Math 162A - Introduction to Differential Geometry

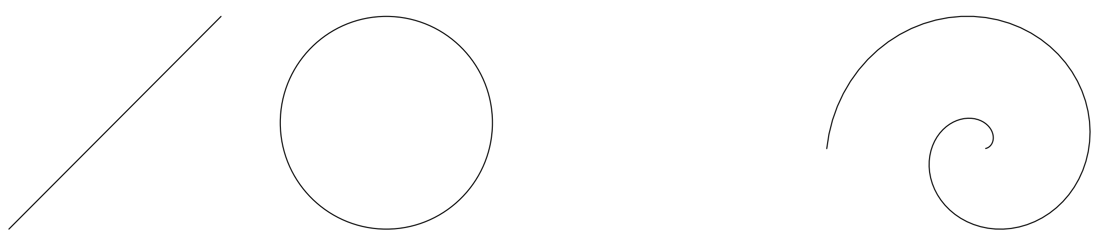
Neil Donaldson

Winter 2024

Introduction

Classical Differential Geometry is the study of curves and surfaces in the plane and three-dimensional space using *multi-variable calculus, linear algebra* & *differential equations*. At a more advanced level, topology, analysis and abstract algebra become more important, but none of this is required for our treatment.

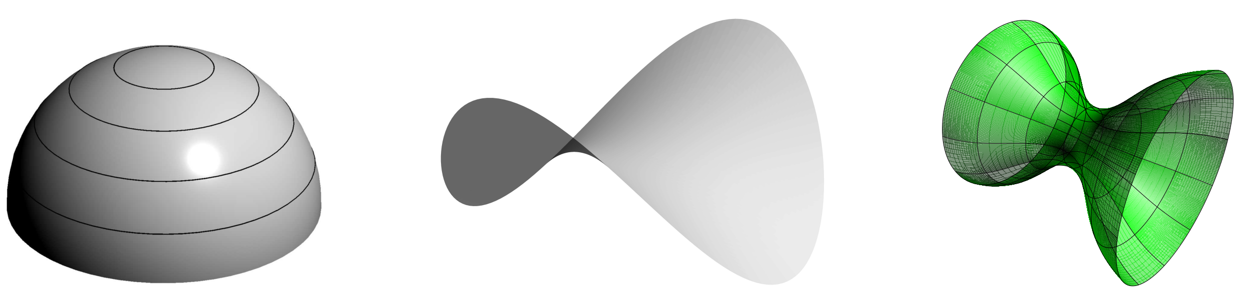
Of particular interest is the notion of *curvature*: a measure of the 'bendiness' of a curve or surface. Intuitively, a straight line should have zero curvature, while the curvature of a circle should vary inversely as the radius: a very large circle should have very small curvature.



Zero curvature   Small curvature   Larger curvature   Variable curvature

Understanding and quantifying this concept for more complicated curves is our first important goal. The rough idea is to imagine a curve as a roller-coaster along which you travel at a constant speed; the curvature is then the *force* necessary to keep you travelling along the curve.

Curvature is a more difficult concept for surfaces. In particular, we will hunt for quantities which measure how much a surface appears to be dome- or saddle-shaped.



Dome-shaped   Saddle-shaped   More complicated

The third surface is saddle-shaped near the narrow neck and dome-shaped away from it.

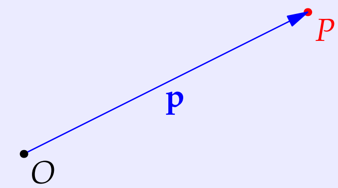
1 Curves in Euclidean Space

1.1 Euclidean Space, Tangent Vectors & Regular Curves

We begin by refreshing and developing a little notation.

Definition 1.1. The set of *n-tuples* of real numbers is denoted $\mathbb{R}^n$.
An element can be thought of either as a *point* P or as its *position vector* $\mathbf{p} = \overrightarrow{OP}$ connecting the origin $O = (0, \ldots, 0)$ to P.
In co-ordinates, points are typically written as row vectors

$$P = (p_1, \ldots, p_n) \text{ where each } p_i \in \mathbb{R}$$



P has position vector $\mathbf{p}$

For vectors, either row or column vector notation is acceptable.
For each i, the *co-ordinate function* $x_i : \mathbb{R}^n \to \mathbb{R}$ returns the i^{th} co-ordinate of a point: $x_i(P) = p_i$.

