Math 140A - Notes

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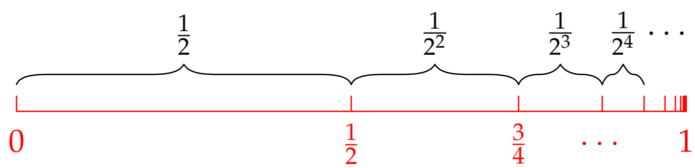
Introduction

Analysis is one of the major sub-disciplines of mathematics, being concerned with continuous functions, limits, calculus and accurate approximations.

Analytic ideas date back over 2000 years. For instance, Archimedes (c. 287–212 BC) used limit-type approaches to approximate the circumference of a circle and compute the area under a parabola.[1] The philosophical objections to such calculations are just as old: how can it make sense to sum up infinitely many infinitesimally small quantities? This was part of a deeper debate among the ancient Greeks: is the matter comprising the natural world *atomic* (consisting of minute, discrete, indivisible pieces) or *continuous* (arbitrarily and infinitesimally divisible). Several of Zeno's famous paradoxes (5^{th} C. BC) grapple with these difficulties: *Achilles and the Tortoise* essentially argues that the infinite series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is meaningless.



As we'll see, with modern definitions it makes sense for this sum to evaluate to 1.

The work of Newton and Leibniz in the late 1600s allowed the easy application of calculus to many important problems in the sciences, but without properly addressing the ancient philosophical challenges. The logical development of calculus necessitated by this became the triumph of 18^{th}–19^{th} century mathematics. The critical notions of limit and continuity only became rigorous in the early 1800s, courtesy of Bolzano, Cauchy and Weierstrass (amongst others), with another 50 years before Riemann provided a thorough description of the definite integral.

The Math 140A/B sequence introduces analysis by focusing on these ideas. In this course we primarily consider sequences, limits, continuity and infinite series. Power series, differentiation and integration are the focus of 140B. We start, however, with something even more basic: to numerically measure continuous quantities, we need to familiarize ourselves with the *real numbers.* Since a concrete description is quite difficult, we build up to it using first the natural numbers and then the rationals…

[1] Archimedes' circle calculation is reminiscent of the Riemann sum approach to integration, whereas his parabolic area method required the evaluation of the infinite series $\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}$.

1 The Set $\mathbb{N}$ of Natural Numbers

You've been using the natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ since you learned to count. In mathematics, these must be axiomatically described. Here is one approach, known as *Peano's Axioms.*

Axioms 1.1. The natural numbers are a set $\mathbb{N}$ satisfying the following properties:

1. (Non-emptiness) $\mathbb{N}$ is non-empty.
2. (Successor function) There exists a function $f : \mathbb{N} \to \mathbb{N}$. This function is usually denoted '+1' so that we may write,

$$n \in \mathbb{N} \implies n+1 \in \mathbb{N}$$

3. (Initial element) f is *not surjective.* Otherwise said, there exists an element $1 \notin \text{range } f$ which is not the successor of any element.[2]
4. (Unique predecessor/order) f is *injective.* If m and n have the same successor, then $m = n$.
5. (Induction) Suppose $A \subseteq \mathbb{N}$ is a subset satisfying
   (a) $1 \in A$
   (b) $n \in A \implies n+1 \in A$.

   Then $A = \mathbb{N}$.

Axioms 1–4 are relatively straightforward, the natural numbers are defined by repeatedly adding 1 to the initial element; for instance

$$3 := f\big(f(1)\big) = (1+1)+1$$

To see why Axiom 5 is so-named, compare an easy example with a standard induction argument.

Example 1.2. Prove that $7^n - 4^n$ is divisible by 3 for all $n \in \mathbb{N}$.
Let A be the set of natural numbers for which $7^n - 4^n$ is divisible by 3. It is required to prove that $A = \mathbb{N}$.

(a) If $n = 1$, then $7^1 - 4^1 = 3$, whence $1 \in A$.

(b) Suppose $n \in A$. Then $7^n - 4^n = 3\lambda$ for some $\lambda \in \mathbb{N}$. But then

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= 7 \cdot 7^n - 4^{n+1} = 7(3\lambda + 4^n) - 4^{n+1} = 3 \cdot 7\lambda + (7-4) \cdot 4^n \\ &= 3\big(7\lambda + 4^n\big) \end{aligned}$$

is divisible by 3. It follows that $n+1 \in A$.

Appealing to axiom 5, we see that $A = \mathbb{N}$, hence result.
The two arguments are precisely the familiar *base case* and *induction step*.

[2] It is purely convention to denote the first natural number by 1; we could use 0, x, α, or any symbol you wish!

What about the integers? It should be clear that the integers satisfy axioms 1, 2 and 4, but not 3 and 5. For instance:

$\not{3}$. The function $f : \mathbb{Z} \to \mathbb{Z} : n \mapsto n+1$ is surjective (indeed bijective/invertible). The number 1 is the successor of 0.

We can reverse this observation to provide an explicit construction of the integers from the natural numbers. Simply extend the function f so that *every* element has a unique predecessor: 0 is the unique predecessor of 1, -1 the unique predecessor of 0, etc. In essence we are forcing $f(n) = n+1$ to be bijective!

Exercises 1. Most of these exercises are to refresh your memory of mathematical induction. You can use either the language of Peano's axiom 5, or the (likely) more familiar base-case/induction-step formulation.

1. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all natural numbers n.
2. Prove that $3 + 11 + \cdots + (8n-5) = 4n^2 - n$ for all $n \in \mathbb{N}$.
3. (a) Guess a formula for $1 + 3 + \cdots + (2n-1)$ by evaluating the sum for $n = 1, 2, 3$, and 4. (*For $n = 1$ the sum is simply 1*)
   (b) Prove your formula using mathematical induction.
4. Prove that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$.
5. The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.
   (a) Prove that $n^2 > n+1$ for all integers $n \geq 2$.
   (b) Prove that $n! > n^2$ for all integers $n \geq 4$.
   (*Recall that $n! = n(n-1)\cdots 2 \cdot 1$*)
6. Prove $(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2$ for all $n \in \mathbb{N}$.
7. For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer".
   (a) Prove that P_{n+1} is true whenever P_n is true.
   (b) For which n is P_n actually true? What is the moral of this exercise?
8. Show that Peano's induction axiom is *false* for the set of integers $\mathbb{Z}$ by exhibiting a *proper subset* $A \subset \mathbb{Z}$ which satisfies conditions (a) and (b).
9. Consider $\mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3. That is,
   $$0+1=1, \quad 1+1=2, \quad 2+1=0$$
   Which of Peano's axioms are satisfied?

2 The Set $\mathbb{Q}$ of Rational Numbers

There are several ways to define the rational numbers from the integers. For instance, we could consider the set of relatively prime ordered pairs

$$\mathbb{Q} = \{(p,q) : p \in \mathbb{Z},\ q \in \mathbb{N},\ \gcd(p,q) = 1\} \subseteq \mathbb{Z} \times \mathbb{N}$$

Things are more familiar once we write $\frac{p}{q}$ instead of (p,q) and adopt the convention that $\frac{\lambda p}{\lambda q} = \frac{p}{q}$ for any non-zero $\lambda \in \mathbb{Z}$. It is easy to define the usual operations $(+, \cdot,$ etc.) consistently with those for the integers (Exercise 7).

An alternative approach involves equations. Each *linear equation* $qx - p = 0$ where $p, q \in \mathbb{Z}$ and $q \neq 0$ corresponds to a rational number! For example

$$13x + 27 = 0 \leftrightsquigarrow x = -\frac{27}{13}$$

Of course $26x + 54 = 0$ *also* corresponds to the same rational number!

Extending this process, we might consider higher degree polynomials.

Definition 2.1. A number x is *algebraic* if it satisfies an equation of the form[3]

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{$*$}$$

for some integers $a_0, \ldots, a_n$.

Examples 2.2. 1. $\sqrt{2}$ is algebraic since it satisfies the equation $x^2 - 2 = 0$.

2. $x = \sqrt[5]{7 + \sqrt{3}}$ is also algebraic:

$$x^5 - 7 = \sqrt{3} \implies (x^5 - 7)^2 = 3 \implies x^{10} - 14x^5 + 46 = 0$$

The next result is helpful for deciding whether a given number is rational and can assist with factorizing polynomials.

Theorem 2.3 (Rational Roots). *Suppose that $a_0, \ldots, a_n \in \mathbb{Z}$ and that $x \in \mathbb{Q}$ satisfies $(*)$. If $x = \frac{p}{q}$ in lowest terms, then $p \mid a_0$ and $q \mid a_n$.*

Proof. Since x satisfies the polynomial, we see that

$$a_n \left(\frac{p}{q}\right)^n + \cdots + a_1 \left(\frac{p}{q}\right) + a_0 = 0 \implies a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

All terms except the last contain a factor of p, whence $p \mid a_0 q^n$. Since $\gcd(p,q) = 1$ it follows that $p \mid a_0$. The result for q is almost identical. ∎

[3]You should be alarmed by this! We have given up on *constructing* new numbers and instead are simply *describing* their properties. No matter, a construction of the real numbers will come later.

Examples 2.4. 1. We show that $\sqrt{2}$ is irrational.[4] Plainly $x = \sqrt{2}$ satisfies the polynomial equation $x^2 - 2 = 0$. If $\sqrt{2} = \frac{p}{q}$ were rational in lowest terms, then the rational roots theorem forces

$$p \mid 2 \quad \text{and} \quad q \mid 1 \implies \sqrt{2} \in \{\pm 1, \pm 2\}$$

Since none of the numbers $\pm 1, \pm 2$ satisfy $x^2 - 2 = 0$, we have a contradiction.

2. $(\sqrt{3} - 1)^{1/3}$ is irrational. It satisfies $(x^3 + 1)^2 = 3$, from which

$$x^6 + 2x^3 - 2 = 0$$

By the theorem, if $x = \frac{p}{q}$ were rational then $p \mid 2$ and $q \mid 1$, whence $x = \pm 1, \pm 2$, none of which satisfies $(x^3 + 1)^2 = 3$.

3. $\left(\frac{4+\sqrt{3}}{5}\right)^{1/2}$ is irrational. It satisfies $5x^2 - 4 = \sqrt{3}$, from which

$$25x^4 - 40x^2 + 13 = 0$$

If $x = \frac{p}{q}$ were rational, then $p \mid 13$ and $q \mid 25$. There are twelve possibilities in all:

$$x = \pm 1, \pm 13, \pm \frac{1}{5}, \pm \frac{13}{5}, \pm \frac{1}{25}, \pm \frac{13}{25}$$

It is tedious to check all cases, but none satisfy the required polynomial.

With this example it is easier to bypass the theorem entirely: if $x \in \mathbb{Q}$ then $\sqrt{3} = 5x^2 - 4$ would also be rational!

4. We factorize the polynomial $3x^3 + x^2 + x - 2 = 0$. By the rational roots theorem, if $x = \frac{p}{q}$ is a rational root, then $p \mid 2$ and $q \mid 3$ which gives several possibilities:

$$x \in \left\{\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}\right\}$$

It doesn't take long to try these and observe that $x = \frac{2}{3}$ is the only rational solution. The polynomial has a factor of $3x - 2$ which we can extract by long division to obtain

$$3x^3 + x^2 + x - 2 = (3x - 2)(x^2 + x + 1)$$

The quadratic has no real roots: absent complex numbers, the factorization is complete.

It is far from clear that there exist non-algebraic (or *transcendental*) numbers, of which e and π are the most famous examples. These satisfy no polynomial equation with integer coefficients, though demonstrating this is tricky.

[4] Compare this to the standard proof of the irrationality of $\sqrt{2}$ as seen in a previous course. Note how easy it is to extend our approach to $\sqrt{3}$, $\sqrt{29}$, $\sqrt[3]{2}$, $\sqrt[5]{8}$, etc.

Exercises 2. 1. Describe all the linear equations which correspond to the rational number $\frac{101}{29}$.

2. Show that $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{24}$ are not rational numbers.
   (*Hint: what are the relevant polynomials?*)
3. Show that $2^{1/3}$ and $13^{1/4}$ are not rational numbers.
4. Show that $(2+\sqrt{2})^{1/2}$ is not rational.
5. Show that $(3+\sqrt{2})^{2/3}$ is not rational.
6. Explain why $4-7b^2$ must be rational if b is rational.
7. Given rational numbers (p,q) and (r,s) as ordered pairs, what are the rational numbers $(p,q)+(r,s)$ and $(p,q)\cdot(r,s)$?
   (*Hint: what is* $\frac{p}{q}+\frac{r}{s}$*?*)
8. Let $n \in \mathbb{N}$. Use the rational roots theorem to prove that $\sqrt{n}$ is rational if and only if it is an *integer.*

3 Ordered Fields

Thus far, we have formally constructed the natural numbers and used them to (loosely) build the integers and rational numbers. It is a significantly greater challenge to *construct* the real numbers. We start by thinking about ordered fields, of which both $\mathbb{Q}$ and $\mathbb{R}$ are examples.

Axioms 3.1. A *field* $\mathbb{F}$ is a set together with two binary operations $+$ and $\cdot$ which satisfy the following (for all $a, b, c \in \mathbb{F}$),[5]

	Addition	Multiplication
Closure	$a + b \in \mathbb{F}$	$ab \in \mathbb{F}$
Associativity	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Commutativity	$a + b = b + a$	$ab = ba$
Identity	$\exists 0 \in \mathbb{F}$ such that $a + 0 = a$	$\exists 1 \in \mathbb{F}$ such that $a \cdot 1 = a$
Inverse	$\exists -a \in \mathbb{F}$ such that $a + (-a) = 0$	If $a \neq 0$, $\exists a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$
Distributivity	$a(b + c) = ab + ac$	

A field $\mathbb{F}$ is *ordered* if we also have a binary relation $\leq$ which satisfies (again for all $a, b, c \in \mathbb{F}$):

O1 $a \leq b$ or $b \leq a$

O2 $a \leq b$ and $b \leq a \implies a = b$

O3 $a \leq b$ and $b \leq c \implies a \leq c$

O4 $a \leq b \implies a + c \leq b + c$

O5 $a \leq b$ and $0 \leq c \implies ac \leq bc$

For an ordered field, the symbol $<$ is used in the usual way: $x < y \iff x \leq y$ and $x \neq y$.

As with Peano's axioms for the natural numbers, these are not worth memorizing. Instead you should quickly check that you believe all of them for your current understanding of the real numbers; you can't *prove* anything since the real numbers haven't yet been defined!

Example 3.2. It is worth considering the rational numbers in a little more detail. Recall (Section 2) how $\mathbb{Q}$ may be defined as a set of ordered pairs $\frac{p}{q} \leftrightsquigarrow (p, q) \in \mathbb{Z} \times \mathbb{N}$. It moreover inherits a natural ordering from $\mathbb{Z}$ and $\mathbb{N}$:

$$\frac{p}{q} \leq \frac{r}{s} \iff ps \leq qr \qquad \text{(remember that } q, s > 0\text{)}$$

[5] We write multiplication $\cdot$ as juxtaposition unless it is helpful for clarity. We also use the common shorthand $a^2 = a \cdot a$. If you know some abstract algebra:

- The addition axioms say that $(\mathbb{F}, +)$ is an abelian group.
- The multiplication axioms say that $(\mathbb{F} \setminus \{0\}, \cdot)$ is an abelian group.
- The distributive axiom describes how addition and multiplication interact.

It is now possible, though tedious, to *prove* that each of the axioms of an ordered field holds for $\mathbb{Q}$, using only basic facts about multiplication, addition and ordering *within the integers*. For instance,

O3 Suppose $a \le b$ and $b \le c$. Write $a = \frac{p}{q}$, $b = \frac{r}{s}$ and $c = \frac{t}{u}$ where all three denominators are positive. By assumption,

$$\begin{aligned} ps \le qr \text{ and } ru \le st &\implies psu \le qru \le qst \implies pu \le qt \\ &\implies a = \frac{p}{q} \le \frac{t}{u} = c \end{aligned}$$

Basic Results about ordered fields

As with the axioms of an ordered field, it is not worth *memorizing* these.

Theorem 3.3. *Let $\mathbb{F}$ be a ordered field with at least two elements $0 \neq 1$. Then:*

1. $a + c = b + c \implies a = b$
2. $a \cdot 0 = 0$
3. $(-a)b = -(ab)$
4. $(-a)(-b) = ab$
5. $ac = bc$ and $c \neq 0 \implies a = b$
6. $ab = 0 \implies a = 0$ or $b = 0$
7. $a \le b \implies -b \le -a$
8. $a \le b$ and $c \le 0 \implies bc \le ac$
9. $0 \le a$ and $0 \le b \implies 0 \le ab$
10. $0 \le a^2$
11. $0 < 1$
12. $0 < a \implies 0 < a^{-1}$
13. $0 < a < b \implies 0 < b^{-1} < a^{-1}$

All of these statements should be intuitive for the fields $\mathbb{Q}$ and $\mathbb{R}$. Try *proving* a few using only the axioms; they are easiest done in the order presented. For instance, part 2 might be proved as follows:

$$\begin{aligned} & a \cdot 0 + 0 = a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0 && \text{(additive identity/distibutive axioms)} \\ \implies & 0 = a \cdot 0 && \text{(part 1)} \end{aligned}$$

We finish with one final useful ingredient.

Definition 3.4. If $\mathbb{F}$ is an ordered field, then the *absolute value* of $a \in \mathbb{F}$ is

$$|a| := \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem 3.5. *In any ordered field:*

1. $|a| \ge 0$
2. $|ab| = |a| \cdot |b|$
3. $|a + b| \le |a| + |b|$ *($\triangle$-inequality)*

All three parts are immediate if you consider the $\pm$-cases separately for a, b.

Exercises 3. 1. Which of the axioms of an ordered field fail for $\mathbb{N}$? For $\mathbb{Z}$?

2. Prove parts 11 and 13 of Theorem 3.3.

   (*Remember you can use any of the parts that come before…*)

3. (a) Prove that $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$.
   (*Hint: Apply the triangle inequality twice. Don't consider eight separate cases!*)

   (b) Use induction to prove

   $$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

   for any $a_1, \ldots, a_n \in \mathbb{R}$.

4. (a) Show that $|b| < a \iff -a < b < a$.

   (b) Show that $|a-b| < c \iff b-c < a < b+c$

   (c) Show that $|a-b| \leq c \iff b-c \leq a \leq b+c$

5. Let $a, b \in \mathbb{R}$. Show that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

   (*Hint: draw a picture if you're stuck. This is a very important example!*)

6. Following Example 3.2, prove that $\mathbb{Q}$ satisfies axiom O5.

   (*Hint: if $a = \frac{p}{q}$, etc., what is meant by $ac \leq bc$?*)

7. (Hard!) The complex numbers $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$ form a field. Consider the lexicographic ordering of $\mathbb{C}$ defined by

   $$x+iy \leq p+iq \iff \begin{cases} x < p \text{ or} \\ x = p \text{ and } y \leq q \end{cases}$$

   Which of the order axioms O1–O5 are satisfied by the lexicographic ordering?

   (*Don't prove your claims if an axiom is satisfied, but provide a counter-example if not*)

4 The Completeness Axiom

While we still haven't provided an explicit *definition* of the real numbers, you should be comfortable with the fact that both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields. The question remains of how to distinguish them? Perhaps surprisingly, only one additional axiom is required: the *completeness axiom* or *least upper bound principle.* To explain this we first need some terminology.

Definition 4.1 (Maxima, Minima & Boundedness). Let $S \subseteq \mathbb{R}$ be non-empty.

1. S is *bounded above* if it has an *upper bound* M:

   $$\exists M \in \mathbb{R} \text{ such that } \forall s \in S,\ s \leq M$$

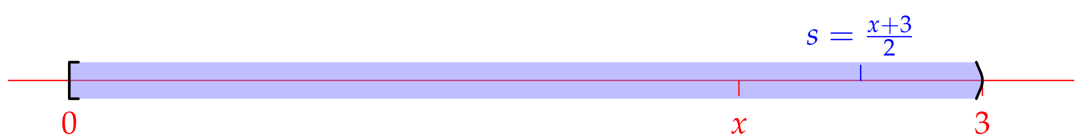
2. We write $M = \max S$, the *maximum* of S, if M is an upper bound for S *and* $M \in S$.
3. S *bounded below*, a *lower bound* m, and the *minimum* $\min S$ are defined similarly.
4. S is *bounded* if it is bounded above and below. We say that S is *bounded by* M if

   $$\forall s \in S,\ |s| \leq M$$

Examples 4.2. 1. If S is a finite set, then it is bounded and has both a maximum and a minimum. For instance, $S = \{-3, \pi, 12\}$ has $\min S = -3$ and $\max S = 12$.

2. $\mathbb{N}$ has minimum 1, but no maximum. $\mathbb{Z}$ and $\mathbb{Q}$ have neither: both are *unbounded.*
3. The interval $S = [0, 3) = \{x \in \mathbb{R} : 0 \leq x < 3\}$ is bounded, for example by $M = 5$, it has minimum 0 and no maximum. While this last is likely intuitive, it worth giving an explicit argument, in this case by contradiction.

   Suppose $x = \max S$ exists. It is helpful to draw a picture to get the lay of the land. Since $x \in S$, we've placed x *inside* the interval, away from 3.

   The crux of the proof is to observe that there exists $s \in S$ which is *larger* than x. The natural choice is the average $s := \frac{1}{2}(x + 3)$. Now observe that

   $$3 - s = s - x = \frac{1}{2}(3 - x) > 0$$

   In particular,

   - $s \in S$ since it is non-negative and $s < 3$.
   - $s > x$.

   Since S contains an element larger than x, it follows that x cannot be the maximum of S. In conclusion, S has no maximum.

The following should be immediate: try proving them yourself.

Lemma 4.3. 1. *If M is an upper bound for S, so is $M + \varepsilon$ for any $\varepsilon \geq 0$.*

2. *If* $\max S$ *exists, then it is unique.*

3. *A set is bounded if and only if it is bounded above and below. In particular, if m, M are lower/upper bounds, then S is bounded by*

$$\forall s \in S, \ |s| \leq \max(|m|, |M|)$$

Example 4.4. Before introducing the key axiom, we consider a variation on the previous example. We show that the following set has no maximum:

$$S = \mathbb{Q} \cap [0, \sqrt{2}) = \{x \in \mathbb{Q} : 0 \leq x < \sqrt{2}\}$$

The approach is similar to before: given a hypothetical maximum x, find an element $s \in S$ between x and $\sqrt{2}$. The challenge is that we can't simply use the *average* $\frac{1}{2}(x + \sqrt{2})$: this isn't rational (*why?*) and so doesn't lie in S!

To fix this, we informally invoke sequences: this might seem quite hard at the moment, but will be made rigorous later. The rough idea is to construct a sequence (s_n) of elements of S which increases to $\sqrt{2}$. Eventually one of these must be larger than x.

Define a sequence of rational numbers (s_n) by $s_n = \frac{1}{10^n}\lfloor 10^n\sqrt{2}\rfloor$, where $\lfloor \ \rfloor$ denotes the *floor function*.[6] The sequence simply recovers the first n decimal places of $\sqrt{2}$:

$$s_0 = 1, \quad s_1 = 1.4 = \frac{14}{10}, \quad s_2 = 1.41 = \frac{141}{100}, \quad s_3 = 1.414 = \frac{1414}{1000}, \quad \ldots$$

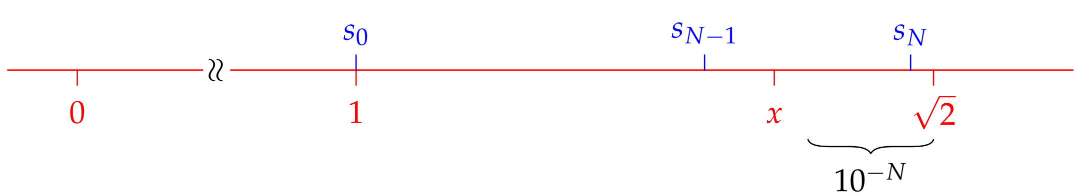
and has the following properties:

- $s_n \in S$ since any truncating decimal is rational and certainly $0 \leq s_n < \sqrt{2}$.
- $\sqrt{2} - s_n < 10^{-n}$ follows since $10^n\sqrt{2} - \lfloor 10^n\sqrt{2}\rfloor < 1$.

Now suppose $x = \max S$ exists. Since $x \in S$, we have $x < \sqrt{2}$. Choose $N \in \mathbb{N}$ large enough so that $10^{-N} < \sqrt{2} - x$. Then $s_N \in S$ and

$$\sqrt{2} - s_N < 10^{-N} < \sqrt{2} - x \implies x < s_N$$

The hypothetical maximum x is not an upper bound for S: contradiction.



[6] $\lfloor y\rfloor$ is the greatest integer less than or equal to y; informally *round down*. For example $\lfloor \pi\rfloor = 3$. This approach is a something of a hack: it can be sped up enormously using the upcoming density of $\mathbb{Q}$ in $\mathbb{R}$ (Corollary 4.12); indeed the Archimedean property on which it depends is necessary for $\lfloor y\rfloor$ to be well-defined.

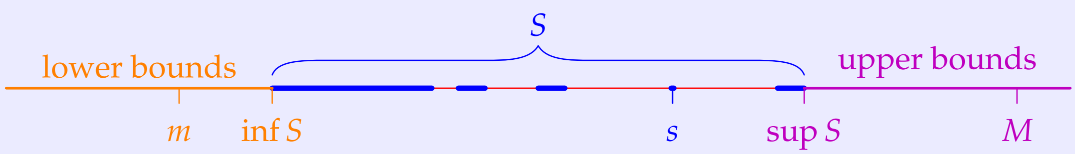
Suprema and Infima

We now generalize the idea of maximum and minimum values for bounded sets.

Example 4.5. The interval $[2,5)$ has *least upper bound* 5: among all upper bounds, 5 is the smallest.

Definition 4.6. Let $S \subseteq \mathbb{R}$ be non-empty.

1. If S is bounded above, its *supremum* $\sup S$ is its *least upper bound.* Otherwise said,
   (a) $\sup S$ is an upper bound: $\quad \forall s \in S,\ s \leq \sup S$,
   (b) $\sup S$ is the least such: $\quad$ if M is an upper bound, then $\sup S \leq M$.
2. If S is bounded below, its *infimum* $\inf S$ is its *greatest lower bound.* Equivalently,
   (a) $\inf S$ is a lower bound: $\quad \forall s \in S,\ \inf S \leq s$,
   (b) $\inf S$ is the greatest such: $\quad$ if m is a lower bound, then $m \leq \inf S$.



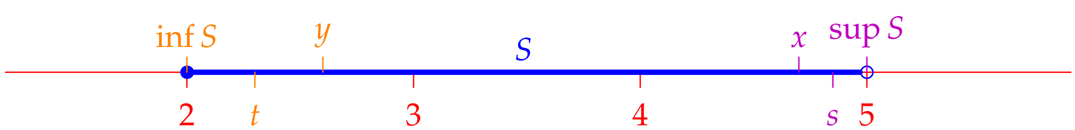
Example (4.5 cont). We verify the supremum and infimum for $S = [2,5)$; parts (a), (b) are the properties in the above definition.

(a) Since $s \in S \iff 2 \leq s < 5$, we see that 5 is an upper bound and 2 a lower bound.

(b) Given $x < 5$, define[7] $s := \max\{\frac{1}{2}(x+5), 4\}$. Observe that $x < s < 5$ from which $s \in S$ is *larger* than x. It follows that x is *not* an upper bound for S, and that 5 is the least such.

Similarly, if $y > 2$, define $t := \min\{\frac{1}{2}(y+2), 4\}$ to see that $t \in S$ is *smaller* than y, which cannot therefore be a lower bound for S.

We conclude that $\sup S = 5$ and $\inf S = 2$.



We are assuming something quite important here!

Axiom 4.7 (Completeness of $\mathbb{R}$). If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then $\sup S$ exists (and is a real number!).

It is this property that distinguishes the real numbers from the rationals.[8] Note that every bounded set S of *rational* numbers has a supremum; the issue is that $\sup S$ *might not be rational*!

[7] The number 4 is merely an arbitrary element to make sure $s \in S$ in case x were huge and negative!

[8] If you've studied abstract algebra, then a more rigorous statement should make sense: every ordered field with $0 \neq 1$ and which satisfies the completeness axiom is isomorphic to the real numbers.

Example (4.4 cont). The set $S = \mathbb{Q} \cap [0, \sqrt{2})$ has $\sup S = \sqrt{2}$. We check conditions (a), (b) in Definition 4.6.

(a) Certainly $\sqrt{2}$ is an upper bound for S, since every element is less than $\sqrt{2}$.

(b) If $x < \sqrt{2}$ is given, then our previous argument says there exists some $s_N \in S$ for which $s_N > x$. Plainly x isn't an upper bound.

In conclusion, $\sqrt{2}$ is the smallest upper bound for S.

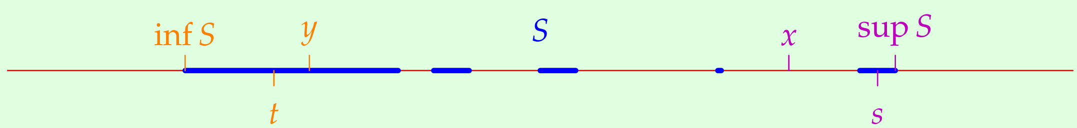
Consider the contrapositive of part (b) of Definition 4.6 after replacing M with x.

> If $x < \sup S$, then x is *not* an upper bound for S.

If we unpack this further, we recover a useful existence result. Indeed this is precisely what we did in both previous examples.

Lemma 4.8. 1. *If $x < \sup S$, then $\exists s \in S$ such that $s > x$.*

2. *If $y > \inf S$, then $\exists t \in S$ such that $t < y$.*



This observation will be used repeatedly, so make sure it is well understood.

