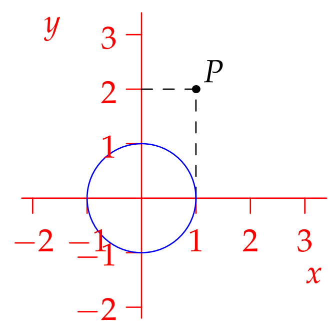
3 Analytic Geometry

Geometry in the style of Euclid and Hilbert is *synthetic*: axiomatic, without co-ordinates or explicit formulæ for length, area, volume, etc. By contrast, the practice of elementary geometry nowadays is typically *analytic*: reliant on co-ordinates & algebra, vectors. The critical invention was the *axis*, developed by René Descartes and Pierre de Fermat in the early 1600s: a fixed reference ruler against which objects can be measured using *co-ordinates.*

3.1 The Cartesian Co-ordinate System

Since Cartesian geometry (*Descartes'* geometry) should be familiar, we merely sketch the core ideas.



- Perpendicular *axes* meet at the *origin* O.
- The *co-ordinates* of a point are measured by projecting onto the axes; since these are real numbers we denote the set of these

  $$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

  E.g., P has co-ordinates $(1, 2)$, we usually just write $P = (1, 2)$.
- Algebra is introduced via *addition* and *scalar multiplication*

  $$P + Q = (p_1, p_2) + (q_1, q_2) = (p_1 + q_1, p_2 + q_2) \qquad \lambda P = (\lambda p_1, \lambda p_2)$$

- The *length* of a segment uses Pythagoras' Theorem

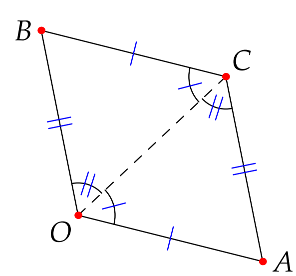
  $$d(P, Q) = |PQ| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

  In the picture $|OP| = \sqrt{1^2 + 2^2} = \sqrt{5}$. As in Section 2.5, segments are congruent if and only if they have the same length.
- *Curves* are defined using *equations.* E.g. $x^2 + y^2 = 1$ describes a circle.

Analytic geometry was conceived as a computational toolkit built on top of Euclid. At first, mathematicians felt the need to justify analytic arguments synthetically lest no-one believe their work.[14] Synthetic geometry is not without its benefits, but its study has increasingly become a fringe activity; co-ordinates are just too useful to ignore.

We may therefore assume anything from Euclid and mix strategies as appropriate. To see this at work, consider a simple result.

Lemma 3.1. *Non-collinear points $O = (0, 0)$, $A = (x, y)$, $B = (v, w)$ and $C := (x + v, y + w)$, form a parallelogram $OACB$.*



Proof. Opposite sides have the same length ($|BC| = \sqrt{x^2 + y^2} = |OA|$, etc.) and are thus congruent. SAS shows $\triangle OAC \cong \triangle CBO$. Euclid's discussion of alternate angles (pages 10–11) forces opposite sides to be parallel. ∎

[14] This attitude persisted for some time. For instance, when Issac Newton published his groundbreaking *Principia* in 1687, his presentation was largely synthetic, even though he had used co-ordinates in his derivations.

Lemma 3.2. *The points X_t on the line $\overleftrightarrow{PQ}$ are in 1–1 correspondence with the real numbers via*

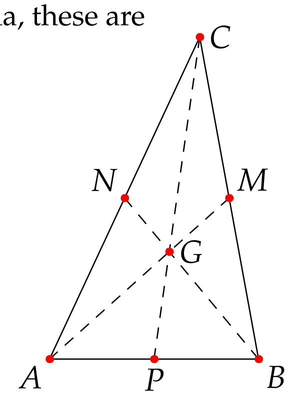
$$X_t = P + t(Q - P) = (1 - t)P + tQ$$

Moreover, $d(P, X_t) = |t|\,|PQ|$ so that t measures the (signed) distance along the line.

