Algebrizations and Quasi-Multipliers of an Opearator Space

What are the Possible Operator Algebra Products a Given Operator Space Can be Equipped With?

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Abstract

One of the most interesting questions in the operator space theory was "What are the possible operator algebra products that a given operator space can be equipped with?". In my Ph.D. thesis, I answered this question using quasi-multipliers and the Haagerup tensor product. Quasi-multipliers of operator spaces were defined by Paulsen in late 2002 as natural variations of one-sided multipliers of operator spaces which had been introduced by Blecher around 1999. However, the significant relation between quasi-multipliers and operator algebra products was discovered and proved by myself in early 2003.

Introduction

A concrete operator space is a subspace of $\mathbb{B}(\mathcal{H})$ the set of bounded linear mapping on a Hilbert space $\mathcal{H}$.

A concrete operator algebra is a (not necessarily self-adjoint) subalgebra of $\mathbb{B}(\mathcal{H})$.

To present Ruan's abstract characterization of operator spaces which is free from Hilbert spaces, we introduce the notion of matrix norms.

A matrix norm on a linear space X is a set $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of functions $\|\cdot\|_n : \mathbb{M}_n(X) \to [0, +\infty)$.

A concrete operator space $X \subset \mathbb{B}(\mathcal{H})$ has a natural matrix norm with the identification

$$\mathbb{M}_n(X) \subset \mathbb{M}_n(\mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H}^n), \ \ \forall n \in \mathbb{N};$$

i.e.,

$$\|[x_{ij}]\|_n := \sup \left\{ \left\| \left[\sum_{j=1}^{n} x_{ij}\xi_j \right] \right\|^{\mathcal{H}^n} ; \xi \in \mathcal{H}^n, \|\xi\|^{\mathcal{H}^n} \leq 1 \right\}.$$

This particular matrix norm has the following properties: $\forall x_1 \in \mathbb{M}_m(X), \forall x_2 \in \mathbb{M}_n(X), \forall \alpha \in \mathbb{M}_{n,m}, \forall \beta \in \mathbb{M}_{m,n}$,

M1: $\|x_1 \oplus x_2\|_{m+n} = \max\{\|x_1\|_m, \|x_2\|_n\}$,

M2: $\|\alpha x_1 \beta\|_n \leq \|\alpha\|\|x_1\|_m\|\beta\|$.

Conversely, Ruan's Theorem says that if a matrix norm on a vector space X satisfies M1 and M2, then X is a concrete operator space "up to complete isometry".

Before stating the precise statement of Ruan's Theorem, let us define a "complete isometry".

Let X and Y be vector spaces with a matrix norm.

Let $\phi : X \to Y$ be a linear mapping. For each $n \in \mathbb{N}$, define $\phi_n : \mathbb{M}_n(X) \to \mathbb{M}_n(Y)$ by $\phi_n([x_{ij}]) := [\phi(x_{ij})], \ \forall [x_{ij}] \in \mathbb{M}_n(X)$.

ϕ is a complete contraction if ϕ_n is a contraction for each $n \in \mathbb{N}$.

ϕ is a complete isometry if ϕ_n is an isometry for each $n \in \mathbb{N}$.

$\boxed{\text{Example}}$ A linear mapping $\mathbb{M}_2 \to \mathbb{M}_2$ defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} b & a \\ d & c \end{bmatrix}$ is a complete isometry. However, the ‘transpose mapping’ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is an isometry, but NOT a complete isometry.

Now we are in a position to state the celebrated theorem by Ruan:

$\boxed{\text{Ruan's Theorem (1988)}}$ Let X be a vector space with a matrix norm satisfying M1 and M2. Then there is a complete isometry from X onto a concrete operator space.

Hence, we define an (abstract) operator space to be a vector space with a matrix norm satisfying M1 and M2. Note that this definition is free from a Hilbert space.

The operator space theory is the study of the category with objects of (abstract) operator spaces and morphisms of “complete contractions”, i.e., we identify two operator spaces up to one-to-one, onto, complete isometry.