Examples 4.9. We state the following without proof or calculation. You should be able to justify all these statements using the definition, or by mirroring the above examples.

1. A bounded set has many possible bounds, but only one supremum or infimum.

2. If S has a maximum, then $\max S = \sup S$. Similarly $\min S = \inf S$ if a minimum exists.

3. $S = \mathbb{Q} \cap (\pi, 4)$ has $\sup S = 4$ and $\inf S = \pi$.

4. $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{\ldots, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$ has $\sup S = \max S = 1$, $\inf S = 0$, and no minimum.

5. $S = \bigcup\limits_{n=1}^{\infty} [n, n + \frac{1}{2}) = [1, 1.5) \cup [2, 2.5) \cup [3, 3.5) \cup \cdots$ has $\inf S = 1$. It is not bounded above.

6. $S = \bigcap\limits_{n=1}^{\infty} [\frac{1}{n}, 1 + \frac{1}{n})$ has $\inf S = 1 = \sup S$ since $S = \{1\}$.

The completeness axiom only asserts the existence of the supremum of a bounded set. By reflecting across zero (see Exercise 9), we obtain the same thing for the infimum.

Theorem 4.10 (Existence of Infima). *If $S \subseteq \mathbb{R}$ non-empty and bounded below, then $\inf S \in \mathbb{R}$ exists.*

The Archimedean Property and the Density of the Rationals

We finish this section by discussing the distribution of the rational numbers among the real numbers.

Theorem 4.11 (Archimedean Property). *If $b > 0$ is a real number, then $\exists n \in \mathbb{N}$ such that $n > b$. Equivalently:*[9] $a, b > 0 \implies \exists n \in \mathbb{N}$ *such that* $an > b$.

In this result we assume nothing about $\mathbb{R}$ except that is an ordered field satisfying the completeness axiom and $0 \neq 1$. The natural numbers in this context are *defined* as the subset

$$\mathbb{N} = \{1, 1+1, 1+1+1, \ldots\} \subseteq \mathbb{R}$$

Proof. Suppose the result were false. Then $\exists b > 0$ such that $n \leq b$ for all $n \in \mathbb{N}$; that is, $\mathbb{N}$ is bounded above! By completeness, $\sup \mathbb{N}$ exists, and we trivially see that

$$0 < 1 \implies \sup \mathbb{N} < \sup \mathbb{N} + 1 \implies \sup \mathbb{N} - 1 < \sup \mathbb{N}$$

By Lemma 4.8, $\exists n \in \mathbb{N}$ such that $n > \sup \mathbb{N} - 1$. But then $\sup \mathbb{N} < n + 1$ which is clearly a natural number! Thus $\sup \mathbb{N}$ is not an upper bound for $\mathbb{N}$: contradiction. ∎

The use of completeness is *necessary*: there exist non-Archimedean ordered fields!

Corollary 4.12 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *Between any two real numbers, there exists a rational number.*

The idea is simple: given $a < b$, stretch the interval by an integer factor n until it contains an integer m, before dividing by n to obtain $a < \frac{m}{n} < b$. The Archimedean property shows the existence of m, n.

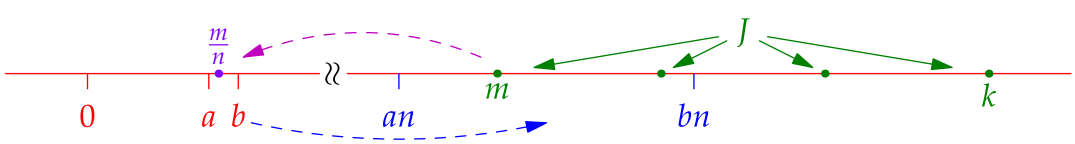
Proof. WLOG suppose $0 \leq a < b$. The Archimedean property applied to $\frac{1}{b-a} > 0$ says

$$\exists n \in \mathbb{N} \text{ such that } n > \frac{1}{b-a}$$

A second application says $\exists k \in \mathbb{N}$ such that $k > an$. Now consider

$$J := \{j \in \mathbb{N} : an < j \leq k\}$$

and define $m = \min J$: this exists since J is a finite non-empty set of natural numbers.[10]



Clearly $m > an > m - 1$, since $m = \min J$. But then $m \leq an + 1 < bn$. We conclude that

$$an < m < bn \implies a < \frac{m}{n} < b$$

∎

It is immediate that any interval (a, b) now contains *infinitely many* rational numbers.

[9] Just replace b with $\frac{b}{a}$.

[10] This part of the argument is needed because, in this context, we haven't established the well-ordering property of $\mathbb{N}$ (equivalent to Peano's fifth axiom).

Exercises 4. 1. Decide if each set is bounded above and/or below. If it is, *state* its supremum and/or infimum (no working is required).

(a) $(0,1)$ (b) $\{2,7\}$ (c) $\{0\}$

(d) $\bigcup\limits_{n=1}^{\infty}[2n,2n+1]$ (e) $\{1-\frac{1}{3^n} : n \in \mathbb{N}\}$ (f) $\{r \in \mathbb{Q} : r^2 < 2\}$

(g) $\bigcup\limits_{n=1}^{\infty}(1-\frac{1}{n},1+\frac{1}{n})$ (h) $\{\frac{1}{n} : n \in \mathbb{N}$ and n is prime$\}$ (i) $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

2. Modelling Example 4.4, *sketch* an argument that $S = \mathbb{Q} \cap (\pi, 4]$ has no minimum.
(*Hint: let s_n be π rounded up to n decimal places*)

3. Let S be a non-empty, bounded subset of $\mathbb{R}$.
   (a) Prove that $\inf S \le \sup S$.
   (b) What can you say about S if $\inf S = \sup S$?

4. Let S and T be non-empty subsets of $\mathbb{R}$ with the property that $s \le t$ for all $s \in S$ and $t \in T$.
   (a) Prove that S is bounded above and T bounded below.
   (b) Prove that $\sup S \le \inf T$.
   (c) Give an example of such sets S, T where $S \cap T$ is non-empty.
   (d) Give an example of such sets S, T where $S \cap T$ is empty, and $\sup S = \inf T$.

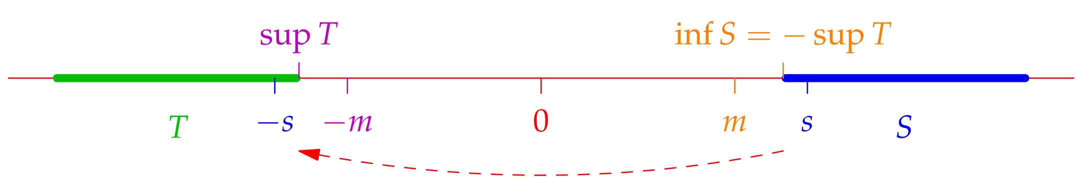
5. Prove that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

6. Let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the set of *irrational* numbers. Given real numbers $a < b$, prove that there exists $x \in \mathbb{I}$ such that $a < x < b$.
(*Hint: First show* $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$)

7. Let A, B be non-empty bounded subsets of $\mathbb{R}$, and let S be the set of all sums
$$S := \{a + b : a \in A, b \in B\}$$
   (a) Prove that $\sup S = \sup A + \sup B$.
   (b) Prove that $\inf S = \inf A + \inf B$.

8. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

9. We prove Theorem 4.10 on the existence of the infimum.
Let $S \subseteq \mathbb{R}$ be non-empty and let m be a lower bound for S. Define $T = \{t \in \mathbb{R} : -t \in S\}$ by reflecting S across zero.



   (a) Prove that $-m$ is an upper bound for T.
   (b) By completeness (Axiom 4.7), $\sup T$ exists. Prove that $\inf S = -\sup T$ by verifying Definition 4.6 parts 2(a) and (b).

5 The Symbols $\pm\infty$

Thus far the only subsets of the real numbers that have a supremum are those which are *non-empty* and *bounded above.* In this very short section, we introduce the ∞-symbol to provide all subsets of the real numbers with both a supremum and an infimum.

Definition 5.1. Let $S \subseteq \mathbb{R}$ be any subset. If S is bounded above/below, then $\sup S/\inf S$ are as in Definition 4.6. Otherwise:

1. We write $\sup S = \infty$ if S is *unbounded above*, that is

$$\forall x \in \mathbb{R}, \exists s \in S \text{ such that } s > x$$

2. We write $\inf S = -\infty$ if S is *unbounded below*,

$$\forall y \in \mathbb{R}, \exists t \in S \text{ such that } t < y$$

3. By convention, $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$, though these will rarely be of use to us.

The symbols $\pm\infty$ have *no other meaning* (as yet): in particular, they are *not numbers*! If one is willing to abuse notation and write $x < \infty$ and $y > -\infty$ for any real numbers x, y, then the conclusions of Lemma 4.8 are precisely statements 1 & 2 in the above definition!

Examples 5.2. 1. $\sup \mathbb{R} = \sup \mathbb{Q} = \sup \mathbb{Z} = \sup \mathbb{N} = \infty$, since all are unbounded above. We also have $\inf \mathbb{R} = \inf \mathbb{Q} = \inf \mathbb{Z} = -\infty$ (recall that $\inf \mathbb{N} = \min \mathbb{N} = 1$).

2. If $a < b$, then *any* interval $[a, b]$, (a, b), $[a, b)$ or $(a, b]$ has supremum b and infimum a, even if one end is infinite. For example,

$$S = (7, \infty) = \{x \in \mathbb{R} : x > 7\}$$

has $\sup S = \infty$ and $\inf S = 7$.

3. Let $S = \{x \in \mathbb{R} : x^3 - 4x < 0\}$. With a little factorization, we see that

$$x^3 - 4x = x(x-2)(x+2) < 0 \iff x < -2 \text{ or } 0 < x < 2$$

It follows that $S = (-\infty, -2) \cup (0, 2)$, from which $\sup S = 2$ and $\inf S = -\infty$.

Exercises 5. 1. Give the infimum and supremum of each of the following sets:

(a) $\{x \in \mathbb{R} : x < 0\}$ (b) $\{x \in \mathbb{R} : x^3 \leq 8\}$

(c) $\{x^2 : x \in \mathbb{R}\}$ (d) $\{x \in \mathbb{R} : x^2 < 8\}$

2. Let $S \subseteq \mathbb{R}$ be non-empty, and let $-S = \{-s : s \in S\}$. Prove that $\inf S = -\sup(-S)$.

3. Let $S, T \subseteq \mathbb{R}$ be non-empty such that $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

4. If $\sup S < \inf S$, what can you say about S?

6 A Development of $\mathbb{R}$ (non-examinable)

The comment in footnote 8 essentially constitutes a *synthetic* definition of the real numbers: there is essentially just one set with the required properties. It is nice, however, to be able to provide an explicit construction. The following approach uses *Dedekind cuts.*

First one defines $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. Use Peano's axioms and proceed as in sections 1 and 2. The operations $+, \cdot$ and $\leq$ are defined, first on $\mathbb{N}$ and then for $\mathbb{Z}$ and $\mathbb{Q}$ building on the concepts for the integers.

Definition 6.1. A *Dedekind cut* α^* is a non-empty proper subset of $\mathbb{Q}$ with the following properties:

1. If $r \in \alpha^*$ and $s \in \mathbb{Q}$ with $s < r$, then $s \in \alpha^*$.
2. α^* has no maximum.

Define $\mathbb{R}$ to be the set of all Dedekind cuts!

The rough idea is that a real number α corresponds to the Dedekind cut α^* of all *rational numbers less than* α. While this is the idea, it doesn't stand up as a *definition* due to circular logic: α cannot be defined in terms of itself!

Examples 6.2. 1. For any *rational number* r, the corresponding *real number* is the Dedekind cut

$$r^* = \{x \in \mathbb{Q} : x < r\}$$

For instance $4^* = \{x \in \mathbb{Q} : x < 4\}$ is the Dedekind cut definition of the *real number* 4.

2. It is a little trickier to explicitly define Dedekind cuts corresponding to irrational numbers, though some are relatively straightforward. For instance the real number $\sqrt{2}$ would be the Dedekind cut

$$\sqrt{2}^* = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$$

It remains to *prove* that the set of Dedekind cuts satisfies all the axioms of a complete ordered field. The full details are too much for us, so here is a rough overview.

- Define the ordering of Dedekind cuts via

  $$\alpha^* \leq \beta^* \iff \alpha^* \subseteq \beta^*$$

  One can now prove axioms O1–O3 and that the ordering corresponds to that of $\mathbb{Q}$.

- Define addition of cuts via

  $$\alpha^* + \beta^* := \{a + b : a \in \alpha^*, b \in \beta^*\}$$

  This suffices to prove the addition axioms and O4: a careful definition of $-\alpha^*$ is required.

- Multiplication is horrible: if $\alpha^*, \beta^* \geq 0$ then

  $$\alpha^*\beta^* := \{ab : a \geq 0, a \in \alpha^*, b \geq 0, b \in \beta^*\} \cup \{q \in \mathbb{Q} : q < 0\}$$

  which may be carefully extended to cover situations when α^* or $\beta < 0$. Once can then prove the multiplication axioms, the final order axiom O5, and the distributive axiom.

- The completeness axiom must also be verified, though it comes almost for free! If $A \subseteq \mathbb{R}$ (so that A is a set of Dedekind cuts), then the supremum of A is

$$\sup A = \bigcup_{\alpha^* \in A} \alpha^*$$

An alternative approach to $\mathbb{R}$ using sequences of rational numbers will be given later in the course.

Exercises 6. 1. Show that if α^*, β^* are Dedekind cuts, then so is

$$\alpha^* + \beta^* = \{r_1 + r_2 : r_1 \in \alpha^*, r_2 \in \beta^*\}$$

2. Let α^*, β^* be Dedekind cuts and define the 'product':

$$\alpha^* \cdot \beta^* = \{r_1 r_2 : r_1 \in \alpha^*, r_2 \in \beta^*\}$$

   (a) Calculate some 'products' using the cuts $0^*, 1^*$ and $(-1)^*$.

   (b) Discuss why this definition of 'product' is unsatisfactory for defining multiplication in $\mathbb{R}$.

3. We verify the Archimedean property (Theorem 4.11) using the Dedekind cut definition of $\mathbb{R}$ (it is somewhat easier since the unboundedness of $\mathbb{N}$ and $\mathbb{Q}$ are baked in).

   (a) Explain why every cut β^* is bounded above by some rational number.
   (*Hint: if β^* satisfies Definition 6.1 parts 1 & 2 but is unbounded above, then what is it?*)

   (b) If $\beta^* > 0^*$ is a positive cut bounded above by $\frac{p}{q}$ with $p, q \in \mathbb{N}$, show that $n := p + 1$ corresponds to a cut for which $n^* > \beta^*$.

7 Limits of Sequences

Sequences are the fundamental tool in our approach to analysis.

Definition 7.1. A *sequence* of real numbers is a list indexed by the natural numbers

$$(s_n) = (s_1, s_2, s_3, \ldots)$$

We call s_1 the *initial term/element.*

This is strictly the definition of an *infinite sequence*; finite sequences don't appear in this course. Other letters may be used (a_n, b_n, etc.), though s_n is most common in the abstract. It is also common to have sequences which start with a different initial term ($n = 0$ is particularly common). If you need to be explicit, describe the range of indices with sub/superscripts, e.g. $(s_n)_{n=0}^{\infty}$.

Examples 7.2. 1. Explicit sequences are often defined by providing a formula for the n^{th} term. For instance, $s_n = \left(1 + \frac{1}{n}\right)^n$ defines a sequence whose first three terms are

$$s_1 = 2, \quad s_2 = \tfrac{9}{4}, \quad s_3 = \tfrac{64}{27}, \quad \ldots$$

Since each term is a rational number, (s_n) could be described as a *rational sequence.*

2. Sequences can be defined inductively. For instance $t_1 = 1$ and $t_{n+1} = 3t_n - 1$ together define the sequence

$$(t_n) = (1, 2, 5, 14, 41, \ldots)$$

3. $u_n = \frac{1}{n^2-4}$ defines a sequence with initial term $u_3 = \frac{1}{5}$:

$$(u_n)_{n=3}^{\infty} = \left(\tfrac{1}{5}, \tfrac{1}{12}, \tfrac{1}{21}, \ldots\right)$$

Limits In analysis we are typically interested in what happens to the terms of a sequence (s_n) when n gets *large* (as such, it is common to be non-explicit as to the initial term). In elementary calculus, you should have become used to writing expressions such as[11]

$$\lim \frac{2n^2 + 3n - 1}{3n^2 - 2} = \frac{2}{3}$$

which encapsulates the idea that the expression $s_n = \frac{2n^2+3n-1}{3n^2-2}$ gets close to $\frac{2}{3}$ when n is large. We can easily convince ourselves of this with a calculator/computer: to 4 decimal places, we have

$$(s_n) = (4,\ 1.3,\ 1.04,\ 0.9348,\ 0.8767,\ 0.8396,\ 0.8138,\ 0.7947,\ \ldots), \qquad s_{1000} = 0.6677$$

Our primary business is to make this idea logically watertight. In the next section we will do so by developing the formal definition of limit. Before seeing this, we quickly refresh a few simple examples from elementary calculus. At the moment, all these rely on your intuition and experience. This is a good thing to practice: in analysis it is often essential to have a good idea of the correct answer *before* you try to prove it!

[11] If there are multiple letters in your expression, then for clarity it can be helpful to write $\lim\limits_{n\to\infty}$ with a subscript.

Examples 7.3. 1. $\lim \frac{1}{n} = 0$. Our instinct is $s_n = \frac{1}{n}$ becomes arbitrarily small as n becomes large.

2. $\lim \frac{7n+9}{2n-4} = \frac{7}{2}$. To convince yourself of this, you might write $\frac{7n+9}{2n-4} = \frac{7+\frac{9}{n}}{2-\frac{4}{n}}$ and observe that the $\frac{1}{n}$ terms become tiny as n increases.

3. The sequence with n^{th} term $s_n = (-1)^n$ does not converge to anything (it *diverges*). Indeed

$$(s_n)_{n=0}^{\infty} = (1, -1, 1, -1, 1, -1, \ldots)$$

isn't getting closer to any real number.

4. If $c_n = \frac{1}{n}\cos\left(\frac{\pi n}{6}\right)$, then $\lim c_n = 0$. To see this, observe that the cosine term lies between ± 1, while $\frac{1}{n}$ has limit 0.

5. The sequence defined inductively by $s_0 = 2,\ s_{n+1} := \frac{1}{2}s_n + 3$ begins

$$(s_n) = \left(2, 4, 5, \tfrac{11}{2}, \tfrac{23}{4}, \tfrac{47}{8}, \ldots\right)$$

This appears to have limit $\lim s_n = 6$. Indeed it is not hard to spot the pattern $s_n = 6 - \frac{4}{2^n}$ which is easily verified by induction: for the induction step, simply observe that

$$\frac{1}{2}s_n + 3 = \frac{1}{2}\left(6 - \frac{4}{2^n}\right) + 3 = 6 - \frac{4}{2^{n+1}}$$

Exercises 7. 1. Decide whether each sequence converges; if it does, give the limit. No proofs are required; if you're unsure what's going on, try writing out the first few terms.

(a) $a_n = \dfrac{1}{3n+1}$ (b) $b_n = \dfrac{3n+1}{4n-1}$ (c) $c_n = \dfrac{n}{3^n}$ (d) $d_n = \sin\left(\dfrac{n\pi}{4}\right)$

2. Repeat the previous question for sequences whose n^{th} term is as follows:

(a) $\dfrac{n^2+3}{n^2-3}$ (b) $1 + \dfrac{2}{n}$ (c) $2^{1/n}$ (d) $(-1)^n n$ (e) $\dfrac{7n^3+8n}{2n^3-31}$

(f) $\sin\left(\dfrac{n\pi}{2}\right)$ (g) $\sin\left(\dfrac{2n\pi}{3}\right)$ (h) $\dfrac{2^{n+1}+5}{2^n-7}$ (i) $\left(1+\dfrac{1}{n}\right)^2$ (j) $\dfrac{6n+4}{9n^2+7}$

3. Give an example of:

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

(b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

4. Prove by induction that the sequence defined in Example 7.2.2 has n^{th} term $t_n = \frac{1}{2}\left(3^{n-1}+1\right)$.

5. In future courses, you'll meet sequences of *functions*. For instance, we could define a sequence (f_n) of functions $f_n : \mathbb{R} \to \mathbb{R}$ inductively via

$$f_0(x) \equiv 1, \qquad f_{n+1}(x) := 1 + \int_0^x f_n(t)\,\mathrm{d}t$$

Compute the functions f_1, f_2 and f_3. The sequence (f_n) should seem familiar if you think back to elementary calculus; why?

8 The Formal Definition of Limit

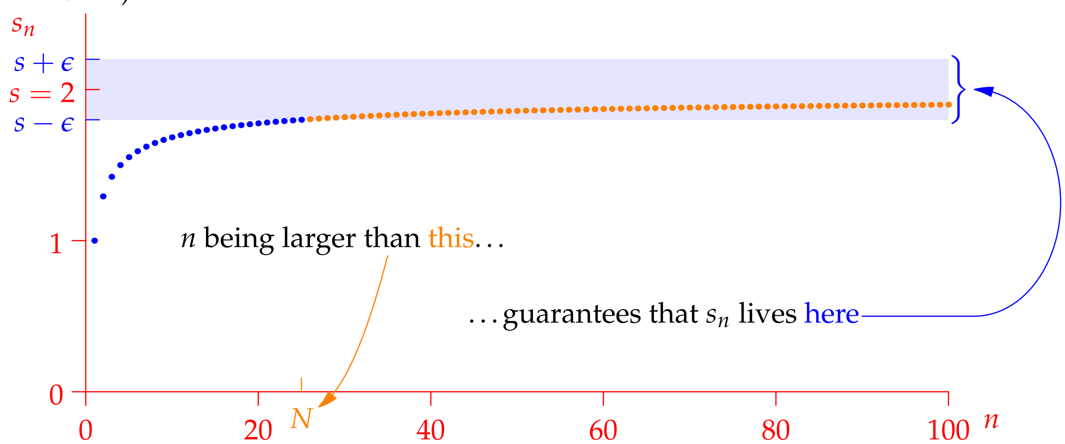
Definition 8.1. A sequence (s_n) *converges to a limit* $s \in \mathbb{R}$, if[12]

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |s_n - s| < \epsilon$$

We write $\lim s_n = s$ or simply $s_n \to s$; both are read "s_n *approaches* (or *tends to*) s."
A sequence *converges* if it has a limit, and *diverges* otherwise.

This isn't as hard as it looks! The best way to understand it is to work a *lot* of examples...

Example 8.2. We show that the sequence with n^{th} term $s_n = 2 - \frac{1}{\sqrt{n}}$ converges to $s = 2$.
If we plot the sequence like a function, we see how ϵ controls the distance from s_n is to the limit s; the definition requires us to show that no matter how small we make ϵ, there is some tail of the sequence (all s_n with $n > N$) whose terms are less than a distance ϵ from the limit.



To verify a 'for all, there exists' statement requires an argument with a specific structure:

- Suppose $\epsilon > 0$ has been provided and describe N, dependent on ϵ (ϵ smaller means N larger).
- Verify algebraically that $n > N \implies |s_n - s| < \epsilon$.

Scratch work. To find a suitable N, start with what you want to be true and let it inspire you.

We want $|s_n - s| = \left|\left(2 - \frac{1}{\sqrt{n}}\right) - 2\right| = \left|\frac{1}{\sqrt{n}}\right| < \epsilon$ (equivalently $n > \frac{1}{\epsilon^2}$) whenever $n > N$.
Choosing $N = \frac{1}{\epsilon^2}$ should be enough to complete the proof!

Warning! We do not yet have a proof: "$N = \frac{1}{\epsilon^2}$" is not the correct conclusion! We finish by rearranging our scratch work to make it clear that the definition is satisfied.

Formal argument. Suppose $\epsilon > 0$ is given, and let $N = \frac{1}{\epsilon^2}$. Then

$$n > N \implies |s_n - s| = \left|2 - \frac{1}{\sqrt{n}} - 2\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \epsilon$$

Thus $s_n \to 2$, as required.

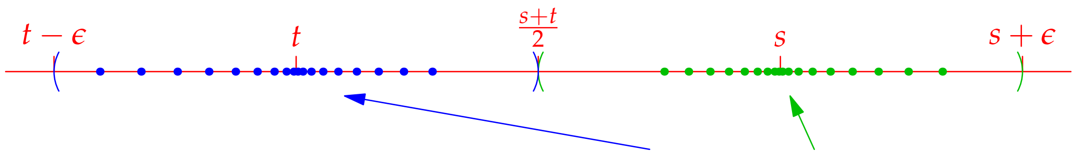
[12] N can be quantified as either a real or a natural number, the definitions being equivalent by the Archimedean property: if $N \in \mathbb{R}$ satisfies the definition, then $\exists \widetilde{N} \in \mathbb{N}$ such that $\widetilde{N} \geq N$; but then $n > \widetilde{N} \implies n > N$... It tends to be easier to use $\mathbb{R}$ for convergence and $\mathbb{N}$ when directly proving *divergence* (see Definition 8.5).

The last three lines are all we need—think of them as the concert performance after much rehearsal! With practice, you might be able to do simple ϵ–N arguments like these without scratch work, though even experts usually require some.

Before seeing more examples, we prove a hopefully intuitive result.

Lemma 8.3 (Uniqueness of Limit). *If (s_n) converges, then its limit is unique.*

The proof structure should be familiar from other uniqueness arguments: assume there are two limits $s \neq t$ and obtain a contradiction. The picture explains the strategy: by choosing $\epsilon = \frac{|s-t|}{2}$ in the definition we obtain a *tail* of the sequence (all terms s_n coming *after* some N) which must be simultaneously close to *both limits*.



For all $n > N$, s_n must lie both here and here!

Proof. Suppose $s \neq t$ are two limits. Take $\epsilon = \frac{|s-t|}{2}$ and apply Definition 8.1 twice: $\exists N_1, N_2$ such that

$$n > N_1 \implies |s_n - s| < \frac{|s-t|}{2} \quad \text{and} \quad n > N_2 \implies |s_n - t| < \frac{|s-t|}{2}$$

Define $N := \max\{N_1, N_2\}$. Then,

$$\begin{aligned} n > N \implies |s-t| = |s - s_n + s_n - t| &\leq |s_n - s| + |s_n - t| && (\triangle\text{-inequality}) \\ &< \frac{|s-t|}{2} + \frac{|s-t|}{2} = |s-t| \end{aligned}$$

Contradiction. ■

Examples 8.4. We give several more examples of using the limit definition. Remember that only the formal arguments needs to be presented; some scratch work is included to show the thought process.

1. For any $k \in \mathbb{R}^+$, we prove that $\lim \frac{1}{n^k} = 0$.

   Scratch work. Given $\epsilon > 0$, we want to choose N such that

   $$n > N \implies \frac{1}{n^k} < \epsilon$$

   This amounts to having $n > \frac{1}{\epsilon^{1/k}}$, so it is enough to choose N to be the right hand side.

   Formal argument. Let $\epsilon > 0$ be given, and let $N = \frac{1}{\epsilon^{1/k}}$. Then

   $$n > N \implies \left|\frac{1}{n^k} - 0\right| = \frac{1}{n^k} < \frac{1}{N^k} = \epsilon$$

   We conclude that $\frac{1}{n^k} \to 0$, as required.

2. We prove that $\lim\left(\sqrt{n+4}-\sqrt{n}\right)=0$.

   Scratch work. Everything follows from a (hopefully) familiar algebraic trick for manipulating surd expressions:

   $$\sqrt{n+4}-\sqrt{n}=\frac{4}{\sqrt{n+4}+\sqrt{n}}<\frac{4}{2\sqrt{n}}=\frac{2}{\sqrt{n}}$$

   Formal argument. Let $\epsilon>0$ be given, and let $N=\frac{4}{\epsilon^2}$. Then

   $$n>N \implies \left|\sqrt{n+4}-\sqrt{n}\right|=\frac{4}{\sqrt{n+4}+\sqrt{n}}<\frac{4}{2\sqrt{n}}=\frac{2}{\sqrt{n}}<\frac{2}{\sqrt{N}}=\epsilon$$

   Thus $\lim\left(\sqrt{n+4}-\sqrt{n}\right)=0$, as required.

3. We prove that $\lim\frac{3n+1}{n-7}=3$.

   Scratch work. Given $\epsilon>0$, we want to choose N such that

   $$n>N \implies \left|\frac{3n+1}{n-7}-3\right|=\left|\frac{(3n+1)-3(n-7)}{n-7}\right|=\left|\frac{22}{n-7}\right|<\epsilon \tag{$*$}$$

   For large n ($n>7$) everything is positive, so it is sufficient for us to have

   $$n-7>\frac{22}{\epsilon}\quad\text{or equivalently}\quad n>7+\frac{22}{\epsilon}$$

   Formal argument 1. Let $\epsilon>0$ be given, and let $N=7+\frac{22}{\epsilon}$. Then

   $$n>N \implies \left|\frac{3n+1}{n-7}-3\right|=\frac{22}{n-7}<\frac{22}{N-7}=\epsilon$$

   The absolute values are dropped since $n>7$. We conclude that $\lim\frac{3n+1}{n-7}=3$.

