_{g-1}\in S$ and z_i, $i=1,\ldots,g-1$, local coordinates at s_i take as local coordinates on $S^{(g-1)}$ the elementary symmetric functions of the z_i in a neighborhood of $s_1+\cdots+s_{g-1}$. Thus the first sentence of the lemma can be reverted to the case $S=\mathbf{P}^1$ and $g-1=n$ is any integer. Identifying $\mathbf{P}^n$ with the space of nonzero polynomials of degree at most n up to multiplication by a constant shows that the map $\Psi:\mathbf{P}^n\times\mathbf{P}^n\to\mathbf{P}^{2n}$ is identified with $(f_1(x),f_2(x))\to f_1\cdot f_2$ and this gives the first sentence.

We now show what it means for $z\in S^{(2g-2)}$ to be suitably general so that (7) holds. By Riemann's theorem (Section 1) (7) holds for $D_1\in S^{(g-1)}$ if and only if $\Psi_{g-1}(D_1)\in\Theta\setminus\Theta_{\mathrm{sing}}$. Let $s_1^{\mathrm{gen}},\ldots,s_{2g-2}^{\mathrm{gen}}$ be algebraically independent generic points of S over $\mathbf{C}$. Then for any subset $i=\{i_1,\ldots,i_{g-1}\}\subseteq\{1,2,\ldots,2g-2\}$ of $g-1$ distinct integers, $s_{i_1}^{\mathrm{gen}}+\cdots+s_{i_{g-1}}^{\mathrm{gen}}=s_i^{\mathrm{gen}}$ is a generic point of $S^{(g-1)}$. In particular $\Psi_{g-1}(s_i^{\mathrm{gen}})\not\subset\Theta_{\mathrm{sing}}$ for any allowable i and the second sentence holds with $z=\sum_{i=1}^{2g-2}s_i^{\mathrm{gen}}$. ■

Now consider the restriction of (2) over Z:

(8) $$\psi:\psi^{-1}(Z)\to Z.$$

We call (8) the $(g-1)$-*support cover* (over the canonical locus). We want to investigate the irreducible components of $\psi^{-1}(Z)$, and in particular, to ask for which S expression (1) (or equivalently, (7)) holds for $z^{\mathrm{gen}}\in Z$ a generic point of Z. Note that since $Z\cong\mathbf{P}^{g-1}$, Z is certainly irreducible.

Lemma 2. *Given any $g-1$ points, $s_1,\ldots,s_{g-1}$ on S (perhaps repeated), there exists a holomorphic differential ϑ on S whose divisor (ϑ) contains $s_1+\cdots+s_{g-1}$ in* *its support. In particular,* $\mathrm{pr}_1(\psi^{-1}(Z)) = S^{(g-1)}$. *In addition, there exists a* ϑ *for which* (ϑ) *consists of* $2g-2$ *distinct points.*

Proof. If $z_1, z_2 \in Z$ are divisors of holomorphic differentials ϑ_1 and ϑ_2 with disjoint support, then $f = \vartheta_1/\vartheta_2$ is a meromorphic function on S of degree $2g-2$. Then f represents S as a cover of $\mathbf{P}^1$ branched over only a finite number of points $x_1, \dots, x_r \in \mathbf{P}^1$. If $c \in \mathbf{P}^1$ is not one of the points x_i then $f^{-1}(c)$ is a divisor whose support consists of $2g-2$ distinct points which is linearly equivalent to (ϑ_i). We need only find two holomorphic differentials with no common support. Let $s_1, \dots, s_r$ be the distinct points of (ϑ_1). If the divisor of each holomorphic differential had common support with (ϑ_1) then $\bigcup_{i=1}^r \Omega(s_i)$ would exhaust the space of holomorphic differentials where $\Omega(s_i)$ is the space of holomorphic differentials whose divisors contain s_i in their support. From the Riemann–Roch theorem, $\dim \Omega(s_i) = g-1$ for $g \geqq 2$. Therefore $\bigcup_{i=1}^r \Omega(s_i)$ cannot exhaust the space of holomorphic differentials. ■