Since the focus of the course is curves and surfaces in 2- and 3-dimensions, we'll mostly restrict to $n \leq 3$ and quote theorems in this context.[1] We typically use x, y, z for the standard (rectangular) co-ordinate functions

$$x(P) = p_1, \quad y(P) = p_2, \quad z(P) = p_3$$

You should be comfortable with this notation from previous classes and, in particular, with *partial derivatives* of functions defined in terms of the co-ordinate functions x, y, z.

Examples 1.2. 1. If $P = (3, 1, 5) \in \mathbb{R}^3$, then $y(P) = 1$.

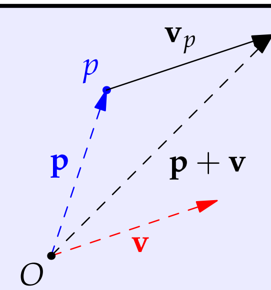
2. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f = x^3 \sin(yz)$ has partial derivatives

$$\frac{\partial f}{\partial x} = 3x^2 \sin(yz) \qquad \frac{\partial f}{\partial y} = x^3 z \cos(yz) \qquad \frac{\partial f}{\partial z} = x^3 y \cos(yz)$$

A *vector* is a directed line segment joining two *points*. We've already seen the *position vector* of a point P, namely $\overrightarrow{OP}$. In differential geometry it is crucial to distinguish the vectors *based* at a given point.

Definition 1.3. A *tangent vector* $\mathbf{v}_p$ is a *pair* of elements of $\mathbb{R}^3$: a *base point* p and a *direction* $\mathbf{v}$. It is the directed line segment from the point with position vector $\mathbf{p}$ to the point with position vector $\mathbf{p} + \mathbf{v}$.

The *tangent space at* p is the set $T_p\mathbb{R}^3$ of all tangent vectors based at p. At each point, $\mathbb{R}^3$ has a *different tangent space*!



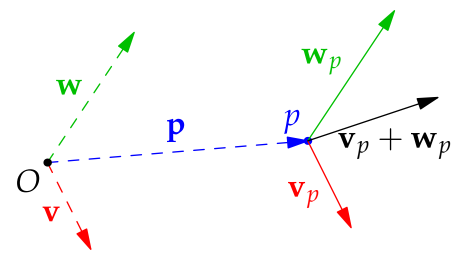
Be aware that $\mathbf{v}_p = \mathbf{w}_q \iff p = q$ *and* $\mathbf{v} = \mathbf{w}$: the same *direction* at different *base points* means a different tangent vector!

[1] For simplicity's sake, we'll almost always state theorems in $\mathbb{R}^3$. The majority are valid in $\mathbb{R}^n$ with a simple notational modification $\{x, y, z\} \rightsquigarrow \{x_1, \ldots, x_n\}$. For $\mathbb{R}^2$ just delete $z = x_3$; many results even make sense in $\mathbb{R} = \mathbb{R}^1$!

The tangent space at p is suitably named, for it is indeed a *vector space*: to add tangent vectors $\mathbf{v}_p, \mathbf{w}_p \in T_p\mathbb{R}^3$, simply sum the *direction vectors*

$$\mathbf{v}_p + \mathbf{w}_p := (\mathbf{v}+\mathbf{w})_p \qquad (*)$$

Scalar multiplication is similar: $\lambda \mathbf{v}_p := (\lambda \mathbf{v})_p$.



In chapter 2 we will see a more abstract discussion of tangent vectors, vector fields, and their application.

Euclidean Space: $\mathbb{E}^n$ versus $\mathbb{R}^n$

To describe curves and surfaces in differential geometry, we *parametrize* using functions.

Example 1.4. There are multiple ways to do this for a given curve: for instance

$$\mathbf{x} : (-\pi, \pi] \to \mathbb{R}^2 : t \mapsto \big(\cos t, \sin t\big) \quad \text{and} \quad \mathbf{y} : \mathbb{R} \to \mathbb{R}^2 : s \mapsto \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}\right)$$

both parametrize (most of) the unit circle in the plane ($\mathbf{y}$ ignores the point $(-1, 0)$).

Plainly the *codomain* $\mathbb{R}^2$ is where the geometric action is: in the above we have the same circle, and concepts such as *length* and *angle* can be measured. This extra structure motivates us to distinguish the codomain with new notation.