The proof is an exercise. As an example of how easy it can be to work in analytic geometry, we repeatedly apply the Lemma to re-establish a famous result.

Theorem 3.3. *The medians of a triangle meet at a point $2/3$ of the way along each median.*

Proof. Given $\triangle ABC$, label the midpoints of each side as shown. By the Lemma, these are



$$M = \frac{1}{2}(B + C), \quad N = \frac{1}{2}(A + B), \quad P = \frac{1}{2}(A + C)$$

The point $\frac{2}{3}$ of the way along median $\overline{AM}$ is then

$$A + \frac{2}{3}(M - A) = A + \frac{2}{3}(B + C - 2A) = \frac{1}{3}(A + B + C)$$

By symmetry (check directly if you like!), this is also the point $\frac{2}{3}$ of the way along the other two medians.

The three points are therefore identical: the medians meet at the centroid $G = \frac{1}{3}(A + B + C)$. ∎

Compare this to Exercise 2.5.8 where we used Ceva's Theorem!

Exercises 3.1. 1. By completing the square, identify the curve described by the equation

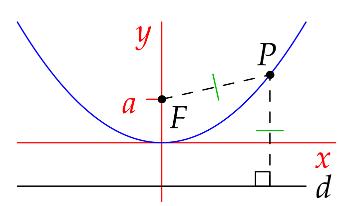
$$x^2 + y^2 - 4x + 2y = 10$$

2. (a) Perform a pure co-ordinate proof of Theorem 3.3. For simplicity, arrange the triangle so that $A = (0, 0)$ is the origin, and B points along the positive x-axis.

   (b) Descartes and Fermat did not have a fixed perpendicular second axis! Their approach was equivalent to choosing a second axis at an angle which made the problem as simple as possible.
   Given $\triangle ABC$, let A be the origin and choose axes which point along the edges $\overline{AB}$ and $\overline{BC}$. What are the co-ordinates of B and C with respect to these axes? Now give an even simpler proof of the centroid theorem.

3. Prove Lemma 3.2.

4. A *parabola* is a curve whose points are equidistant from a fixed point F, the *focus*, and a fixed line d (the *directrix*). Choose axes as shown in the picture so that $F = (0, a)$ and d has equation $y = -a$. Find the equation of the parabola.



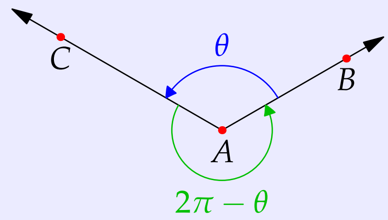
3.2 Angles and Trigonometry

Angles are defined differently to Section 2.5, though the approach should feel familiar.

Definition 3.4. Suppose A, B, C are distinct points in the plane. Take any circular arc centered at A and define the *radian measure*

$$\measuredangle BAC := \frac{\text{arc-length}}{\text{radius}} \in [0, 2\pi)$$

where arc-length is measured *counter-clockwise* from $\overrightarrow{AB}$ to $\overrightarrow{AC}$.



Since arc-length scales with radius, the definition is independent of the radius of the circular arc. It is important to appreciate the difference between angle measures in our two geometries.

Euclidean geometry All angles $< 180°$. Reversed legs $\rightsquigarrow$ *congruent* angles and *same degree measure*:

$$\angle CAB \cong \angle BAC \iff m\angle CAB = m\angle BAC$$

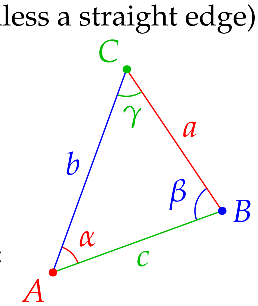
Analytic geometry Reflex angles exist ($\geq \pi$). Reversed legs $\rightsquigarrow$ *different radian measure*:

$$\measuredangle CAB = 2\pi - \theta = 2\pi - \measuredangle BAC \neq \measuredangle BAC \qquad \text{(unless a straight edge)}$$