We take the “embedding property” as the definition of an (abstract) operator algebra; i.e., a vector space with a matrix norm is an (abstract) operator algebra if there is a completely isometric homomorphism onto a concrete operator algebra.

One of the main results in this presentation is the striking characterization of (abstract) operator algebras.

In the GPOTS 2002

Kaneda: What are the possible operator algebra products that a given operator space can be equipped with?

Blecher: We can always equip any operator space X with the trivial product $xy = 0$, $\forall x, y \in X$.

Kaneda: I know. I am asking what "all" possible operator algebra products are, that a given operator space can be equipped with.

Blecher: ……

Therefore, this seemed to be an interesting question to pursue.

Let us state the question clearly:

Question Let X be an (abstract) operator space which is also an algebra with a product $\varphi : X \times X \to X$. When is X an (abstract) operator algebra?

Since X is an (abstract) operator space, by Ruan's theorem, there always exists (at least one) complete isometry ϕ from X onto a concrete operator space which is a subspace of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The essence of the question is; Can ϕ also be taken as an (algebraic) homomorphism?

Example The mapping $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \to \mathbb{M}_3$ defined by $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix}$ is a completely isometric homomorphism. However, the mapping defined by $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \begin{bmatrix} 0 & a & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}$ is a complete isometry, but NOT a homomorphism.

Preliminaries

Ruan, Hamana (independently)

Let $X \subset \mathbb{B}(\mathcal{H})$ be an operator space, and let $\mathcal{S}_X$ be Paulsen's operator system. Take a minimal completely positive (= unital completely contractive) $\mathcal{S}_X$-projection Φ on $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$.

$$\begin{array}{rcc} & & \mathbb{M}_2(\mathbb{B}(\mathcal{H})) \\ & & \cup \\ \mathrm{Im}\Phi = & I(\mathcal{S}_X) = & \begin{bmatrix} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{bmatrix} \\ & & \cup \\ & \mathcal{S}_X := & \begin{bmatrix} \mathbb{C}1_{\mathcal{H}} & X \\ X^* & \mathbb{C}1_{\mathcal{H}} \end{bmatrix} \end{array}$$

Φ can be factored as $\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}$, where ψ_1 and ψ_2 are completely positive, and ϕ is completely contractive, and $\phi^*(x^*) := \phi(x)^* \ \ \forall x \in X$.

ImΦ is an injective envelope of $\mathcal{S}_X$ and also a unital C*-algebra with the new product $\circledcirc$ defined by $\xi \circledcirc \eta := \Phi(\xi\eta) \ \ \forall \xi, \eta \in$ ImΦ.

$I_{11}(X)$ and $I_{22}(X)$ are injective C*-algebras.

$I(X)$ is an injective envelope of X.

$\circledcirc$ induces new products $\bullet$ between elements of $I_{11}(X)$, $I_{22}(X)$, $I(X)$, and $I(X)^*$. For example, $x \bullet y^* := \psi_1(xy^*)$ for $x \in I(X)$, $y^* \in I(X)^*$.

Also one may write the C^*-subalgebra $I(\mathcal{S}_X)$ generated by $\begin{bmatrix} O & X \\ O & O \end{bmatrix}$ as

$$\begin{bmatrix} \mathcal{T}(X) \bullet \mathcal{T}(X)^* & \mathcal{T}(X) \\ \mathcal{T}(X)^* & \mathcal{T}(X)^* \bullet \mathcal{T}(X) \end{bmatrix},$$

where $\mathcal{T}(X)$ is a triple envelope of X, i.e., the "minimum" TRO (ternary ring of operators) that contains X.

$\boxed{\text{Definition (Paulsen)}}$ The quasi-multiplier space of an operator space X is the set

$$\mathcal{QM}(X) := \{z \in I(X)^*; X \bullet z \bullet X \subset X\}.$$

We call an element of $\mathcal{QM}(X)$ a quasi-multiplier of X.

The universal property of $QM(X)$ is summarized as follows.