   Scratch work (cont). An alternative approach is available if we play with ($*$) a little. By insisting that $n\geq 14$, we may simplify the denominator

   $$n-7\geq\frac{1}{2}n \implies \frac{22}{n-7}\leq\frac{22}{\frac{1}{2}n}=\frac{44}{n}$$

   Formal argument 2. Let $\epsilon>0$ be given, and let $N=\max\{14,\frac{44}{\epsilon}\}$. Then

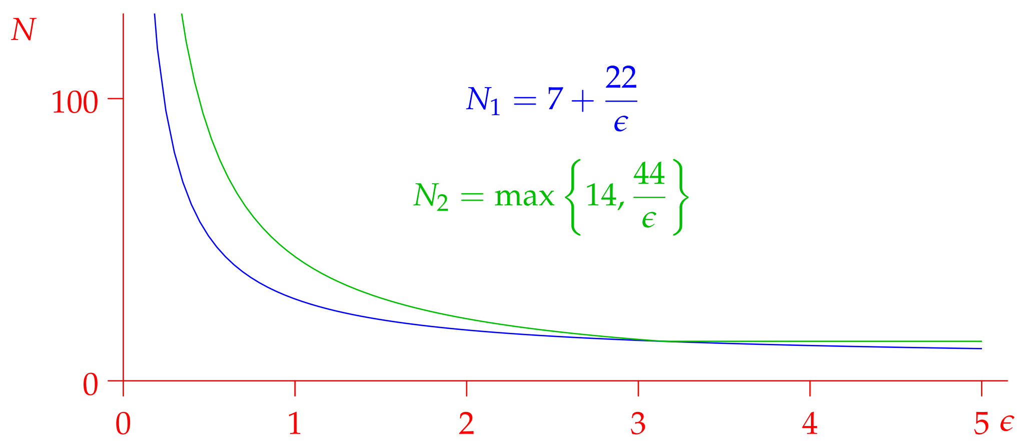
   $$\begin{aligned} n>N \implies \left|\frac{3n+1}{n-7}-3\right|&=\left|\frac{22}{n-7}\right|\leq\frac{22}{\frac{1}{2}n}=\frac{44}{n} && \text{(since } n\geq 14\text{)}\\ &<\frac{44}{N}\leq\epsilon && \text{(since } N\geq \tfrac{44}{\epsilon}\text{)}\end{aligned}$$

   We again conclude that $\lim\frac{3n+1}{n-7}=3$.

The plot illustrates the two choices of N as functions of ϵ. Observe how the second is always larger than the first: if $N = N_1(\epsilon)$ works in a proof, then any larger choice $N_2(\epsilon)$ will also,

$$n > N_2 \geq N_1 \implies n > N \implies |s_n - s| < \epsilon$$

Use this to your advantage to produce simpler arguments.



4. Given $s_n = \frac{2n^4-3n+1}{3n^4+n^2+4}$, we prove that $\lim s_n = \frac{2}{3}$.

Scratch work. We want to conclude that

$$n > N \implies \left|\frac{2n^4-3n+1}{3n^4+n^2+4} - \frac{2}{3}\right| = \left|\frac{-2n^2-9n-5}{3(3n^4+n^2+4)}\right| < \epsilon$$

Attempting to solve for n (as in the first method previously) is crazy! Instead we simplify the fraction by observing that since $n \geq 1$, we have

$$\begin{aligned}\left|\frac{-2n^2-9n-5}{3(3n^4+n^2+4)}\right| &\leq \frac{16n^2}{3(3n^4+n^2+4)} && (1 \leq n \leq n^2 \text{ and the } \triangle\text{-inequality})\\ &< \frac{16n^2}{9n^4} < \frac{2}{n^2} && (n^2+4>0 \implies 3n^4+n^2+4 > 3n^4)\end{aligned}$$

The final simplification is merely for additional tidying.

Formal argument. Let $\epsilon > 0$ be given, and let $N = \sqrt{\frac{2}{\epsilon}}$. Then

$$\begin{aligned}n > N \implies \left|s_n - \frac{2}{3}\right| &= \left|\frac{-2n^2-9n-5}{3(3n^4+n^2+4)}\right| < \frac{16n^2}{9n^4} && (\text{since } n \geq 1)\\ &< \frac{2}{n^2} < \frac{2}{N^2} = \epsilon\end{aligned}$$

Other choices of N are feasible (see e.g. Exercise 5); everything depends on how you want to simplify things in your scratch work.

Divergent sequences

By negating Definition 8.1, we obtain a new definition.

Definition 8.5. A sequence (s_n) *does not converge to* $s \in \mathbb{R}$ if,

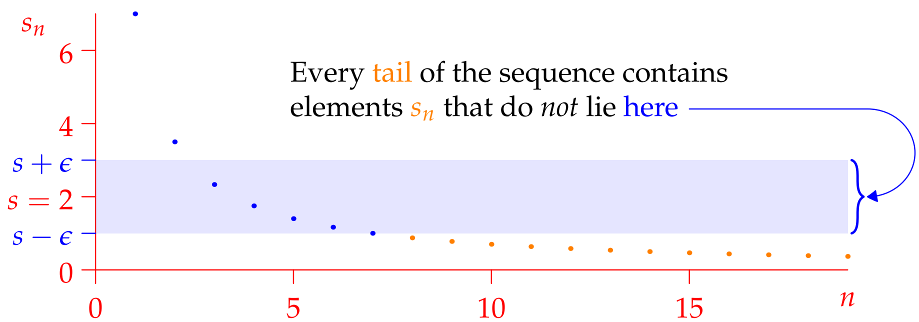
$$\exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

A sequence is *divergent* if it does not converge to *any* limit $s \in \mathbb{R}$. Otherwise said,

$$\forall s \in \mathbb{R},\ \exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

Examples 8.6. 1. We prove that the sequence with $s_n = \frac{7}{n}$ does not converge to $s = 2$.

Visualization. We intuitively know that $s_n \to 0$. If ϵ is anything smaller than 2, then the terms s_n will eventually be further than this from $s = 2$.



Direct argument. Let $\epsilon = 1$. Since we are only concerned with *large* values of n, we see that

$$|s_n - s| = \left|\frac{7}{n} - 2\right| = 2 - \frac{7}{n} \geq \epsilon = 1 \iff \frac{7}{n} \leq 1 \iff n \geq 7$$

Given $N \in \mathbb{N}$, let[13] $n = \max\{7, N+1\}$. But then $|s_n - s| = \left|\frac{7}{n} - 2\right| \geq \epsilon$, from which we conclude that $s_n \not\to 2$.

Contradiction argument. For an alternative approach, we suppose $s_n \to 2$ and let $\epsilon = 1$ in Definition 8.1. Then $\exists N$ such that

$$n > N \implies \left|\frac{7}{n} - 2\right| < 1 \implies 1 < \frac{7}{n} < 3 \implies \frac{7}{3} < n < 7$$

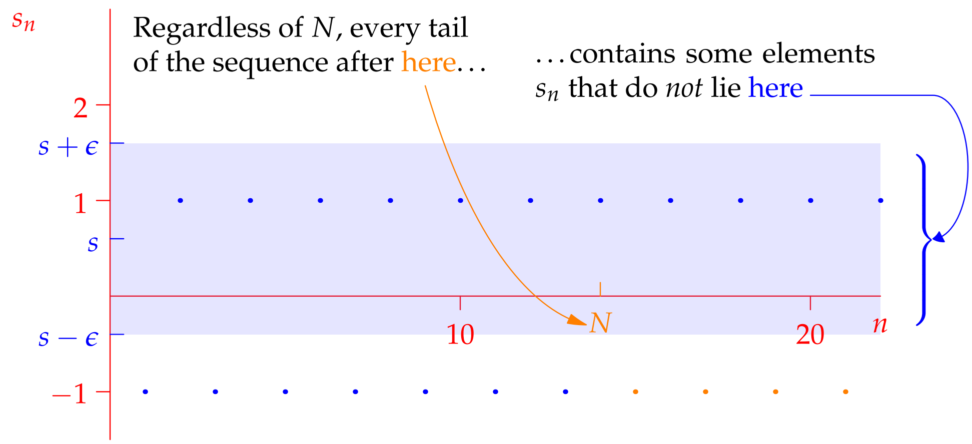
Regardless of the value of N, this cannot hold *for all* $n > N$: in particular $n := \max\{7, N+1\}$. Contradiction.

The two arguments are very similar, though consider that a significant advantage of the contradiction approach is that you only have to remember *one definition*!

[13] This is why we prefer to let N be a natural number when proving divergence. If $N \in \mathbb{R}$, then we'd have to use the ceiling function ($n = \max\{14, \lceil N \rceil + 1\}$), or resort to the Archimedean property on which it depends ($\exists n > \max\{14, N\}$). Either way is ugly and potentially confusing, so better avoided.

2. We prove that the sequence defined by $s_n = (-1)^n$ is divergent.

   Suppose, for contradiction, that $s_n \to s$. The picture shows the case $s \geq 0$ and strongly suggests that $\epsilon = 1$ will lead to a contradiction (why?).

   Let $\epsilon = 1$ in the definition of limit. Then $\exists N \in \mathbb{N}$ such that

   $$n > N \implies |(-1)^n - s| < 1$$

   One each of the values $\{n_e, n_o\} = \{N, N+1\}$ is even and the other odd. There are two cases:

   - If $s \geq 0$ then $|(-1)^{n_o} - s| = |-1-s| = s+1 \geq 1 = \epsilon$.
   - If $s < 0$ then $|(-1)^{n_e} - s| = |1-s| = 1-s \geq 1 = \epsilon$.

   Either way we have a contradiction. We conclude that (s_n) is divergent.

3. We show that the sequence defined by $s_n = \ln n$ is divergent.[14]

   Our intuition from calculus is that logarithms increase unboundedly. For any $s \in \mathbb{R}$, letting $\epsilon = 1$ should be enough, for eventually $\ln n \geq s+1$. This time we prove directly using the definition of divergence (8.5).

   Suppose $s \in \mathbb{R}$, let $\epsilon = 1$, and suppose that $N \in \mathbb{N}$ is given. Define $n = \max\{N+1, e^{s+1}\}$ and observe that. Then

   $$n > N \text{ and } \ln n \geq \ln(e^{s+1}) = s+1 \qquad \text{(ln is \textit{increasing}, and so respects inequalities!)}$$

   In particular,

   $$|s_n - s| = \ln n - s \geq 1 = \epsilon$$

   We conclude that (s_n) is divergent.

[14] In the next section we'll have a definition of what it means for a sequence to *diverge to* ∞: this is what's happening for $s_n = \ln n$, but it's not (yet) what we're trying to demonstrate.

A Little Abstraction

Working explicitly with the limit definition is tedious. In the next section we'll develop and summarize the *limit laws* so we can combine limits of sequences without providing new ϵ-N proofs. To start working towards this, here are three general results.

Lemma 8.7. *If* $\lim s_n = s$*, then* $\lim s_n^2 = s^2$*.*

The challenge is that we want to bound $\left|s_n^2 - s^2\right| = |s_n - s|\,|s_n + s|$, which means we need some control over $|s_n + s|$. One way uses the triangle-inequality,

$$|s_n + s| = |s_n - s + 2s| \le |s_n - s| + 2\,|s|$$

Assuming $|s_n - s| \le 1$ gives us a fixed bound $|s + s_n| \le 1 + 2\,|s|$. We may now begin a proof.

Proof. Suppose $s_n \to s$. Let $\epsilon > 0$ be given, and let $\delta = \min\{1, \frac{\epsilon}{1+2|s|}\}$. Since $s_n \to s$, $\exists N$ such that

$$n > N \implies |s_n - s| < \delta$$

But then

$$\begin{aligned} n > N \implies \left|s_n^2 - s^2\right| &= |s_n - s|\,|s_n + s| \\ &\le |s_n - s|\,(|s_n - s| + 2\,|s|) && (\triangle\text{-inequality}) \\ &< \delta(1 + 2\,|s|) && (\text{since } |s_n - s| < \delta \le 1) \\ &\le \epsilon \end{aligned}$$

■

Theorem 8.8. *Suppose* $\lim s_n = s$*.*

1. *If* $s_n \ge m$ *for all except finitely many* n*, then* $s \ge m$*.*
2. *If* $s_n \le M$ *for all except finitely many* n*, then* $s \le M$*.*

Proof. We prove part 1 by contradiction—part 2 is similar.
Suppose $s_n \to s < m$ and let $\epsilon = \frac{m-s}{2} > 0$. Then $\exists N$ such that

$$\begin{aligned} n > N &\implies |s_n - s| < \frac{m-s}{2} \implies s_n - s < \frac{m-s}{2} \\ &\implies s_n - m < \frac{s-m}{2} < 0 && (\text{add } s - m \text{ to both sides}) \end{aligned}$$

By assumption, $s_n < m$ holds for *at most finitely many* n. Contradiction.

■

The expression *for all but finitely many* n can be added to many abstract limit theorems; other common variants are *for all large* n, and *for some tail of the sequence.* To avoid cumbersome language, the expression is often omitted. Remember that convergence/divergence is concerned with what happens when n is *large*: we can change or delete the first million terms of (s_n) without altering it's convergence status!

Theorem 8.9 (Squeeze Theorem). *Suppose three sequences satisfy $a_n \le s_n \le b_n$ (for all large n) and that (a_n) and (b_n) both converge to s. Then $\lim s_n = s$.*

Proof. By subtracting s from our assumed inequality, we see that

$$a_n - s \le s_n - s \le b_n - s \implies |s_n - s| \le \max\{|a_n - s|, |b_n - s|\}$$

It remains to bound the right hand side by ϵ. Let $\epsilon > 0$ be given, then there exist N_a, N_b such that

$$n > N_a \implies |a_n - s| < \epsilon \quad \text{and} \quad n > N_b \implies |b_n - s| < \epsilon$$

Finally let $N = \max\{N_a, N_b\}$ to see that

$$n > N \implies |s_n - s| \le \max\{|a_n - s|, |b_n - s|\} < \epsilon$$

■

Example 8.10. If $s_n = \frac{1+\sin n}{n}$, then $0 \le s_n \le \frac{2}{n}$. The squeeze theorem quickly forces $\lim s_n = 0$.

Exercises 8. 1. For each sequence, determine the limit and prove your claim.

(a) $a_n = \frac{n}{n^2+1}$ (b) $b_n = \frac{7n-19}{3n+7}$ (c) $c_n = \frac{4n+3}{7n-5}$

(d) $d_n = \frac{2n+4}{5n+2}$ (e) $e_n = \frac{1}{n}\sin n$ (f) $f_n = \frac{n^2+n-1}{3n^2-10}$

2. Let (t_n) be a bounded sequence (there exists M such that $|t_n| \le M$ for all n), and let (s_n) be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.
(*Hint: given ϵ, note that $\frac{\epsilon}{|M|}$ is also a small number...*)

3. Prove the following

(a) $\lim[\sqrt{n^2+1} - n] = 0$ (b) $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$ (c) $\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$

4. Let (s_n) be a convergent sequence, and suppose $\lim s_n > a$. Prove that there exists N such that $n > N \implies s_n > a$.

5. (a) Show that $n \ge 2 \implies 2n^2 + 9n + 5 \le 9n^2$.
(b) (Recall Example 8.4.4) Provide another argument that $\lim \frac{2n^4-3n+1}{3n^4+n^2+4} = \frac{2}{3}$ by choosing $N = \max\{2, \frac{1}{\sqrt{\epsilon}}\}$.

6. (a) Prove that the sequence with n^{th} term $s_n = \frac{2}{n^2}$ does not converge to -1.
(b) Prove that (s_n) does not converge to 1.

7. Prove that the sequence defined by $t_n = n^2$ diverges.

8. Provide a contradiction argument to justify Example 8.6.3: $(\ln n)$ diverges.

9. (Recall Theorem 8.8) Suppose $\lim s_n = s$ where every $s_n > m$. Can we conclude that $s > m$? Explain your answer.

10. (a) If $|s_n - s| < 1$, explain why $|s_n^2 + s s_n + s^2| < 1 + 3|s| + 3|s|^2$
(b) Suppose $s_n \to s$. Prove that $s_n^3 \to s^3$.

9 Limit Theorems for Sequences

We'd like to develop some rules for working with limits so that we don't have to resort to an ϵ–N proof every time. The rough idea is that limits respect the basic rules of algebra. For instance...

Example 9.1. If $\lim s_n = s$, it seems natural that a new sequence $(5s_n)$ obtained by multiplying the original terms by 5 should have limit $\lim 5s_n = 5s$. Consider what we have to prove to confirm this:

$$\forall \epsilon > 0, \exists N \text{ such that } n > N \implies |5s_n - 5s| < \epsilon$$

This last amounts to observing that $|s_n - s| < \frac{\epsilon}{5}$. The challenge here is to see that we're essentially done: this is just the statement $\lim s_n = s$ in disguise! Here is a more formal argument.

Let $\epsilon > 0$ be given. Since $\lim s_n = s$, we know that

$$\exists N \text{ such that } n > N \implies |s_n - s| < \frac{\epsilon}{5} \implies |5s_n - 5s| < \epsilon$$

The trick in the example will be used repeatedly in the proofs that follow. What's critical is that you read the limit definition correctly: given any small number (ϵ, $\frac{\epsilon}{5}$, etc.) there is some tail of the sequence which remains closer to the limit than this.

Theorem 9.2 (Limit laws). *Limit calculations respect algebraic operations: $\pm, \times, \div$ and roots. More specifically, if (s_n) converges to s and (t_n) to t, then,*

1. $\lim(s_n \pm t_n) = s \pm t$
2. $\lim(s_n t_n) = st$; *as a special case, if k is constant, then* $\lim ks_n = ks$
3. *If $t \neq 0$, then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$
4. *If $k \in \mathbb{N}$, then* $\lim \sqrt[k]{s_n} = \sqrt[k]{s}$*, provided the roots exist ($s_n, s \geq 0$ if k even)*

Our first example was the special case of part 2 with $k = 5$. Note also how parts 2 and 4 extend Lemma 8.7: by induction we now have $s_n^q \to s^q$ for any $q \in \mathbb{Q}$.

Proving the limit laws takes a little work, including a small lemma. To advertise their benefit, we repeated apply them to a limit calculation as you might have seen it in elementary calculus.

Examples 9.3. 1. We evaluate $\lim \frac{3n^2+2\sqrt{n}-1}{5n^2-2}$ using the limit laws.

$$\begin{aligned}
\lim \frac{3n^2 + 2\sqrt{n} - 1}{5n^2 - 2} &= \lim \frac{3 + \frac{2}{n^{3/2}} - \frac{1}{n^2}}{5 - \frac{2}{n^2}} = \frac{\lim \left(3 + \frac{2}{n^{3/2}} - \frac{1}{n^2}\right)}{\lim \left(5 - \frac{2}{n^2}\right)} && \text{(part 3)} \\
&= \frac{\lim 3 + \lim \frac{2}{n^{3/2}} - \lim \frac{1}{n^2}}{\lim 5 - \lim \frac{2}{n^2}} && \text{(part 1)} \\
&= \frac{3 + 0 - 0}{5 - 0} = \frac{3}{5} && \text{(parts 2, 4 and } \lim \tfrac{1}{n} = 0 \text{ (Example 8.4.1))}
\end{aligned}$$

This calculation involves some generally accepted sleight of hand; formally we're working from the bottom up since $\lim \frac{3n^2+2\sqrt{n}-1}{5n^2-2}$ shouldn't really be written until you know it exists!

2. Suppose (s_n) is defined inductively via $s_1 = 2$ and $s_{n+1} = \frac{1}{2}(s_n + \frac{2}{s_n})$:

   $(s_n) = (2, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots)$

   This sequence in fact converges, though we'll need to wait until the next section to see why. Given this fact, the limit laws allow us to compute the limit s:

   $$s = \lim s_{n+1} = \frac{1}{2}\left(\lim s_n + \frac{2}{\lim s_n}\right) = \frac{1}{2}\left(s + \frac{2}{s}\right) \implies \frac{1}{2}s = \frac{1}{s} \implies s^2 = 2$$

   Since s_n is plainly always positive, we conclude that $\lim s_n = \sqrt{2}$.

We now commence our assault on the limit laws. The strategy for the first is simple: control both sequences so that both $|s_n - s|, |t_n - t| < \frac{\epsilon}{2}$, then add. The only challenge is writing it formally.

Proof of Theorem 9.2, part 1. Let $\epsilon > 0$ be given. Since $s_n \to s$ and $t_n \to t$, we see that $\exists N_1, N_2$ such that

$$\exists N_1 \text{ such that } n > N_1 \implies |s_n - s| < \frac{\epsilon}{2} \quad \text{and,}$$
$$\exists N_2 \text{ such that } n > N_2 \implies |t_n - t| < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$, then

$$n > N \implies |s_n + t_n - (s+t)| \overset{\triangle}{\leq} |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

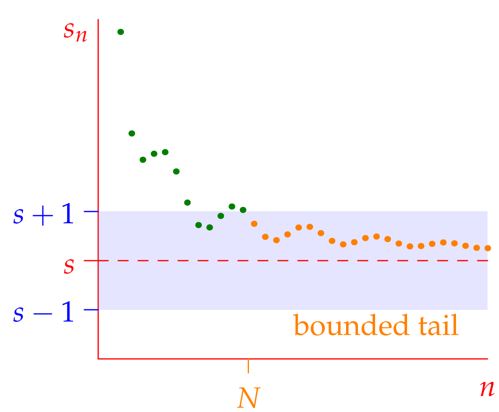
The argument for $s_n - t_n$ is almost identical. ■

The multiplicative limit law requires a preparatory result.

Lemma 9.4. *(s_n) convergent $\implies$ (s_n) bounded ($\exists M$ such that $\forall n, |s_n| \leq M$).*

The converse to this statement is *false*: why?

The picture shows the strategy: taking $\epsilon = 1$ in the limit definition bounds an infinite tail of the sequence; the finitely many terms that come before are a non-issue.



Proof. Suppose $\lim s_n = s$ and let $\epsilon = 1$ in the definition of limit. Then $\exists N$ such that

$$\begin{aligned} n > N &\implies |s_n - s| < 1 \implies s - 1 < s_n < s + 1 \\ &\implies |s_n| < \max\{|s-1|, |s+1|\} \end{aligned}$$

It follows that every term of the sequence is bounded by

$$M := \max\{|s-1|, |s+1|, |s_n| : n \leq N\}$$

■

The approach to part 2 is similar to part 1, we just need to be a bit cleverer to break up $|s_n t_n - st|$.

Proof of Theorem 9.2, part 2. Exercise 8.2 deals with (and extends) the case when $s = 0$. Instead suppose $s \neq 0$, and let $\epsilon > 0$ be given. Since $s_n \to s$ and $t_n \to t$,

$$(t_n) \text{ is bounded (Lemma)} : \exists M \text{ such that } \forall n, |t_n| \leq M$$

$$\exists N_1, N_2 \text{ such that } n > N_1 \implies |s_n - s| < \frac{\epsilon}{2M} \quad \text{and} \quad n > N_2 \implies |t_n - t| < \frac{\epsilon}{2|s|}$$

Again let $N = \max\{N_1, N_2\}$, then

$$|s_n t_n - st| = |s_n t_n - st_n + st_n - st| \overset{\triangle}{\leq} |s_n - s| |t_n| + |s| |t_n - t| < \frac{\epsilon}{2M} M + |s| \frac{\epsilon}{2|s|} = \epsilon$$ ■

The proofs of parts 3 and 4 are in Exercise 6.

More basic examples With a few simple general examples, the limit laws allow us to rapidly compute the limits of a great variety of sequences.

Theorem 9.5. 1. *If* $k > 0$ *then* $\lim \frac{1}{n^k} = 0$

2. *If* $|a| < 1$ *then* $\lim a^n = 0$
3. *If* $a > 0$ *then* $\lim a^{1/n} = 1$
4. $\lim n^{1/n} = 1$

Examples 9.6. 1. $\lim(3n)^{2/n} = (\lim 3^{1/n})^2 (\lim n^{1/n})^2 = 1.$

2. $$\lim \frac{n^{2/n} + \left(3 - n^{-1} \sin n\right)^{1/5}}{4n^{-3/2} + 7} = \frac{(\lim n^{1/n})^2 - \left(3 - \lim \frac{\sin n}{n}\right)^{1/5}}{4 \lim \frac{1}{n^{3/2}} + 7} = \frac{1 - \sqrt[5]{3}}{7}$$

   Note that $\lim \frac{\sin n}{n} = 0$ follows from the squeeze theorem: $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \to 0$.

Proof. 1. This is Example 8.4.1.

2. The $a = 0$ case is trivial. Otherwise, given $\epsilon > 0$, let $N = \log_{|a|} \epsilon$ and observe that

$$n > N \implies |a^n| < \left|a^N\right| = |a|^N = \epsilon$$

3. Suppose $a \geq 1$, and let $s_n := a^{1/n} - 1$. Since $s_n > 0$, the binomial theorem[15] shows that

$$a = (1 + s_n)^n \geq 1 + ns_n \implies 0 < s_n \leq \frac{a - 1}{n}$$

   The squeeze theorem (8.9) shows that $s_n \to 0$, whence $\lim a^{1/n} = 1$.

   We leave the $a < 1$ case and part 4 to Exercise 7.

■

[15] $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{2 \cdot 3} x^3 + \cdots + x^n.$

Divergence to $\pm\infty$ and the 'divergence laws'

We now consider unbounded sequences and provide a positive definition of a type of divergence.

Definition 9.7. We say that (s_n) *diverges to* ∞ if,

$$\forall M > 0,\ \exists N \text{ such that } n > N \implies s_n > M$$

We write $s_n \to \infty$ or $\lim s_n = \infty$. The definition for $s_n \to -\infty$ is similar.
If (s_n) neither converges nor diverges to $\pm\infty$, we say that it *diverges by oscillation*.[16]

Consider how M is trying to describe "closeness" to infinity similarly to how ϵ measures closeness to s in the original definition of limit (8.1).

Examples 9.8. As with convergence proofs, it is a good idea to try some scratch work first!

1. We show that $\lim(n^2 + 4n) = \infty$.
   Let $M > 0$ be given, and let $N = \sqrt{M}$. Then
   $$n > N \implies n^2 + 4n > n^2 > N^2 = M$$
2. Prove that $s_n = n^5 - n^4 - 2n + 1 \to \infty$.
   The negative terms cause some trouble, though our solution should be familiar from previous calculations:
   $$s_n > \frac{1}{2}n^5 \iff n^5 > 2(n^4 + 2n - 1) \iff n > 2 + \frac{4}{n^3} - \frac{1}{n^4}$$
   Certainly this holds if $n > 6$. We can now complete the proof.
   Let $M > 0$ be given, and let $N = \max\{6, \sqrt[5]{2M}\}$. Then
   $$n > N \implies s_n > \frac{1}{2}n^5 > \frac{1}{2}(2M) = M$$
3. Prove that the sequence defined by $s_n = n^2 - n^3$ diverges to $-\infty$.
   First observe that
   $$s_n = n^2(1-n) < -\frac{1}{2}n^3 \iff 1 - n < -\frac{1}{2}n \iff n \geq 2$$
   Now let $M > 0$ be given,[17] and define $N = \max\{2, \sqrt[3]{2M}\}$. Then
   $$n > N \implies n > 2 \implies s_n < -\frac{1}{2}n^3 < -\frac{1}{2}N^3 \leq -M$$

[16] In such cases $\lim s_n$ is meaningless; you likely wrote $\lim s_n = \text{DNE}$ ("does not exist") in elementary calculus.
[17] The notion that $s_n \to -\infty$ can be phrased in multiple ways: some prefer

$$\forall m < 0,\ \exists N \text{ such that } n > N \implies s_n < m \qquad \text{(in our argument } M = -m\text{)}$$

Several of the limit laws can be adapted to sequences which diverge to $\pm\infty$.

Theorem 9.9. *Suppose* $\lim s_n = \infty$.

1. *If* $t_n \geq s_n$ *for all (large)* n*, then* $\lim t_n = \infty$
2. *If* $\lim t_n$ *exists and is* finite*, then* $\lim s_n + t_n = \infty$.
3. *If* $\lim t_n > 0$ *then* $\lim s_n t_n = \infty$.
4. $\lim \frac{1}{s_n} = 0$
5. *If* $\lim t_n = 0$ *and* $t_n > 0$ *for all (large)* n*, then* $\lim \frac{1}{t_n} = \infty$

Similar statements when $s_n \to -\infty$ *should be clear.*

Proof. We prove two of the results: try the rest yourself.

2. Since (t_n) converges, it is bounded (below): $\exists m$ such that $\forall n,\ t_n \geq m$. Let M be given. Since $\lim s_n = \infty$, $\exists N$ such that

$$n > N \implies s_n > M - m \implies s_n + t_n > M - m + m = M$$

4. Let $\epsilon > 0$ be given, and let $M = \frac{1}{\epsilon}$. Then $\exists N$ such that

$$n > N \implies s_n > M = \frac{1}{\epsilon} \implies \frac{1}{s_n} < \epsilon$$

■

Rational Sequences We can now find the limit of any rational sequence: $\frac{p_n}{q_n}$ where (p_n), (q_n) are polynomials in n.

Example 9.10. By applying Theorem 9.9 (part 3) to

$$s_n := 3n + 4n^{-2} \to \infty \quad \text{and} \quad t_n = \frac{1}{2 - n^{-2}} \to \frac{1}{2}$$

we see that

$$\lim \frac{3n^3 + 4}{2n^2 - 1} = \lim \frac{3n + 4n^{-2}}{2 - n^{-2}} = \lim(3n + 4n^{-2}) \cdot \lim \frac{1}{2 - n^{-2}} = \infty$$

Indeed, you should be able to confirm the familiar result from elementary calculus:

Corollary 9.11. *If* p_n, q_n *are polynomials in* n *with leading coefficients* p, q *respectively then*

$$\lim \frac{p_n}{q_n} = \begin{cases} 0 & \text{if } \deg(p_n) < \deg(q_n) \\ \frac{p}{q} & \text{if } \deg(p_n) = \deg(q_n) \\ \operatorname{sgn}(\frac{p}{q})\infty & \text{if } \deg(p_n) > \deg(q_n) \end{cases}$$

Exercises 9. 1. Suppose $\lim x_n = 3$, $\lim y_n = 7$ and that all y_n are non-zero. Determine the following:

(a) $\lim(x_n + y_n)$ (b) $\lim \frac{3y_n - x_n}{y_n^2}$ (c) $\lim \sqrt{x_n y_n + 4}$

2. Consider $s_n = (100n)^{\frac{100}{n}}$. Describe s_1 (1 followed by how many zeros?). Repeat for s_{10}. Now compute the limit $\lim s_n$.