In the picture, $\measuredangle CAB$ is not the radian measure (θ) of $\angle CAB$! However,

Angles congruent $\iff$ radian measures equal and $< \pi$

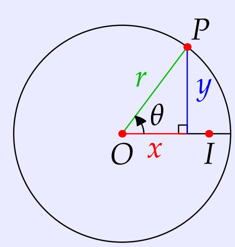
As such, it is common to label angles in a triangle by their radian measure; standard convention is shown: e.g., (A, a, α) for (point,length,angle).



Definition 3.5 (Trigonometric Functions). Let O be the origin and $I = (1, 0)$. Let $P = (x, y)$ lie on a circle of radius r and $\theta = \measuredangle IOP$. We define:

$$\cos\theta := \frac{x}{r} \qquad \sin\theta := \frac{y}{r} \qquad \tan\theta := \frac{y}{x} \quad (x \neq 0)$$

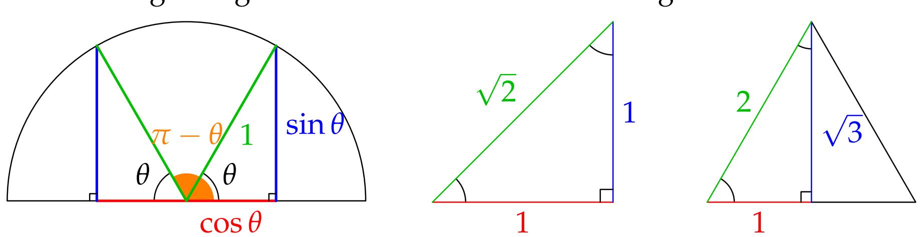
AAA similarity (Thm. 2.42) says these are well-defined, independent of r.



Example 3.6. Basic trig identities should be obvious from the picture: e.g.,

$$\cos^2\theta + \sin^2\theta = 1 \text{ (Pythagoras!)} \quad \text{and} \quad \sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right)$$

What well-known facts regarding sine and cosine do the following illustrate?

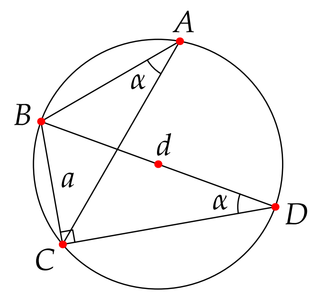


Solving Triangles A triangle is described by six values: three side lengths and three angle measures. Euclid's triangle congruence theorems (SAS, ASA, SSS, SAA) say that three of these in suitable combination is enough to recover the rest. In analytic geometry, these calculations typically use the sine and cosine rules.

Theorem 3.7. *Label the sides/angles of $\triangle ABC$ following the standard convention (page 37):*

Sine Rule *If d is the diameter of the circumcircle (Defn. 2.30), then* $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = \dfrac{1}{d}$

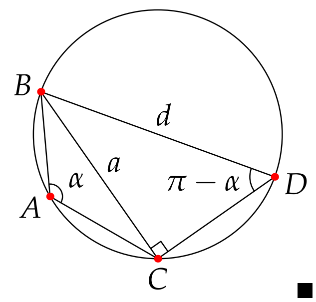
Cosine Rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$



Proof. We prove the sine rule and leave the cosine rule as an exercise. Everything relies on Corollary 2.32. Draw the circumcircle of $\triangle ABC$. Construct $\triangle BCD$ with diameter $\overline{BD}$; this is right-angled at C by Thales' Theorem. There are two cases:

1. If A lies on the same side of $\overleftrightarrow{BC}$ as D, then A and D share the same arc, whence $\measuredangle BDC = \alpha$ and

$$a = d\sin\measuredangle BDC = d\sin\alpha$$

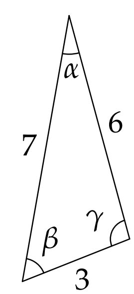


2. If A lies on the opposite side, then the quadrilateral $ABDC$ lies on a circle. Opposite angles at A, D are supplementary, whence

$$\sin\alpha = \sin(\pi - \alpha) = \sin\measuredangle BDC = \frac{a}{d}$$

The two other angle-side combinations follow by permutation. ■

Examples 3.8. 1. The SSS congruence corresponds to solving a triangle using the cosine rule. For instance, the given triangle has angles

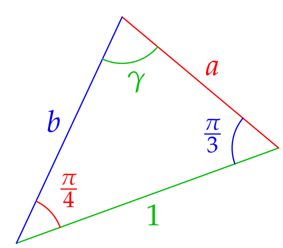


$$\alpha = \frac{6^2+7^2-3^2}{2\cdot 6\cdot 7} = \cos^{-1}\frac{19}{21} \approx 25° \qquad \beta = \frac{3^2+7^2-6^2}{2\cdot 3\cdot 7} = \cos^{-1}\frac{11}{21} \approx 58°$$

$$\gamma = \frac{3^2+6^2-7^2}{2\cdot 3\cdot 6} = \cos^{-1}\frac{-1}{9} \approx 96°$$

Once you have α, you could alternatively switch to the sine rule to find β, before computing $\gamma = \pi - \alpha - \beta$.

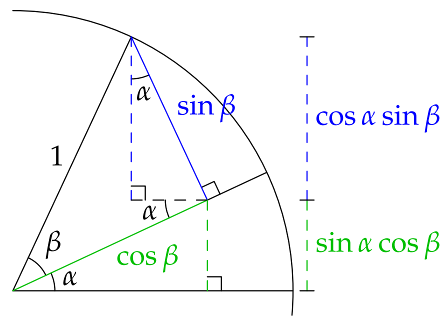
2. To solve a triangle with data corresponding to the ASA congruence, find the remaining angle $\gamma = \pi - \frac{\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$ and apply the sine rule



$$\frac{\sin\frac{\pi}{4}}{a} = \frac{\sin\frac{\pi}{3}}{b} = \sin\frac{5\pi}{12} = \cos\frac{\pi}{12} \implies a = \frac{1}{\sqrt{2}\cos\frac{\pi}{12}} \approx 0.732$$

$$b = \frac{\sqrt{3}}{2\cos\frac{\pi}{12}} \approx 0.897$$

Multiple-angle formulæ The picture provides a very simple proof of the expressions



$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

at least when $\alpha+\beta < \frac{\pi}{2}$. A little algebraic manipulation produces the double-angle and difference formulæ, and verifies that these hold for all possible angle inputs.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Exercises 3.2. 1. A triangle has angle of $\frac{2\pi}{3}$ radians between sides of lengths 2 and $\sqrt{3}-1$. Find the length of the remaining side, and the remaining angles.

2. Describe how to solve a triangle given data in line with the SAA congruence theorem.

3. Two measurements for the height of a mountain are taken at sea level 5000 ft apart in a line pointing away from the mountain. The angles of elevation to the mountain top from the horizontal are 15° and 13° respectively. What is the height of the mountain?

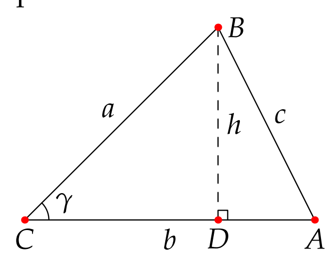
4. Use a multiple angle formula to find an exact value for $\cos\frac{\pi}{12}$ and thus exact values for the side lengths of the triangle in Exercise 3.8.2.

5. The area of a triangle is $\frac{1}{2}$(base)·(height). By using each side of the triangle alternately as the 'base,' find an alternative proof of the sine rule without the relationship to the circumcircle.

6. (a) By dropping a perpendicular from B to $\overleftrightarrow{AC}$ at D, construct a proof of the cosine rule.