Example 3. *Hyperelliptic surfaces.* Let S be a hyperelliptic surface, and let $\tau : S \to S$ be the canonical involution. For any holomorphic differential φ on S, $\tau(\varphi) = -\varphi$. Thus each point of Z is fixed by τ (i.e., for $z \in Z$, $z = s_1 + \cdots + s_{g-1} + \tau(s_1) + \cdots + \tau(s_{g-1})$ for some points $s_1, \dots, s_{g-1} \in S$). Indeed, let $s_1^{\mathrm{gen}}, \dots, s_{g-1}^{\mathrm{gen}}$ be algebraically independent generic points on S. Then the following points,

$$(s_1^{\mathrm{gen}} + \cdots + s_{g-1}^{\mathrm{gen}}, \tau(s_1^{\mathrm{gen}}) + \cdots + \tau(s_{g-1}^{\mathrm{gen}})) = w_1,$$

$$(s_1^{\mathrm{gen}} + \cdots + s_{g-2}^{\mathrm{gen}} + \tau(s_1^{\mathrm{gen}}),\ \tau(s_2^{\mathrm{gen}}) + \cdots + \tau(s_{g-1}^{\mathrm{gen}}) + s_{g-1}^{\mathrm{gen}}) = w_2,$$

$$(s_1^{\mathrm{gen}} + \cdots + s_{g-3}^{\mathrm{gen}} + \tau(s_1^{\mathrm{gen}}) + \tau(s_2^{\mathrm{gen}}),\ \tau(s_3^{\mathrm{gen}}) + \cdots + \tau(s_{g-1}^{\mathrm{gen}}) + s_{g-1}^{\mathrm{gen}} + s_{g-2}^{\mathrm{gen}}) = w_3,$$

etc., are generic points for distinct components $W_1, W_2, W_3, \dots, W_{[g+1/2]}$ of $\psi^{-1}(Z)$, and each is of dimension $g-1$. ■

Lemma 4. *Let* $s_1^{\mathrm{gen}}, \dots, s_{g-1}^{\mathrm{gen}}$ *be* $g-1$ *algebraically independent generic points for* S *over* $\mathbf{C}$. *Let* ϑ *be a holomorphic differential on* S *for which* $(\vartheta) = s_1^{\mathrm{gen}} + \cdots + s_{g-1}^{\mathrm{gen}} + s_1' + \cdots + s_{g-1}'$. *Then* $s_1', \dots, s_{g-1}'$ *are also algebraically independent generic points on* S *and the support of* (ϑ) *consists of* $2g-2$ *distinct points. Furthermore*

$$\mathbf{C}(s_1^{\mathrm{gen}} + \cdots + s_{g-1}^{\mathrm{gen}}) = \mathbf{C}(s_1' + \cdots + s_{g-1}')$$

(*where, as in the introduction, we regard* $s_1^{\mathrm{gen}} + \cdots + s_{g-1}^{\mathrm{gen}}$ *as a generic point of* $S^{(g-1)}$).

Proof. Since $s_1^{\mathrm{gen}}, \dots, s_{g-1}^{\mathrm{gen}}$ are algebraically independent generic points, the Riemann–Roch theorem implies that, up to constant multiples, there is only one holomorphic differential ϑ whose divisor has $s_1^{\mathrm{gen}}, \dots, s_{g-1}^{\mathrm{gen}}$ in its support. Lemma 2 shows that the support of (ϑ) consists of $2g-2$ distinct points. We must show that $s_1', \dots, s_{g-1}'$ are algebraically independent generic points of S. Suppose not.

The field $\mathbf{C}(s'_1,\ldots,s'_{g-1}) = L'$ is of transcendence dimension $g-1$ if and only if $s'_1,\ldots,s'_{g-1}$ are algebraically independent generic points. Thus L' is of transcendence dimension at most $g-2$. In particular, not only is $L^{\text{gen}} = \mathbf{C}(s_1^{\text{gen}},\ldots,s_{g-1}^{\text{gen}})$ not equal to L', but their composite $L^{\text{gen}}L'$ has transcendence dimension at least 1 over L'. Therefore there exist infinitely many distinct field isomorphisms $\lambda : L^{\text{gen}}L' \to T_{\text{un}}$ fixed on L'. Each λ is determined by its application to (the coordinates of) the points $s_1^{\text{gen}},\ldots,s_{g-1}^{\text{gen}}$. Thus we can choose λ so that

$$\lambda(s_1^{\text{gen}}) + \cdots + \lambda(s_{g-1}^{\text{gen}}) \neq s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}.$$