3. Define (s_n) inductively via $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for $n \geq 1$.

   (a) List the first four terms of (s_n).

   (b) It turns out that (s_n) converges. Assume this and prove that $\lim s_n = \frac{1}{2}(1 + \sqrt{5})$.

4. Prove the following:

   (a) $\lim(n^3 - 98n) = \infty$ (b) $\lim\left(\sqrt{n} - n + \frac{4}{n}\right) = -\infty$

5. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

   (a) Show that if (x_n) converges with limit a, then $a = \frac{1}{3}$ or $a = 0$.

   (b) What is $\lim x_n$? Prove your assertion and explain what is going on.

6. We prove parts 3 and 4 of the limit laws (Theorem 9.2). Assume $\lim s_n = s$ and $\lim t_n = t$.

   (a) Suppose $t \neq 0$. Explain why $\exists N_1$ such that $n > N_1 \implies |t_n| > \frac{1}{2}|t|$.

   (b) Let $\epsilon > 0$ be given. Since $t_n \to t$, $\exists N_2$ such that $n > N_2 \implies |t_n - t| < \frac{1}{2}|t|^2 \epsilon$. Combine N_1 and N_2 to provide a proof that $\lim \frac{1}{t_n} = \frac{1}{t}$.

   (c) Explain how to conclude part 3: $\lim \frac{s_n}{t_n} = \frac{s}{t}$.

   (d) Use the following inequality (valid when $s_n, s > 0$) to help construct a proof for part 4

$$\left|s_n^{1/k} - s^{1/k}\right| = \frac{|s_n - s|}{s_n^{\frac{k-1}{k}} + s_n^{\frac{k-2}{k}} s^{\frac{1}{k}} + \cdots + s^{\frac{k-1}{k}}} \leq \frac{|s_n - s|}{s^{\frac{k-1}{k}}}$$

7. We finish the proof of Theorem 9.5.

   (a) Suppose $0 < a < 1$. Prove that $\lim a^{1/n} = 1$.
   (*Hint: consider* $b = \frac{1}{a} \ldots$)

   (b) Let $s_n = n^{1/n} - 1$. Apply the binomial theorem to $n = (1 + s_n)^n$ to prove that $s_n < \sqrt{\frac{2}{n-1}}$.
   Hence conclude that $\lim n^{1/n} = 1$.

8. Prove the remaining parts of Theorem 9.9.

9. Assume $s_n \neq 0$ for all n, and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

   (a) Show that if $L < 1$, then $\lim s_n = 0$.
   (*Hint: if* $L < a < 1$, *obtain* N *so that* $n > N \implies |s_n| < a^{n-N}|s_N|$)

   (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.
   (*Hint: apply (a) to the sequence* $t_n = \frac{1}{|s_n|}$)

   (c) Let $p > 0$ and $a \in \mathbb{R}$ be given. How does $\lim_{n\to\infty} \frac{a^n}{n^p}$ depend on the value of a?

10 Monotone and Cauchy Sequences

The definition of limit (Definition 8.1) exhibits a major weakness; to demonstrate the convergence of a sequence, we must already know its limit! What we'd like is a method for determining whether a sequence converges *without* first guessing a suitable limit.[18] In this section we consider two important classes of sequence for which this can be done.

Definition 10.1. A sequence (s_n) is:

- *Monotone-up*[19] if $s_{n+1} \geq s_n$ for all n.
- *Monotone-down* if $s_{n+1} \leq s_n$ for all n.
- *Monotone* if either of the above is true.

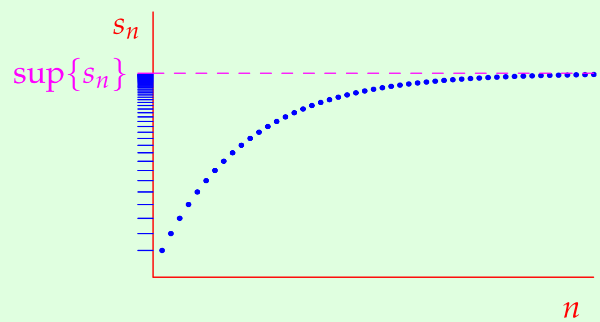
Examples 10.2. 1. The sequence with n^{th} term $s_n = \frac{7}{n} + 4$ is (strictly) monotone-down:

$$s_{n+1} = \frac{7}{n+1} < \frac{7}{n} = s_n$$

2. A constant sequence $(s_n) = (s, s, s, s, \ldots)$ is *both* monotone-up and monotone-down.

Theorem 10.3 (Monotone Convergence). *Every bounded monotone sequence is convergent. Specifically:*

- *If (s_n) is bounded above and monotone-up, then $\lim s_n$ exists and equals $\sup\{s_n\}$.*
- *If (s_n) is bounded below and monotone-down, then $\lim s_n$ exists and equals $\inf\{s_n\}$.*



In fact the conclusion $\lim s_n = \sup\{s_n\}$ holds for all monotone-up sequences: if unbounded above, then the result is ∞ (see Exercise 5). The statement is $\lim s_n = \inf\{s_n\}$ for monotone-down sequences.

Proof. If (s_n) is bounded above, then $s := \sup\{s_n\}$ exists by the completeness axiom (s is finite!). Let $\epsilon > 0$ be given. By Lemma 4.8, there exists some $s_N > s - \epsilon$. Since (s_n) is monotone-up, we have

$$n > N \implies s_n \geq s_N > s - \epsilon \implies 0 \leq s - s_n < \epsilon \implies |s - s_n| < \epsilon$$

The monotone-down case is similar. ∎

[18] This gets at the typical role of sequences in analysis: to demonstrate the existence of and define a new object (the limit) and, more broadly, to transfer useful properties from the sequence to the limit. For instance, if (f_n) is a sequence of differentiable functions, we'd like to know if $\lim f_n(x)$ exists and is itself differentiable with derivative $\lim f_n'(x)$: discussions of this ilk will dominate Math 140B.

[19] Some authors describe such as sequence as either *non-decreasing* or *increasing*. We prefer *monotone-up/down* since this directly describes the direction of any potential movement in the sequence and prevents confusion over whether the inequality is strict. A sequence with $s_{n+1} > s_n$ may be described as *strictly increasing* or *strictly monotone-up*.

Examples 10.4. 1. Define (s_n) via $s_n = 1$ and $s_{n+1} = \frac{1}{5}(s_n + 8)$:

$$(s_n) = \big(1, 1.8, 1.96, 1.992, 1.9984, 1.99968, \ldots\big)$$

The sequence certainly appears to be monotone-up and converging to 2. We prove this:

Bounded above: $s_n < 2 \implies s_{n+1} < \frac{1}{5}[2+8] = 2$. By induction, (s_n) is bounded above by 2.

Monotone-up: $s_{n+1} - s_n = \frac{4}{5}[2 - s_n] > 0$ since $s_n < 2$.

Convergence: By monotone convergence, $s = \lim s_n$ exists. Now use the limit laws to find s:

$$s = \lim s_{n+1} = \frac{1}{5}(\lim s_n + 8) = \frac{1}{5}(s+8) \implies s = 2$$

2. (Example 9.3.2, cont.) Let $s_1 = 2$ and $s_{n+1} = \frac{1}{2}\big(s_n + \frac{2}{s_n}\big)$.

Bounded below: The sequence is plainly always positive and thus bounded below by zero.

Monotone-down: We first obtain an improved lower bound:

$$s_{n+1}^2 = \frac{1}{4}\left(s_n + \frac{2}{s_n}\right)^2 = 2 + \frac{1}{4}\left(s_n - \frac{2}{s_n}\right)^2 \geq 2$$

shows[20] that $s_n^2 \geq 2$ for all n. It follows that

$$\frac{s_{n+1}}{s_n} = \frac{1}{2}\left(1 + \frac{2}{s_n^2}\right) \leq 1 \implies s_{n+1} \leq s_n$$

Convergence: By monotone convergence, $s = \lim s_n$ exists. Example 9.3.2 provides the limit:

$$s = \frac{1}{2}\left(s + \frac{2}{s}\right) \implies s = \sqrt{2}$$

This shows the necessity of completeness: (s_n) is a monotone, bounded sequence of *rational* numbers, but its limit is *irrational.*

3. A *decimal* $0.d_1d_2d_3\ldots$ may be viewed as the limit of a monotone-up sequence of *rational numbers*:

$$0.d_1d_2d_3\ldots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{d_k}{10^k}$$

This is bounded above by 1 and so converges. Compare this with Example 4.4.

4. The sequence with $s_n = \big(1 + \frac{1}{n}\big)^n$ is particularly famous. In Exercise 10 we show that (s_n) is monotone-up and bounded above. The limit provides, arguably, the oldest definition of e:

$$e := \lim\left(1 + \frac{1}{n}\right)^n$$

[20] In case you've seen it before, this is the famous AM–GM inequality $\frac{x+y}{2} \geq \sqrt{xy}$ with $x = s_n$ and $y = \frac{2}{s_n}$.

Limits Superior and Inferior

One interpretation of $\lim s_n$ is that it approximately describes s_n for large n. Even when a sequence does not have a limit, it remains useful to be able to describe its long-term behavior.

Definition 10.5. Let (s_n) be a sequence and define two related sequences (v_N) and (u_N):

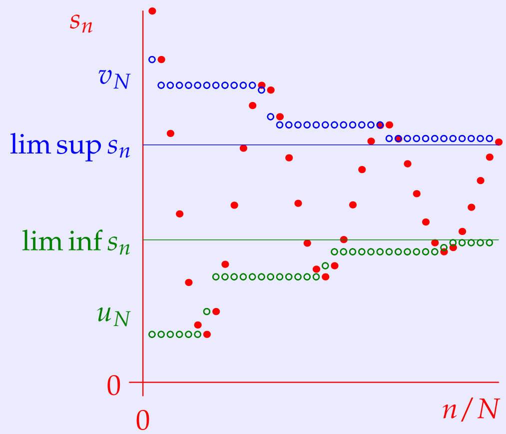
$$v_N := \sup\{s_n : n > N\}, \qquad u_N := \inf\{s_n : n > N\}$$

1. The *limit superior* of (s_n) is

$$\limsup s_n = \begin{cases} \lim\limits_{N\to\infty} v_N & \text{if } (s_n) \text{ bounded above} \\ \infty & \text{if } (s_n) \text{ unbounded above} \end{cases}$$

2. The *limit inferior* of (s_n) is

$$\liminf s_n = \begin{cases} \lim\limits_{N\to\infty} u_N & \text{if } (s_n) \text{ bounded below} \\ -\infty & \text{if } (s_n) \text{ unbounded below} \end{cases}$$



The original sequence (s_n) is *almost* wedged between[21] (v_n) and (u_n) in a situation reminiscent of the squeeze theorem (except lim sup and lim inf need not be equal). The next result summarizes the situation more formally; we omit the proof since these claims should be clear from the definition and previous results, particularly the monotone convergence theorem.

Lemma 10.6. 1. *(v_N) is monotone-down, (u_N) is monotone-up, and $u_N \le s_{N+1} \le v_N$.*

2. *$\limsup s_n$ and $\liminf s_n$ exist for any sequence (they might be infinite).*

3. *$\liminf s_n \le \limsup s_n$.*

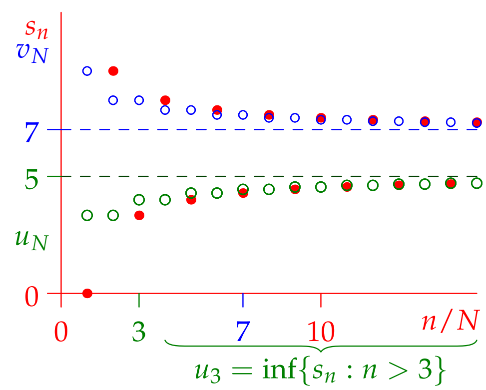
Examples 10.7. 1. The picture shows sequences (s_n), (u_N) and (v_N) when $s_n = 6 + (-1)^n \left(1 + \frac{5}{n}\right)$

We won't compute everything precisely, but the picture suggests (s_n) has two "sub"sequences: the odd terms increase while the even terms decrease towards, respectively

$$\liminf s_n = 5, \qquad \limsup s_n = 7$$

Here is one value from each derived sequence:

$$u_3 = \inf\{s_n : n > 3\} = s_5 = 4$$
$$v_7 = \sup\{s_n : n > 7\} = s_8 = 7.625$$



2. If $s_n = \frac{1}{n}$, then $v_N = s_{N+1}$ and $u_N = 0$ for all N, whence $\limsup s_n = \liminf s_n = 0$.

[21] A minor redefinition would remove the 'almost,' but at the cost of making some subsequent arguments a little messier. It is still reasonable to think of (u_N) and (v_N) as providing a long-term envelope for the original sequence.

3. Let $s_n = (-1)^n$. This time the calculation is easy:

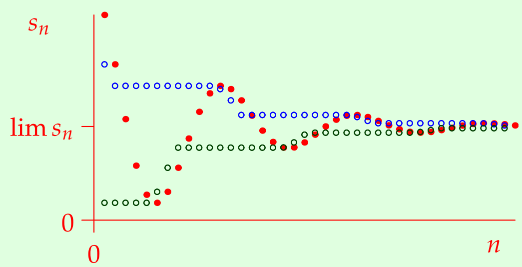
$$u_N = \inf\{s_n : n > N\} = -1 \quad \text{and} \quad v_N = \sup\{s_n : n > N\} = 1$$

Therefore $\limsup s_n = 1$ and $\liminf s_n = -1$.

Lemma 10.8. *For any sequence (s_n),*

$$\liminf s_n = \limsup s_n \implies \lim s_n \text{ exists}$$

(the limit can be infinite!).
In such a case all three values are equal.



In fact the converse to this is also true: we could prove it now, but it will come for free a little later...

Proof. (s finite) Since $u_{n-1} \le s_n \le v_{n-1}$ for all n, the squeeze theorem tells us that $\lim s_n = s$.

($s = \infty$) Since $u_{n-1} \le s_n$ for all n and $\lim u_{n-1} = \infty$, it follows (Theorem 9.9.1) that $\lim s_n = \infty$.

($s = -\infty$) This time $s_n \le v_{n-1} \to -\infty \implies \lim s_n = -\infty$. ■

Cauchy Sequences

We now come to a class of sequences whose analogues will dominate your study of analysis.

Definition 10.9. A sequence (s_n) is *Cauchy*[22] if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon$$

A sequence is Cauchy when terms in the tails of the sequence are constrained to stay close to one another. As we'll see shortly, this will provide an alternative way to detect and describe *convergence*.

Examples 10.10. 1. Let $s_n = \frac{1}{n}$. Let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then[23]

$$m > n > N \implies |s_m - s_n| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon$$

Thus (s_n) is Cauchy. A similar argument works for any $s_n = \frac{1}{n^k}$ for positive k.

2. Suppose $s_1 = 5$ and $s_{n+1} = s_n + \frac{1}{n(n+1)}$. As before, let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then,

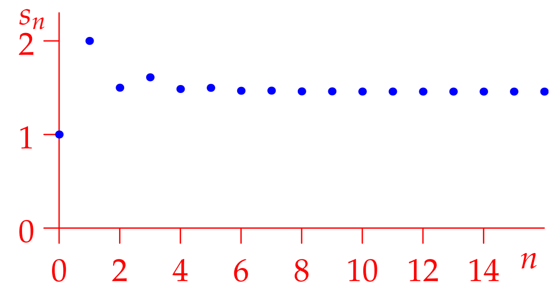
$$|s_{n+1} - s_n| = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\implies |s_m - s_n| \overset{\triangle}{\le} |s_{n+1} - s_n| + \cdots + |s_m - s_{m-1}| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon$$

Again we have a Cauchy sequence.

[22] Augustin-Louis Cauchy (1789–1857) was a French mathematician, responsible (in part) for the ϵ-N definition of limit.

3. Define $(s_n)_{n=0}^{\infty}$ inductively:



$$s_0 = 1, \quad s_{n+1} = \begin{cases} s_n + 3^{-n} & \text{if } n \text{ even} \\ s_n - 2^{-n} & \text{if } n \text{ odd} \end{cases}$$

$$(s_n) = \left(1, 2, \frac{3}{2}, \frac{29}{18}, \frac{107}{72}, \ldots\right)$$

Since $|s_{n+1} - s_n| \le 2^{-n}$, we see that

$$m > n \implies |s_m - s_n| \overset{\triangle}{\le} |s_{n+1} - s_n| + \cdots + |s_m - s_{m-1}| = \sum_{k=n}^{m-1} |s_{k+1} - s_k|$$
$$\le \sum_{k=n}^{m-1} 2^{-k} = \frac{2^{-n} - 2^{-m}}{1 - 2^{-1}} < 2^{1-n}$$

where we used the familiar geometric sum formula from calculus: $\sum\limits_{k=a}^{b-1} r^k = \frac{r^a - r^b}{1-r}$.
Suppose $\epsilon > 0$ is given, and let $N = 1 - \log_2 \epsilon = \log_2 \frac{2}{\epsilon}$. Then

$$m > n > N \implies |s_m - s_n| < 2^{1-n} < 2^{1-N} = \epsilon$$

We conclude that (s_n) is Cauchy.

The picture in the last example illustrates the essential point regarding Cauchy sequences: (s_n) appears very much to converge…

Theorem 10.11 (Cauchy Completeness). *A sequence of real numbers is convergent if and only if it is Cauchy.*

Proof. $(\Rightarrow)$ Suppose $\lim s_n = s$ (finite). Given $\epsilon > 0$ we may choose N such that

$$m, n > N \implies |s_n - s| < \frac{\epsilon}{2} \quad \text{and} \quad |s_m - s| < \frac{\epsilon}{2}$$
$$\implies |s_n - s_m| = |s_n - s + s - s_m| \overset{\triangle}{\le} |s_n - s| + |s - s_m| < \epsilon$$

whence (s_n) is Cauchy.

$(\Leftarrow)$ To discuss the convergence of (s_n) we first need a potential limit! In view of Lemma 10.8, the obvious candidates are $\limsup s_n$ and $\liminf s_n$. We have two goals: show that (s_n) is bounded whence the limits superior and inferior are *finite*; then show that these are *equal*.

(Boundedness of (s_n)) Take $\epsilon = 1$ in Definition 10.9:

$$\exists N \text{ such that } m, n > N \implies |s_n - s_m| < 1$$

It follows that

$$n > N \implies |s_n - s_{N+1}| < 1 \implies s_{N+1} - 1 < s_n < s_{N+1} + 1$$

whence (s_n) is bounded; it follows that $\limsup s_n$ and $\liminf s_n$ are *finite*.

[23] Since m, n are arbitrary, WLOG we may assume $m > n$; equality is never interesting in these situations. This assumption is very common and we'll use it repeatedly without comment.

($\limsup s_n = \liminf s_n$) Since (s_n) is Cauchy, given $\epsilon > 0$,

$$\exists N \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon \implies s_n < s_m + \epsilon$$

Taking $v_N = \sup\{s_n : n > N\}$, we see that

$$m > N \implies v_N \leq s_m + \epsilon$$

Taking the infimum of the right hand side yields

$$v_N \leq u_N + \epsilon \qquad \text{(since } u_N = \inf\{s_m : m > N\}\text{)}$$

Since (v_N) is monotone-down and (u_N) monotone-up, we see that

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon \implies \limsup s_n \leq \liminf s_n + \epsilon$$

This last holds for all $\epsilon > 0$, whence $\limsup s_n \leq \liminf s_n$. By Lemma 10.6 we have equality.

By Lemma 10.8, we conclude that (s_n) converges to $\limsup s_n = \liminf s_n$. ∎

In view of the Theorem, the previous examples converge. All three limits can be found precisely (for instance, see Exercise 7). With a small modification to the second example, however, we obtain something genuinely new:

Example (10.10.2 cont). Let $s_1 = 5$ and , for each n, define $s_{n+1} = s_n + \frac{\sin n}{n(n+1)}$. Since $|\sin n| \leq 1$, the computation proceeds almost the same as before:

$$|s_{n+1} - s_n| = \frac{|\sin n|}{n(n+1)} \leq \frac{1}{n(n+1)} = \cdots$$

The new sequence is Cauchy and therefore convergent; good luck explicitly finding its limit though!

The main point is easy to miss: Cauchy Completeness provides a powerful tool for determining whether a sequence converges *without first guessing a limit.* While the result depends on monotone convergence (via limit superior/inferior), it is more powerful in that it applies even to non-monotone sequences. We finish with an application of this idea.

An Alternative Definition of $\mathbb{R}$ Cauchy sequences suggest a *definition* of the real numbers which does not rely on Dedekind cuts (Section 6).

Define an equivalence relation $\sim$ on the collection $\mathcal{C}$ of all Cauchy sequences of rational numbers:[24]

$$(s_n) \sim (t_n) \iff \lim(s_n - t_n) = 0$$

We then define $\mathbb{R} := {}^{\mathcal{C}}/_{\sim}$. All this is done without reference to Cauchy Completeness, though it certainly informs our intuition that (s_n) and (t_n) have the same limit. Some work is still required to define $+, \cdot, \leq$, etc., and to verify the axioms of a complete ordered field—we won't pursue this.

[24] We don't need real numbers to define the limit of the *rational* sequence $(s_n - t_n)$: $\forall \epsilon \in \mathbb{Q}^+$ is enough...

Exercises 10. 1. Use the definition to show that the sequence with n^{th} term $s_n = \frac{1}{n^2}$ is Cauchy. Repeat for $t_n = \frac{1}{n(n-2)}$.

2. Let $s_1 = 1$ and $s_{n+1} = \frac{n}{n+1}s_n^2$ for $n \geq 1$.
   (a) Find s_2, s_3 and s_4.
   (b) Show that $\lim s_n$ exists and hence prove that $\lim s_n = 0$.

3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.
   (a) Find s_2, s_3 and s_4.
   (b) Use induction to show that $s_n > \frac{1}{2}$ for all n, and conclude that (s_n) is monotone-down.
   (c) Show that $\lim s_n$ exists and find $\lim s_n$.

4. (a) Let (s_n) be a sequence such that $\forall n,\ |s_{n+1} - s_n| \leq 3^{-n}$. Prove that (s_n) is Cauchy.
   (b) Let $s_1 = 10$ and, for each n, let $s_{n+1} = s_n + \frac{\cos n}{3^n}$. Explain why (s_n) is convergent.
   (c) Is the result in (a) true if we only assume that $|s_{n+1} - s_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$?

5. Suppose (s_n) is *unbounded* and monotone-up. Prove that $\lim s_n = \infty$.
   (*Thus* $\lim s_n = \sup\{s_n\}$ *for any monotone-up sequence*)

6. Let $s_n = \frac{(-1)^n}{n}$. Find the sequences $(u_N), (v_N)$ and explicitly compute $\limsup s_n$ and $\liminf s_n$.

7. Consider the sequence in Example 10.10.3. Explain why $s_{2n} = s_{2n-2} - \frac{2}{4^n} + \frac{9}{9^n}$.
   Now use the geometric sum formula to evaluate $\lim s_{2n}$.
   (*Since* (s_n) *converges, this means the original sequence has the same limit*)

8. Let S be a bounded nonempty set for which $\sup S \notin S$. Prove that there exists a monotone-up sequence (s_n) of points in S such that $\lim s_n = \sup S$.
   (*Hint: for each* n*, use* $\sup S - \frac{1}{n}$ *to build* s_n)

9. Let (s_n) be a monotone-up sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove that (σ_n) is monotone-up.

10. (Hard!) We prove that the sequence defined by $s_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.
   (a) Show that
   $$\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}} = 1 - \frac{1}{(n+1)^2} \quad \text{and} \quad \frac{1 + \frac{1}{n}}{1 + \frac{1}{n+1}} = 1 + \frac{1}{n(n+2)}$$
   (b) Prove *Bernoulli's inequality* by induction.
   For all real $x > -1$ and $n \in \mathbb{N}_0$ we have $(1 + x)^n \geq 1 + nx$.
   (c) Use parts (a) and (b) to prove that (s_n) is monotone-up.
   (*Hint: consider* $\frac{s_{n+1}}{s_n}$)
   (d) Similarly, show that $t_n := \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)s_n$ defines a monotone-down sequence.
   (e) Prove that (s_n) and (t_n) converge, and to the *same* limit (this limit is e).
   (*Hint: compute* $t_n - s_n$)

11 Subsequences

The general behavior of a sequence is often hard to ascertain, but if we delete some of its terms we might obtain a *subsequence* with interesting behavior.

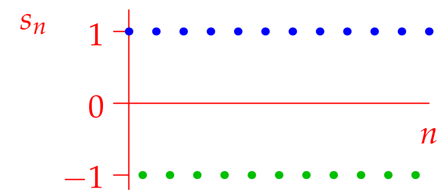
Definition 11.1. Let (s_n) be a sequence. A *subsequence* (s_{n_k}) is a subset $(s_{n_k}) \subseteq (s_n)$, where

$$n_1 < n_2 < n_3 < \cdots$$

A subsequence is simply an infinite subset, with order inherited from the original sequence.

Example 11.2. Take $s_n = (-1)^n$ (recall Example 8.6.2) and let $n_k = 2k$. Then $s_{n_k} = 1$ for all k. Note two important facts:

- The subsequence $(s_{n_k})_{k=0}^{\infty}$ is indexed by k, not n.
- The subsequence is constant and thus *convergent*.



Our main goal in this section is to prove the result illustrated in the example, that every bounded sequence has a convergent subsequence (the famous Bolzano–Weierstraß theorem).

Lemma 11.3. *If* $\lim\limits_{n\to\infty} s_n = s$*, then every subsequence* (s_{n_k}) *also satisfies* $\lim\limits_{k\to\infty} s_{n_k} = s$.

Proof. Suppose s is finite and let $\epsilon > 0$ be given. Then $\exists N$ such that $n > N \implies |s_n - s| < \epsilon$. Since $n_k \geq k$ for all k, we see that

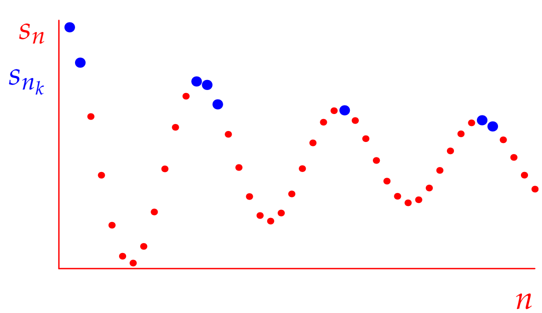
$$k > N \implies n_k > N \implies |s_{n_k} - s| < \epsilon$$

The case where $s = \pm\infty$ is an exercise. ∎

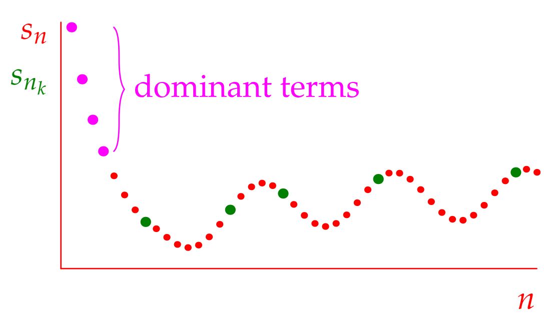
Lemma 11.4. *Every sequence has a monotonic subsequence.*

Proof. Given (s_n), we call the term s_n 'dominant' if $m > n \implies s_m < s_n$. There are two cases:

1. If there are infinitely many dominant terms, then the subsequence of such is monotone-down.
2. If there are finitely many dominant terms, choose s_{n_1} after all such. Since s_{n_1} is not dominant, $\exists n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Induct to obtain a monotone-up subsequence.



Case 1: monotone-down subsequence



Case 2: monotone-up subsequence

∎

Theorem 11.5. *Given a sequence* (s_n)*, there exist subsequences* (s_{n_k}) *and* (s_{n_l}) *such that*

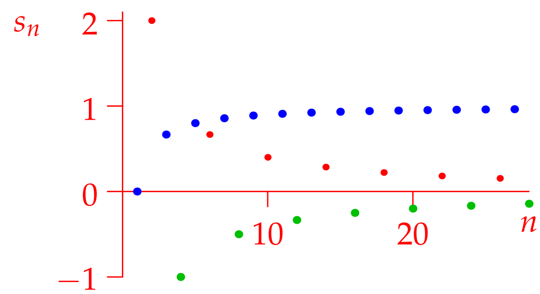
$$\lim s_{n_k} = \limsup s_n \quad \text{and} \quad \lim s_{n_l} = \liminf s_n$$

Combining with the lemmas, we may assume these subsequences are monotonic.

Example 11.6. The picture shows the sequence with n^{th} term

$$s_n = \begin{cases} \frac{4}{n}(-1)^{\frac{n}{2}+1} & \text{when } n \text{ is even} \\ 1 - \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$$

Monotonic subsequences with limits $\limsup s_n = 1$ and $\liminf s_n = 0$ are indicated.



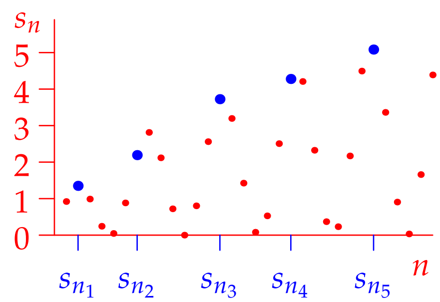
Proof. We prove only the lim sup claim since the other is similar. There are three cases to consider; visualizing the third is particularly difficult and may take several readings.