   (*Hint: apply Pythagoras' to the two right-triangles*)

   (b) Is your argument valid if D is not interior to $\overline{AC}$?



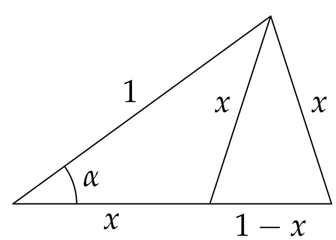
7. The *dot product* of $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is $A \cdot B := a_1b_1 + a_2b_2$. Apply the cosine rule to $\triangle OAB$ to prove that

$$A \cdot B = |OA|\,|OB|\cos\measuredangle AOB$$

8. Derive the multiple-angle formula for $\sin(\alpha-\beta)$.

   (*Remember that* $0 \le \alpha, \beta, \alpha-\beta < 2\pi$ *so you can't simply switch the sign of* β*!*)

9. Given the arrangement pictured, find x, the radian-measure α and the exact value of $\cos\alpha$.
   (*Hint: first show that you have similar isosceles triangles*)



3.3 Isometries

At the heart of elementary geometry is *congruence*, the idea that geometric figures can be essentially the same without necessarily being equal. In analytic geometry, congruence may be described algebraically using *functions*. This follows from the idea that two segments have the same length if and only if they are congruent.

Definition 3.9. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a *(Euclidean) isometry* if it preserves lengths:[15]

$$\forall P, Q \in \mathbb{R}^2,\ d\big(f(P), f(Q)\big) = |PQ|$$

Two figures (segments, angles, triangles, etc.) are *congruent* precisely when there is an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ mapping one to the other.

Example 3.10. We check that the map $f(x,y) = \frac{1}{5}(3x+4y, 4x-3y) + (3,1)$ is an isometry. If $P = (x,y)$ and $Q = (v,w)$, then

$$d\big(f(P), f(Q)\big)^2 = \left(\frac{3v+4w-3x-4y}{5}\right)^2 + \left(\frac{4v-3w-4x+3y}{5}\right)^2$$
$$= \frac{3^2+4^2}{5^2}\big((v-x)^2 + (w-y)^2\big) = |PQ|^2$$

Isometric segments are certainly congruent. We should make sure the same holds for angles.

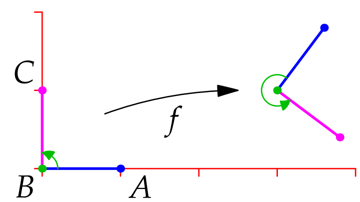
Lemma 3.11. *Isometries preserve (non-oriented) angles: if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then*

$$\angle PQR \cong \angle f(P) f(Q) f(R)$$

Proof. Since f is an isometry, the sides of $\triangle PQR$ and $\triangle f(P)f(Q)f(R)$ are mutually congruent in pairs. The SSS triangle congruence theorem says that the angles are also mutually congruent. ∎

Example (3.10, cont). Warning: Isometries can *reverse orientation*! In the picture,

$$\measuredangle ABC = \frac{\pi}{2} \quad \text{but} \quad \measuredangle f(A)f(B)f(C) = \frac{3\pi}{2} = 2\pi - \measuredangle ABC$$



Our next task is to confirm our intuition that isometries are rotations, reflections and translations. Given an isometry f, define $g(X) = f(X) - f(O)$, where O is the origin. Then g is an isometry

$$g(P) - g(Q) = f(P) - f(Q) \implies d\big(g(P), g(Q)\big) = d\big(f(P), f(Q)\big) = |PQ|$$

which moreover *fixes the origin*: $g(O) = O$. We conclude that every isometry f is the composition of an *origin-preserving* isometry g followed by a *translation* "+C:"

$$f(X) = g(X) + C$$

[15] In ancient Greek, *iso-metros* is literally *same measure (length/distance)*.