But, since λ is fixed on L', $\lambda(s'_i) = s'_i$, $i = 1,\ldots,g-1$. Extend λ to any field extension over which ϑ is defined, so that ϑ^λ is a new holomorphic differential on S for which

$$(\vartheta^\lambda) = \lambda(s_1^{\text{gen}}) + \cdots + \lambda(s_{g-1}^{\text{gen}}) + s'_1 + \cdots + s'_{g-1}.$$

Thus $\vartheta^\lambda/\vartheta$ is a non-trivial function on S whose divisor of poles has support in $s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}$. Since, however, these are algebraically independent generic points, no such function can exist. Actually the same argument works if

$$\mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}) \cdot \mathbf{C}(s'_1 + \cdots + s'_{g-1})$$

is of degree greater than 1 over $\mathbf{C}(s'_1 + \cdots + s'_{g-1})$: there exists λ as above (see Section 1). Thus $\mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}) \subseteq \mathbf{C}(s'_1 + \cdots + s'_{g-1})$. But, now that we know that $s'_1,\ldots,s'_{g-1}$ are algebraically independent generic points of S, we can interchange the roles of $s_1^{\text{gen}},\ldots,s_{g-1}^{\text{gen}}$ and $s'_1,\ldots,s'_{g-1}$ to conclude that

$$\mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}) = \mathbf{C}(s'_1 + \cdots + s'_{g-1}). \qquad \blacksquare$$

For $s_1^{\text{gen}},\ldots,s_{g-1}^{\text{gen}}$ algebraically independent generic points of S, consider the variety $V_{g-1} \subseteq S^{(g-1)} \times S^{(g-1)}$ that has $(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}, s'_1 + \cdots + s'_{g-1})$ as a generic point (in the notation of Lemma 4). As in the introduction consider $(\Lambda')^{-1}(V_{g-1})$, the subvariety of $S^{g-1} \times S^{g-1}$ which has $(s_1^{\text{gen}},\ldots,s_{g-1}^{\text{gen}}, s'_1,\ldots,s'_{g-1})$ as a generic point. From Lemma 4 we know that

$$\mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}) = \mathbf{C}(s'_1 + \cdots + s'_{g-1}) = \mathbf{C}(S^{(g-1)}). \tag{9}$$

Let κ denote the canonical class on S (i.e., image of Z in $\text{Pic}(S)^{(2g-2)}$). Clearly, the birational automorphism h of $\mathbf{C}(S^{(g-1)})$ induced by mapping $s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}$ to $s'_1 + \cdots + s'_{g-1}$ can be identified with the automorphism of $\Theta = \Psi_{g-1}(S^{(g-1)})$ obtained by mapping $x \in \Theta$ to $\kappa - x \in \Theta$ (i.e., the *canonical involution*; see Section 1).

Now consider the morphisms of (4)

$$\lambda_i : S^{g-1} \to S^{(g-1)}, \qquad i = 1,2. \tag{10}$$

Indeed, since S^{g-1} is nonsingular (and therefore normal) it is clear that we identify λ_1 (resp., λ_2) by taking the *normalization* of $S^{(g-1)}$ in the field $\mathbf{C}(s_1^{\text{gen}},\ldots,s_{g-1}^{\text{gen}})$ (resp., $\mathbf{C}(s'_1,\ldots,s'_{g-1})$) as described in [4, Part III, Chapter 8].

§3. Characterizations of hyperellipticity

Denote by D_{g-1} the discriminant locus of $\Lambda_{g-1}: S^{g-1} \to S^{(g-1)}$:

$$D_{g-1} = \{\Lambda_{g-1}(s_1, \ldots, s_{g-1}) \mid s_i = s_j \text{ for some } i \neq j\}.$$

Note that Λ_{g-1} is a Galois cover with group S_{g-1} (the symmetric group of degree $g-1$). In generic point notation $\mathbf{C}(s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})/\mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}})$ is an extension of degree $(g-1)!$ whose group of automorphisms, denoted $G(\mathbf{C}(s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})/\mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}))$ (or by $G(S^{g-1}/S^{(g-1)})$ when there can be no confusion) is identified with the collection of permutations of $s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$. Let Δ_{ij}, $1 \leqq i < j \leqq g-1$ be the subvariety of S^{g-1} which has generic point $(m_1, \ldots, m_{g-1})$ where $m_k = s_k^{\text{gen}}$ for $k \neq j$ and $m_j = s_i^{\text{gen}}$. Since $G(S^{g-1}/S^{(g-1)})$ maps the Δ_{ij} transitively among each other, D_{g-1} is the image of the irreducible variety Δ_{12}, and thus D_{g-1} is irreducible.