($\limsup s_n = \infty$) Since (s_n) is unbounded above, for any $k > 0$ there exist *infinitely many* terms $s_n > k$. We may therefore inductively choose a subsequence (s_{n_k}) via

$$n_1 = \min\{n \in \mathbb{N} : s_{n_1} > 1\}$$
$$n_k = \min\{n \in \mathbb{N} : n_k > n_{k-1},\ s_{n_k} > k\}$$

Choosing the minimum isn't necessary here, but it at least keeps the subsequence explicit. Clearly

$$s_{n_k} > k \implies \lim_{k\to\infty} s_{n_k} = \infty = \limsup s_n$$



Example: $\limsup \frac{\sqrt{n}}{2}(1 + \sin n) = \infty$

($\limsup s_n = -\infty$) Since $\liminf s_n \le \limsup s_n = -\infty$, Lemma 10.8 says that $\lim s_n = -\infty$, whence (s_n) itself is a suitable subsequence.

($\limsup s_n = v$ finite) We follow an inductive construction: let $n_1 = 1$ and define s_{n_k} for $k \ge 2$ via,

- Since (v_N) is monotone-down and converges to v, take $\epsilon = \frac{1}{2k}$ to see that[25]

$$\exists N_k \ge n_{k-1} \text{ such that } v \le v_{N_k} < v + \frac{1}{2k}$$

- Since $v_{N_k} = \sup\{s_n : n > N_k\}$, Lemma 4.8 says

$$\exists n_k > N_k \text{ such that } s_{n_k} > v_{N_k} - \frac{1}{2k}$$

But then $|v - s_{n_k}| \overset{\triangle}{\le} |v - v_{N_k}| + |v_{N_k} - s_{n_k}| < \frac{1}{k}$. The squeeze theorem says that $\lim_{k\to\infty} s_{n_k} = v$. ∎

[25] (v_N) being monotone-down is crucial: if N satisfies $v_N - v < \frac{1}{2k}$, so does $N_k := \max\{N, n_{k-1}\}$.

Example (11.6 cont.). The example shows why the two-step construction is necessary. It may seem that we should simply be able to modify subsequences of (u_N) and (v_N). Indeed,

$$(u_N) = \left(-1, -1, -1, -1, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \ldots\right)$$

contains a monotonic subsequence of (s_n) converging to $\liminf s_n = 0$. Unfortunately, the same isn't true for $(v_N) = (2, 1, 1, 1, 1 \ldots)$, where $v_{N_k} = 1$ for all $k \geq 2$; taking $n_k = 2k - 1$ results in

$$s_{n_k} = 1 - \frac{1}{2k-1} > 1 - \frac{1}{2k} = v_{N_k} - \frac{1}{2k}$$

The above discussion rapidly provides two results, the first of which is Exercise 3.

Theorem 11.7 (Lemma 10.8 with converse). *For any sequence,*

$$\limsup s_n = \liminf s_n \iff \lim s_n \text{ exists}$$

Theorem 11.8 (Bolzano–Weierstraß). *Every bounded sequence has a convergent subsequence.*

Proof 1. Lemma 11.4 says there exists a monotone subsequence. This is bounded and thus converges by the monotone convergence theorem. ∎

Proof 2. By Theorem 11.5, there exists a subsequence converging to the *finite* value $\limsup s_n$. ∎

For a third proof(!) we present the classic 'shrinking-interval' argument which has the benefit of generalizing to higher dimensions (rather than intervals, take boxes...).

Proof 3. Suppose (s_n) is bounded by M. One of the intervals $[-M, 0]$ or $[0, M]$ must contain infinitely many terms of the sequence (perhaps both!). Call this interval E_0 and choose any $n_0 \in E_0$.

Split E_0 into left- and right half-intervals, one of which must contain infinitely many terms of the sequence for which $n > n_0$;[26] call this half-interval E_1 and choose any $s_{n_1} \in E_1$ for which $n_1 > n_0$.

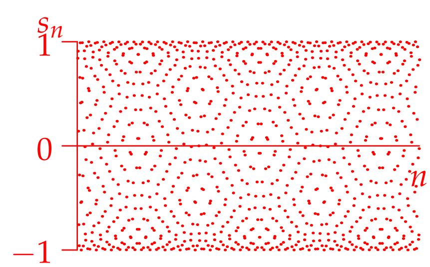
Repeat this *ad infinitum* to obtain a subsequence (s_{n_k}) and a family of nested intervals

$$[-M, M] \supset E_0 \supset E_1 \supset E_2 \supset \cdots \quad \text{of width} \quad |E_k| = \frac{M}{2^k} \quad \text{with} \quad s_{n_k} \in E_k$$

It remains only to see that (s_{n_k}) converges; we leave this to Exercise 5. ∎

Example 11.9. $(s_n) = (\sin n)$ is bounded and therefore has a convergent subsequence! Its limit s must lie in the interval $[-1, 1]$.

The picture shows the first 1000 terms—remember that n is measured in *radians*. It is not at all clear from the picture what s or our mystery subsequence should be! There is a reason for this, as we'll see momentarily...



[26] Only *finitely many* terms in (s_n) come before s_{n_0} ...

Subsequential Limits, Divergence by Oscillation & Closed Sets

Recall Definition 9.7, where we stated that a sequence (s_n) *diverges by oscillation* if it neither converges nor diverges to $\pm\infty$. We can now give a positive statement of this idea.

$$\begin{aligned}(s_n) \text{ diverges by oscillation} &\overset{\text{Thm 11.7}}{\iff} \liminf s_n \neq \limsup s_n \\ &\overset{\text{Thm 11.5}}{\iff} (s_n) \text{ has subsequences tending to different limits}\end{aligned}$$

The word *oscillation* comes from the third interpretation: if $s_1 \neq s_2$ are limits of two subsequences, then any tail of the sequence $\{s_n : n > N\}$ contains infinitely many terms arbitrarily close to s_1 and infinitely many (other) terms arbitrarily close to s_2. The original sequence (s_n) therefore *oscillates* between neighborhoods of s_1 and s_2. Of course there could be many other *subsequential limits*…

Definition 11.10. We call $s \in \mathbb{R} \cup \{\pm\infty\}$ a *subsequential limit* of a sequence (s_n) if there exists a subsequence (s_{n_k}) such that $\lim\limits_{k\to\infty} s_{n_k} = s$.

Examples 11.11. 1. The sequence defined by $s_n = \frac{1}{n}$ has only one subsequential limit, namely zero. Recall Lemma 11.3: $\lim s_n = 0$ implies that every subsequence also converges to 0.

2. If $s_n = (-1)^n$, then the subsequential limits are ± 1.

3. The sequence $s_n = n^2(1 + (-1)^n)$ has subsequential limits 0 and ∞.

4. All positive even integers are subsequential limits of $(s_n) = (2, 4, 2, 4, 6, 2, 4, 6, 8, 2, 4, 6, 8, 10, \ldots)$.

5. (Hard!) Recall the countability of $\mathbb{Q}$ from a previous class: the standard argument enumerates the rationals by constructing a sequence

$$(r_n) = \Big(\frac{0}{1}, \underbrace{\frac{1}{1}, -\frac{1}{1}}_{|p|+q=2}, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}}_{|p|+q=3}, \underbrace{\frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}}_{|p|+q=4}, \underbrace{\frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1}}_{|p|+q=5}, \ldots\Big)$$

We claim that the set of subsequential limits of (r_n) is in fact the full set of $\mathbb{R} \cup \{\pm\infty\}$!

To see this, let $a \in \mathbb{R}$ be given and choose a subsequence (r_{n_k}) inductively:

- By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Corollary 4.12), the set $S_n = \mathbb{Q} \cap (a - \frac{1}{n}, a + \frac{1}{n})$ contains infinitely many rational numbers and thus infinitely many terms of the sequence (r_n).
- Choose any $r_{n_1} \in S_1$ and, for each $k \geq 2$, choose any[27]

$$r_{n_k} \in S_k \text{ such that } n_k > n_{k-1}$$

- Since $|r_{n_k} - a| < \frac{1}{k}$, we conclude that $\lim\limits_{k\to\infty} r_{n_k} = a$.

An argument for the subsequential limits $\pm\infty$ is in the Exercises. Somewhat amazingly, the *specific* sequence (r_n) is irrelevant: the conclusion is the same for *any* sequence enumerating $\mathbb{Q}$!

6. (Even harder—Example 11.9, cont.) We won't prove it, but the set of subsequential limits of $(s_n) = (\sin n)$ is the *entire interval* $[-1, 1]$! Otherwise said, for any $s \in [-1, 1]$ there exists a subsequence $(\sin n_k)$ such that $\lim\limits_{k\to\infty} \sin n_k = s$.

Theorem 11.12. *Let (s_n) be a sequence in $\mathbb{R}$ and let S be its set of subsequential limits. Then*

1. *S is non-empty (as a subset of $\mathbb{R} \cup \{\pm\infty\}$).*
2. *$\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.*
3. *$\lim s_n$ exists iff S has only one element: namely $\lim s_n$.*

Proof. 1. By Theorem 11.5, $\limsup s_n \in S$.

2. By part 1, $\limsup s_n \leq \sup S$. For any convergent subsequence (s_{n_k}), we have $n_k \geq k$, whence

$$\forall N,\ \{s_{n_k} : k > N\} \subseteq \{s_n : n > N\} \implies \lim s_{n_k} = \limsup s_{n_k} \leq \limsup s_n$$

Since this holds for *every* convergent subsequence, we have $\sup S \leq \limsup s_n$, and therefore equality. The result for $\inf S$ is similar.

3. Applying Theorem 11.7, we see that $\lim s_n$ exists if and only if

$$\limsup s_n = \liminf s_n \iff \sup S = \inf S \iff S \text{ has only one element}$$

∎

Closed Sets You should be comfortable with the notion of a *closed interval* (e.g. $[0,1]$) from elementary calculus. Using sequences, we can make a formal definition.

Definition 11.13. Let $A \subseteq \mathbb{R}$.

- We say that $s \in \mathbb{R}$ is a *limit point* of A if there exists a sequence $(s_n) \subseteq A$ converging to s.
- The *closure* $\overline{A}$ is the set of limit points of A.
- A is *closed* if it equals its closure: $A = \overline{A}$.

Examples 11.14. 1. The interval $[0,1]$ is closed. If $(s_n) \subseteq [0,1]$ has $\lim s_n = s$, then

$$0 \leq s_n \leq 1 \overset{\text{Thm 8.8}}{\Longrightarrow} s \in [0,1]$$

More generally, every 'closed interval' $[a,b]$ is closed, as are *finite* unions of closed intervals, for instance $[1,5] \cup [7,11]$.

2. The interval $(0,1]$ is not closed since its closure is $\overline{(0,1]} = [0,1]$. In particular, the sequence $s_n = \frac{1}{n}$ lies in the original interval but has limit 0. Indeed this example shows that an *infinite* union of closed intervals need not be closed.

Theorem 11.15. *If (s_n) is a sequence, then its set of (finite) subsequential limits is closed.*

We omit the proof since it is hard to read, involving unpleasantly many subscripts (subsequences of subsequences...).

[27]As in the proof of Theorem 11.5, we could make this more explicit by choosing minimums, but there is no need: if there are infinitely many r_n in S_k, then only finitely many of them can come before $r_{n_{k-1}}$.

Exercises 11. 1. Consider the sequences with the following n^{th} terms:

$$a_n = (-1)^n \qquad b_n = \frac{1}{n} \qquad c_n = n^2 \qquad d_n = \frac{6n+4}{7n-3}$$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, state its set of subsequential limits.

(c) For each sequence, state its lim sup and lim inf.

(d) Which of the sequences converge? diverge to $+\infty$? diverge to $-\infty$?

(e) Which of the sequences are bounded?

2. Prove the case of Lemma 11.3 when $\lim s_n = \infty$

3. Suppose that $\lim s_n = s$ (could be $\pm\infty$). Use Theorem 11.5 and Lemma 11.3 to prove that $\limsup s_n = s = \liminf s_n$.

   (*This completes the proof of Theorem 11.7*)

4. Suppose that $L = \lim s_n^2$ exists and is finite.

   (a) Given an example of such a sequence where (s_n) is *divergent.*

   (b) Prove that (s_n) contains a convergent *subsequence.* What are the possible limits of this subsequence? Why?
   (*Hint: use Bolzano–Weierstraß*)

5. Complete the third proof of Bolzano–Weierstraß (Theorem 11.8) by proving that the constructed subsequence (s_{n_k}) is Cauchy.

6. (a) Show that the closed interval $[a,b]$ is a closed set in the sense of Definition 11.13.

   (b) Is there a sequence (s_n) such that $(0,1)$ is its set of subsequential limits?

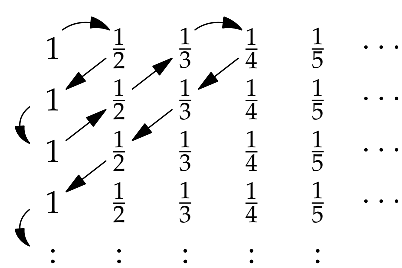
7. Let (r_n) be any sequence enumerating of the set $\mathbb{Q}$ of rational numbers. Show that there exists a subsequence (r_{n_k}) such that $\lim\limits_{k\to\infty} r_{n_k} = +\infty$.

   (*Hint: modify the argument in Example 11.11.5*)

8. (Hard) Let (s_n) be the sequence of numbers defined in the figure, listed in the indicated order.

   (a) Find the set S of subsequential limits of (s_n).

   (b) Determine $\limsup s_n$ and $\liminf s_n$.



12 Lim sup and Lim inf

In this section we collect a couple of useful results, mostly for later use. First, we observe that the limit laws do not work as tightly for limits superior and inferior.

Theorem 12.1. *Let $(s_n), (t_n)$ be bounded sequences.*

1. $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$
2. *If, in addition, (s_n) is convergent to s, then we have equality*

$$\limsup(s_n + t_n) = s + \limsup t_n$$

Modifications can be made infima and products of sequences (Exercise 3).

Example 12.2. To see that equality is unlikely, take $s_n = (-1)^n = -t_n$, then

$$\limsup(s_n + t_n) = 0 < 2 = \limsup s_n + \limsup t_n$$

Proof. 1. For each N, the set $\{s_n + t_n : n > N\}$ is bounded above by

$$\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

from which

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

Simply take limits as $N \to \infty$ for the first result.

2. By part 1, we already know that

$$\limsup(s_n + t_n) \le s + \limsup t_n$$

For the other direction, let $a_n = s_n + t_n$ and apply part 1 again:

$$\begin{aligned}\limsup t_n &= \limsup\big((s_n + t_n) - s_n\big) \le \limsup(s_n + t_n) + \limsup(-s_n)\\ &= \limsup(s_n + t_n) - s\end{aligned}$$

■

The next result will be critical when we study infinite series, particularly the ratio and root tests.

Theorem 12.3. *Let (s_n) be a non-zero sequence. Then*

$$\liminf\left|\frac{s_{n+1}}{s_n}\right| \le \liminf|s_n|^{1/n} \le \limsup|s_n|^{1/n} \le \limsup\left|\frac{s_{n+1}}{s_n}\right|$$

In particular, $\lim\left|\frac{s_{n+1}}{s_n}\right| = L \implies \lim|s_n|^{1/n} = L$ (†)

Examples 12.4. 1. Here is a quick proof that $\lim n^{1/n} = 1$ (recall Theorem 9.5): let $s_n = n$, then

$$\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \frac{n+1}{n} = 1 \implies \lim n^{1/n} = \lim |s_n|^{1/n} = 1$$

2. Let $s_n = n!$ and apply the corollary to see that

$$\lim(n!)^{1/n} = \lim \left|\frac{s_{n+1}}{s_n}\right| = \lim(n+1) = \infty$$

Proof. Assume $\limsup \left|\frac{s_{n+1}}{s_n}\right| = L \neq \infty$ (otherwise the third inequality is trivial) and let $\epsilon > 0$. Then

$$\lim_{N\to\infty} \sup \left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L + \epsilon \implies \exists N \text{ such that } \sup \left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L + \epsilon$$

For brevity, denote $a = L + \epsilon$ and $b = a^{-N-1} |s_{N+1}|$. For any $n > N$, we therefore have

$$\left|\frac{s_{n+1}}{s_n}\right| < a \implies |s_n| < a^{n-N-1} |s_{N+1}| \implies |s_n|^{1/n} < a \left(a^{-N-1} |s_{N+1}|\right)^{1/n} = ab^{1/n}$$
$$\implies \limsup |s_n|^{1/n} \leq a \lim b^{1/n} = a = L + \epsilon$$

Since this holds for all $\epsilon > 0$, we conclude the third inequality: $\limsup |s_n|^{1/n} \leq L$.
The second inequality is trivial and the first is similar to the third. ∎

Exercises 12. 1. Compute $\lim \frac{1}{n}(n!)^{1/n}$
(*Hint: let* $s_n = \frac{n!}{n^n}$ *in Theorem 12.3 and recall that* $\lim \left(1 + \frac{1}{n}\right)^n = e$)

2. Evaluate $\lim \left(\frac{(2n)!}{(n!)^2}\right)^{1/n}$

3. Let (s_n) and (t_n) be non-negative, bounded sequences.
   (a) Prove that $\limsup(s_n t_n) \leq (\limsup s_n)(\limsup t_n)$
   (b) Give an example which shows that we do not expect equality in part (a).
   (c) If, in addition, $\lim s_n = s$, prove that $\limsup(s_n t_n) = s \limsup t_n$.

4. Consider the sequence with $s_{2m} = s_{2m+1} = 2^{-m}$:

$$(s_n)_{n=0}^{\infty} = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots\right)$$

Compute $|s_n|^{1/n}$ and $\left|\frac{s_{n+1}}{s_n}\right|$ when n is even and then when it is odd. Thus find all expressions in Theorem 12.3 and hence conclude that the converse of (†) is *false*.

14 Series

What should be meant by the following expression?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The ∞-symbol in the summation should lead you to suspect a role for *limits*...

Definition 14.1. Define the n^{th} *partial sum* s_n of a sequence $(a_n)_{n=m}^{\infty}$ via

$$s_n := \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

- The *(infinite) series*[28] $\sum_{n=m}^{\infty} a_n$ is the limit $\lim s_n$ of the sequence (s_n) of partial sums.
- A series *converges, s to* $\pm\infty$ or *diverges by oscillation* as does the sequence (s_n).
- $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.
- $\sum a_n$ *converges conditionally* if it converges but not absolutely ($\sum |a_n|$ diverges to ∞).

To return to our motivating example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim s_n \quad \text{where} \quad s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2}$$

We don't (yet) know whether the series converges or diverges to ∞...

Theorem 14.2 (Basic Series Laws). *Infinite series behave nicely with respect to addition and scalar multiplication. For instance:*

1. *If $\sum a_n$ is convergent and k is constant, then $\sum ka_n = k\sum a_n$ is convergent.*
2. *If $\sum a_n$ and $\sum b_n$ are convergent, then $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$ are also convergent.*
3. *If $\sum a_n = \infty$ and $k > 0$, then $\sum ka_n = \infty$.*
4. *If $\sum a_n = \infty$ and $\sum b_n$ converges, then $\sum(a_n + b_n) = \infty$.*

Proof. Simply apply the limit/divergence laws to the sequence of partial sums. E.g. for 1,

$$\sum ka_n = \lim_{n\to\infty} \sum_{j=m}^{n} ka_j \overset{\substack{\text{finite}\\\text{sum}}}{=} \lim_{n\to\infty} k \sum_{j=m}^{n} a_j \overset{\substack{\text{limit}\\\text{laws}}}{=} k \lim_{n\to\infty} \sum_{j=m}^{n} a_j = k\sum a_n$$

The others may be proved similarly. ∎

Note that series do not behave nicely with respect to multiplication (see also Exercise 3):

$$a_1b_1 + a_2b_2 + \cdots = \sum a_nb_n \neq \left(\sum a_n\right)\left(\sum b_n\right) = (a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots)$$

[28] It is common to denote a series $\sum a_n$ if the initial term is understood (typically a_0 or a_1), or is irrelevant to the situation.

Series which may be evaluated exactly

Our major goal is to develop techniques for answering the binary question, "Does $\sum a_n$ converge?" Even when the answer is *yes*, a precise computation of the limit is usually beyond us. However, our techniques (the upcoming *series tests*) will typically rely on comparing $\sum a_n$ to a 'standard' series whose properties are completely understood. You have met two such series in elementary calculus.

Definition 14.3 (Geometric series). A sequence (a_n) is *geometric* if the ratio of successive terms is constant: $a_n = ba^n$ for some constants a, b. A *geometric series* is the sum of a geometric sequence.

The computation of the sequence of partial sums should be familiar (for simplicity assume $b = 1$)

$$(1-a)s_n = \left(a^m + a^{m+1} + \cdots + a^n\right) - \left(a^{m+1} + a^{m+2} + \cdots + a^n + a^{n+1}\right) = a^m - a^{n+1}$$

from which we quickly conclude:

Theorem 14.4. *If a is constant, then*

$$s_n = \sum_{k=m}^{n} a^k = \begin{cases} \dfrac{a^m - a^{n+1}}{1-a} & \text{if } a \neq 1 \\ n+1-m & \text{if } a = 1 \end{cases} \implies \sum_{n=m}^{\infty} a^n \begin{cases} \text{converges to } \dfrac{a^m}{1-a} & \text{if } |a| < 1 \\ \text{diverges to } \infty & \text{if } a \geq 1 \\ \text{diverges by oscillation} & \text{if } a \leq -1 \end{cases}$$

In particular, $\sum a^n$ converges absolutely if $|a| < 1$ and diverges otherwise.

Examples 14.5. 1. $\displaystyle\sum_{n=-1}^{\infty} 2\left(-\frac{4}{5}\right)^n = 2\frac{\left(-\frac{4}{5}\right)^{-1}}{1+\frac{4}{5}} = -\frac{5}{2}\cdot\frac{5}{9} = -\frac{25}{18}$

2. Consider the series $\sum a_n = \sum\limits_{n=3}^{\infty} \left(\frac{2}{5}\right)^n + 2^n$. If this were convergent, then

$$\sum 2^n = \sum a_n - \sum \left(\frac{2}{5}\right)^n$$

would converge (Theorem 14.2); a contradiction.

Telescoping series A rarer type of series can be evaluated using the algebra of partial fractions.

Example 14.6. To compute $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$, first observe that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim\left(1 - \frac{1}{n+1}\right) = 1$$

Similar arguments can be made for other series such as $\sum \frac{1}{n(n+2)}$.

The Cauchy Criterion

The starting point for general series convergence uses Cauchy completeness.

Example 14.7. Consider again the series $\sum \frac{1}{n^2}$. We show that the sequence of partial sums (s_n) is Cauchy. Let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then,

$$m > n > N \implies |s_m - s_n| = \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k}$$
$$= \frac{1}{n} - \frac{1}{m} < \frac{1}{N} = \epsilon$$

where we follow the telescoping series approach to cancel most terms. By Cauchy completeness (Theorem 10.11), (s_n) converges and we conclude

$$\boxed{\text{The series } \sum \frac{1}{n^2} \text{ is convergent}}$$

Computing the value of this series rigorously is significantly harder, though a sketch argument for why $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ is in Exercise 10.

Theorem 14.8 (Cauchy Criterion for Series). *A series $\sum a_n$ converges if and only if*

$$\forall \epsilon > 0,\ \exists N \text{ such that } m \geq n > N \implies |s_m - s_{n-1}| = \left| \sum_{k=n}^{m} a_k \right| < \epsilon$$

In the previous example we essentially verified the Cauchy criterion for the series $\sum \frac{1}{n^2}$.

Proof. Let (s_n) be the sequence of partial sums. Then

$$\sum a_n \text{ converges} \iff (s_n) \text{ converges} \iff (s_n) \text{ is Cauchy} \qquad \text{(Thm 10.11)}$$
$$\iff \Big(\forall \epsilon > 0,\ \exists N \text{ such that } m > n > N \implies |s_m - s_n| < \epsilon\Big)$$

To finish, simply replace n with $n-1$. ∎

Example 14.9. Assume, for contradiction, that the *harmonic series* $\sum \frac{1}{n}$ converges. Now take $\epsilon = \frac{1}{2}$ in the Cauchy criterion:

$$\exists N \text{ such that } m \geq n > N \implies \left| \sum_{k=n}^{m} \frac{1}{k} \right| < \frac{1}{2}$$

However, taking $m = 2(n-1) \geq n$ (true since $n > N \geq 1$) results in a contradiction:

$$\frac{1}{2} > \left| \sum_{k=n}^{m} \frac{1}{k} \right| = \left| \frac{1}{n} + \cdots + \frac{1}{m} \right| \geq \frac{m-(n-1)}{m} = 1 - \frac{n-1}{m} = \frac{1}{2}$$

We conclude that the harmonic series diverges to ∞.

The Series Tests

For the remainder of this section we develop several standard tests for the convergence/divergence of an infinite series: the divergence, comparison, root and ratio tests. The first of these follow quickly from the Cauchy criterion.

Theorem 14.10 (Divergence/n^{th}-term test). *If $\lim a_n \neq 0$ then $\sum a_n$ is divergent.*

Proof. We prove the contrapositive. Suppose $\sum a_n$ is convergent, let $\epsilon > 0$ be given and take $m = n$ in the Cauchy criterion. Then

$$\exists N \text{ such that } n > N \implies |a_n| < \epsilon$$

Otherwise said, $\lim a_n = 0$. ∎

Examples 14.11.

1. The series $\sum \sin(\frac{n\pi}{9})$ diverges.
2. The divergence test tells us that the geometric series $\sum a^n$ diverges whenever $|a| \geq 1$. We still need our earlier analysis for when $|a| < 1$.
3. The *converse of the n^{th}-term test is false!* For the canonical example, consider the *divergent* harmonic series $\sum \frac{1}{n}$ (Example 14.9, even though $\lim \frac{1}{n} = 0$.

Theorem 14.12 (Comparison test).

1. *Let $\sum b_n$ be a convergent series of non-negative terms and assume $|a_n| \leq b_n$ for all (large n). Then both $\sum b_n$ and $\sum |b_n|$ are convergent.*
2. *If $\sum a_n = \infty$ and $a_n \leq b_n$ for all (large) n, then $\sum b_n = \infty$.*

Proof. Suppose "large n" means $n > M$.

1. Let $\epsilon > 0$ be given. Then $\exists N \geq M$ such that

$$m \geq n > N \implies \left| \sum_{k=n}^{m} a_k \right| \overset{\triangle}{\leq} \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} b_k < \epsilon$$

2. The n^{th} partial sum of $\sum b_n$ is

$$\sum_{k=M}^{n} b_k \geq \sum_{k=M}^{n} a_k \to +\infty$$

∎

Corollary 14.13.

1. *Take $|a_n| = b_n$ in part 1 to see that $\sum |a_n|$ converges $\implies$ $\sum a_n$ converges. Thus absolute convergence implies convergence.*
2. *If $\sum b_n$ is a convergent series of non-negative terms and $|a_n| \leq b_n$ for <u>all</u> n, then*

$$\sum a_n \leq \sum |a_n| \leq \sum b_n$$

Examples 14.14. 1. Since $\frac{2n+1}{(n+2)3^n} \leq 2 \cdot 3^{-n}$ and the geometric series $\sum 2 \cdot 3^{-n}$ converges, we see that the resulting series converges (absolutely), to some value

$$\sum_{n=0}^{\infty} \frac{2n+1}{(n+2)3^n} \leq 2\sum_{n=0}^{\infty} 3^{-n} = \frac{2}{1-\frac{1}{3}} = 3$$

2. One can usually find a sensible series to compare with just by thinking about how a_n behaves when n is very large. For instance, $a_n = \frac{(n^2+1)^{1/2}}{(1+\sqrt{n})^4}$ behaves like $\frac{n}{n^2} = \frac{1}{n}$ and we see that

$$a_n > \frac{n}{(1+\sqrt{n})^4} > \frac{n}{(2\sqrt{n})^4} = \frac{1}{16n}$$

Comparison with $\frac{1}{16}\sum \frac{1}{n}$ shows that $\sum a_n$ diverges to ∞.

3. Since $\ln n < n \implies \frac{1}{\ln n} > \frac{1}{n}$, we see that $\sum \frac{1}{\ln n}$ diverges to ∞ by comparison with $\sum \frac{1}{n}$.

4. $\sum \frac{\sin n}{n^2}$ converges absolutely by comparison to $\sum \frac{1}{n^2}$. Corollary 14.13 gives an estimation for the value of the series, though it is not accurate! The n^{th} partial sums satisfy

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \leq \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(approximately } 1.014 \leq 1.280 \leq 1.645\text{)}$$

5. The *alternating harmonic series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges via a sneaky comparison.

Consider the series $t = \sum\limits_{n=1}^{\infty} \frac{1}{2n(2n-1)}$ which converges by comparison with $\sum \frac{1}{4(n-1)^2}$. Its n^{th} partial sum is

$$t_n = \sum_{k=1}^{n} \frac{1}{2k(2k-1)} = \sum_{k=1}^{n} \left[\frac{1}{2k-1} - \frac{1}{2k}\right]$$

which is the *even* partial sum of the alternating harmonic series $s_{2n} = \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m}$. Take limits of $s_{2n+1} = s_{2n} + \frac{1}{2n+1}$, we see that $\lim s_{2n+1} = t$ from which $\lim s_n = t$. Since the harmonic series $\sum \frac{1}{n}$ diverges (Example 14.9), we conclude that the alternating harmonic series converges conditionally. We will revisit this discussion in the next section.