It thus suffices to describe the origin-preserving isometries g. For these, we make two observations.

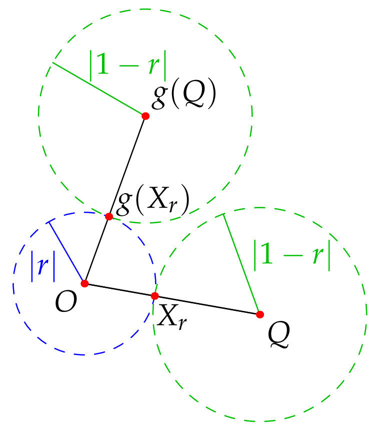
1. Suppose $|OQ| = 1$ and let $X_r = rQ$ for some $r \in \mathbb{R}$. Then

   - $g(X_r)$ is a distance $|r| = |OX_r|$ from the origin $O = g(O)$.
   - $g(X_r)$ is a distance $|1-r| = |QX_r|$ from $g(Q)$.

   $g(X_r)$ therefore lies on the intersection of two circles, which intersect at a single point: we conclude that

   $$g(rQ) = rg(Q)$$

   The picture shows the case $0 < r < 1$, where the uniqueness of intersection follows from $1 = |r| + |1-r|$.



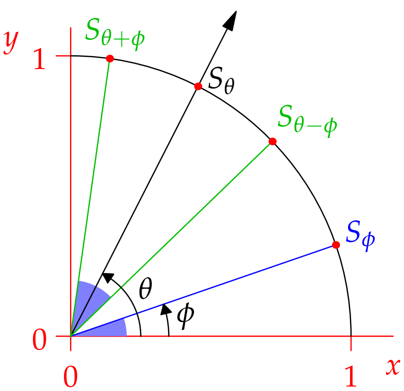
2. $g(1,0)$ lies on the unit circle and therefore has the form

   $$g(1,0) = S_\theta := (\cos\theta, \sin\theta)$$

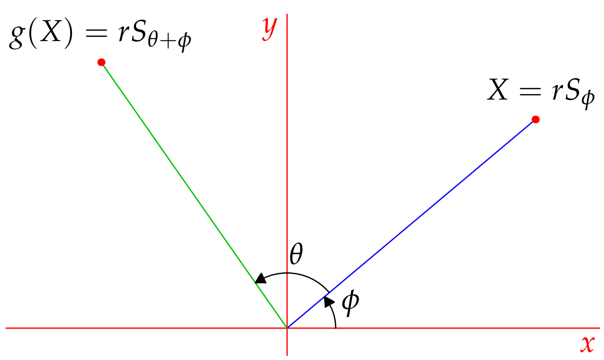
   for some $\theta \in [0, 2\pi)$. By preservation of length and angle (Lemma 3.11), any other point $S_\phi = (\cos\phi, \sin\phi)$ on the unit circle must therefore be mapped to one of two points

   $$g(S_\phi) = S_{\theta\pm\phi} = (\cos(\theta\pm\phi), \sin(\theta\pm\phi))$$

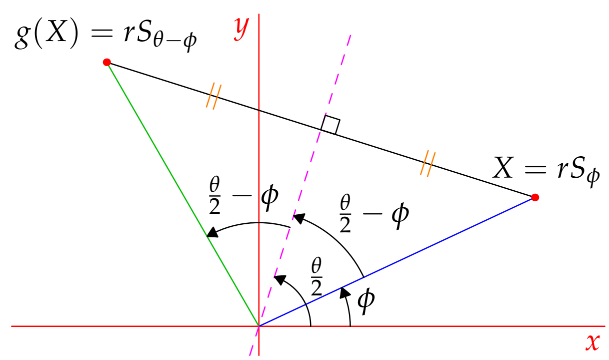
   The angle ϕ is transferred to one side of the ray $\overrightarrow{OS_\theta}$.