Proposition (Kaneda): Let X and Y be operator spaces. If $X, Y \subset \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $XYX \subset X$, then there exists a unique complete contraction $\sigma : Y \to QM(X)$ such that

$$\forall x_1, x_2 \in X, y \in Y, \ \ x_1 \bullet \sigma(y) \bullet x_2 = x_1 y x_2.$$

Note: The quasi-multipliers of operator spaces were defined by Paulsen in late 2002 as natural variations of one-sided multipliers which had been introduced by Blecher around 1999. However, the significant relation between quasi-multipliers and operator algebra products were discovered and proved by myself in early 2003. These results have been published in [Kaneda-Paulsen, Journal of Functional Analysis 217 (2) (2004), 347-365].

Furthermore, I also developed an elegant geometric characterization of operator algebra products using the Haagerup tensor product. So let us recall the Haagerup tensor product.

Haagerup Tensor Product

For the algebraic tensor product $X \otimes Y$ of two operator spaces X and Y with matrix norms $\{\| \cdot \|_n\}_{n\in\mathbb{N}}$ and $\{|\| \cdot \||_n\}_{n\in\mathbb{N}}$, respectively, there are many matrix norms that make it an operator space. Among them, here we use the Haagerup tensor norm, which plays a crucial role in our theorem and in many other places in the operator space theory.

For given $n \in \mathbb{N}$ and $w \in \mathbb{M}_n(X \otimes Y)$, we define

$$\|w\|_n^h := \inf\{\|x\| \cdot |||y||| : w = x \odot y, \\ x \in \mathbb{M}_{n,m}(X), y \in \mathbb{M}_{m,n}(Y), m \in \mathbb{N}\}. \tag{1}$$

Here, $x \odot y$ is an element of $\mathbb{M}_n(X \otimes Y)$ whose (i,j)-entry is given by $\sum_{k=1}^{m} x_{i,k} \otimes y_{k,j}$. It is known that such an expression exists for each element in $\mathbb{M}_n(X \otimes Y)$, so the right hand side of (1) is defined.

$\{\| \cdot \|_n^h\}_{n \in \mathbb{N}}$ becomes an operator space matrix norm on $X \otimes Y$, and call it the Haagerup tensor norm on $X \otimes Y$. We call a tensor product of operator spaces X and Y together with the Haagerup tensor norm a Haagerup tensor product, and denote it by $X \otimes_h Y$.

Now we are in a place to state the main theorem which answers our question.

Theorem (Kaneda): Let X be an operator space with a product $\varphi : X \times X \to X$, and let 1 be the identity of $I(\mathcal{S}(X))$. Let

$$\begin{array}{cccc} & \mathbb{M}_2\left(I(\mathcal{S}(X)) \otimes_h I(\mathcal{S}(X))\right) & & \mathbb{M}_2(X) \\ & \cup & & \cup \\ \Gamma_\varphi : & \begin{bmatrix} X \otimes_h \mathbb{C}1 & X \otimes_h X \\ O & \mathbb{C}1 \otimes_h X \end{bmatrix} & \to & \begin{bmatrix} X & X \\ O & X \end{bmatrix} \end{array}$$

be defined by

$$\Gamma_\varphi\left(\begin{bmatrix} x_1 \otimes 1 & x \otimes y \\ 0 & 1 \otimes x_2 \end{bmatrix}\right) := \begin{bmatrix} x_1 & \varphi(x, y) \\ 0 & x_2 \end{bmatrix}$$

and their linear extension.

Then, the following are equivalent;

(i) (X, φ) is an abstract operator algebra (i.e., there is a completely isometric homomorphism from X onto a concrete operator algebra),

(ii) there exists a $z \in QM(X)$ with $\|z\| \leq 1$ such that $\forall x, y \in X$, $\varphi(x, y) = x \bullet z \bullet y$,

(iii) Γ_φ is a complete contraction.

Moreover, such a z is unique.

When these conditions hold, we call φ an operator algebra product on X, and denote φ by m_z.

$\boxed{\text{Example}}$ Let us try to equip with a product $\varphi : X \times X \to X$ on the operator space $X := \begin{bmatrix} * & O \\ * & O \end{bmatrix}$. Let us consider the following candidates.

(1) $\varphi_1\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) := \begin{bmatrix} ac & 0 \\ bc & 0 \end{bmatrix}$.