6. $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges by comparison with the geometric series $\sum 2^{-n}$. To see this, note that

$$\left(\frac{n}{n+1}\right)^n = \frac{n+1}{n}\left(1-\frac{1}{n+1}\right)^{n+1} \xrightarrow[n\to\infty]{} e^{-1} < \frac{1}{2}$$

from which we see that, for all large n,

$$\left(\frac{n}{n+1}\right)^n < \frac{1}{2} \implies \left(\frac{n}{n+1}\right)^{n^2} < 2^{-n}$$

In fact (compare Exercise 10.10), $\left(\frac{n}{n+1}\right)^n$ is monotone-down, whence $e^{-1} \leq \left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$ and

$$0.58198 \approx \frac{1}{e-1} = \frac{e^{-1}}{1-e^{-1}} = \sum_{n=1}^{\infty} e^{-n} \leq \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} \leq \sum_{n=1}^{\infty} 2^{-n} = \frac{1/2}{1-1/2} = 1$$

A computer estimate yields $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \approx 0.8174$.

Our final two tests in this section are less powerful, but have the advantage of being easier to use.

Theorem 14.15 (Root test). *Let* $\limsup |a_n|^{1/n} = L$.

1. *If* $L < 1$, *then* $\sum a_n$ *converges absolutely.*
2. *If* $L > 1$, *then* $\sum a_n$ *diverges.*

If $L = 1$, *then no conclusion can be drawn.*

We defer the proof until after seeing some examples.

Corollary 14.16 (Ratio test). *Suppose* (a_n) *is a sequence of non-zero terms.*

1. *If* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $\sum a_n$ *converges absolutely*
2. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $\sum a_n$ *diverges*

Proof. This follows directly from the root test and Theorem 12.3:

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

■

The versions of these tests familiar from elementary calculus are the special cases when

$$L = \lim |a_n|^{1/n} = \lim \left|\frac{a_{n+1}}{a_n}\right|$$

Our versions are more general since these limits are *not guaranteed to exist.*

Examples 14.17. 1. The ratio test is particularly useful for series involving *factorials* and *exponentials.*

(a) $\sum \frac{n^4}{2^n}$ converges: just observe that $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)^4}{2n^4} = \frac{1}{2} < 1$.

(b) $\sum \frac{n!}{2^n}$ diverges: in this case $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)!}{2n!} = \lim \frac{n+1}{2} = \infty$.

2. Both tests are inconclusive for rational sequences: if $a_n = \frac{b_n}{c_n}$ where b_n, c_n are polynomials, then

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = 1 = \lim |a_n|^{1/n}$$

For example,

$$\sum \frac{n+5}{n^2} \rightsquigarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+6)n^2}{(n+5)(n+1)^2} = 1$$

In fact this example is divergent by comparison with the harmonic series $\sum \frac{1}{n}$.

3. Recall Example 14.14.6: our use of the comparison test was really the root test in disguise

$$a_n = \left(\frac{n}{n+1}\right)^{n^2} \implies \lim |a_n|^{1/n} = \lim \left(\frac{n}{n+1}\right)^n = e^{-1} < 1 \implies \sum a_n \text{ converges}$$

In this case the root test was much easier to apply!

4. The ratio test is the weakest test thus far; certainly it does not apply if any of the terms a_n are zero! For a more subtle example, consider:

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even} \\ 3^{-n} & \text{if } n \text{ is odd} \end{cases}$$

First we try applying the ratio test:

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^n & \text{if } n \text{ is even} \\ \frac{1}{2}\left(\frac{3}{2}\right)^n & \text{if } n \text{ is odd} \end{cases} \implies \liminf \left|\frac{a_{n+1}}{a_n}\right| = 0, \quad \limsup \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

The ratio test is therefore inconclusive. However

$$|a_n|^{1/n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{3} & \text{if } n \text{ is odd} \end{cases} \implies \limsup |a_n|^{1/n} = \frac{1}{2} < 1$$

By the root test, the series $\sum a_n$ converges. We need not even have used the root test: $\sum a_n$ converges by comparison with $\sum 2^{-n}$!

Proof of the Root Test. 1. Suppose $\limsup |a_n|^{1/n} = L < 1$. Recall that $v_N = \sup\{|a_n|^{1/n} : n > N\}$ defines a *monotone-down* sequence converging to L. Choose any $\epsilon > 0$ such that $L + \epsilon < 1$ (say $\epsilon = \frac{1-L}{2}$) to see that

$$\exists N \text{ such that } v_N - L < \epsilon$$

But then

$$n > N \implies |a_n|^{1/n} - L < \epsilon \implies |a_n| < (L+\epsilon)^n$$

$\sum |a_n|$ therefore converges by comparison with the geometric series $\sum (L+\epsilon)^n$.

2. If $L > 1$ then there exists some subsequence (a_{n_k}) such that $|a_{n_k}|^{1/n_k} \to L > 1$. In particular, infinitely many terms of this subsequence must be greater than 1. Clearly a_n does not converge to zero whence the series diverges by the n^{th} term test.

∎

Summary The logical flow of the tests in this section is as follows:

$$\begin{tikzcd}
\text{(divergence tests)} & & n^{\text{th}} \text{ term} \arrow[r, Rightarrow] & \text{Root} \arrow[r, Rightarrow] & \text{Ratio} \\
\text{(testing both)} & \text{Definition of } \sum a_n \arrow[r, Leftrightarrow] & \text{Cauchy Criterion} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{Comparison} \arrow[d, Rightarrow] & \\
\text{(convergence tests)} & & & \text{Root} \arrow[r, Rightarrow] & \text{Ratio}
\end{tikzcd}$$

The ratio test is typically the easiest to use, but the least powerful. Every series which converges by the ratio test can be seen to converge by the root and comparison tests, etc. If a series diverges by the ratio test, then it in fact diverges by the n^{th} term test.

Exercises 14. 1. Determine which of the following sequences converge. Justify your answers.

(a) $\sum \frac{n-1}{n^2}$ (b) $\sum(-1)^n$ (c) $\sum \frac{3^n}{n^3}$ (d) $\sum \frac{n^3}{3^n}$

(e) $\sum \frac{n^2}{n!}$ (f) $\sum \frac{1}{n^n}$ (g) $\sum \frac{n}{2^n}$ (h) $\sum \frac{n!}{n^n}$

(i) $\sum\limits_{n=2}^{\infty} \frac{1}{\left[n+(-1)^n\right]^2}$ (j) $\sum\left[\sqrt{n+1}-\sqrt{n}\right]$

2. Let $\sum a_n$ and $\sum b_n$ be convergent series of non-negative terms. Prove that $\sum \sqrt{a_n b_n}$ converges.
(*Hint: start by showing that* $\sqrt{a_n b_n} \leq a_n + b_n$)

3. (a) If $\sum a_n$ converges absolutely, prove that $\sum a_n^2$ converges.
   (b) More generally, if $\sum |a_n|$ converges and (b_n) is a bounded sequence, prove that $\sum a_n b_n$ converges absolutely.

4. Find a series $\sum a_n$ which diverges by the root test but for which the ratio test is inconclusive.

5. (Hard) Let (a_n) be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence (a_{n_k}) such that $\sum a_{n_k}$ converges.
(*Hint: Try to construct a subsequence which converges to zero faster than* $\frac{1}{k^2}$.

6. Prove that the harmonic series $\sum \frac{1}{n}$ diverges by comparing with the series $\sum a_n$, where

$$(a_n) = \left(1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{32}, \ldots\right)$$

7. Suppose $b_n \leq a_n$ for all n and that $\sum b_n$ and $\sum a_n$ converge. Prove that $\sum b_n \leq \sum a_n$.
(*This also proves part 2 of Corollary 14.13*)

8. Use the basic series laws to find the values of $\sum \frac{1}{(2n)^2}$, $\sum \frac{1}{(2n+1)^2}$ and $\sum \frac{(-1)^{n+1}}{n^2}$.

9. The *limit comparison test* states:

   Suppose $\sum a_n, \sum b_n$ are series of positive terms and that $L = \lim \frac{a_n}{b_n} \in (0, \infty)$. Then the series have the same convergence status (both converge or both diverge to ∞).

   (a) Use the limit comparison test with $b_n = \frac{1}{n^2}$ to show that the series $\sum \frac{1}{n} \ln\left(1+\frac{1}{n}\right)$ converges.
   (*Hint: Recall that* $e = \lim\left(1+\frac{1}{n}\right)^n$)
   (b) Prove the limit comparison test.
   (*Hint: first show that* $\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$ *for large* n)
   (c) What can you say about the series $\sum a_n$ and $\sum b_n$ if $L = 0$ or $L = \infty$? Explain.

10. Euler asserted that the sine function, written as an infinite polynomial in the form of a Maclaurin series, could also be expressed as an infinite product,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

By considering the solutions to $\sin x = 0$, give some weight to Euler's claim. By comparing coefficients in these expressions, deduce the fact $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.
(*As we've presented it, this argument is non-rigorous!*)

15 The Integral and Alternating Series Tests

In this section we develop two further standalone series tests, both with narrower applications than our previous tests.

The first a little out of place given that it requires (improper) integration.[29]

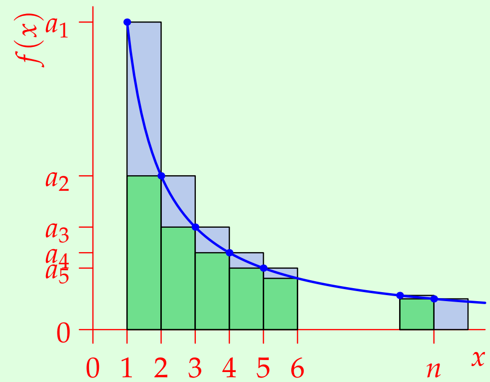
Theorem 15.1 (Integral test). *Let $a_n = f(n)$, where f is non-negative, non-increasing and integrable on $[1,\infty)$. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

in which case

$$\int_1^{\infty} f(x)\,\mathrm{d}x \le \sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,\mathrm{d}x$$

The statement is easily modified if the initial term is a_m.



Proof. We need only interpret the picture:

$$\int_1^{n+1} f(x)\,\mathrm{d}x \le \sum_{k=1}^{n} a_k = s_n = a_1 + \sum_{k=2}^{n} a_k \le a_1 + \int_1^{n} f(x)\,\mathrm{d}x \tag{$*$}$$

Taking limits gives the result. ∎

Even for divergent sums, $(*)$ allows us to estimate s_n and analyze its rate of growth. For greater accuracy, explicitly evaluate the first few terms and use a modified integral test to estimate the remainder.

The big application of the integral test is a complete description of the convergence status of p-series, another useful collection of series to which others may be compared.

Corollary 15.2 (p-series). *Let $p > 0$. The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.*

Examples 15.3. 1. $\sum \frac{1}{n^3}$ converges (it is a p-series with $p > 1$). For a simple estimate, observe that

$$\int_1^{\infty} \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^b = \frac{1}{2} \implies \frac{1}{2} \le \sum_{n=1}^{\infty}\frac{1}{n^3} \le \frac{3}{2}$$

This is a poor estimate, especially the lower bound. For a quick improvement, we could explicitly evaluate the first term and re-run the test starting at $n = 2$:

$$1 + \int_2^{\infty}\frac{1}{x^3}\,\mathrm{d}x \le \sum_{n=1}^{\infty}\frac{1}{n^3} \le 1 + \frac{1}{8} + \int_2^{\infty}\frac{1}{x^3}\,\mathrm{d}x \implies 1 + \frac{1}{8} \le \sum_{n=1}^{\infty}\frac{1}{n^3} \le 1 + \frac{1}{4}$$

If greater accuracy is required, more terms can be explicitly evaluated.

[29]Which in turn requires limits of functions: $\int_1^{\infty} f(x)\,\mathrm{d}x := \lim_{b\to\infty}\int_1^b f(x)\,\mathrm{d}x$. Even though we haven't developed these concepts, the relevant computations should be familiar from elementary calculus.

2. In Example 14.9, we used the Cauchy criterion to show that the harmonic series diverges to ∞. The integral test makes this much easier! The integral test also allows us to estimate how many terms are required for the partial sum to s_n to reach a certain threshold, say 10. Since

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \le \sum_{k=1}^{n} \frac{1}{k} \le 1 + \int_1^{n} \frac{1}{x}\,\mathrm{d}x = 1 + \ln n$$

we see that

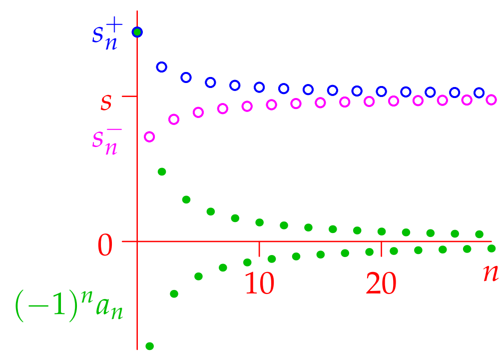
$$s_n \approx 10 \implies \ln(n+1) \le 10 \le 1 + \ln n \implies e^9 \le n \le e^{10} - 1 \implies 8104 \le n \le 22025$$

Somewhere between 8 and 22 *thousand* terms are required! The harmonic series diverges to infinity, but it does so *very slowly.*

3. The integral test shows that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$ and moreover that, to exceed 10, somewhere between 10^{3223} and 10^{6631} terms are required!

4. The series $\sum \frac{2n+1}{\sqrt{4n^3-1}}$ diverges to ∞ by comparison with the p-series $\sum \frac{1}{\sqrt{n}}$.

Alternating Series and Conditional Convergence



Our final test is the only one capable of detecting *conditional convergence*, the canonical example of which is the *alternating harmonic series* (recall Example 14.14.5). With an eye on generalization, we re-index so that the first term is $a_0 = 1$:

$$\begin{aligned} s &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots \end{aligned}$$

The *alternating* $\pm$-signs give the series its name. For us, however, the behavior of the sequence (s_n) of partial sums is more interesting. Consider two subsequences $(s_n^+) = (s_{2n})$ and $(s_n^-) = (s_{2n-1})$:

$$s_n^+ = \sum_{k=0}^{2n} (-1)^k a_k = 1 - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_1 - a_2} - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_3 - a_4} - \cdots - \underbrace{\left(\frac{1}{2n} - \frac{1}{2n+1}\right)}_{a_{2n-1} - a_{2n}} \qquad (n \ge 0)$$

$$s_n^- = \sum_{k=0}^{2n-1} (-1)^k a_k = \underbrace{\left(1 - \frac{1}{2}\right)}_{a_0 - a_1} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{a_2 - a_3} + \cdots + \underbrace{\left(\frac{1}{2n-1} - \frac{1}{2n}\right)}_{a_{2n-2} - a_{2n-1}} \qquad (n \ge 1)$$

Since the brackets are non-negative, (s_n^+) is monotone-down and (s_n^-) monotone-up. Moreover,

$$\frac{1}{2} = s_1^- \le s_n^- \le s_n^- + a_{2n} = s_n^+ \le s_0^+ = 1 \qquad (\dagger)$$

from which both subsequences are *bounded* and thus *convergent.* Not only this, but

$$\lim(s_n^+ - s_n^-) = \lim a_{2n} = 0$$

shows that the limits of both subsequences are *identical* (of course both are s).

The above discussion depended only on two simple properties of the sequence (a_n); we've therefore proved a general statement.

Theorem 15.4 (Alternating series test). *Let (a_n) be monotone-down with $\lim a_n = 0$.*

1. *The series $\sum(-1)^n a_n$ converges.*
2. *If (s_n) is the sequence of partial sums converging to $s = \sum(-1)^n a_n$, then $|s - s_n| \le a_{n+1}$.*

Think about where our assumptions on (a_n) are used in the proof. It can, in fact, be shown that the alternating harmonic series converges to $\ln 2$, although the estimates provided by the alternating series test make this a terrible method of approximation. Even summing the first 100 terms only results in 2 decimal places of accuracy!

Examples 15.5. 1. Since $a_n = \frac{1}{n!}$ converges monotone-down to zero, the alternating series $\sum \frac{(-1)^{n+1}}{n!}$ converges. By taking the first 9 and 10 terms of this series, we see that

$$0.9010498898\ldots \le \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \le 0.90105016538\ldots$$

which at least yields the estimate 0.9015 to 5 decimal places. The alternating series test is not required for this example, since it in fact converges absolutely.

2. The series $\sum_{n=2}^{\infty} \frac{\sin\frac{\pi}{2}n}{\ln n}$ can be viewed as an alternating series since every *even* term is zero. Writing $m = 2n+1$, we obtain

$$\sum_{n=2}^{\infty} \frac{\sin\frac{\pi}{2}n}{\ln n} = \sum_{m=1}^{\infty} \frac{\sin(\pi m + \frac{\pi}{2})}{\ln(2m+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m}{\ln(2m+1)}$$

Since $\frac{1}{\ln(2m+1)}$ decreases to zero, the alternating series test shows convergence.

Rearranging Infinite Series

A *rearrangement* of an infinite series $\sum a_n$ arises when we change the order of the terms of the sequence (a_n) *before* computing the sequence of partial sums. We still have to use every term a_n in the new series. Since the resulting sequence of partials sums is likely completely different, we shouldn't assume that the new series has the same convergence properties as the old.

Example 15.6. We rearrange the alternating harmonic series by summing two positive terms before each negative term:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} + \cdots$$

Every term in the original sequence is used here, so this is a genuine rearrangement. It is perhaps surprising to discover that the new series converges, though its limit is *not the same* as the original alternating harmonic series! We leave the details to Exercise 9.

This behavior is quite different to that of finite sums, where the order of summation makes no difference at all. The situation can be summarized in a famous result of Riemann.

Theorem 15.7 (Riemann rearrangement). 1. *If $\sum a_n$ is conditionally convergent and $s \in \mathbb{R} \cup \{\pm\infty\}$, then there exists a rearrangement of the series which tends*[30] *to s.*

2. *If $\sum a_n$ converges absolutely, then all rearrangements converge to the same limit.*

The second part says that absolutely convergent series behave just like finite sums! We omit the proofs since they are lengthy and require nasty notation. Instead we illustrate the rough idea of part 1 via an example.

Example 15.8. We show how to construct a rearrangement of the alternating harmonic series which converges to $s = \sqrt{2} - 1 = 0.41421\ldots$

First we convince ourselves that the sum of the positive terms $\sum a_n^+$ diverges to infinity. In this case the comparison test comes to our rescue:

$$\frac{1}{2n-1} > \frac{1}{2n} \implies \sum a_n^+ = \sum \frac{1}{2n-1} > \frac{1}{2}\sum \frac{1}{n} = \infty$$

The negative terms also diverge $\sum a_n^- = -\infty$. Construction of the rearrangement is inductive.

1. Sum just enough positive terms until the partial sum exceeds s: plainly $S_1 = 1$ will do.
2. Sum negative terms starting at the beginning of the sequence until the sum is less than s:

$$S_2 = 1 - \frac{1}{2} - \frac{1}{4} = 0.25 < s$$

3. Repeat: add positive terms until the sum just exceeds s, then add negative terms, etc.,

$$S_3 = S_2 + \frac{1}{3} = 0.583\ldots > s, \qquad S_4 = S_3 - \frac{1}{6} - \frac{1}{8} = 0.291\ldots < s$$

Continuing the process ad infinitum, we claim that

$$s = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} + \frac{1}{7} - \frac{1}{12} - \frac{1}{14} + \frac{1}{9} - \frac{1}{16} - \frac{1}{18} + \frac{1}{11} - \frac{1}{20} + \frac{1}{13} - \cdots$$

To see why, observe:

- Since $\sum a_n^+ = \infty$ and $\sum a_n^- = -\infty$, at each stage we only add/subtract *finitely many terms.*
- All terms of the original sequence (a_n) are eventually used since we add the positive (negative) terms *in order.* E.g., $a_{495} = \frac{1}{495}$ appears, *at the latest*, during the 495^{th} positive-addition phase.
- $|S_n - s| \le |a_{m_n}|$, where a_{m_n} is the last term used at the n^{th} stage. This plainly converges to zero, whence $\lim S_n = s$.

[30] Riemann's result is in fact even stronger. Rearrangements also exist which diverge by oscillation between any given lim inf and lim sup!

Exercises 15. 1. Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.

2. Prove Corollary 15.2 regarding the convergence/divergence of p-series.

3. Let $s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Estimate how many terms are required before $s_n \geq 100$.

4. (Example 15.3.3) Verify the claim that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$. If you want a challenge, verify the estimate claim also.

5. (a) Give an example of a series $\sum a_n$ which converges, but for which $\sum a_n^2$ diverges. (*Exercise 14.3 really requires that $\sum a_n$ be absolutely convergent!*)
   (b) Give an example of a divergent series $\sum b_n$ for which $\sum b_n^2$ converges.

6. Suppose (a_n) satisfies the hypotheses of the alternating series test *except* that $\lim a_n = a > 0$. What can you say about the sequences (s_n^+) and (s_n^-) and the series $\sum (-1)^n a_n$?

7. We know that the harmonic series has a growth rate comparable to $\ln n$. Let $a_n = \frac{1}{n}$ and define a new sequence (t_n) by

$$t_n = s_n - \ln n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

   where $s_n = \sum_{k=1}^{n} a_n$ is the n^{th} partial sum. Prove that (t_n) is a positive, monotone-down sequence, which therefore converges.[31]
   (*Hint: you'll need the mean value theorem from elementary calculus*)

8. (a) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is conditionally convergent to some real number s.
   (b) How many terms are required for the partial sum s_n to approximate s to within 0.01.
   (c) Following Example 15.8, use a calculator to state the first 15 terms in a rearrangement of the series in part (a) which converges to 0.

9. In Example 15.6 we rearranged the terms of the alternating harmonic series by taking two positive terms before each negative term.

   (a) Verify, for each $n \in \mathbb{N}$, that

$$b_n := \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0$$

   whence the *subsequence* of partial sums (s_{3n}) is monotone-up.
   (b) Use the comparison test to show that $\sum b_n$ converges.
   (c) Prove that the rearranged series converges, to some value $s > \frac{5}{6}$. (*Thus $s > \ln 2 \approx 0.69$, the limit of the original alternating harmonic series*)

[31] The limit $\gamma := \lim t_n \approx 0.5772$ is the *Euler–Mascheroni constant.* It appears in many mathematical identities, and yet very little about it is known; it is not even known whether γ is irrational!

17 Continuous Functions

For the rest of the course, we discuss continuous functions. Functions themselves should be familiar. For reference, we begin with a review of some basic concepts and conventions.

We are concerned with functions $f : U \to V$ where both U, V are subsets of the real numbers $\mathbb{R}$ and f is some *rule* assigning to each real number $x \in U$ a real number $f(x) \in V$. For instance

$$f(x) = \frac{x^2(x-7)}{(x-2)(x^2-9)} \quad \text{assigns to } x = 1 \text{ the value} \quad f(1) = \frac{1(-6)}{(-1)(-8)} = \frac{3}{4}$$

Domain $\operatorname{dom} f = U$ is the set of *inputs* to f. When f is defined by a formula, its *implied domain* is the largest set on which the formula is defined: for the above example, $\operatorname{dom} f = \mathbb{R} \setminus \{2, 3, -3\}$. In examples, the domain is typically a union of intervals of *positive length.*

Codomain $\operatorname{codom} f = V$ is the set of *possible outputs.* In real analysis, we often take $V = \mathbb{R}$ by default.

Range $\operatorname{range} f = f(U) = \{f(x) : x \in U\}$; is the set of *realized outputs.* It is a subset of V.

Injectivity f is *injective/one-to-one* if distinct inputs produce distinct outputs. This is usually stated in the contrapositive: $f(x) = f(u) \implies x = u$.

Surjectivity f is *surjective/onto* if every possible output is realized: otherwise said, $f(U) = V$.

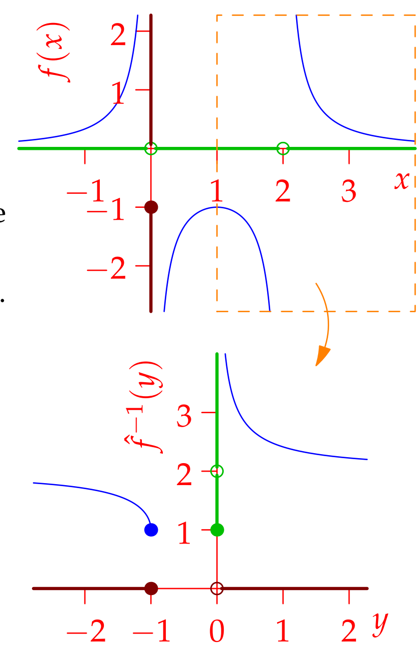
Inverses f is *bijective/invertible* if it is both injective and surjective. Equivalently, f has an *inverse function* $f^{-1} : V \to U$ defined as follows:

- Given $y \in V$, f surjective $\implies \exists x \in U$ such that $f(x) = y$.
- Since f is injective, $f(x) = f(u) \implies x = u$, so x is unique. We *define* $f^{-1}(y) = x$.

Example 17.1. The function defined by $f(x) = \frac{1}{x(x-2)}$ has implied

$$\operatorname{dom} f = \mathbb{R} \setminus \{0, 2\} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

$$\operatorname{range} f = (-\infty, -1] \cup (0, \infty)$$



The function is neither injective (e.g., $f(3) = f(-1)$) nor surjective (e.g., $0 \notin \operatorname{range} f$).

We can remedy both issues by restricting the domain and codomain. For instance, the same rule/formula but with

$$\operatorname{dom} \hat{f} = [1, 2) \cup (2, \infty)$$
$$\operatorname{codom} \hat{f} = (-\infty, -1] \cup (0, \infty)$$

defines a bijection with inverse function

$$\hat{f}^{-1}(y) = \begin{cases} 1 + y^{-1}\sqrt{y+1} & \text{if } y > 0 \\ 1 - y^{-1}\sqrt{y+1} & \text{if } y \leq -1 \end{cases}$$

Observe that $\operatorname{dom} \hat{f}^{-1} = \operatorname{codom} \hat{f}$ and $\operatorname{codom} \hat{f}^{-1} = \operatorname{dom} \hat{f}$.

To introduce continuity, we consider two common naïve notions.

The graph of f can be drawn without removing one's pen from the page This is intuitive but unusable: *drawn* is poorly defined, so how could we *calculate* or *prove* anything with this concept? Moreover, it cannot reasonably be extended to other situations (e.g., multivariable functions) where drawing a graph is meaningless.

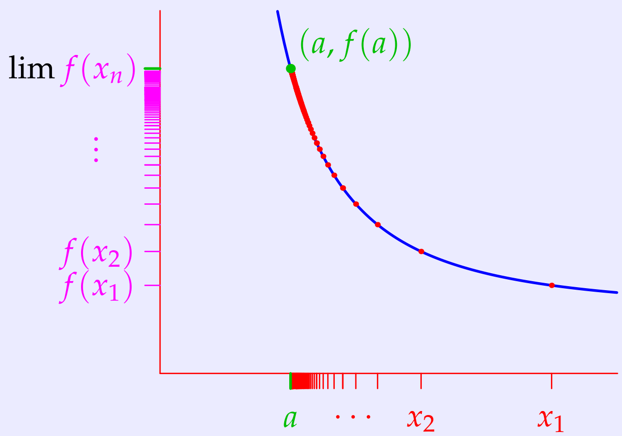
If x is close to a, then $f(x)$ is close to $f(a)$ This is better and can be generalized to other situations. The major issue is the unclear meaning of *close.* Our formal definition of continuity addresses this using *sequences* and *limits.*

Definition 17.2 (Sequential continuity). Let $f : U \to V \subseteq \mathbb{R}$ be a function. We say that f *is continuous at* $u \in U$ if,

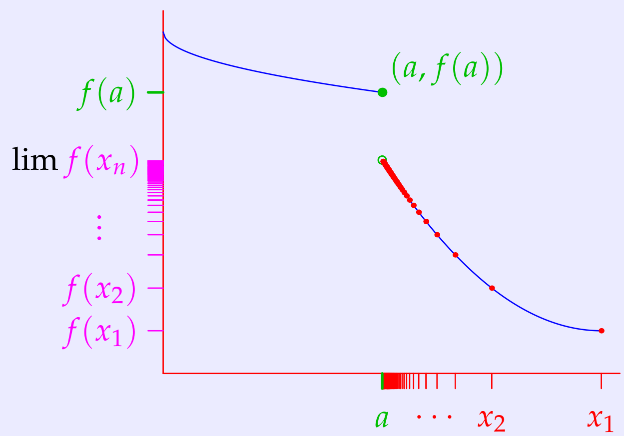
$$\forall (x_n) \subseteq U, \text{ we have } \lim x_n = a \implies \lim f(x_n) = f(a)$$

f is *continuous (on U)* if it is continuous at every point $a \in U$.
A *discontinuity* of f is a value $a \in U$ at which f is *discontinuous* (not continuous),

$$\exists (x_n) \subseteq U, \text{ such that } \lim x_n = a \text{ and } \big(f(x_n)\big) \text{ does not converge to } f(a)$$



Continuity at a: *every sequence* with $\lim x_n = a$ has $\lim f(x_n) = f(a)$



Discontinuity at a: *at least one sequence* with $\lim x_n = a$ has $\lim f(x_n) \neq f(a)$

Examples 17.3. 1. $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is continuous (at every $a \in \mathbb{R}$). To see this, suppose (x_n) converges to a, then, by the limit laws,

$$\lim f(x_n) = \lim x_n^2 = (\lim x_n)^2 = a^2 = f(a)$$

2. The function with $g(x) = 1 + \frac{4}{x^2}$ is continuous. Choose any $a \in \operatorname{dom} g = \mathbb{R} \setminus \{0\}$ and any $(x_n) \subseteq \operatorname{dom} g$ with $\lim x_n = a$. Again, by the limit laws,

$$\lim g(x_n) = \lim \left(1 + \frac{4}{x_n^2}\right) = 1 + \frac{4}{(\lim x_n)^2} = 1 + \frac{4}{a^2} = f(a)$$

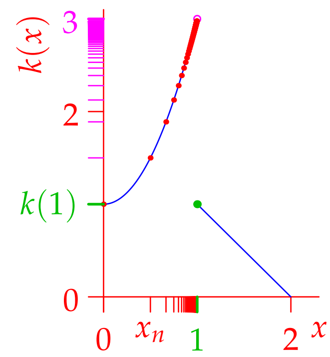
This example (with $a = 1$ and $x_n = 1 + \frac{2}{n}$) is the first picture in the definition.