Putting these together by writing $X = rS_\phi = (r\cos\phi, r\sin\phi)$ in polar co-ordinates, we conclude that g has one of two forms:



Rotation counter-clockwise by θ



Reflection across the line making angle $\frac{\theta}{2}$ with positive x-axis

Theorem 3.12. *Every isometry of $\mathbb{R}^2$ has the form*

$$f(X) = g(X) + C$$

where g is either a rotation about the origin, or a reflection across a line through the origin.

Calculating with isometries

This benefits from column-vector notation and matrix multiplication. Writing $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} r\cos\phi \\ r\sin\phi \end{smallmatrix}\right)$ for the position vector of $X_r = (x, y) = rS_\phi$ and applying the multiple-angle formulæ, rotation becomes

$$g(\mathbf{x}) = r\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix} = r\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}$$

For reflections, the sign of the second column is reversed: $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$. Every isometry therefore has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ where A is an *orthogonal matrix*.[16]

Examples 3.13. 1. We revisit Example 3.10 in matrix format:

$$f(\mathbf{x}) = \frac{1}{5}\begin{pmatrix} 3x+4y \\ 4x-3y \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

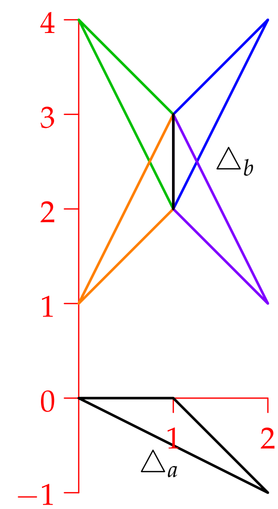
Since $\frac{\sin\theta}{\cos\theta} = \frac{4/5}{3/5} = \frac{4}{3}$, we see that its effect is to *reflect* across the line through the origin making angle $\frac{1}{2}\tan^{-1}\frac{4}{3} \approx 26.6°$ with the positive x-axis, before *translating* by $(3, 1)$.

2. $\triangle_a$ has vertices $(0, 0)$, $(1, 0)$, $(2, -1)$ and is congruent to $\triangle_b$, two of whose vertices are $(1, 2)$ and $(1, 3)$. Find all isometries transforming $\triangle_a$ to $\triangle_b$ and the location(s) of the third vertex of $\triangle_b$.

Let $f = A\mathbf{x} + \mathbf{c}$ be the isometry. Since $d\big((1, 2), (1, 3)\big) = 1$ these points must be the images under f of $(0, 0)$ and $(1, 0)$. There are *four* distinct isometries:

Cases 1, 2: If $f(0, 0) = (1, 2)$ and $f(1, 0) = (1, 3)$, then $\mathbf{c} = f\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \implies A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies A = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}$$



for some a_{12}, a_{22}. Since A is orthogonal, the options are $A = \left(\begin{smallmatrix} 0 & \mp 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we obtain two possible isometries:

- $f_1(\mathbf{x}) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\mathbf{x} + \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ rotates by 90°, then translates by $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$.
- $f_2(\mathbf{x}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\mathbf{x} + \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ reflects across $y = x$, then translates by $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$.

The third point of $\triangle_b$ is $f_1(2, -1) = (2, 4)$ or $f_2(2, -1) = (0, 4)$.

Cases 3, 4: $f(0, 0) = (1, 3)$ and $f(1, 0) = (1, 2)$ results in two further isometries f_3 and f_4. The details are an exercise.

All four possible triangles $\triangle_b$ are drawn in the picture.

In 1872, Felix Klein suggested that the geometry of a set is the study of its *invariants*: properties preserved by its *group* of structure-preserving transformations. In Euclidean geometry, this is the group of *Euclidean isometries* (Exercise 9). Klein's approach provided a method for analyzing and comparing the non-Euclidean geometries beginning to appear in the late 1800s. By the mid 1900s, the resulting theory of *Lie groups* had largely classified classical geometries. Klein's algebraic approach remains dominant in modern mathematics and physics research.