Theorem 5. *Denote the discriminant locus of λ_i in* (10) *by $D_{g-1}^{(i)}$. Then $D_{g-1}^{(i)}$ is irreducible and of dimension $g-2$, $i = 1, 2$. Furthermore, S is hyperelliptic if and only if $D_{g-1}^{(1)} = D_{g-1}^{(2)}$.*

Proof. The first sentence follows from the observations made prior to the theorem about D_{g-1}. Note, also, that the curve C_1 on D_{g-1} which has generic point $\Lambda_{g-1}(s_1^{\text{gen}}, \ldots, s_1^{\text{gen}}) = (g-1)s_1^{\text{gen}}$ (as a divisor) can be characterized as the unique irreducible curve in D_{g-1} all of whose points appear in D_{g-1} with multiplicity $g-3$. Indeed, the proof of Lemma 1 turns this into a similar computation for the discriminant locus of the cover

$$(A^1)^n \xrightarrow[\Lambda_n]{} A^n \qquad \text{with } n = g-1$$

(i.e., replace S by the affine line A^1). If $(t_1, \ldots, t_n)$ is an n-tuple of functions that parametrize a neighborhood of a point $(s_0, \ldots, s_0) \in (A^1)^n = A^n$ then we count the multiplicity of the image point under Λ_n on $0 = \Pi_{i \neq j}(t_i - t_j) = D_t$ by expressing D_t in terms of the elementary symmetric functions $x_1 = t_1 + \cdots + t_n, \ldots, x_n = t_1 \cdots t_n$. This is a classical resultant computation [3] and the lowest homogeneous part is a polynomial of degree $n-1$ in $x_1, \ldots, x_n$. Thus the largest integer k for which all the partial derivatives of D_t of order k in $x_1, \ldots, x_n$ vanish at $\mathbf{x} = (0, \ldots 0)$ is $n-2$. Similar computation for points on $S^{(g-1)}$ that are not in the curve C_1 shows that their multiplicity is less than $g-3$ on D_{g-1}, and this completes the characterization of C_1.

Denote the corresponding curve for $D_{g-1}^{(i)}$ by $C_1^{(i)}$, $i = 1, 2$. Therefore $D_{g-1}^{(1)} = D_{g-1}^{(2)}$ if and only if

$$C_1^{(1)} = C_1^{(2)}. \tag{11}$$

The general point on S is not a Weierstrass point. Therefore there is a unique (up to constant multiple) holomorphic differential ϑ on S having $(g-1)s_1^{\text{gen}}$ in its support. Since $\lambda_1 = \lambda_2 \circ h$ (as in the introduction) $D^{(1)}_{g-1} = D^{(2)}_{d-2}$ if and only if h maps D_{g-1} into itself. We conclude that (11) holds if and only if the divisor of ϑ is

$$(12) \qquad (g-1)s_1^{\text{gen}} + (g-1)s_1''$$

where s_1'' is also a generic point for S. If $\mathbf{C}(s_1^{\text{gen}}) \neq \mathbf{C}(s_1'')$ (say $\mathbf{C}(s_1'') \not\supseteq \mathbf{C}(s_1^{\text{gen}})$) then there exists a field isomorphism $\lambda : \mathbf{C}(s_1^{\text{gen}}, s_1'') \to T_{\text{un}}$ fixed on $\mathbf{C}(s_1'')$ such that $\lambda(s_1^{\text{gen}}) \neq (s_1^{\text{gen}})$. The argument of the proof of Lemma 4 thereby gives a nonconstant function f on S whose divisor of poles is contained in $(g-1)s_1^{\text{gen}}$, contrary to our understanding that s_1^{gen} is not a Weierstrass point. Thus we conclude that (11) holds if and only if