Definition 1.5. *Euclidean space* $\mathbb{E}^n$ is $\mathbb{R}^n$ equipped with the usual *dot product*. Specifically in $\mathbb{E}^3$:

The *dot product* of $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{p} \cdot \mathbf{q} = \mathbf{p}^T \mathbf{q} = (p_1\ p_2\ p_3) \left(\begin{smallmatrix} q_1 \\ q_2 \\ q_3 \end{smallmatrix}\right) = p_1q_1 + p_2q_2 + p_3q_3$

The *length* of $\mathbf{p}$ is $||\mathbf{p}|| = \sqrt{\mathbf{p}\cdot\mathbf{p}} = \sqrt{p_1^2 + p_2^2 + p_3^2}$

The *angle* θ between $\mathbf{p}$ and $\mathbf{q}$ satisfies $\cos\theta = \dfrac{\mathbf{p}\cdot\mathbf{q}}{||\mathbf{p}||\,||\mathbf{q}||}$

Vectors are *orthogonal*/*perpendicular* if $\mathbf{p}\cdot\mathbf{q} = 0$, equivalently $\theta = \frac{\pi}{2}$; we write $\mathbf{p} \perp \mathbf{q}$.

Curves in $\mathbb{E}^2$ and $\mathbb{E}^3$

This course is primarily concerned with *functions* $\mathbf{x} : U \subseteq \mathbb{R}^m \to \mathbb{E}^n$. In particular:

Plane curves: $m = 1$ and $n = 2$; for example the above circle.

Spacecurves: $m = 1$ and $n = 3$; we'll see several momentarily.

Surfaces: $m = 2$ and $n = 3$. For instance, the parametrization $\mathbf{x} : \mathbb{R}^2 \mapsto \mathbb{E}^3 : (u, v) \mapsto (u, v, u^2 + v^2)$ of a *paraboloid* should be familiar.

Surfaces are the focus of Chapter 3. It is now time for the formal definition of a curve.

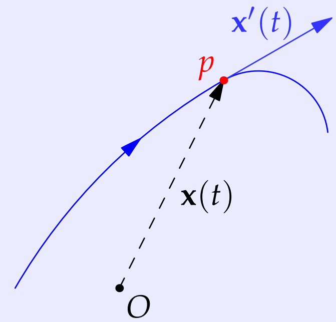
Definition 1.6. A *(smooth parametrized) curve* is a function, $\mathbf{x} : I \to \mathbb{E}^3$, $\mathbf{x}(t) = \big(x(t), y(t), z(t)\big)$, defined on an interval I and whose components x, y, z are infinitely differentiable[2] everywhere on I. Its *derivative* (also *velocity* or *tangent vector*) is denoted

$$\mathbf{x}'(t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \big(x'(t), y'(t), z'(t)\big)$$

The curve's *speed* is the continuous scalar function

$$v(t) = ||\mathbf{x}'(t)|| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

A curve is *regular* if its *tangent vector* $\mathbf{x}'(t)$ is everywhere non-zero.



In the context of Definitions 1.1 and 1.3, note that for each $t \in I$:

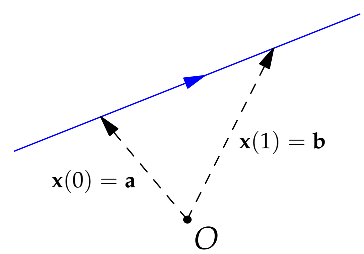
$\mathbf{x}(t)$ is a *position vector* whose nose describes the location of a point on the curve.

$\mathbf{x}'(t) \in T_p\mathbb{E}^3$ is a *tangent vector* based at the point p with position vector $\mathbf{x}(t)$.

A parametrized curve has an *orientation* (indicated by the blue arrow): as t increases along the interval I, the point $\mathbf{x}(t)$ moves in a particular direction along the curve.