(2) $\varphi_2\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) := \begin{bmatrix} ad & 0 \\ bd & 0 \end{bmatrix}$.

(3) $\varphi_3\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) := \begin{bmatrix} ac & 0 \\ ad & 0 \end{bmatrix}$.

(4) $\varphi_4\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) := \begin{bmatrix} \frac{ac}{\sqrt{2}} & 0 \\ \frac{bc}{\sqrt{2}} & 0 \end{bmatrix}$.

(5) $\varphi_5\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) := \begin{bmatrix} 2ac & 0 \\ 2bc & 0 \end{bmatrix}$.

(6) $\varphi_6\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) := \begin{bmatrix} ac & 0 \\ bd & 0 \end{bmatrix}$.

Conclusion: (1), (2), (4) are operator algebras, but (3), (5), (6) are NOT operator algebras.

To see this, first note that in this case, $X = I(X)$ and $\phi : X \to I(X)$ can be take as the identity. Moreover, $QM(X) = \begin{bmatrix} * & * \\ O & O \end{bmatrix}$. And use (i)⇔(ii) of the theorem.

Hence each product in (1), (2), (4) can be written as follows:

(1): $\begin{bmatrix} ac & 0 \\ bc & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$
with $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in QM(X), \left\| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\| = 1.$

(2): $\begin{bmatrix} ad & 0 \\ bd & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$
with $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in QM(X), \left\| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\| = 1.$

(4): $\begin{bmatrix} \frac{ac}{\sqrt{2}} & 0 \\ \frac{bd}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$
with $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \in QM(X), \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \right\| = \frac{1}{\sqrt{2}} = \leq 1.$

However,

(5): $\begin{bmatrix} 2ac & 0 \\ 2bd & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$
with $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \in QM(X)$, but $\left\| \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\| = 2 > 1.$

It is easy to check that there does not exist $q \in QM(X)$ such that $\varphi(x_1, x_2) = x_1 q x_2$ in (3), (6).

Note that (1) is simply the matrix product, so it is obvious that this is an operator algebra.

Another way to see that (2) is an operator algebra is; consider the following 'translation': $\begin{bmatrix} * & O \\ * & O \end{bmatrix} \ni \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in \begin{bmatrix} O & * \\ O & * \end{bmatrix}$ which is a complete isometry. Now the original product is 'translated' to the matrix product since $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & ad \\ 0 & bd \end{bmatrix}$.

It is interesting to note that (3) is similar to (1) and (2), but it is not an operator algebra.

To explicitly understand that (4) is an operator algebra, let us embed it into another matrix like (1) and (2). $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{\sqrt{2}}a & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}a & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}b & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}b & 0 & 0 & 0 \end{bmatrix}$ is a complete isometry. Then the original product is 'transfered' to the matrix product

$$\begin{bmatrix} \frac{1}{\sqrt{2}}a & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}a & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}b & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}b & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}c & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}c & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}d & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}d & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{ac}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\frac{ac}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\frac{bc}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\frac{bc}{\sqrt{2}} & 0 & 0 & 0 \end{bmatrix}.$$

Finally, we remark that such a reasonable product as (6) is NOT an operator algebra in spite that the product is completely contractive, i.e., $\|\varphi(x_1, x_2)\|_n \leq \|x_1\|_n \|x_2\|_n \ \forall x_1, x_2 \in \mathbb{M}_n(X), \forall n \in \mathbb{N}$.

$\boxed{\text{Pedersen's Question in the GPOTS 2003}}$ What are ext$(Ball(\mathcal{QM}(X)))$ the extreme points of the unit ball of a quasi-multiplier space $\mathcal{QM}(X)$?

My answer to this question is that ext$(Ball(\mathcal{QM}(X)))$ is strongly connected to approximate identities. We introduce the new notion "(approximate) quasi-identity" which is a generalization of one-sided identity.

Definition (Kaneda) Let $\mathcal{A}$ be an operator algebra.

(1) A quasi-identity of $\mathcal{A}$ is an element $e \in \mathcal{A}$ such that $a = ea + ae - eae$, $\forall a \in \mathcal{A}$.

