MATH 13 DISCUSSION PROBLEMS

THURSDAY MARCH 10, 2016

1. Consider the functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\begin{aligned} f(\langle a, b\rangle) &= a \\ g(\langle a, b\rangle) &= b \end{aligned}$$

Also consider the following sets:

$$\begin{aligned} L &= \{\langle a, b\rangle \in \mathbb{N} \times \mathbb{N} \mid a < b\} \\ D &= \{\langle a, b\rangle \in \mathbb{N} \times \mathbb{N} \mid a \text{ divides } b\} \\ E &= \{a \in \mathbb{N} \mid a \text{ is even}\} \\ O &= \{a \in \mathbb{N} \mid a \text{ is odd}\} \end{aligned}$$

Compute the following images/preimages:

(a) $f^{-1}[E], g^{-1}[O]$.
(b) $g[\{\langle a, b\rangle \in D \mid a = 2\}]$.
(c) Given a fixed $d \in \mathbb{N}$ find a useful description of $g[\{\langle a, b\rangle \in D \mid a = d\}]$.
(d) $f[\{\langle a, b\rangle \in D \mid b = 24\}]$.
(e) Given a fixed $m \in \mathbb{N}$ find a useful description of $g[\{\langle a, b\rangle \in D \mid b = m\}]$.

Do problems analogous to (b)-(d) with L in place of D.

2. Consider the functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} f(\langle a, b\rangle) &= a \\ g(\langle a, b\rangle) &= b \end{aligned}$$

Also consider the following sets:

$$\begin{aligned} S &= \{\text{the square in the plane } \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ with edges } \langle 1, 0\rangle, \langle 0, 1\rangle, \langle -1, 0\rangle, \langle 0, -1\rangle\} \\ Q &= \{\text{the quadrangle in the plane } \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ with edges } \langle 1, 0\rangle, \langle -1, 2\rangle, \langle -2, 1\rangle, \langle 0, -1\rangle\} \\ P &= \{\langle a, b\rangle \in \mathbb{R} \times \mathbb{R} \mid b = 1/(1 + a^2)\} \end{aligned}$$

Compute:

(a) $f[S], g[S], f[Q], g[Q], f[P], g[P]$
(b) $f^{-1}[g[P]], g^{-1}[f[Q]]$

3. Let A, B be sets. Consider the following maps.

(a) $f : A \to A \times A$ defined by $f(a) = \langle a, a\rangle$.
(b) Let $b \in B$. Then $g : A \to A \times B$ is defined by $g(a) = \langle a, b\rangle$.
(c) Let $u : A \to B$ be a function. Then $h : A \to A \times B$ is defined by

$$h(a) = \langle a, u(a)\rangle.$$

(d) $w : A \times B \to A \times B \times A$ defined by $w(\langle a, b\rangle) = \langle a, b, a\rangle$.

In each case determine whether the function is injective/surjective.

4. Consider the folloiwing binary relations.

(a) Relation R is on the 2-dimensional plane $P = \mathbb{R} \times \mathbb{R}$ and is defined by

$$\langle a, b\rangle \in R \iff \mathsf{dist}(a, \mathbf{c}) = \mathsf{dist}(b, \mathbf{c})$$

where $\mathbf{c}$ is the center of the plane (so $\mathbf{c} = \langle 0, 0\rangle$) and dist is the distance.

(b) Relation S is on the set of numbers

$$K = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$$

and is defined by

$$\langle x, y\rangle \in S \iff x - y \in \mathbb{Q}$$

(c) Relation T is on the set $K \setminus \{0\}$ where K as in (b) and is defined by

$$\langle x, y\rangle \in T \iff x/y \in \mathbb{Q}$$

(d) Relation V is on the 2-dimensional plane $P = \mathbb{R} \times \mathbb{R}$ and is defined by

$$\langle a, b\rangle \in V \iff a = b \ \vee \ (b_1 - a_1)/(b_0 - a_0) = 1$$

where $a = \langle a_0, a_1\rangle$ and $b = \langle b_0, b_1\rangle$.

(e) Relation "equinumerosity" $\sim$ is on $\mathcal{P}(U)$ where U is a large set chosen in advance. Recall that if $A, B \in \mathcal{P}(U)$ then

$$A \sim B \iff \text{there exists a bijection } f : A \to B$$

In each case prove that the relation in question is an equivalence relation, and then compute the equivalence classes. Draw pictures whenever possible to visualize the situation.

5. Given an equivalence relation R on a set A, the set of all equivalence classes

$$I_R = \{[a]_R \mid a \in A\}$$

is called the **partition of** A **induced by** R. To justify the term "partition", prove:

(a) $a \in [a]_R$.

(b) $\langle a, b\rangle \in R$ iff $b \in [a]_R$ iff $a \in [b]_R$.

(c) If $a, b \in A$ then either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \varnothing$.