PROPOSAL TO AMS: VON NEUMANN SYMPOSIUM
ARITHMETIC FUNDAMENTAL GROUPS AND NONCOMMUTATIVE ALGEBRA: AUGUST 1999

MIKE FRIED, DAVID HARBATER, LANCE SMALL

1. Organizing committees

1.1. **Arithmetic Fundamental Groups.** Eva Bayer, Michael Fried, David Harbater (chair), Yasutaka Ihara, B. Heinrich Matzat, Michel Raynaud and John Thompson.

1.2. **Non-commutative Algebra.** Michael Artin, Susan Montgomery, Claudio Procesi, Lance Small (chair), Toby Stafford and Efim Zelmanov.

2. Introduction.

This document explains the events for a two week von Neumann symposium on Arithmetic Fundamental groups and Non-Commutative Algebraic Geometry, Monday, August 16 to Friday, August 27, 1999. Two main themes link these areas: group actions and deformations. Geometric considerations have dominated recent activity in both areas, counter to the classical view of both subjects as primarily algebraic. This proposed symposium will join algebraic geometry, field theory, number theory, representation theory, topology, ring theory, and group theory. A related document will appear at this site giving details on speakers.

The symposium will present reviews of current mathematical developments including many special talks appropriate for non-experts (especially students and postdoctorals). These will take the form of interrelated minicourses supplemented by individual up-to-date research lectures. Our proposal is to hold this symposium at MSRI, which has agreed to host such an event and to provide logistic support. Most participants in the symposium would come to MSRI only for this event. Still, a solid core of participants would stay longer (using other funding), to be part of two corresponding special programs:

- Fall 1999 semester at MSRI: a one-semester program on Galois theory and arithmetic fundamental groups, and
- the first semester of a year-long program on noncommutative algebraic geometry.

The von Neumann symposium offers both MSRI semesters an opportunity to take advantage of important connections between the areas.

Date: December 31, 1998.

3. Subject areas

This proposal emphasizes the geometry of the two areas. Still, to simplify terminology we use *Galois theory* and *noncommutative algebra* to refer, respectively, to Arithmetic Fundamental Groups and noncommutative algebraic geometry.

3.1. Galois theory topics.

Extensions of function fields give Galois theory its fundamentally geometric flavor. This holds even when the main consideration is to decipher information about the absolute Galois group of the rational numbers $G_{\mathbb{Q}}$. For one, the main progress on the Inverse Galois problem in recent years has been through geometric (*regular*) realizations that find the field of definition of a Galois curve cover (with its automorphisms) of the Riemann sphere. This is a moduli space approach. It operates by applying intricate knowledge of representations of the *Artin braid group*, and various of its quotients, like the *mapping class group*.

Geometric information appears from considering quotients of several classical fundamental groups. The Inverse Galois problem is the most well-known application. Still, myriad others arise in large-expander graph theory, cryptography, constructions coming from polynomial mappings over finite fields, the study of variables separated equations, explicit forms on Hilbert's irreducibility theorem, and other constructive aspects of algebra.

The absolute Galois group of $\mathbb{Q}$ acts on the fundamental group of a variety over a number field. This gives geometric information about covers from quotient actions of $G_{\mathbb{Q}}$. The classical model was the study of explicit abelian extensions of complex quadratic fields, coming from the action of $G_{\mathbb{Q}}$ on division points of elliptic curves: the theory of *complex multiplication*. A complementary deformation approach uses degenerate covers and their deformations. This has been especially valuable in analyzing fundamental groups over fields of positive characteristic and to understand the complete absolute Galois group of classical function fields. Some achievements here include Grothendieck's anabelian conjecture, Abhyankar's conjecture, and the geometric case of Shafarevich's conjecture.

3.2. Noncommutative algebra topics.

Areas of strong recent interest in noncommutative ring theory include Hopf algebras and quantum groups. Physics motivated the idea of quantum equivalence of field theories by noting how to reconstruct a conformal field theory from its operator algebra. Here, the very laws of associativity and group-like elements introduce old objects in a new form. This, for example, is how the Grothendieck-Teichmüller group $\widehat{GT}$ arose with the companion revelation that $\widehat{GT}$ contains $G_{\mathbb{Q}}$. The general study is of noncommutative analogs of rings of functions on affine group schemes in algebraic geometry. These often arise as deformations of algebras in the commutative case, where there really is an underlying group scheme.

Even projective varieties have noncommutative generalizations through the deformation of the structural formulas for their functions fields. This was an idea started by Artin, van den Bergh, Stafford, et al.. Example: Deformation of the function field of an abelian variety to a *noncommutative abelian variety* encodes variation of the canonical symplectic structure on the abelian variety.