[16] An orthogonal matrix satisfies $A^TA = I$. All such have the form $\left(\begin{smallmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \mp b \\ b & \pm a \end{smallmatrix}\right)$ where $a^2 + b^2 = 1$.

Exercises 3.3. 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry, "reflect across the line through the origin making angle $\frac{\pi}{3}$ with the positive x-axis." Find a 2×2 matrix A such that $f(\mathbf{x}) = A\mathbf{x}$.

2. Describe the geometric effect of the isometry $f(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

3. Find the remaining isometries f_3, f_4 and the third points of $\triangle_b$ in Exercise 3.13.2.

4. Find the reflection of the point $(4,1)$ across the line making angle $\frac{1}{2}\tan^{-1}\frac{12}{5} \approx 33.7°$ with the positive x-axis.

   (*Hint: if* $\tan\theta = \frac{12}{5}$, *what are* $\cos\theta$ *and* $\sin\theta$*?*)

5. An origin-preserving isometry $f(\mathbf{v}) = A\mathbf{v}$ moves the point $(7,4)$ to $(-1,8)$.

   (a) If f is a rotation, find the matrix A. Through what angle does it rotate?

   (b) If f is a reflection, find the matrix A. Across which line does it reflect?

6. Let $ABCD$ be the rectangle with vertices $A = (0,0)$, $B = (4,0)$, $C = (4,3)$, $D = (0,3)$. Suppose an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps $ABCD$ to a new rectangle $PQRS$ where

   $$P = f(A) := (2,4) \quad \text{and} \quad R = f(C) := (2,9)$$

   Find all possible isometries f and the remaining points $Q = f(B)$ and $S = f(D)$.

7. (a) If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\mathbf{p}$ is constant, explain why $f(\mathbf{x}) = A(\mathbf{x} - \mathbf{p}) + \mathbf{p} = A\mathbf{x} + (I - A)\mathbf{p}$ rotates by θ around the point with position vector $\mathbf{p}$.

   (b) Suppose $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ rotates the plane around the point $P = (-2,1)$ by an angle $\theta = \tan^{-1}\frac{3}{4}$. Find A and $\mathbf{c}$.

   (c) Suppose f rotates by θ around $\mathbf{p}$ and g rotates by ϕ around $\mathbf{q}$ where θ, ϕ are non-zero.

      i. If $\theta + \phi \neq 2\pi$, show that $f \circ g$ is a rotation: by what angle and about which point?

      ii. What happens instead if $\theta + \phi = 2\pi$?

8. Make an argument involving circle intersections (see page 41) to prove that for any isometry f,

   $$f\big((1-t)P + tQ\big) = (1-t)f(P) + tf(Q)$$

9. Throughout this question, we use the notation $f_{A,\mathbf{c}} : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{c}$.

   (a) Prove that isometries obey the composition law $f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}} = f_{AB,\mathbf{c}+A\mathbf{d}}$.

   (b) Find the inverse function of the isometry $f_{A,\mathbf{c}}$. Otherwise said, if $f_{A,\mathbf{c}} \circ f_{C,\mathbf{d}} = f_{I,\mathbf{0}}$, where I is the identity matrix, how do $B, \mathbf{d}$ depend on $A, \mathbf{c}$?

   (c) Verify that the following composition $f_{A,\mathbf{c}} \circ f_{I,\mathbf{d}} \circ f_{A,\mathbf{c}}^{-1}$ is a translation.

   Part (a) can be written using augmented matrices: $(A \,|\, \mathbf{c})(B \,|\, \mathbf{d}) := (AB \,|\, \mathbf{c} + A\mathbf{d})$.
   If you know group theory, parts (a) and (b) are the closure and inverse properties of the group of Euclidean isometries E. Part (c) says the translations T form a normal subgroup. We may therefore write E as a semi-direct product of T and the orthogonal group of origin-preserving isometries

   $$E = T \rtimes O_2(\mathbb{R})$$