

Algebrizations and Quasi-Multipliers of an Operator Space

What are the Possible Operator Algebra
Products a Given Operator Space Can be
Equipped With?

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Introduction

One of the most interesting questions in the operator space theory was “What are the possible operator algebra products that a given operator space can be equipped with?”. In my Ph.D. thesis, I answered this question using quasi-multipliers and the Haagerup tensor product. Quasi-multipliers of operator spaces were defined by Paulsen in late 2002 as a natural variant of one-sided multipliers of operator spaces which had been introduced by Blecher around 1999. However, the significant relation between quasi-multipliers and operator algebra products was discovered and proved by myself in early 2003.

Question Let X be an operator space, and let $m : X \times X \rightarrow X$ be a bilinear mapping. When is m an operator algebra product? That is, when does there exist a complete isometry ϕ from X into $\mathbb{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} such that $\phi(m(x, y)) = \phi(x)\phi(y)$, $\forall x, y \in X$?

Preliminaries

Recall that X can be embedded in an off-diagonal corner of a sequence of C^* -algebras:

$$\begin{aligned} \begin{bmatrix} O & X \\ O & O \end{bmatrix} &\subset \begin{bmatrix} \mathcal{T}(X) \bullet \mathcal{T}(X)^* & \mathcal{T}(X) \\ \mathcal{T}(X)^* & \mathcal{T}(X)^* \bullet \mathcal{T}(X) \end{bmatrix} \\ &\subset \begin{bmatrix} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{bmatrix} =: I(S_X), \end{aligned}$$

where $\mathcal{T}(X)$ is a triple envelope of X and $I(X)$ is an injective envelope of X .

Definition (Paulsen) The **quasi-multiplier space** of an operator space X is the set

$$\mathcal{QM}(X) := \{z \in I(X)^*; XzX \subset X\}.$$

We call an element of $\mathcal{QM}(X)$ a **quasi-multiplier** of X .

Theorem (Kaneda): Let X be an operator space with a bilinear mapping $m : X \times X \rightarrow X$, and let 1 be the identity of $I(\mathcal{S}_X)$. Let

$$\Gamma_m : \begin{array}{c} \mathbb{M}_2(I(\mathcal{S}_X) \otimes_h I(\mathcal{S}_X)) \\ \cup \\ \begin{bmatrix} X \otimes_h \mathbb{C}1 & X \otimes_h X \\ O & \mathbb{C}1 \otimes_h X \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \mathbb{M}_2(X) \\ \cup \\ \begin{bmatrix} X & X \\ O & X \end{bmatrix} \end{array}$$

be defined by

$$\Gamma_m \left(\begin{bmatrix} x_1 \otimes 1 & x \otimes y \\ 0 & 1 \otimes x_2 \end{bmatrix} \right) := \begin{bmatrix} x_1 & m(x, y) \\ 0 & x_2 \end{bmatrix}$$

and their linear extension. Then, the following are equivalent:

- (i) (X, m) is an operator algebra (i.e., there is a completely isometric homomorphism from (X, m) onto a concrete operator algebra),
- (ii) there exists a $z \in QM(X)$ with $\|z\| \leq 1$ such that $m(x, y) = xzy, \forall x, y \in X,$
- (iii) Γ_m is a complete contraction.

Moreover, such a z is unique.

When these conditions hold, we call m an **operator algebra product** on X , and denote m by m_z .

Some of the above results were published in [Kaneda-Paulsen, *Journal of Functional Analysis* 217 (2) (2004), 347-365].

Pedersen's Question in the GPOTS 2003 What are $\text{ext}(Ball(QM(X)))$ the extreme points of the unit ball of a quasi-multiplier space $QM(X)$?

My answer to this question is that $\text{ext}(Ball(QM(X)))$ is strongly connected to (approximate) identities. We introduce the new notion (approximate) “quasi-identity” which is a generalization of one-sided identity.

