Quasi-Multipliers and Algebrizations of an Operator Space

Masayoshi KANEDA
E-Mail: kaneda@math.uh.edu
URL: http://www.math.uh.edu/∼kaneda/

Department of Mathematics
University of Houston

Thursday, May 29, 2003

Motivation

Theorem (Blecher) Let X be an operator space with a product φ. The product is completely bounded if and only if for every $\epsilon > 0$ there exists a Hilbert space $\mathcal{H}$ and a one-to-one completely bounded homomorphism $\pi : X \to \mathbb{B}(\mathcal{H})$ such that $\pi^{-1} : \pi(X) \to X$ is also completely bounded with $\|\pi\|_{cb}\|\pi^{-1}\|_{cb} \leq 1 + \epsilon$.

Questions

(1) When can $\epsilon = 0$ be attained?

(2) When can π be taken to be a completely isometric homomorphism? That is; when is (X, φ) an (abstract) operator algebra?

Quasi-multipliers answer the questions!

Recall (Ruan, Hamana (independently))

Let $X \subset \mathbb{B}(\mathcal{H})$ be an operator space, and let $\mathcal{S}(X)$ be Paulsen's operator system. Consider a minimal completely positive $\mathcal{S}(X)$-projection Φ on $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$.

$$
\begin{array}{rcl}
 & & \mathbb{M}_2(\mathbb{B}(\mathcal{H})) \\
 & & \cup \\
\mathrm{Im}\Phi = & I(\mathcal{S}(X)) = & \begin{bmatrix} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{bmatrix} \\
 & & \cup \\
 & \mathcal{S}(X) := & \begin{bmatrix} \mathbb{C}1_{\mathcal{H}} & X \\ X^* & \mathbb{C}1_{\mathcal{H}} \end{bmatrix}
\end{array}
$$

Φ can be factored as $\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}$, where ψ_1 and ψ_2 are completely positive, and ϕ is completely contractive, and $\phi^*(x^*) := \phi(x)^* \ \ \forall x \in X$.

ImΦ is an injective envelope of $\mathcal{S}(X)$ and also a unital C*-algebra with a new product $\odot$ defined by $\xi \odot \eta := \Phi(\xi\eta) \ \ \forall \xi, \eta \in$ ImΦ.

$I_{11}(X)$ and $I_{22}(X)$ are injective unital C*-algebras.

$I(X)$ is an injective envelope of X.

$\odot$ induces new products $\circ$ between elements of $I_{11}(X)$, $I_{22}(X)$, $I(X)$, and $I(X)^*$. For example, $x \circ y^* := \psi_1(xy^*)$ for $x \in I(X)$, $y^* \in I(X)^*$.

Definition The quasi-multiplier space for an operator space X is the set

$$IM_q(X) := \{z \in I(X)^*; X \circ z \circ X \subset X\}.$$

We call an element of $IM_q(X)$ a quasi-multiplier of X.

The quasi-multiplier space has the following universal property.

Theorem (M. Kaneda) If $X, Y \subset \mathbb{B}(\mathcal{H})$ are operator spaces such that $XYX \subset X$, then there exists a unique completely contractive linear mapping $\sigma : Y \to IM_q(X)$ such that $j(x) \circ \sigma(y) \circ j(x') = xyx'$, where $j : X \to I(X)$ is the canonical inclusion.

The case of o.a.'s with a two-sided c.a.i.

Definition Let A be an operator algebra. The quasi-centralizer space for A is the set

$$QC(A) := \{\rho : A \times A \to A;\ \rho(a, \cdot) \in CB_A(A) \text{ and } \rho(\cdot, a) \in {}_ACB(A)\ \forall a \in A\}.$$

We call an element of $QC(A)$ a quasi-centralizer.

Theorem (M. Kaneda) Let A be an operator algebra with a two-sided c.a.i.. Then each of the following sets contains a copy of A by the natural inclusion, and (I)-(IV) are all completely isometric via a mapping that fixes the copies of A.