3. $h : [0, \infty) \to \mathbb{R} : x \mapsto 3x^{1/4}$ is continuous. Again, everything follows from the limit laws. If $x_n \to a$ where $x_n \geq 0$ and $a \geq 0$, then

$$\lim h(x_n) = \lim 3x_n^{1/4} = 3(\lim x_n)^{1/4} = 3a^{1/4} = h(a)$$

4. The function defined by

$$k(x) = \begin{cases} 1 + 2x^2 & \text{if } x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$



is discontinuous at $a = 1$. This seems obvious from the picture, but we need to use the definition. The sequence with $x_n = 1 - \frac{1}{n}$ converges to 1 from below, whence

$$\lim k(x_n) = \lim \left(1 + 2\left(1 - \frac{2}{n} + \frac{1}{n^2}\right)\right) = 3 \neq 1 = k(1)$$

Basic examples and combinations of continuous functions

By appealing to the limit laws for sequences (Theorem 9.2), we can combine continuous functions in natural ways. For instance, if f, g are continuous at a, then

$$\lim x_n = a \implies \lim f(x_n) + g(x_n) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a)$$

whence $f + g$ is continuous at a. Here is a general summary.

Theorem 17.4. 1. *Suppose f, g are continuous and that k is constant. Then the following functions are continuous (on their domains)*

$$kf, \quad |f|, \quad f+g, \quad f-g, \quad fg, \quad \frac{f}{g}, \quad \max(f,g), \quad \min(f,g)$$

2. *If $n \in \mathbb{N}$ then the function $f : x \mapsto x^{1/n}$ is continuous on its domain.*

3. *Compositions of continuous functions are continuous. Specifically, if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

4. *Algebraic functions are continuous (this includes all polynomials and rational functions).*

Proof. Parts 1, 2 are the limit laws; for the maximum and minimum, see Exercise 2. For part 3:

$$\lim x_n = a \overset{g \text{ cont}}{\Longrightarrow} \lim g(x_n) = g(a) \overset{f \text{ cont}}{\Longrightarrow} \lim f(g(x_n)) = f(g(a))$$

Part 4 follows from parts 1, 2 and 3. ∎

Example 17.5. Part 3 of the theorem says that the following algebraic function is continuous

$$f : (7, \infty) \to \mathbb{R} : x \mapsto \sqrt{\frac{3x^{5/2} + 7x^2 + 4}{(x-7)^{1/3}}}$$

Theorem 17.6 (Squeeze theorem). *Suppose $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$, that f, h are continuous at a, and that $f(a) = g(a) = h(a)$. Then g is continuous at a.*

Proof. This is simply the squeeze theorem (8.9) for sequences: if $\lim x_n = a$, then

$$f(x_n) \leq g(x_n) \leq h(x_n) \implies \lim g(x_n) = g(a)$$

■

To provide more interesting examples, we state the following without proof.

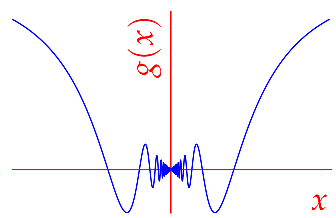
Theorem 17.7. *The common trigonometric, exponential and logarithmic functions are continuous.*

It is possible, though slow and ugly to address some of this now, though it is not very profitable. It is better to define these functions later using power series[32] when their continuity (and differentiability/integrability!) come for free.

Examples 17.8. 1. $f(x) = \frac{\sqrt{x}}{\sin e^x}$ is continuous on its domain $\mathbb{R} \setminus \{\ln(n\pi) : n \in \mathbb{N}_0\}$.

2. The function defined by $g(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $g(0) = 0$ is continuous on $\mathbb{R}$. When $x \neq 0$, this follows from Theorems 17.4 and 17.7, while at $a = 0$ we rely on the squeeze theorem:

$$x \neq 0 \implies -x \leq x \sin \frac{1}{x} \leq x$$



The ϵ–δ Definition of Continuity

The sequential definition of continuity uses limits *twice*. By stating each of these using the ϵ-definition of limit, we can reformulate continuity without ever mentioning sequences.

To motivate this, consider $f(x) = x^2$ at $a = 2$. By continuity, if (x_n) is a sequence with $\lim x_n = 2$, then $\lim f(x_n) = 4$. We restate each of these using the definition of limit:

(a) $(\lim x_n = 2)$ $\quad \forall \delta > 0, \exists M$ such that $n > M \implies |x_n - 2| < \delta$

(b) $(\lim x_n^2 = 4)$ $\quad \forall \epsilon > 0, \exists N$ such that $n > N \implies \left|x_n^2 - 4\right| < \epsilon$

Here is a short argument that shows how (a) $\Rightarrow$ (b) (we'll revisit this formally in a moment).

Assume (a), let $\epsilon > 0$ be given, and *define* $\delta = \min(1, \frac{\epsilon}{5})$. Since $\lim x_n = 2$, $\exists M$ such that

$$\begin{aligned} n > M \implies \left|x_n^2 - 4\right| &= |x_n - 2|\,|x_n + 2| < \delta\,|(x_n - 2) + 4| && \text{(by (a))} \\ &\leq \delta\big(|x_n - 2| + 4\big) && (\triangle\text{-inequality}) \\ &< \delta(\delta + 4) \leq 5\delta \leq \epsilon && \text{((a) again)} \end{aligned}$$

Let $N = M$ to conclude (b).

It turns out not to be very important that (x_n) be a *sequence*. In fact we can dispense with it entirely...

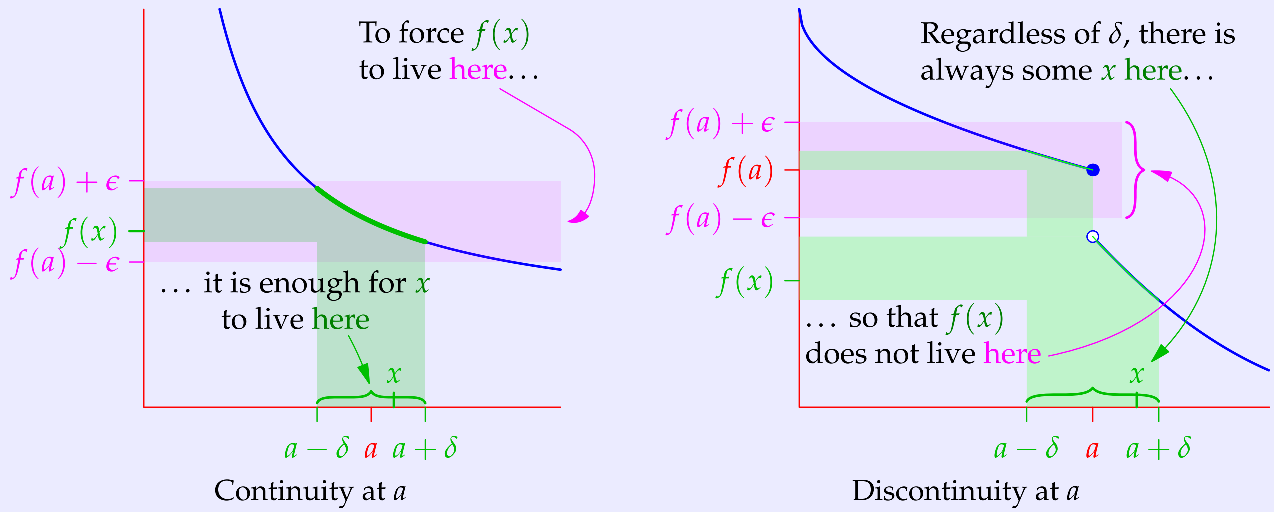
[32] For instance via the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Definition 17.9 (ϵ–δ continuity). A function $f : U \to V \subseteq \mathbb{R}$ is continuous at a if[33]

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x \in U)\ |x-a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{$*$}$$

A *discontinuity* of f is a value $a \in U$ for which,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x \in U \text{ with } |x-a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon \tag{$\dagger$}$$



This is the intuitive interpretation of continuity: if x is close to a, then $f(x)$ is close to $f(a)$; ϵ and δ are merely measures of closeness. Most mathematicians consider the ϵ–δ version to be *the* definition of continuity. Thankfully, it doesn't matter which you prefer...

Theorem 17.10. *The sequential and ϵ–δ definitions of continuity (17.2 & 17.9) are equivalent.*

Examples (17.3, cont). Before seeing the proof, we repeat our earlier examples using the ϵ–δ definition. As with ϵ–N arguments for limits, it is often useful to do some scratch work first.

1. Suppose $f(x) = x^2$ and $a \in \mathbb{R}$. Our goal is to control the size of $|x^2 - a^2|$ whenever $|x-a|$ is small. To keep things simple, assume $|x - a| < 1$, then,

$$\begin{aligned} |x^2 - a^2| &= |x-a|\,|x+a| = |x-a|\,|(x-a) + 2a| \\ &\overset{\triangle}{\leq} |x-a|\,(|x-a| + 2\,|a|) = |x-a|\,(1 + 2\,|a|) \end{aligned}$$

Now let $\epsilon > 0$ be given and define $\delta = \min(1, \frac{\epsilon}{1+2|a|})$. Then

$$|x-a| < \delta \implies |f(x) - f(a)| = |x^2 - a^2| < \delta(1 + 2\,|a|) \leq \epsilon$$

Thus f is continuous at a. This is simply a general version of the argument on page 66 with all mention of sequences removed!

[33] The bracketed statement $\forall x \in U$ is often omitted in ($*$), since the implication requires x to be universally quantified. It is important that $x \in U = \operatorname{dom} f$ rather than merely $x \in \mathbb{R}$! By contrast, the expression $\exists x \in U$ in ($\dagger$) is *always* written.

2. Let $g(x) = 1 + \frac{4}{x^2}$ and $a \neq 0$. The first challenge is to control $\frac{1}{x}$ by staying away from zero: to do this, we start by insisting that $\delta \le \frac{|a|}{2}$. But now,

$$|x-a| < \delta \implies |x| > \frac{|a|}{2} \implies \frac{1}{|x|} < \frac{2}{|a|} \tag{$*$}$$

Now consider the required difference; if $|x-a| < \delta$, then

$$\begin{aligned} |g(x)-g(a)| &= \left|1 + \frac{4}{x^2} - 1 - \frac{4}{a^2}\right| = \frac{4\left|a^2-x^2\right|}{a^2x^2} = \frac{4\,|a+x|}{a^2x^2}\,|x-a| \\ &\overset{\triangle}{\le} 4\left(\frac{1}{|a|\,x^2} + \frac{1}{a^2\,|x|}\right)|x-a| \overset{(*)}{<} 4\left(\frac{4}{|a|^3} + \frac{2}{|a|^3}\right)|x-a| = \frac{24}{|a|^3}\delta \end{aligned}$$

Given $\epsilon > 0$, it suffices to let $\delta = \min(\frac{1}{2}\,|a|\,, \frac{1}{24}|a|^3\epsilon)$. Then $|x-a| < \delta \implies |f(x)-f(a)| < \epsilon$.

3. For $h(x) = 3x^{1/4}$, there are two cases. Suppose $\epsilon > 0$ is given.

   - If $a = 0$, let $\delta = \left(\frac{\epsilon}{3}\right)^4$, then[34]

   $$|x-a| < \delta \implies 0 \le x < \delta \implies |h(x)-h(a)| = 3x^{1/4} < 3\delta^{1/4} = \epsilon$$

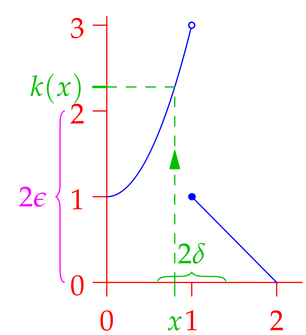
   - If $a > 0$, let $\delta = \frac{1}{3}a^{3/4}\epsilon$. Then, if $|x-a| < \delta$, we have

   $$|h(x)-h(a)| = 3\left|x^{1/4} - a^{1/4}\right| = \frac{3\,|x-a|}{x^{\frac{3}{4}} + a^{\frac{1}{4}}x^{\frac{2}{4}} + a^{\frac{2}{4}}x^{\frac{1}{4}} + a^{\frac{3}{4}}} \le \frac{3\,|x-a|}{a^{3/4}} < \frac{3\delta}{a^{3/4}} = \epsilon$$

4. Suppose k is continuous at 1 and let $\epsilon = 1$. Then $\exists \delta > 0$ for which

$$\begin{aligned} |x-1| < \delta &\implies |k(x)-k(1)| = |k(x)-1| < 1 \\ &\implies 0 < k(x) < 2 \end{aligned}$$

However, if we choose $x = \max(\frac{1}{\sqrt{2}}, 1 - \frac{\delta}{2})$, then $|x-1| \le \frac{\delta}{2} < \delta$ and $k(x) \ge k(\frac{1}{\sqrt{2}}) = 1 + \frac{2}{2} = 2$. Contradiction.



The basic rules for combining continuous functions may also be proved using ϵ–δ arguments. E.g.,

ϵ–δ proof of the squeeze theorem. Given $\epsilon > 0$, we know there exist $\delta_1, \delta_2 > 0$ for which

$$|x-a| < \delta_1 \implies |f(x)-f(a)| < \epsilon \quad \text{and} \quad |x-a| < \delta_2 \implies |h(x)-h(a)| < \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$, then

$$|x-a| < \delta \implies |g(x)-g(a)| \le \max\big(|f(x)-f(a)|\,, |h(x)-h(a)|\big) < \epsilon$$

whence g is continuous at 0. ∎

[34] Remember the hidden quantifier: $|x-a| < \delta$ *for all* $x \in \operatorname{dom} f = [0,\infty)$, thus $x \ge 0$ for the duration of this example.

Several other arguments are in the exercises. Finally, here is the promise proof of equivalence.

Proof of Theorem 17.10. (sequential $\Rightarrow \epsilon$–δ) We prove the contrapositive. Suppose a is an ϵ–δ discontinuity (†) and let $\delta = \frac{1}{n}$. Then there exists $x_n \in U$ such that

$$|x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(a)| \geq \epsilon$$

Repeating for all $n \in \mathbb{N}$ results in a sequence (x_n) for which $\lim x_n = a$ and $\lim f(x_n) \neq f(a)$: otherwise said, a is a sequential discontinuity.

(ϵ–$\delta \Rightarrow$ sequential) Assume $(*)$, let $(x_n) \subseteq U$ and suppose $\lim x_n = a$; we must prove that $\lim f(x_n) = f(a)$. Let $\epsilon > 0$ be given so that a suitable δ satisfying $(*)$ exists. Since $\lim x_n = a$,

$$\begin{aligned} \exists N \text{ such that } n > N &\implies |x_n - a| < \delta && \text{(since } x_n \to a \text{ and } \delta > 0 \text{ is given)} \\ &\implies |f(x_n) - f(a)| < \epsilon && \text{(by } (*)) \end{aligned}$$

We conclude that $\lim f(x_n) = f(a)$, as required. ∎

Examples 17.11. We finish with a couple of esoteric examples on the same theme.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the *indicator function* for the rational numbers:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Suppose f is continuous at a and let $\epsilon = 1$. Then $\exists\delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < 1 \qquad (*)$$

There are two cases; these rely on the fact that any interval contains both rational and irrational numbers (Corollary 4.12, etc.).

(a) If $a \in \mathbb{Q}$, then $f(a) = 1$. There exists an irrational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |0 - 1| = 1 \not< 1$.

(b) If $a \notin \mathbb{Q}$, then $f(a) = 0$. There exists a rational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |1 - 0| = 1 \not< 1$.

Either way, we have contradicted $(*)$. We conclude that f is *nowhere continuous.*

2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since $0 \leq |g(x)| \leq |x|$, the squeeze theorem tells us that g is continuous at $x = 0$.

Now suppose g is continuous at $a \neq 0$ and let $\epsilon = |a|$. Then $\exists\delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < |a|$$

The same two cases as in the previous example provide contradictions. We conclude that g *is continuous at precisely one point!*

Exercises 17. 1. Consider the function with $f(x) = \frac{1}{\sqrt{x^2+2x-3}}$.

(a) The implied domain of f has the form $\operatorname{dom} f = (-\infty, a) \cup (b, \infty)$. Find a and b.

(b) What is the range of f?

(c) Show that $f : (b, \infty) \to \operatorname{range} f$ is *bijective* and compute its inverse function.

(d) Find the inverse function when we instead restrict the domain to $(-\infty, a)$.

(e) Briefly explain why f is continuous on its domain.

2. Let f and g be continuous functions at a.

(a) Show that $\max(f, g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and deduce that $\max(f, g)$ is continuous at a.

(b) How might you show continuity of $\min(f, g)$?

3. Use ϵ–δ arguments to prove the following.

(a) $f(x) = x^2 - 3x$ is continuous at $x = 1$

(b) $g(x) = x^3$ is continuous at $x = a$.

(c) $h : [0, \infty) \to \mathbb{R} : x \mapsto \sqrt{x}$ is continuous.

(d) $j(x) = 3x^{-1}$ is continuous on $\mathbb{R} \setminus \{0\}$.

4. Prove that $x = 0$ is a discontinuity of each function: use *both* definitions of continuity.

(a) $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.

(b) $g(x) = \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$.

5. Suppose f and g are continuous at a. Prove the following using ϵ–δ arguments.

(a) $f - g$ is continuous at a.

(b) If h is continuous at $f(a)$, then $h \circ f$ is continuous at a.

6. Suppose $f : U \to V \subseteq \mathbb{R}$ is a function whose domain U contains an *isolated point* a: i.e. $\exists r > 0$ such that $(a-r, a+r) \cap U = \{a\}$. Prove that f is continuous at a.

7. In Example 17.11.2, provide the details of the required contradiction.

8. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which $f(x) = 0$ whenever $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all rational x. Prove that $f = g$.

9. (Hard) Consider $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{q} & \text{whenever } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q > 0 \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For example, $f(1) = f(2) = f(-7) = 1$, and $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, etc. Prove that f is continuous at each irrational number, and discontinuous at each rational number.

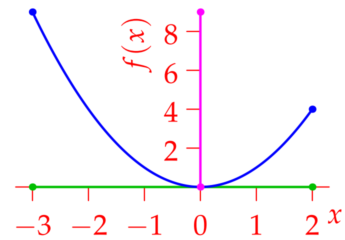
18 Properties of Continuous Functions

In this section consider how continuous functions transform *intervals*.

Example 18.1. $f(x) = x^2$ maps $[-3, 2]$ onto $[0, 9]$. In particular:

- f transforms one interval into another.
- f transforms one closed bounded set into another.

The purpose of this section is to see that these are general properties exhibited by *any* continuous function.



Before stating our first result, recall a couple of definitions.

Definition 18.2. Let $U, V \subseteq \mathbb{R}$ and $f : U \to V$.

1. (a) U is *bounded* if $\exists M$ such that $\forall x \in U, |x| \le M$.
   (b) f is *bounded* if its range is a bounded set: $\exists M$ such that $\forall x \in U,\ |f(x)| \le M$.
2. (Definition 11.13) U is *closed* if every convergent sequence in U has its limit in U:

$$\forall (x_n) \subseteq U,\ \lim x_n = s\ (\in \mathbb{R}) \implies s \in U$$

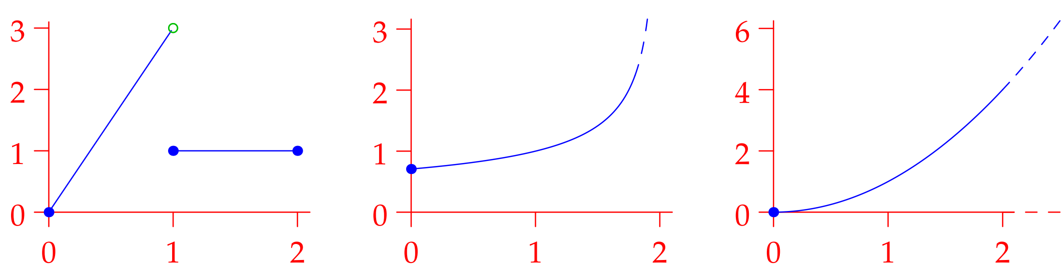
Theorem 18.3 (Extreme Value Theorem). *Suppose $f : U \to V$ is continuous where U is closed and bounded. Then f is bounded and attains its bounds:*

$$\exists s, i \in U \quad \text{such that} \quad f(s) = \sup\big(f(U)\big) \quad \text{and} \quad f(i) = \inf\big(f(U)\big)$$

In fact $f(U)$ is also closed and bounded.

In Example 18.1, if $U = [-3, 2]$, then $s = -3$ and $i = 0$.

Examples 18.4. Before seeing the proof, here are three examples where we weaken one of the hypotheses and see that the result fails.



1. f discontinuous 2. U not closed 3. U not bounded

1. $\sup(\text{range } f) = 3$ is not attained by $f(x) = \begin{cases} 3x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2 \end{cases}$
2. If $f(x) = \frac{1}{\sqrt{2-x}}$ and $U = [0, 2)$, then range $f = [\frac{1}{\sqrt{2}}, \infty)$ is unbounded.
3. If $f(x) = x^2$ and $U = [0, \infty)$, then range $f = [0, \infty)$ is unbounded.

The goal of the proof is to show that every limit point of $f(U) = \operatorname{range} f$ lies in $f(U)$. The proof is broken into simple steps: observe where each hypothesis is used.

Proof.

1. Suppose M is a limit point of $f(U)$: that is, $M = \lim\big(f(x_n)\big)$ for some sequence $(x_n) \subseteq U$. *A priori* M need not be finite, but it is possible[35] that $M = \sup\big(f(U)\big)$ or $\inf\big(f(U)\big)$.
2. Since $(x_n) \subseteq U$ is *bounded*, Bolzano–Weierstraß (Theorem 11.8) says it has a convergent subsequence, $\lim_{k\to\infty} x_{n_k} = x$.
3. Since U is *closed*, we have $x \in U$. This means $f(x)$ makes sense (it is a *real number*!).
4. Since f is *continuous*, we have $\lim f(x_{n_k}) = f(x)$.
5. Finally, $M = f(x)$ since all subsequences of a convergent (or divergent to $\pm\infty$) sequence tend to the same limit (Lemma 11.3). It follows that all limit points M are *finite* and lie in $f(U)$: otherwise said, $f(U)$ is closed and bounded.

In particular, choosing $M = \sup\big(f(U)\big)$ yields $x = s \in U$ as in the Theorem. ∎

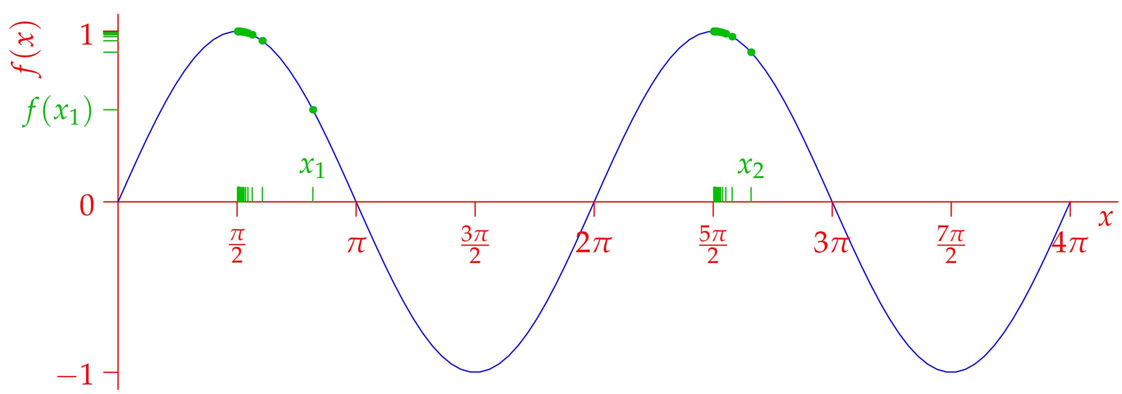
Example 18.5. It is worth thinking about why we needed to use a *subsequence* in the proof. The reason is that it is possible for the bounds of f to be attained multiple times. For example, consider

$$f : [0, 4\pi] \to \mathbb{R} : x \mapsto \sin x$$

This satisfies the hypotheses of the extreme value theorem: $[0, 4\pi]$ is closed and bounded and f is continuous. Indeed $\max(\operatorname{range} f) = 1$ is attained at *both* $x = \frac{\pi}{2}$ and $\frac{5\pi}{2}$. The sequence defined by

$$x_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n} & \text{if } n \text{ odd} \\ \frac{5\pi}{2} + \frac{1}{n} & \text{if } n \text{ even} \end{cases} \quad \text{has} \quad f(x_n) = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) \xrightarrow[n\to\infty]{} 1 = \sup(\operatorname{range} f)$$

and therefore satisfies step 1 of the proof. However, (x_n) itself is *divergent by oscillation*. Bolzano–Weierstraß is used to force the existence of a convergent subsequence; in this case the subsequence of odd terms $(x_{n_k}) = (x_{2k-1})$ satisfies the remaining steps of the proof.



[35] If $M = \sup\big(f(U)\big)$, then a suitable (x_n) might be constructed as follows:

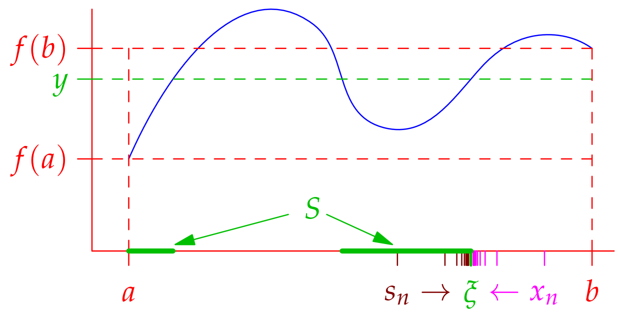
- If $M \in \mathbb{R}$, then for each $n \in \mathbb{N}, \exists x_n \in U$ such that $M - \frac{1}{n} < f(x_n) \le M$ (Lemma 4.8).
- If $M = \infty$, then for each $n \in \mathbb{N}, \exists x_n \in U$ such that $f(x_n) \ge n$.

The Intermediate Value Theorem and its Consequences

This result should be familiar from elementary calculus, even if its proof is not! It is also intuitive: if you climb a hill, then at some point you must be half-way up the hill...

Theorem 18.6 (Intermediate Value Theorem (IVT)). *Let f be continuous on the interval $[a, b]$ and let y lie strictly between $f(a)$ and $f(b)$. Then $\exists \xi \in (a, b)$ such that $f(\xi) = y$.*

Proof. WLOG assume $f(a) < y < f(b)$. Now let $S = \{x \in [a, b] : f(x) < y\}$ and *define* $\xi := \sup S$.



Since S is non-empty ($a \in S$) and bounded above (by b), we see that ξ exists and is finite. It remains to prove that $a < \xi < b$ and $f(\xi) = y$.

First choose any $(s_n) \subseteq S$ such that[36] $\lim s_n = \xi$. Continuity forces $\lim f(s_n) = f(\xi)$. Moreover

$$f(s_n) \le y \implies f(\xi) \le y$$

This also shows that $\xi < b$.

We now play a similar game from the other side: define $x_n := \min(\xi + \frac{1}{n}, b)$, then $\lim x_n = \xi$ and $x_n > \xi = \sup S \implies x_n \notin S$, whence

$$f(x_n) \ge y \implies f(\xi) = \lim f(x_n) \ge y$$

This also shows that $\xi > a$. Putting it all together, we conclude that $f(\xi) = y$ and $\xi \in (a, b)$. ∎

Note how the value of ξ in the proof is always the *largest* of potentially several choices.

Examples 18.7. The intermediate value theorem was typically used in elementary calculus to show the existence of solutions to equations. Here are a couple of examples of this process.

1. We show that the equation $x^7 + 3x = 1 + 4\cos(\pi x)$ has a solution.

   The trick is to express the equation in the form $f(x) = y$ where f is continuous, then choose suitable a, b to fit the theorem. In this case,

   $$f(x) = x^7 + 3x - 4\cos(\pi x) \quad \text{and} \quad y = 1$$

   are suitable choices. Now observe that

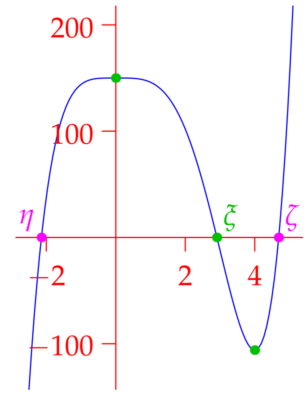
   $$f(0) = -4 < y \quad \text{and} \quad f(1) = 1 + 3 + 4 = 8 > y \qquad \text{(i.e., } a = 0 \text{ and } b = 1\text{)}$$

   whence $\exists \xi \in (0, 1)$ such that $f(\xi) = 01$. Otherwise said, ξ is a solution to the original equation.

   The function f is in fact continuous on $\mathbb{R}$, a much larger interval that $[a, b]$, but no matter!

[36] Similarly to step 1 of the proof of the Extreme Value Theorem.