3.3. Research collaboration between the two conference areas.

The symposium would increase interaction between the rubrics of Galois groups and noncommutative algebra. Some of this is heavy mathematics. Still, as the objects of

Galois theory find increasing application, they benefit from the systematic computational tools of noncommutative algebra, including cyclic cohomology and various forms of nonabelian Hodge theory.

Classical Lie algebras and noncommutative group rings appear in both subjects. One appearance comes from the profinite fundamental group acting through modular representations, the other from an L^2 norm on the classical fundamental group. The former stems from objects called *Modular Towers*, generalizing classical systems of modular curve covers. The latter produces characteristic classes on algebraic varieties whose structure has been previously a mystery. One benefit of a joint symposium would be to have these two groups of researchers educate each other on the disparate techniques for practical understanding of both modular and $\mathbb{C}^*$ representations.

The proposed symposium seeks to emphasize potential progress in applications and to focus attention on common mathematical goals and problems. One example here is to combine expertise from both areas to decide cohomological obstructions to the unirationality of reduced Hurwitz spaces. This practical application would conclude the Inverse Galois problem in a strong form. The leaders at this conference expect to encourage sharing of techniques and approaches and to make problems and progress more widely known. In addition, we seek to bring new people into the field, by systematically presenting the key ideas.

3.4. **The intended audience.** The symposium has three intended audiences:

- researchers from related fields interested in an overview of this area;
- postdoctorals and advanced graduate students in this area who wish to gain a broader perspective on the whole area; and
- senior researchers in the area who could broaden their perspective from hearing more about the full scope of the subject.

4. Structure and Content of the symposium

Introductory lectures would present a program overview, the current state of Galois theory research, and the current state of noncommutative algebra research. These serve also as preparation for the conjoining MSRI semesters (§6). Lectures emphasizing the potential for mutual research benefits (especially through group actions and deformation theory) will get special billing. The organizing committee will remind all speakers there are few general experts and it will make suggestions for assuring that speakers consider their intended audiences.

The bulk of the lectures will fall into one of a series of mini-courses, each of three or four lectures, and each systematically presenting one mathematical aspect. These mini-courses plan to expose connections between topics coherently so speakers can take advantage of other presentations to remain in the grasp of their audience. Several organizing committee members will help speakers plan this. In addition, there will be several individual lectures on special topics. Some would be more advanced, on current research topics, and one or more mini-courses would preceed these to provide the necessary background. Again, the substance of the talks would be coordinated, so as to avoid gaps or overlaps. Each day we plan to have four or five lectures, with no parallel sessions. The rest of the time will be left open for mathematical discussions.

5. Main minicourse topics

There will be no no overlap between mini courses and advanced research seminars, and there will be no parallel sessions. The organizers will attach advanced seminars to mini-courses. Some mini-courses will act as prerequisites to several advanced seminars. Mini-course speakers will divide their time so those experienced (though maybe not expert) in the area can calculate the proportion of the mini-course valuable to their expertise.

Example: A student in arithmetic geometry may have had experience with Galois groups, fundamental groups and Hilbert's irreducibility Theorem, so may not need the first two hours of the mini-course on that topic. The 3rd hour, however, including projective profinite groups, might be necessary to assure understanding on an advanced seminar talk on fundamental groups of projective curves in positive characteristic.

5.1. **Arithmetic Fundamental Groups.**

5.1.1. *Galois groups, fundamental groups, and Hilbert's Irreducibility Theorem.* This will focus on geometric aspect of Galois groups, and on how Hilbert's theorem allows the realization of arithmetic Galois groups by specialization from a cover of spaces defined over a number field. Other topics will include: Noether's problem, the approach via group actions, invariant theory and generic extensions. Galois module structure and Galois cohomology will appear in discussions related to projective profinite groups.

5.1.2. *Hurwitz spaces and deformation of covers.* This will start with a classical approach to Hurwitz spaces including the irreducibility of various moduli space of curves. Then lectures on the arithmetic properties of Hurwitz spaces will show how to investigate the fine structure of fields of moduli and fields of definition of covers and their associated families. Special cases will tie to modular curves. Lectures will conclude with information about geometric and arithmetic fundamental groups (in part using Hilbert's Irreducibility Theorem). The Inverse Galois problem is just one problem that reverts to finding $\mathbb{Q}$ points on a Hurwitz space. Compatible with this, we will show how modular curve computations generalize to include finding Hurwitz spaces with many rational points.