Examples 1.7. *Straight line*: The line through points with position vectors $\mathbf{a}, \mathbf{b}$ may be parametrized by

$$\mathbf{x}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1-t)\mathbf{a} + t\mathbf{b}$$



The tangent vector at $\mathbf{x}(t)$ is the constant $\mathbf{x}'(t) = \mathbf{b} - \mathbf{a}$ and the parametrization has constant speed $||\mathbf{b} - \mathbf{a}||$. For instance,

$$\mathbf{x}(t) = \big(2 + t, 3 - 2t\big)$$

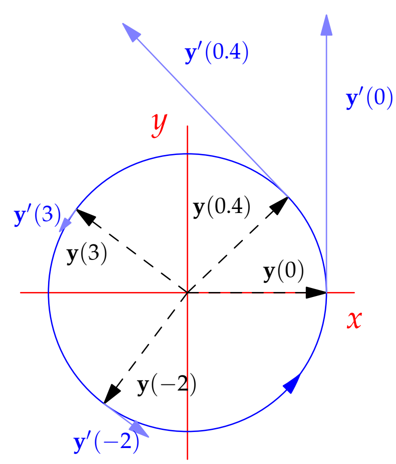
has constant velocity $\mathbf{x}'(t) = (1, -2)$ and speed $v(t) = \sqrt{5}$.

Circle (Example 1.4) The parametrization $\mathbf{x}(t) = \big(\cos t, \sin t\big)$ has velocity $\mathbf{x}'(t) = \big(-\sin t, \cos t\big)$ and constant speed $v(t) = 1$.

By contrast, $\mathbf{y}(s) = \frac{1}{1+s^2}\big(1 - s^2, 2s\big)$ has non-constant speed

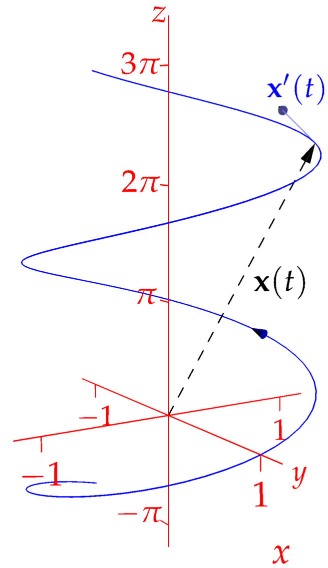
$$\mathbf{y}'(s) = \frac{2}{(1+s^2)^2}\big(-2s, 1 - s^2\big) \qquad v(s) = \frac{2}{1+s^2}$$

A particle moves quickest at $s = 0$ when $v(0) = 2$ and the speed tends to zero as $s \to \pm\infty$ (see the linked animation).

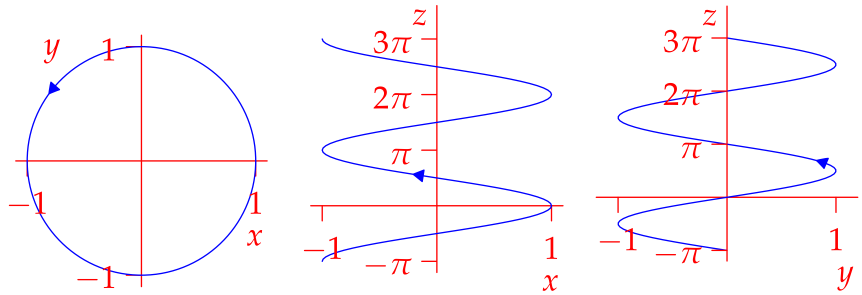


[2]The meaning of *smooth* depends on the author: at a minimum it means that x, y, z must be *differentiable with continuous derivative.* We take the maximal approach for simplicity.

Helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ parametrizes a *helix* (ascending spiral).

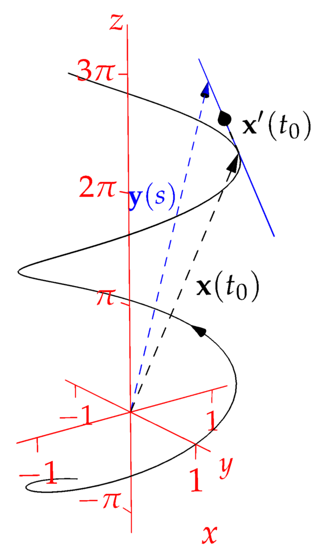


To help visualize this, imagine sitting on top of the z-axis and looking down; you'd see its horizontal projection $t \mapsto (\cos t, \sin t)$ (a counter-clockwise circle). Since $z(t) = t$, the curve moves upwards at constant speed. One can similarly project onto the xz- and yz-planes.