Definition (Kaneda) Let \mathcal{R} be a ring. A **quasi-identity** of \mathcal{R} is an element $e \in \mathcal{R}$ such that $r = er + re - ere, \forall r \in \mathcal{R}$.

The idea of the definition

One may rewrite the definition of quasi-identity as

$$(1 - e)\mathcal{R}(1 - e) = \{0\}.$$

Compare with Kadison's theorem:

Theorem (Kadison) Let \mathcal{A} be a C^* -algebra. Then $a \in \text{ext}(Ball(\mathcal{A}))$ if and only if

$$(1 - aa^*)\mathcal{A}(1 - a^*a) = \{0\}.$$

Identities, left identities, right identities of rings are quasi-identities.

Furthermore, in the operator algebra case, a quasi-identity has the following nice properties. These convince us that at least in the operator algebra case, the notion of quasi-identities is a natural generalization of identities or one-sided identities.

Proposition (Kaneda)

- (1) A contractive quasi-identity is unique. In particular, if an operator algebra \mathcal{A} has an identity or one-sided identity, then it is the only contractive quasi-identity of \mathcal{A} .
- (2) A contractive quasi-identity of an operator algebra $\mathcal{A} (\subset \mathbb{B}(\mathcal{H}))$ is necessarily an idempotent, and hence self-adjoint.

Theorem (Kaneda)

Let X be an injective operator space, and $z \in \text{Ball}(\mathcal{QM}(X))$, and (X, m_z) be the corresponding operator algebra. Let 1_{11} and 1_{22} be the identities of the C^* -algebras $I_{11}(X)$ and $I_{22}(X)$, respectively. Then the following statements hold.

1. (X, m_z) has a quasi-identity of norm 1 if and only if $z \in \text{ext}(\text{Ball}(X))$.
2. (X, m_z) has a left identity of norm 1 if and only if $zz^* = 1_{11}$.
3. (X, m_z) has a right identity of norm 1 if and only if $z^*z = 1_{22}$.
4. (X, m_z) has a two-sided identity of norm 1 if and only if $zz^* = 1_{11}$ and $z^*z = 1_{22}$.

Together with the theorem below noting that $QM(X) = X^*$, it follows that an injective operator space always can be made an operator algebra with a quasi-identity for some operator algebra product.

Theorem (Kaneda) Let X be an injective operator space. Then $\text{ext}(Ball(X)) \neq \emptyset$.

Question Is $\text{ext}(Ball(QM(X))) \neq \emptyset$ for any operator space X ?

C^* -algebras and their one-sided ideals

The following gives a characterization of C^* -algebras.

Theorem (Kaneda) Let X be an operator space, and $z \in \text{Ball}(\mathcal{QM}(X))$. Then (X, m_z) is a C^* -algebra with a certain involution \sharp if and only if $zz^* = 1_{22}$, $z^*z = 1_{11}$, and $X \bullet z = z^* \bullet X^*$. Involution \sharp is uniquely given by $x^\sharp = z^* \bullet x^* \bullet z^*$, $\forall x \in X$.

The following gives a characterization of one-sided ideals in C^* -algebras.

Theorem (Kaneda) Let X be an operator space, and $z \in \text{Ball}(\mathcal{QM}(X))$. Then (X, m_z) is completely isometrically isomorphic to a left (respectively, right) ideal in some C^* -algebra if and only if $zz^* = 1_{22}$ (respectively, $z^*z = 1_{11}$) and $z \bullet X = X^* \bullet X$ (respectively, $X \bullet z = X \bullet X^*$).

Further Projects

Problem 1: Characterize subalgebras of an operator algebra in terms of quasi-multipliers.

This will be useful in classifying operator algebras.

Problem 2: Characterize commutative operator algebras in terms of quasi-multipliers.

Solving Problems 1 & 2 will result in answering the 55 year old famous open question by Kaplansky: Are all AW^* -algebras monotone complete?