(2) An approximate quasi-identity of $\mathcal{A}$ is a net $\{e_\alpha\} \subset \mathcal{A}$ such that

- (i) $\lim_\alpha e_\alpha a$ exists in $\mathcal{A}$, $\forall a \in \mathcal{A}$,
- (ii) $\lim_\alpha a e_\alpha$ exists in $\mathcal{A}$, $\forall a \in \mathcal{A}$,
- (iii) $a = \lim_{\alpha'} \lim_\alpha (e_\alpha a + a e_\alpha - e_\alpha a e_{\alpha'})$ $= \lim_\alpha \lim_{\alpha'} (e_\alpha a + a e_\alpha - e_\alpha a e_{\alpha'})$, $\forall a \in \mathcal{A}$.

The idea of the definition

One may rewrite the definition of quasi-identity as

$$(1-e)\mathcal{A}(1-e)=\{0\}.$$

Compare with Kadison's theorem:

Theorem (Kadison) Let $\mathcal{A}$ be a C^*-algebra. Then $a \in \mathrm{ext}(Ball(\mathcal{A}))$ if and only if

$$(1-aa^*)\mathcal{A}(1-a^*a)=\{0\}.$$

(Approximate) identities, left identities, right identities of a normed algebra are (approximate) quasi-identities.

In the operator algebra case, a quasi-identity has the following nice properties. These convince us that at least in the operator algebra case, the notion of quasi-identities is a natural generalization of identities or one-sided identities.

Proposition (Kaneda)

(1) A contractive quasi-identity is unique. In particular, if an operator algebra $\mathcal{A}$ has an identity or one-sided identity, then it is the only contractive quasi-identity of $\mathcal{A}$.

(2) A contractive quasi-identity of an operator algebra $\mathcal{A}$ $(\subset \mathbb{B}(\mathcal{H}))$ is necessarily an idempotent, and hence self-adjoint.

Many normed algebras do not have (approximate) one-sided or two-sided identities, though they have quasi-identities.

$\boxed{\text{Example}}$

$\mathcal{A}$: normed algebra with a left approximate identity $\{e_\alpha\}$, but no right approximate identity.

$\mathcal{B}$: normed algebra with a right approximate identity $\{f_\beta\}$, but no left approximate identity.

Then $\mathcal{A} \oplus_p \mathcal{B}$ has neither left nor right approximate identity, but do have an approximate quasi-identity $\{e_\alpha \oplus f_\beta\}_{(\alpha,\beta)}$, where $\{(\alpha, \beta)\}$ is a direct set ordered by $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$.

But there is an easy example an algebra with quasi-identity which we cannot decompose to a direct sum of two algebras one of which has a left approximate identity and the other has a right approximate identity.

$\boxed{\text{Example}}$ $\begin{bmatrix} * & * & * \\ O & O & * \\ O & O & * \end{bmatrix}$ cannot be decomposed to two algebras with one-sided identities, but do have a quasi-identity $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

However, if an operator space is a TRO, then we can always decompose as follows.

$\boxed{\boxed{\text{Theorem (Kaneda)}}}$ Let X be a TRO. Then there is a decomposition

$$X = X_C \oplus X_L \oplus X_R$$

and a complete isometry ϕ from X into some C^*-algebra $\mathcal{A}$ such that $\phi(X_C)$ is a two-sided ideal in $\mathcal{A}$, and $\phi(X_L)$ is a left ideal in $\mathcal{A}$, and $\phi(X_R)$ is a right ideal in $\mathcal{A}$. Moreover, if X is injective, then each of X_C, X_L, and X_R are also injective, and $\phi(X_C)$ has an identity, and $\phi(X_L)$ has a right identity, and $\phi(X_R)$ has a left identity.

Note that a left (resp. right) ideal in a C^*-algebra always has a right (resp. left) contractive approximate identity.

The above decomposition is not unique.

The above theorem follows from the following theorem.

Theorem (Kaneda) Let X be an injective operator space. Then ext$(Ball(X)) \neq \varnothing$.