(I) $\{z \in A^{**}; \hat{A}z\hat{A} \subset \hat{A}\}$,

(II) $\{T \in \mathbb{B}(\mathcal{H}); \pi(A)T\pi(A) \subset \pi(A)\}$ where $\pi : A \to \mathbb{B}(\mathcal{H})$ is a completely isometric nondegenerate representation,

(III) $IM_q(A)$,

(IV) $\mathcal{Q}C(A)$ (equipped with the completely multiplicative norm).

Moreover, these are maximum essential quasi-multiplier extensions in a certain sense.

Corollary In the C*-algebra case, our definition coincides with the existing one.

Theorem 1. (M. Kaneda) Let X be an operator space with a bilinear mapping $\varphi : X \times X \to X$. We regard $X \subset I(\mathcal{S}(X))$. Let

$$\begin{array}{cccc} & \mathbb{M}_2(I(\mathcal{S}(X)) \otimes_h I(\mathcal{S}(X))) & & \mathbb{M}_2(X) \\ & \cup & & \cup \\ \Gamma_\varphi : & \begin{bmatrix} X \otimes_h \mathbb{C}1 & X \otimes_h X \\ O & \mathbb{C}1 \otimes_h X \end{bmatrix} & \to & \begin{bmatrix} X & X \\ O & X \end{bmatrix} \end{array}$$

be defined by

$$\Gamma_\varphi \left(\begin{bmatrix} x_1 \otimes 1 & x \otimes y \\ 0 & 1 \otimes x_2 \end{bmatrix} \right) := \begin{bmatrix} x_1 & \varphi(x, y) \\ 0 & x_2 \end{bmatrix},$$

where 1 is the identity of $I(\mathcal{S}(X))$.

Then the following are equivalent:

(i) (X, φ) is an abstract operator algebra (i.e., there is a completely isometric homomorphism from X into a concrete operator algebra),

(ii) there exits a $z \in IM_q(X)$ with $\|z\| \leq 1$ such that $\forall x, y \in X, \;\; \varphi(x, y) = x \circ z \circ y$,

(iii) Γ_φ is completely contractive.

Moreover, such a z is unique.

When these conditions hold, we say that φ is an operator algebra product (OAP) on X, and denote the set of OAP on X by $OAP(X)$.

Remark

(1) We did not assume an identity or an approximate identity. So this theorem can be considered as a very general characterization of operator algebras.

(2) (i)$\Leftrightarrow$(iii) says that algebraic property (products) is completely characterized by the underlying geometric (matrix norm) structure. This can be considered as "quasi" version of Blecher-Effros-Zarikian's theorem which characterizes left (resp. right) multipliers in terms of the norm of column (resp. row) matrices.

(3) Associativity is automatic and $OAP(X)$ is a convex set.

Theorem 2. (M. Kaneda) Let X be an operator space with a bilinear mapping $\varphi : X \times X \to X$. Then the following are equivalent.

(i) there exists a Hilbert space $\mathcal{H}$ and a one-to-one completely bounded homomorphism $\pi : X \to \mathbb{B}(\mathcal{H})$ such that $\pi^{-1} : \pi(X) \to X$ is also completely bounded with $\|\pi\|_{cb}\|\pi^{-1}\|_{cb} = 1$,

(ii) there exists a $z \in IM_q(X)$ such that $\forall x, y \in X$ $\varphi(x, y) = x \circ z \circ y$.

When these conditions hold, we say that φ is a scaled operator algebra product (SOAP) on X, and denote the set of SOAP on X by $SOAP(X)$.

The following is an example of a completely contractive product, which is neither OAP nor SOAP.

Example Let A be any injective C*-algebra. Let $X := \begin{bmatrix} A \\ A \end{bmatrix}$, then $IM_q(X) = \begin{bmatrix} A & A \end{bmatrix}$. Give X a product $\varphi : X \times X \to X$ defined by $\varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) := \begin{bmatrix} ac \\ bd \end{bmatrix}$. Then it is easy to see that φ is completely contractive, but there is no $z \in IM_q(X)$ such that $\forall x, y \in X \;\; \varphi(x, y) = x \circ z \circ y$.

Final Remark Using these characterizations and “bootstrap type” another characterization of OAP by Paulsen imply many classical theorems (Blecher-Ruan-Sinclair representation theorem, Kadison’s isometry theorem, non-commutative Banach-Stone theorem, etc.) as corollaries, on which we are working now...