5.1.3. *Constructive Galois theory and various forms of rigidity.* Rigidity is a special property of groups and their generators. Rigid Hurwitz spaces are a particularly simple type. General results on Hurwitz spaces come from interpretations of the action of the Artin braid group. A choice of generators of the group in question determines this action, and if the generators are suitably *nice* the group has a regular realization as a Galois group over $\mathbb{Q}$. These lectures will discuss this method, applications to such realizations as the *Monster group* and realizations of the alternating group that eliminate Serre's obstruction to realizations of the spin group extension of the alternating group. Identifying components of a Hurwitz space reverts to generalizing a technique using the Schur multiplier of a group. Continuing previous lectures, we introduce techniques for investigating the cohomology of Hurwitz spaces, and for using *cyclic cohomology* to produce their characteristic classes.

5.1.4. *Comparison theorems in mixed characteristic.* Paths don't exist in positive characteristic. Still, unramified covers do. Grothendieck supplied comparison theorems between characteristic 0 and characteristic p that are the forerunner of how to use the profinite fundamental group in positive characteristic. Applications to finite fields abound from extending these theorems, particularly in cases of semistable reduction or in which there is other control on singularities.

5.1.5. *Patching methods.* The most successful technique for constructing quotients of fundamental groups in positive characteristic applies *patching methods*. This is an algebraic analog of the 19th century *cut and paste* constructions for Riemann surfaces. Using rigid analytic spaces or formal schemes this approach combines with comparison theorems. The lectures will discuss recent contributions of this method to solving embedding problems in Galois theory, the structure of π_1 as a profinite group, and Grothendieck's anabelian conjecture.

5.1.6. *Galois actions on moduli spaces.* Moduli spaces of curves appear throughout this theory. The data may including markings, or a fixed cover to the projective line, or some kind of differential. When such spaces have $\mathbb{Q}$ as field of definition, the absolute Galois group $G_{\mathbb{Q}}$ acts on their fundamental groups, providing geometric properties of $G_{\mathbb{Q}}$. These lectures will discuss this approach and how it produces the Grothendieck-Teichmüller group $\widehat{GT}$. In particular, this will include use of Deligne's tangential base points, and the explicit view they give of $G_{\mathbb{Q}}$ and various of its near nilpotent quotients.

5.2. **Non-commutative algebra.**

5.2.1. *Classical ring theory.* Non-commutative ring theory, including rings satisfying a polynomial identity, group rings, Gelfand-Kirillov dimension, Noetherian rings, constitutes the *infra-structure* of the other components of our program. These lectures will include more recent applications and developments of these mature subjects, including infinite dimensional division rings, spectra of quantum groups, and enveloping algebras of Lie superalgebras.

5.2.2. *Hopf algebras.* A surge of activity in Hopf algebras started with the introduction of quantum groups by Drinfel'd and Jimbo in 1986. Further, several old conjectures of Kaplansky on the structure of finite-dimensional Hopf algebras have been solved. The lectures will include basics on quantum groups, finite dimensional Hopf algebras, and actions of Hopf algebras on algebras (including connections to invariant theory and Galois theory). These results unify and extend earlier results on automorphism groups, derivations, and group gradings of rings.

5.2.3. *Algebraic aspects of quantum groups.* Extending the previous minicourse, we consider the following quantum group topics: quantization of Poisson groups, quantum groups at roots of one, representations of quantum groups, and Yang Baxter equations. As time permits we discuss connections to quantum cohomology, knot invariants, representation theory of Kac Moody algebras, and string theory.

5.2.4. *Non-commutative projective geometry.* The previous two mini-courses studied noncommutative affine geometry. This mini-course considers how to adapt classical *projective* algebraic geometry to the study of a restricted class of noncommutative rings. It will consider the foundations of noncommutative projective geometry though Sklyanin's discovery of rings based on the geometry of elliptic curves and the eight-vertex model. This will include classification results and open problems.

5.2.5. *Lie algebras and group theory.* These lectures will emphasize the relation between *asymptotic group theory* and the theory of noncommutative infinite dimensional algebras. Topics include the restricted Burnside problem, p-adic analytic groups, infinite dimensional algebras of finite growth, and pro-p groups in topology and number theory.

6. Costs and participants

We have $30,000 to cover travel costs and some local expenses for most of the more junior participants and for some senior participants. MSRI will cover hosting expenses. They will also cover the expenses of those participants who will stay at MSRI afterwards for the for Arithmetic Fundamental Groups and Non-commutative algebra workshops. We expect to have about 70 participants in the symposium, and many mathematicians (both junior and senior) have so far expressed interest in participating. In general, though, it will be necessary to cover at least part of their expenses to enable them to participate.

6.0.6. *General participant philosophy.* The organizers will actively seek participants from a wide variety of backgrounds, including recent Ph.D.'s and students, and individuals who are not currently working in the area but who are interested in learning more about the material to be presented at this symposium. Especially, they will try to include among the participants women and underrepresented minorities.

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