$$(13) \qquad \mathbf{C}(s_1^{\text{gen}}) = \mathbf{C}(s_1'').$$

The remainder of the proof consists of showing that (13) holds if and only if S is hyperelliptic. Let C be the curve in $S^{(2)}$ with generic point $s_1^{\text{gen}} + s_1''$. Since each point $c \in C$ represents a divisor of degree 2 for which $(g-1)c$ is in the class of κ (as in (9)), the class $[c]$ of c in $\text{Pic}(S)^{(2)}$ is constant. Thus $C \cong \mathbf{P}^1$ and $S \to C$ by mapping s_1^{gen} to $s_1^{\text{gen}} + s_1''$ is a degree 2 map. Clearly, therefore, S is hyperelliptic. ■

Remark 6. The curve C in the proof of Theorem 5 is the curve with property (5) that appears in the introduction.

§4. Proof of the main theorem

We continue the notation of Section 3.

Lemma 7. *The algebraic set $\psi^{-1}(Z)$ is irreducible if and only if $s_j', s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ is an algebraically independent set of generic points for S, $j = 1, \ldots, g-1$.*

Proof. First note that $\psi^{-1}(Z)$ is irreducible if and only if each point of $\psi^{-1}(Z)$ lying above generic point $D^{\text{gen}} = s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}} + s_1' + \cdots + s_{g-1}'$ of Z is a generic point of $\psi^{-1}(Z)$. That is, for (D_1, D_2) and $(D_1^*, D_2^*) \in \psi^{-1}(Z)$ with $D_1 + D_2 = D_1^* + D_2^* = D^{\text{gen}}$, there exists a field isomorphism that takes the coordinates of D_1 to those of D_1^* and is fixed on the coordinates of D^{gen}. If $\psi^{-1}(Z)$ is irreducible,

$$D_1 = s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}, \qquad D_1^* = s_j' + s_2^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}},$$

then there is a field isomorphism between $\mathbf{C}(D_1)$ (the field generated by ratios of projective coordinates for D_1) and $\mathbf{C}(D_1^*)$ that takes D_1 to D_1^*. Since $\mathbf{C}(s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})/\mathbf{C}(D_1)$ is a finite extension (of degree $(g-1)!$), the transcendence dimensions of $\mathbf{C}(s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$ and $\mathbf{C}(D_1)$ are the same. But the isomorphism between $\mathbf{C}(D_1)$ and $\mathbf{C}(D_1^*)$ implies that they, too, have the same transcendence dimensions. Thus $s_j', s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ are algebraically independent generic points.

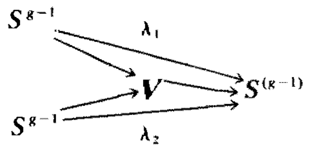
Conversely, if $s'_j, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ are generic points, then there is a field isomorphism $\lambda : \mathbf{C}(s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}) \to \mathbf{C}(s'_j, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$ that leaves $s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ fixed and maps s_1^{gen} to s'_j. Since $s'_j, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ are algebraically independent generic points, there is only one holomorphic differential ϑ (up to a constant multiple) on S having $s'_j, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ in its support, and this must therefore be ϑ. The extension of λ to a field over which ϑ is defined must map ϑ to a constant multiple of itself. In particular λ leaves D^{gen} (in the paragraph above) fixed. Continue this argument inductively, to show that such a λ can be constructed that maps (D_1, D_2) to any $(D_1^*, D_2^*) \in \psi^{-1}(Z)$ that lies over D^{gen}. ■

Theorem 8. *If S is a nonhyperelliptic surface of genus $g \geqq 2$, then $\psi^{-1}(Z)$ is irreducible. In particular, for $z \in Z$ suitably general, for each $(x_1, x_2) \in \psi^{-1}(z)$,*

(14) *x_1 is the only positive divisor linearly equivalent to x_1.*

Proof. Use the notation of Lemmas 4 and 7. To prove the irreducibility of $\psi^{-1}(Z)$ we have only to show that $s'_j, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ are algebraically independent generic points for each $j = 1, \ldots, g-1$. For simplicity we assume that $j = 1$, and that $s'_1, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ are *not* algebraically independent. That is, $L = \mathbf{C}(s'_1, s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$ is an algebraic extension of $M = \mathbf{C}(s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$.

