PRACTICE PROBLEMS ON DIAGONALIZATION

First recall the recipe for diagonalization. Given a matrix A, here are the steps.

Step 1. Compute the characteristic polynomial $\mathsf{det}(A-\lambda I)$. Then compute the eigenvalues; these are the roots of the characteristic polynomial.

Step 2. For each eigenvalue λ compute all eigenvalue. This amounts to solving the linear system $A-\lambda I=0$. You need to find maximal linearly independent list of solutions $\mathbf{z}_1,\ldots,\mathbf{z}_\ell$; here maximal means that any other solution to the system $A-\lambda I$ is a linear combination of $\mathbf{z}_1,\ldots,\mathbf{z}_\ell$. Here is the rule: Convert the matrix $A-\lambda I$ into Row Echelon form. Then:

$$\#\text{ of free variables } = \#\text{ of linearly independent solutions.}$$

This number is always at most the multiplicity of the root λ.

Step 3. If there is an eigenvalue λ such that the multiplicity of the root λ is strictly larger than the number of free variables in Step 2 then the matrix A is not diagonalizable. Otherwise the matrix A is diagonalizable. **Tip:** If each eigenvalue has multiplicity 1 then matrix A is diagonalizable.

Step 4. Write down the ist all eigenvalues $\lambda_1,\ldots,\lambda_k$ and linearly independent eigenvectors $\mathbf{x}_1,\ldots,\mathbf{x}_n$ such that $\mathbf{x}_k$ belongs to λ_k. That is, $A\mathbf{x}_k=\lambda_k\mathbf{x}_k$. Then $A=XDX^{-1}$ where

$$X=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\quad\text{and}\quad D=\begin{pmatrix}\lambda_1 & 0 & 0 & \cdots & 0\\ 0 & \lambda_2 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_n\end{pmatrix}$$

TYPE I PROBLEMS: FIND A DIAGONALIZATION OF THE GIVEN MATRIX

Find diagonalizations for the following matrices.

$$\begin{pmatrix}2 & 1 & -1\\ -1 & 4 & -1\\ -1 & 1 & 2\end{pmatrix}\quad\begin{pmatrix}3 & -1 & 1\\ 1 & 1 & 1\\ 1 & -1 & 3\end{pmatrix}\quad\begin{pmatrix}3 & 2 & -2\\ 2 & 3 & -2\\ 2 & 2 & -1\end{pmatrix}$$

$$\begin{pmatrix}3 & 4 & -2\\ 2 & 5 & -2\\ 4 & 8 & -3\end{pmatrix}\quad\begin{pmatrix}1 & 1 & -1\\ -4 & 5 & -2\\ -2 & 1 & 2\end{pmatrix}\quad\begin{pmatrix}4 & 1 & -1\\ 4 & 0 & 2\\ 2 & -1 & 3\end{pmatrix}$$

TYPE II PROBLEMS: DECIDE IF TWO MATRICES ARE CONJUGATE

Two matrices are conjugate if and only if they have a common diagonalization: To see this, notice that $A=XDX^{-1}$ and $B=YDY^{-1}$ is equivalent to $X^{-1}AX=D=Y^{-1}BY$, which in turn is equivalent to $B=YX^{-1}AXY^{-1}$.

Decide if any two of matrices in Set I are conjugate. Caution: It is not sufficient to verify that the eigenvalues of the two matrices are same. You also have to verify that both matrices are diagonalizable. It may happen that two matrices have same eigenvalues, one of them is diagonalizable and the other one not.

TYPE III PROBLEMS: COMPUTE A POWER OF THE MATRIX

If D is the diagonal matrix in Step 4 above then D^ℓ is the diagonal matrix with λ_k^ℓ on the diagonal. So computing power of diagonal matrices is easy. Now if A is diagonalizable and $A = XDX^{-1}$ then

$$A^\ell = XD^\ell X^{-1}.$$

If ℓ is large, this formula significantly reduces the computation, as we only need to compute X^{-1} and two matrix products, one of which is very easy.

To see that this formula works, say for $\ell = 2$ we have

$$A^2 = A \cdot A = XDX^{-1}XDX^{-1} = XD^2X^{-1}.$$

Inductively you can verify this for every ℓ.

Compute $A^{50}, B^{100}, B^{101}, C^{100}, C^{101}$ where

$$A = \begin{pmatrix} -1 & -1 & 5 \\ -2 & 1 & 4 \\ -2 & -1 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 2 \\ -1 & -3 & 5 \\ -1 & -2 & 4 \end{pmatrix} \quad C = \begin{pmatrix} -5 & 4 & 4 \\ -4 & 3 & 4 \\ -4 & 4 & 3 \end{pmatrix}$$

TYPE IV PROBLEMS: COMPUTE A "ROOT" OF THE MATRIX

This prolem is similar to computing the power. Say if we want to compute "square root" of A, i.e. find B such that $A = B^2$, we first diagonalize A, so find X, D such that $A = XDX^{-1}$. Then for each eigenvalue λ_k find β_k such that $\lambda_k = \beta_k^2$. The desired matrix B is then the diagonal matrix with β_k on the diagonal. Notice that for each λ_k there are two numbers β_k such that $\lambda_k = \beta_k^2$, so if $\lambda_k > 0$ we have in total 2^n real matrices B such that $A = B^2$ (here A is of type $n \times n$).

Find two distinct matrices B, C such that $B^2 = A = C^2$ where A is any of the following matrices.

$$\begin{pmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{pmatrix} \quad \begin{pmatrix} -7 & 8 & 8 \\ -8 & 9 & 8 \\ -8 & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} 6 & -5 & 3 \\ -3 & 4 & 3 \\ 5 & -5 & 4 \end{pmatrix}$$