2. The existence of a root ξ of the (continuous) polynomial



$$f(x) = x^5 - 5x^4 + 150$$

follows from the intermediate value theorem by observing that

$$f(0) = 150 > 0 \quad \text{and} \quad f(4) = -256 + 150 = -106 < 0$$

We conclude that such a root exists and that $\xi \in (0, 4)$.
As the graph suggests, there are other roots (η, ζ), the existence of which may be shown by also evaluating, say,

$$f(-3) = -798 < 0 \quad \text{and} \quad f(5) = 150 > 0$$

With an eye on generalizing, consider a slightly different approach. Define two sequences (s_n) and (t_n) via

$$s_n := \frac{f(-n)}{n^5} = -1 - \frac{5}{n} + \frac{150}{n^5} \qquad t_n := \frac{f(n)}{n^5} = 1 - \frac{5}{n} + \frac{150}{n^5}$$

Since $\lim s_n = -1$ and $\lim t_n = 1$, we see that

$$\exists a \text{ such that } s_a < -\frac{1}{2} \implies f(-a) = a^5 s_a < -\frac{1}{2}a^5 < 0$$
$$\exists b \text{ such that } t_b > \frac{1}{2} \implies f(b) = b^5 t_b > \frac{1}{2}b^5 > 0$$

Applying the intermediate value theorem on $[-a, b]$ shows the existence of a root.

The second approach in Example 18.5.2 may be applied to prove the general result.

Corollary 18.8. *A polynomial function of odd degree has at least one real root.*

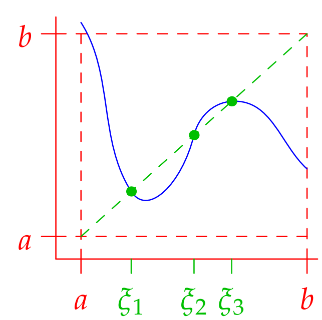
The proof is an exercise. An even simpler exercise shows the existence of a *fixed point* for a particular type of continuous function.

Corollary 18.9 (Fixed Point Theorem). *Suppose a and b are finite and that $f : [a, b] \to [a, b]$ is continuous. Then f has a fixed point:*

$$\exists \xi \in [a, b] \text{ such that } f(\xi) = \xi$$

As the picture shows, a function could have several fixed points.

This is the first of several fixed-point theorems you'll meet if your studies of analysis continue. Many important consequences flow from these, including a common fractal construction and the standard existence/uniqueness result for differential equations.



For our final corollary, we first note a straightforward characterization: a set $I \subseteq \mathbb{R}$ is an interval if

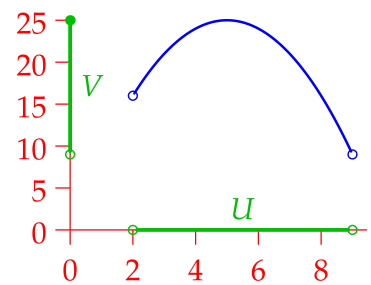
$$a, b \in I \text{ and } a < y < b \implies y \in I \tag{$*$}$$

Corollary 18.10 (Preservation of Intervals). *Suppose $f : U \to V$ is continuous where $U = \operatorname{dom} f$ is an interval (of positive length) and $V = \operatorname{range} f$.*

1. *V is an interval or a point.*
2. *If f is strictly increasing (decreasing), then:*
   (a) *V is an interval (it has positive length).*
   (b) *f is* injective *(and thus bijective).*
   (c) *The inverse function $f^{-1} : V \to U$ is also continuous and strictly increasing (decreasing).*

Example 18.11. In part 1, note that the interval V *need not be of the same type* as U. For instance, if $f(x) = 10x - x^2$, then f maps the *open* interval $U = (2, 9)$ to the *half-open* interval $V = (9, 25]$.
The extreme value theorem, however, guarantees that if U is a closed bounded interval, then V is also, for instance,

$$f([2, 9]) = [9, 25]$$



Proof. 1. If V is not a point, then $\exists a, b \in U$ such that $f(a) < f(b)$. Let $y \in \big(f(a), f(b)\big)$; IVT says $\exists \xi$ between a and b such that $y = f(\xi)$. That is, $y \in \operatorname{range} f$. By $(*)$, $V = \operatorname{range} f$ is an interval.

2. (a,b) If f is strictly increasing, then $\forall a, b \in U$, $a < b \implies f(a) < f(b)$. It follows that f is injective and that V contains at least 2 points; by part 1 it has positive length.

(c) Let $y_1 < y_2$ where both lie in V, and define $x_i = f^{-1}(y_i)$ for $i = 1, 2$. Since f is increasing,

$$x_2 \le x_1 \implies y_2 = f(x_2) \le f(x_1) = y_1$$

is a contradiction. Thus $x_1 < x_2$ and f^{-1} is also strictly increasing.
It remains to show that f^{-1} is continuous at $b = f^{-1}(a)$. Assume first that a is not an endpoint of U. Given $\epsilon > 0$ for which $[a - \epsilon, a + \epsilon] \subseteq U$, let

$$\delta := \min\big(b - f(a - \epsilon), f(a + \epsilon) - b\big)$$

and observe that

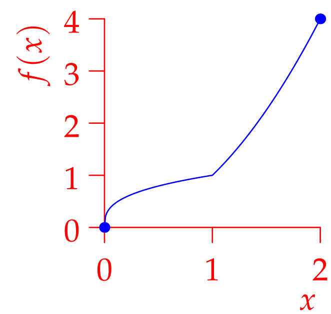
$$\begin{aligned} |y - b| < \delta &\implies f(a - \epsilon) - b < y - b < f(a + \epsilon) - b \implies f(a - \epsilon) < y < f(a + \epsilon) \\ &\implies a - \epsilon < y < a + \epsilon \qquad (f \text{ strictly increasing}) \\ &\implies \left| f^{-1}(y) - a \right| < \epsilon \end{aligned}$$

If a is an endpoint of U, instead use $[a - \epsilon, a] \subseteq U$ or $[a, a + \epsilon] \subseteq U$ and only the corresponding half of the expression δ.

■

Example 18.12. The function $f : [0,2] \to [0,4]$ defined by

$$f(x) = \begin{cases} \sqrt[3]{x} & \text{if } 0 \le x \le 1 \\ x^2 & \text{if } 1 < x \le 2 \end{cases}$$



is continuous and strictly increasing. It therefore has a continuous inverse function $f^{-1} : [0,4] \to [0,2]$.
Compare this with the result from elementary calculus: $f' > 0 \implies f$ injective. We cannot apply this here since f is not differentiable!

Exercises 18. 1. Give an example of a *discontinuous* function $f : [0,1] \to \mathbb{R}$ which is *not bounded.*

2. Let $a < b$ be given. Give examples of *continuous* functions $g, h : (a,b) \to \mathbb{R}$ such that:
   (a) g is *not bounded.*
   (b) h is bounded but *does not attain its bounds.*

3. Compute the inverse of the function f in Example 18.12.

4. Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S that converges to a number $x_0 \notin S$. Show that there exists an unbounded continuous function on S.

5. Prove that $x = \cos x$ for some x in $(0, \frac{\pi}{2})$.

6. Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists x between a, b such that $f(x) = 0$.

7. Suppose that f is continuous on $[0,2]$ and that $f(0) = f(2)$. Prove that there exist $x, y \in [0,2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.
   (*Hint: consider* $g(x) = f(x+1) - f(x)$ *on* $[0,1]$)

8. (a) Prove the fixed point theorem (Corollary 18.9).
   (*Hint: If neither a nor b are fixed points, consider* $g(x) = f(x) - x$)
   (b) Prove Corollary 18.8 for a general odd-degree monic polynomial $f(x) = x^{2m+1} + \sum_{k=0}^{2m} \alpha_k x^k$.

9. Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.
   (a) Explain why f is continuous on any interval U.
   (b) Suppose $a < 0 < b$ and that $f(a), f(b)$ have opposite signs. If $y = 0$, show that the intermediate value theorem is satisfied by *infinitely many* distinct values ξ.

10. (a) Suppose $f : U \to \mathbb{R}$ is continuous and that $U = \bigcup_{k=1}^{n} I_k$ is the union of a finite sequence (I_k) of closed bounded intervals. Prove that f is bounded and attains its bounds.
    (b) Let $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for each $n \in \mathbb{N}$. Give an example of a continuous function $f : U \to \mathbb{R}$ which is either unbounded or does not attain its bounds. Explain.

(*This is related to the idea that* *finite* *unions of closed sets remain closed, but* *infinite* *unions need not*)

19 Uniform Continuity

Suppose $f : U \to V$ is continuous. By the ϵ-δ definition (17.9),

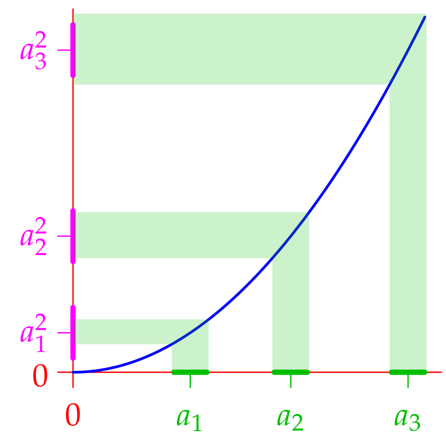
$$\forall a \in U,\ \forall \epsilon > 0,\ \exists \delta(a,\epsilon) > 0 \text{ such that } (\forall x \in U)\ |x-a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{$*$}$$

We write $\delta(a,\epsilon)$ to stress the fact that δ can depend both on the *location* a and the *distance* ϵ. The goal of this section is to understand if/when it is possible to choose δ *independently of the location* a.

Example 19.1. We start by considering how this desire might be impossible to satisfy.

Consider $f(x) = x^2$ with domain $U = [0,\infty)$. Given $\epsilon > 0$ and $a_1 \in U$, we can certainly find some[37] δ such that

$$|x - a_1| < \delta \implies |f(x) - f(a_1)| = \left|x^2 - a_1^2\right| < \epsilon$$



Visualize what happens if we try to use the *same* δ for different a_i: imagine sliding the fixed-width δ-interval along the x-axis while simultaneously sliding the ϵ-interval vertically. As a_i increases, the image of the δ-interval eventually becomes too large for the ϵ-interval to contain:

$$\text{length}\big(f(a_i - \delta, a_i + \delta)\big) = (a_i+\delta)^2 - (a_i - \delta)^2 = 4a_i\delta$$

increases unboundedly with a_i.

For fixed ϵ, as a increases, the increasing *gradient* of f means that we need to choose a *smaller* δ.

By contrast, if we consider the same formula $f(x) = x^2$ but on a restricted *finite* domain $[0,b]$, then any δ that suffices to demonstrate continuity at $x = b$ will also do so everywhere else on $[0,b]$. We'll check this explicitly in a moment.

To formalize things, consider rewriting $(*)$, where we additionally assume that δ may be chosen independently of the location a; this amounts simply to moving the quantifier $\forall a \in U$ after δ.

Definition 19.2. A function $f : U \to V \subseteq \mathbb{R}$ is *uniformly continuous* if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x, y \in U)\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \tag{$\dagger$}$$

For reasons of symmetry we use y instead of a. Note how δ now depends only on ϵ since it is quantified *before* x and y; as previously, the quantifiers for x, y are usually hidden. Note also how uniform continuity is only relevant on the entire domain U; it makes no sense to speak of uniform continuity at a point a.

For the sake of tidiness, we make one more observation before seeing some examples.

Lemma 19.3. *If f is uniformly continuous on U, then it is continuous on U.*

This should be trivial: $(\dagger)$ is the ϵ-δ continuity of f at $y \in U$, *for all y simultaneously*! The special feature of the definition is that the same δ works for all y.

[37] For instance $\delta = \min(1, \frac{\epsilon}{1+2a_1})$, as we saw on page 67.

Examples 19.4. 1. We re-analyze $f(x) = x^2$ in view of the definition. Recall first that

$$|f(x) - f(y)| = \left|x^2 - y^2\right| = |x-y|\,|x+y|$$

where $|x-y|$ is easily controlled by δ. We consider the behavior of $|x+y|$ in two cases.

Bounded domain If $U = \operatorname{dom} f \subseteq [-T, T]$ for some $T > 0$, we show that f is uniformly continuous. This follows because $|x+y| \le 2T$ is also easily controlled.
Let $\epsilon > 0$ be given and define $\delta = \frac{\epsilon}{2T}$, then

$$|x-y| < \delta \implies |f(x) - f(y)| < \delta \cdot 2T = \epsilon$$

Compare with Example 19.1. Our approach works for *this* function because the gradient (and therefore potential discrepancy between $x^2 - y^2$ and $x - y$) is greatest at the endpoints of the interval. The same approach may not work for other functions!

Unbounded domain We show that f is not uniformly continuous when $\operatorname{dom} f = [0, \infty)$.
For contradiction, assume f is uniformly continuous: let $\epsilon = 1$ and suppose $\delta > 0$ satisfies the definition. Supposing $x - y = \frac{\delta}{2}$, we see that

$$|x+y| = 2y + \frac{\delta}{2} \implies |f(x) - f(y)| = \frac{\delta}{2}\left(2y + \frac{\delta}{2}\right) = \delta\left(y + \frac{\delta}{4}\right) > \delta x$$

Letting $y = \frac{1}{\delta}$ $\left(x = \frac{1}{\delta} + \frac{\delta}{2}\right)$ yields the contradiction $|f(x) - f(y)| > 1 = \epsilon$.

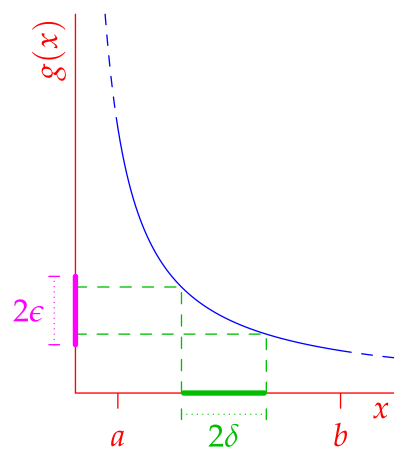
2. Let $g(x) = \frac{1}{x}$; we again consider two domains.

Uniform continuity on $[a, b)$ whenever $0 < a < b \le \infty$.

Let $\epsilon > 0$ be given and let $\delta = a^2\epsilon$. Then,

$$\begin{aligned} |x-y| < \delta \implies |g(x) - g(y)| &= \left|\frac{y-x}{xy}\right| \\ &< \frac{\delta}{xy} \le \frac{\delta}{a^2} = \epsilon \end{aligned}$$

where the last inequality follows because $x, y \ge a$.



Non-uniform continuity on $(0, b)$ whenever $0 < b \le \infty$.

As before, let $\epsilon = 1$ and suppose $\delta > 0$ is given. Let

$$x = \min\left(\delta, 1, \frac{b}{2}\right) \quad \text{and} \quad y = \frac{x}{2}$$

Certainly $x, y \in (0, b)$ and $|x - y| = \frac{x}{2} \le \frac{\delta}{2} < \delta$. However,

$$|f(x) - f(y)| = \frac{1}{x} \ge 1 = \epsilon$$

Think about how ϵ and δ must relate as one slides the intervals in the picture up/down and left/right. In this case, large values of x, y are not the problem, it's the vertical asymptote at zero that causes trouble.

General Conditions for Uniform Continuity

For the remainder of this section, we develop a few general ideas related to uniform continuity. The first is a little out of order since it depends on differentiation and the mean value theorem.

Theorem 19.5. *Suppose f is continuous on an interval U (finite or infinite) and differentiable except perhaps at its endpoints. If f' is bounded, then f is uniformly continuous on U.*

Proof. Suppose $|f'(x)| \le M$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{M}$. Then

$$|x-y| < \delta \implies |f(x)-f(y)| = \left|f'(\xi)\right| |x-y| < M\delta = \epsilon$$

where the existence of ξ between x, y follows from the Mean Value Theorem.[38] ■

Examples 19.6. 1. Compare the arguments in the previous exercise. For instance, if $\operatorname{dom} f \subseteq [-T, T]$,

$$f(x) = x^2 \implies f'(x) = 2x \implies \left|f'(x)\right| \le 2T$$

The derivative is bounded, whence f is uniformly continuous on $[-T, T]$.

2. Any polynomial is uniformly continuous on any bounded interval.
3. The function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$ since $f'(x) = \cos x$ is bounded (by 1).
4. Consider $f(x) = \frac{1}{x} - \frac{5}{x^2}$ on $(1, \infty)$. We have

$$f'(x) = -\frac{1}{x^2} + \frac{10}{x^3} \implies \left|f'(x)\right| \le 11$$

We conclude that f is uniformly continuous on $(1, \infty)$.

The approach is often useful when you are asked to show *using the definition* that a function is uniformly continuous; provided f' is bounded by M, you may always choose $\delta = \frac{\epsilon}{M}$ to obtain an argument. For instance, with our function:
Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{11}$. If $x, y \in (1, \infty)$ and $|x-y| < \delta$, then

$$\begin{aligned} |f(x)-f(y)| &= \left|\frac{1}{x} - \frac{1}{y} + \frac{5}{y^2} - \frac{5}{x^2}\right| = |x-y| \left|\frac{5(x+y)}{x^2y^2} - \frac{1}{xy}\right| \\ &= |x-y| \left|\frac{5}{xy^2} + \frac{5}{x^2y} - \frac{1}{xy}\right| \\ &< 11\,|x-y| && (\triangle\text{-inequality, since } x, y > 1) \\ &< 11\delta = \epsilon \end{aligned}$$

As we'll see very shortly, the above result isn't a biconditional: non-differentiable functions and functions with unbounded derivatives can be uniformly continuous.

[38] If $x < y$ then $\exists \xi \in (x, y)$ such that $f'(\xi) = \dfrac{f(x)-f(y)}{x-y}$.

Our remaining conditions are variations on a theme: uniform continuity on a bounded interval U is roughly the same thing as continuity on its *closure* $\overline{U}$ (recall Definition 11.13).

Theorem 19.7. *A continuous function on a closed bounded domain is uniformly continuous.*

Proof. Assume f is continuous, but not uniformly so, on a closed bounded domain U. Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x, y \in U \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon \qquad (*)$$

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$ to obtain sequences $(x_n), (y_n) \subseteq U$ satisfying $(*)$.[39]
Since $(x_n) \subseteq U$ is bounded, Bolzano–Weierstraß says there exists a convergent subsequence (x_{n_k}) which, since U is closed, converges to some $x_0 \in U$.
Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$, we see that $\lim\limits_{k\to\infty} y_{n_k} = x_0$. Finally, the continuity of f contradicts $(*)$:

$$\epsilon \leq \lim |f(x_{n_k}) - f(y_{n_k})| = |f(x_0) - f(x_0)| = 0$$

■

Both hypotheses are crucial: Examples 19.4 provide counter-examples if either is weakened.

Example 19.8. $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$. This cannot be concluded from Theorem 19.5, since its derivative $f'(x) = \frac{1}{2}x^{-1/2}$ is unbounded on $(0,1)$.

Our next goal is to develop a partial converse, for which we first need a lemma.

Lemma 19.9. *If f is uniformly continuous on U and $(x_n) \subseteq U$ is Cauchy, then $\big(f(x_n)\big)$ is also Cauchy.*

To apply the result, consider a convergent (Cauchy) sequence in U whose limit is *not* itself in U.

Example 19.10. Let $f(x) = \frac{1}{x}$ be defined on $U = (0, \infty)$ and consider the sequence defined by $x_n = \frac{1}{n}$. This is plainly Cauchy since it converges; note crucially that its limit 0 *does not lie in U*. Moreover,

$$\lim f(x_n) = \lim n = \infty$$

$\big(f(x_n)\big)$ is not Cauchy, whence f is not uniformly continuous. This is a far simpler argument than that presented previously!

Proof. Let $\epsilon > 0$ be given. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ such that } \forall x, y \in U,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Now use this δ in the definition of (x_n) being Cauchy:[40]

$$\exists N \text{ such that } m, n > N \implies |x_n - x_m| < \delta \implies |f(x_n) - f(x_m)| < \epsilon$$

Otherwise said, $\big(f(x_n)\big)$ is Cauchy.

■

[39] These arguments should feel familiar: compare this line to the proof of Theorem 17.10 and the rest to Theorem 18.3.

[40] The Cauchy condition is important here: we cannot apply the uniform continuity condition directly to a convergent sequence ($|x_n - x| < \delta \ldots$) if we do not already know that its limit (here x) lies in U!

We apply the Lemma to show that a continuous function on a *bounded* interval is uniformly continuous if and only if has a *continuous extension.*

Theorem 19.11. *Suppose f is continuous on a bounded interval (a,b). Define $g : [a,b] \to \mathbb{R}$ via*

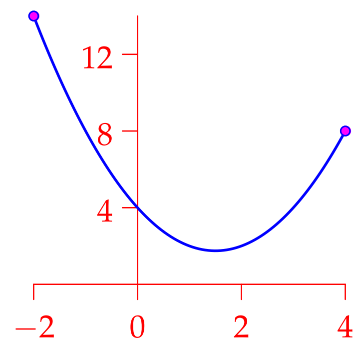
$$g(x) := \begin{cases} f(x) & \text{if } x \in (a,b) \\ \lim f(x_n) & \text{whenever } (x_n) \subseteq (a,b) \text{ and } \lim x_n = a \text{ or } b \end{cases}$$

Then f is uniformly continuous if and only if g is well-defined; in such a case g is automatically continuous.

Examples 19.12. 1. $f(x) = x^2 - 3x + 4$ is uniformly continuous on $(-2,4)$ since it has a continuous extension

$$g : [-2,4] \to \mathbb{R} : x \mapsto x^2 - 3x + 4$$

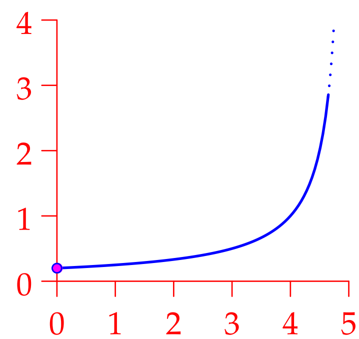
It should be obvious what is happening from the picture: to create the extension g, we simply fill in the holes at the endpoints of the graph.



2. The function $f(x) = \frac{1}{5-x}$ is continuous, but not uniformly, on the interval $(0,5)$. This follows since

$$\lim f\left(5 - \frac{1}{n}\right) = \lim n = \infty$$

means we cannot define $g(5)$ unambiguously. Again the picture is helpful; while we can fill in the hole at the left endpoint ($a = 0$), the vertical asymptote at $b = 5$ means that there is no hole to fill in and thereby extend the function.



Proof. ($\Leftarrow$) Suppose g is well-defined; we leave the claim that it is continuous as an exercise, but by Theorem 19.7 it is uniformly so. Since $f = g$ on a subset $(a,b) \subseteq \operatorname{dom} g$, the same choice of δ will work for f as it does for g: f is therefore uniformly continuous.

($\Rightarrow$) Suppose f is uniformly continuous on (a,b). Let $(x_n), (y_n) \subseteq (a,b)$ be sequences converging to a. To show that $g(a)$ is unambiguously defined, we must prove that $\big(f(x_n)\big)$ and $\big(f(y_n)\big)$ are convergent, and to the same limit.

Define a sequence

$$(u_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$$

Plainly $\lim u_n = a$ since (x_n) and (y_n) have the same limit. But then (u_n) is Cauchy; by Lemma 19.9, $\big(f(u_n)\big)$ is also Cauchy and thus convergent. Since $\big(f(x_n)\big)$ and $\big(f(y_n)\big)$ are subsequences of a convergent sequence, they also converge to the same (finite!) limit.

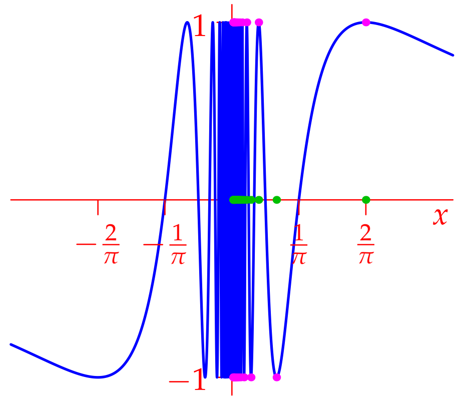
The case for $g(b)$ is similar. ∎

Examples 19.13. We finish with three related examples of continuous functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$; these will appear repeatedly as you continue to study analysis.

1. $f(x) = \sin \frac{1}{x}$ is continuous but *not uniformly so.* To see this, note that $x_n = \frac{1}{(n+\frac{1}{2})\pi}$ defines a Cauchy sequence ($\lim x_n = 0$), and yet

   $$f(x_n) = \sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$$

   is not Cauchy since it diverges by oscillation.
   Consequently, there is no way to extend f to a continuous function on any interval containing $x = 0$.



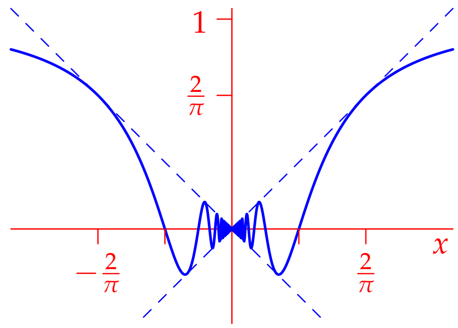
2. $f(x) = x \sin \frac{1}{x}$ is *uniformly continuous.* One way to see this is to extend the function to the origin by defining

   $$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   By the squeeze theorem, $\lim x_n = 0 \implies \lim f(x_n) = 0$, so g is well-defined and continuous on $\mathbb{R}$. By Theorem 19.11, f is uniformly continuous on any bounded interval. Moreover, the derivative

   $$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

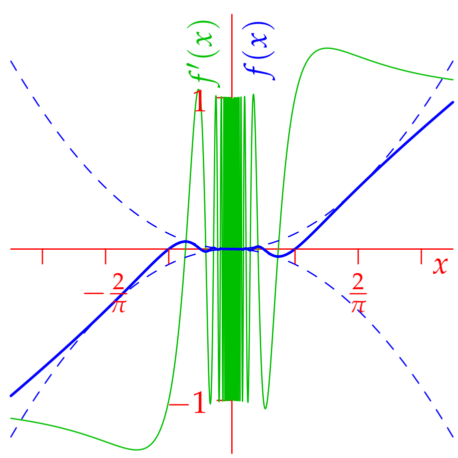
   is bounded whenever x is large; together with Exercise 6 we could use this to conclude uniform continuity of $f(x)$. Note however that $f'(x)$ is unbounded when x small ($\lim f'\left(\frac{1}{2\pi n}\right) = \lim(-2\pi n) = -\infty$) so we can't use Theorem 19.5 to conclude that f is uniformly continuous on its entire domain.



3. $f(x) = x^2 \sin \frac{1}{x}$ is also *uniformly continuous*: again extend by $g(0) = 0$. This time however, we could argue that the derivative is bounded

   $$\left|f'(x)\right| = \left|2x \sin \frac{1}{x} - \cos \frac{1}{x}\right| \leq 3$$

   since $|\sin y| \leq |y|$ for all y.
   In fact something stranger is going on. As you may verify (see Exercise 3), the *extended function g is everywhere differentiable* with $g'(0) = 0$, and yet the derivative $g'(x)$ itself is *discontinuous* at $x = 0$!



Exercises 19. 1. Decide whether each function is uniformly continuous on the given interval. Explain your answers.

(a) $f(x) = x^3$ on $[-2, 4]$ (b) $f(x) = x^3$ on $(-2, 4)$
(c) $f(x) = x^{-3}$ on $(0, 4]$ (d) $f(x) = x^{-3}$ on $(1, 4]$
(e) $f(x) = e^x$ on $(-\infty, 100)$ (f) $f(x) = e^x$ on $\mathbb{R}$

2. Prove that each function is uniformly continuous on the indicated domain by verifying the ϵ-δ property.

(a) $f(x) = 3x + 11$ on $\mathbb{R}$ (b) $f(x) = x^2$ on $[0, 3]$
(c) $f(x) = \frac{1}{x^2}$ on $[\frac{1}{2}, \infty)$ (d) $f(x) = \frac{x+2}{x+1}$ on $[0, 1]$

3. Verify the claim in Example 19.13.3 that the function $g(x)$ is differentiable at zero[41] but that the derivative $g'(x)$ is discontinuous there.

4. (a) If f is uniformly continuous on a bounded set U, prove that f is bounded on U.
(*Hint: for contradiction, assume* $\exists (x_n) \subseteq U$ *for which* $|f(x_n)| \to \infty \ldots$)
(b) Use (a) to give another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.
(c) Give an example to show that a uniformly continuous function on an *unbounded* set U could be unbounded.

5. Suppose g is defined on U and $a \in U$. Give *very brief* (one line!) arguments for the following.

(a) Prove that g is continuous at a provided

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |g(x) - g(a)| < \epsilon$$

(b) Prove that g is continuous at a provided

$$\forall (x_n) \subseteq U \setminus \{a\},\ \lim x_n = a \implies \lim g(x_n) = g(a)$$

(c) Verify that the function g defined in Theorem 19.11 is indeed continuous whenever it is well-defined.

6. (a) Suppose f is uniformly continuous on intervals U_1, U_2 for which $U_1 \cap U_2$ is non-empty. Prove that f is uniformly continuous on $U_1 \cup U_2$.
(*Hint: if* x, y *do not lie in the same interval* U_i*, choose some* $a \in U_1 \cap U_2$ *between* x *and* y)
(b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.
(c) More generally, prove that any root function $f(x) = x^{1/n}$ ($n \in \mathbb{N}$) is uniformly continuous on its domain ($\mathbb{R}$ if n is odd and $[0, \infty)$ if n is even).
(d) (Hard) Given $f(x) = x^{1/n}$, show that $\delta = \epsilon^n$ demonstrates uniform continuity when n is even and $\delta = \left(\frac{\epsilon}{2}\right)^n$ when n is odd.
(*Hint: use the binomial theorem to prove that* $0 \le y < x + \delta \implies y^{1/n} < x^{1/n} + \delta^{1/n}$)

[41] Use the definition $g'(0) = \lim\limits_{x \to 0} \frac{g(x) - g(0)}{x - 0}$. Limits of functions are covered formally in the next section (course!), but you should be familiar with the idea from elementary calculus.