Moreover, for X is an injective operator space (in which case $\mathcal{QM}(X) = X^*$), we can completely characterize ext$(Ball(\mathcal{QM}(X)))$ in terms of quasi-identities.

Before we state this characterization theorem, we need the following definition.

Definition (Kaneda)

Let X be an operator space.

1. $\mathcal{LU}_{loc}(\mathcal{QM}(X))$
   $:= \{z \in \mathcal{QM}(X) \ ; \ z^* \bullet z = 1_{11}\}$.

2. $\mathcal{RU}_{loc}(\mathcal{QM}(X))$
   $:= \{z \in \mathcal{QM}(X) \ ; \ z \bullet z^* = 1_{22}\}$.

3. $\mathcal{U}_{loc}(\mathcal{QM}(X))$
   $:= \mathcal{LU}_{loc}(\mathcal{QM}(X)) \cap \mathcal{RU}_{loc}(\mathcal{QM}(X))$.

We call an element of $\mathcal{LU}_{loc}(\mathcal{QM}(X))$ (respectively, $\mathcal{RU}_{loc}(\mathcal{QM}(X))$, $\mathcal{U}_{loc}(\mathcal{QM}(X))$) a local isometry (or, local left unitary) (respectively, local co-isometry (or, local right unitary), local unitary.

Theorem (Kaneda)

Let X be an injective operator space, and $z \in Ball(\mathcal{QM}(X))$, and (X, m_z) be the corresponding operator algebra.

1. (X, m_z) has a quasi-identity of norm 1 if and only if $z \in$ext$(Ball(\mathcal{QM}(X)))$.
2. (X, m_z) has a left identity of norm 1 if and only if $z \in \mathcal{LU}_{loc}(\mathcal{QM}(X))$.
3. (X, m_z) has a right identity of norm 1 if and only if $z \in \mathcal{RU}_{loc}(\mathcal{QM}(X))$.
4. (X, m_z) has a two-sided identity of norm 1 if and only if $z \in \mathcal{U}_{loc}(\mathcal{QM}(X))$.

C^*-algebras and their one-sided ideals The following gives a characterization of C^*-algebras.

Theorem (Kaneda) Let X be an operator space, and $z \in Ball(\mathcal{QM}(X))$. Then (X, m_z) is a C^*-algebra with a certain involution $\sharp$ if and only if $z \in \mathcal{U}_{loc}(\mathcal{QM}(X))$ and $X \bullet z = z^* \bullet X^*$. Involution $\sharp$ is uniquely given by $x^\sharp = z^* \bullet x^* \bullet z^*$, $\forall x \in X$.

The following gives a characterization of one-sided ideals in C^*-algebras.

Theorem (Kaneda) Let X be an operator space, and $z \in Ball(\mathcal{QM}(X))$. Then (X, m_z) is completely isometrically isomorphic to a left (respectively, right) ideal in some C^*-algebra if and only if $z \in \mathcal{RU}_{loc}(\mathcal{QM}(X))$ (respectively, $z \in \mathcal{LU}_{loc}(\mathcal{QM}(X))$) and $z \bullet X = X^* \bullet X$ (respectively, $X \bullet z = X \bullet X^*$).

Answer to Blecher's Open Question

In his paper (2004), Blecher defined Property (L) (resp. (R)) as follows:

If an operator algebra $\mathcal{A}$ with a left (resp. right) contractive approximate identity has such a contractive approximate identity $\{e_\alpha\}$ that $\lim_\alpha e_{\alpha'} e_\alpha = e_{\alpha'}$ (resp. $\lim_\alpha e_\alpha e_{\alpha'} = e_{\alpha'}$) for each α' then we say that $\mathcal{A}$ has Property (L) (resp. (R)).

Operator algebras having such properties is tame in his paper, but it was not know that any operator algebra that has a left (resp. right) contractive approximate identity has Property (L) (resp. (R)). So he left the following as an open question.

Blecher's Question Are there any operator algebras with r.c.a.i. which do not have Property (R)?

My answer is “no”, i.e., all operator algebras with r.c.a.i. has Property (R). The proof involves the technique of quasi-multipliers.