Let $L^{\text{gen}} = \mathbf{C}(s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$, $L' = \mathbf{C}(s'_1, \ldots, s'_{g-1})$ and $K = \mathbf{C}(s_1^{\text{gen}} + \cdots + s_{g-1}^{\text{gen}}) = \mathbf{C}(s'_1 + \cdots + s'_{g-1})$ (as in the proof of Lemma 4). We claim that $L^{\text{gen}} \cap L' = K$. If not, then $L^{\text{gen}} \cap L'$ is a finite extension of K that corresponds to a variety V that is a quotient of S^{g-1} by the action of a (normal) subgroup H of S_{g-1}. Thus V sits in a commutative diagram (notation as prior to Theorem 5).



But, if the degree of V over $S^{(g-1)}$ exceeds 1, then $V \to S^{(g-1)}$ must be ramified (again, revert this to the argument of Lemma 1 for $(A')^n \to A^n$, as in the first paragraph of the proof of Theorem 5). Thus, from Theorem 5, the degree of V over $S^{(g-1)}$ is 1 and $[L^{\text{gen}} \cap L' : K] = 1$ (i.e., $L^{\text{gen}} \cap L' = K$).

From [3; p. 198] the Galois group $G(L^{\text{gen}} L'/K)$ is isomorphic to the product $G(L^{\text{gen}}/K) \times G(L'/K) = S_{g-1} \times S_{g-1}$.

In particular for any integer i, $\sigma = (1, (1i)) \in G(L^{\text{gen}} L'/K)$. On the field $L^{\text{gen}} L'$, σ acts by leaving $s_1^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}}$ fixed, and by sending s'_1 to s'_i. Thus s'_i is also algebraic over $M = \mathbf{C}(s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$, $i = 1, \ldots, g-1$. But $\mathbf{C}(s_2^{\text{gen}}, \ldots, s_{g-1}^{\text{gen}})$ has transcendence dimension $g-2$ and $\mathbf{C}(s'_1, \ldots, s'_{g-1})$ has transcendence dimension $g-1$. This is a contradiction that establishes that $\psi^{-1}(Z)$ is irreducible.

Condition (14) follows easily from the irreducibility of $\psi^{-1}(Z)$. Indeed, the map $\Psi_{g-1} \circ \mathrm{pr}_i : \psi^{-1}(Z) \rightarrow \Theta$ is one-one off of Θ_{sing} by Riemann's theorem. Since all the points of $\psi^{-1}(Z)$ that lie over $s_1^{\mathrm{gen}} + \cdots + s_{g-1}^{\mathrm{gen}} + s_1' + \cdots + s_{g-1}'$ map by pr_i to generic points of $S^{(g-1)}$, the statement before Riemann's theorem in Section 1 shows that (14) holds. ■

Corollary 9 (Noether's Theorem). *Let $\omega_1, \ldots, \omega_g$ be a basis for the space of holomorphic differentials. Then $3g-3$ of the products $\omega_i \cdot \omega_j$ can be chosen to serve as a basis for the space of holomorphic quadratic differentials.*

Proof. Let ϑ be a holomorphic differential on S with property (1). Then the space of meromorphic functions on S whose poles are included in the support of (ϑ) is g dimensional. Let the $2g-2$ distinct zeros of ϑ be denoted by $P_1 \cdots P_{g-1} Q_1 \cdots Q_{g-1}$. Then a basis for the above-mentioned space or functions can be chosen as $1, f_1, \ldots, f_{g-1}$ where f_i has simple poles at precisely the points $P_1 \cdots P_{g-1} Q_i$.

It is now immediate that $1, f_1, \ldots, f_{g-1}, f_1^2, f_1 f_2, \ldots, f_1 f_{g-1}, f_2^2, f_3^2, \ldots, f_g^2$ are $3g-3$ linearly independent functions on S so that $\vartheta^2, f_1\vartheta^2, \ldots, f_g^g\vartheta^2$ are $3g-3$ linearly independent holomorphic quadratic differentials. These $3g-3$ quadratic differentials have been chosen from the product of the basis $\vartheta, f_1\vartheta, \ldots, f_{g-1}\vartheta$ of the holomorphic differentials. ■

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