MATH 13 WINTER 2016 HOMEWORK 3

Due: Friday, February 12 Please turn in at the lecture.

Each group please turn in only one paper.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS: It is crucial that you write your arguments clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.

When preparing the homeworks, please follow the **Rules for homeworks** on the course webpage under **Course information and policies** and also the guidelines under **Grading**. In particular, keep in mind the **Aspects of grading** in the **Grading** section.

1. (5pt) Consider the sequence $a_1, a_2, a_3, \ldots$ of integers such that

- $a_1 = 3$, $a_2 = 6$ and $a_3 = 9$.
- $a_{n+1} = a_n + a_{n-1} \cdot a_{n-2}$.

Use strong induction to prove that a_n is divisible by 9 whenever $n \geq 3$.

2. (5pt) Recall that in the lecture we will prove/have proved the following property of prime numbers: If p is a prime number and a, b are arbitrary integers then

$$p \,|\, a \cdot b \quad \Longrightarrow \quad (p \,|\, a \;\vee\; p \,|\, b) \tag{1}$$

Recall also that in the discussion you proved, using strong induction on n, the following generalization of (1): If p is prime number and $a_1, a_2, \ldots, a_n$ are arbitrary integers then

$$p \,|\, a_1 \cdot a_2 \cdot \cdots \cdot a_n \quad \Longrightarrow \quad p \,|\, a_k \text{ for some } k \text{ such that } 1 \leq k \leq n. \tag{2}$$

Notice the resemblance between (2) and (1): Statement (2) can be written as

$$p \,|\, a_1 \cdot a_2 \cdot \cdots \cdot a_n \quad \Longrightarrow \quad (p \,|\, a_1 \;\vee\; p \,|\, a_2 \;\vee\; \ldots \;\vee\; p \,|\, a_n) \tag{3}$$

Prove (2) using the least counterexample argument: Assume (2) is false for some positive integer n, some prime p and some integers $a_1, a_2, \ldots, a_n$, then look at the smallest such n and derive a contradiction.

3. (5pt) By calculating the remainder, decide whether the number

$$42^{113} + 53^{21} \cdot 85^{235} \tag{4}$$

is divisible by 43. Justify and explain all your steps clearly.

4. (5pt) (Book, Exercise 3.1.6) Assume $7x \equiv 28 \pmod{42}$. By Theorem 3.9 in the book, it follows that $x \equiv 4 \pmod{6}$.

(a) **(0.5pt)** Use Theorem 3.6 from the book to verify this.

(b) **(1pt)** If $7x \equiv 28 \pmod{42}$, is it possible that $x \equiv 4 \pmod{42}$? (That means, is there such an integer x ?) Justify your answer.

(c) **(1.5pt)** Is it always the case that

$$7x \equiv 28 \pmod{42} \quad \Longrightarrow \quad x \equiv 4 \pmod{42}?$$

(That means, does the implication holds for all integers x ?) Justify your answer.

(d) **(2pt)** Prove Theorem 3.9 in the book, that is, prove that for any integers k, n, a, b the following implication holds:

$$k \cdot a \equiv k \cdot b \pmod{k \cdot n} \quad \Longrightarrow \quad a \equiv b \pmod{n}$$