The tangent vector at $\mathbf{x}(t)$ is $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$ and the speed is constant $v(t) = \sqrt{2}$.

Tangent Line Let $\mathbf{x} : I \to \mathbb{E}^3$ be regular and $t_0 \in I$ be fixed. The *tangent line at* $\mathbf{x}(t_0)$ is simply the straight line through the point with position vector $\mathbf{x}(t_0)$ oriented in the direction of the tangent vector $\mathbf{x}'(t_0)$. It is itself a parametrized curve, $\mathbf{y} : \mathbb{R} \to \mathbb{E}^3$:



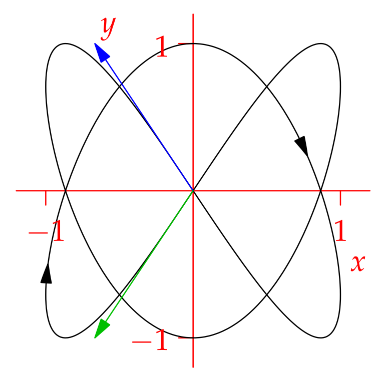
$$\mathbf{y}(s) = \mathbf{x}(t_0) + s\mathbf{x}'(t_0)$$

For example, the tangent line to the above helix at $t_0 = \frac{7\pi}{3}$ is

$$\mathbf{y}(s) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{7\pi}{3}\right) + \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right) s$$

The tangent line has the same speed as the helix $\sqrt{2}$.

Self-intersections These are no problem for our formulation! The curve



$$\mathbf{x}(t) = \left(\sin \tfrac{2t}{3}, \cos t\right), \quad t \in [0, 6\pi)$$

passes through the origin at both $t_1 = \frac{3\pi}{2}$ and $t_2 = \frac{9\pi}{2}$, with corresponding tangent vectors

$$\mathbf{x}'\left(\tfrac{3\pi}{2}\right) = \left(-\tfrac{2}{3}, 1\right), \qquad \mathbf{x}'\left(\tfrac{9\pi}{2}\right) = \left(-\tfrac{2}{3}, -1\right)$$

In this example, we shouldn't talk about *the* tangent vector to the curve *at the origin*, since it is non-unique. Rather we should refer to the *co-ordinates* $\frac{3\pi}{2}$ or $\frac{9\pi}{2}$.

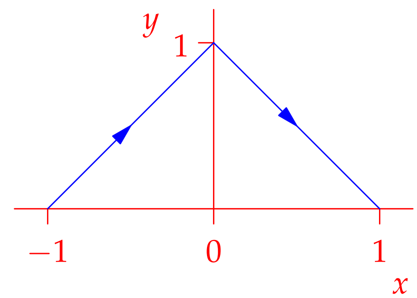
The linked animation shows the variable speed $v(t) = \sqrt{\frac{4}{9}\cos^2 \frac{2t}{3} + \sin^2 t}$ of this curve.

Corners and Cusps To ensure that a tangent direction exists, a regular curve has everywhere non-zero derivative. Here are a couple of examples of curves with non-regular points.

Examples 1.8. *Corner* A curve might enter and leave a point in different directions. For example, $\mathbf{x}(t) = \big(t, 1 - |t|\big)$ has derivative

$$\mathbf{x}'(t) = \begin{cases} (1,1) & \text{if } t < 0 \\ (1,-1) & \text{if } t > 0 \end{cases}$$

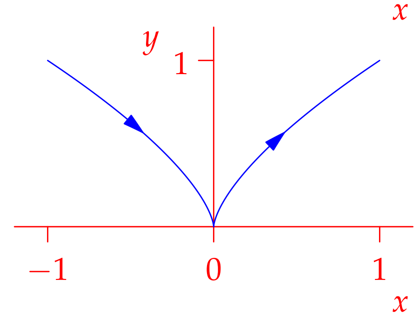
At $\mathbf{x}(0) = (0,1)$ the curve is non-differentiable and thus non-smooth and non-regular.



Cusp The curve $\mathbf{x}(t) = \big(t^3, t^2\big)$ has derivative

$$\mathbf{x}'(t) = \big(3t^2, 2t\big)$$

The origin is a *cusp*, a special type of corner where the curve leaves the point in the opposite direction to how it entered. In this case the curve is differentiable at the origin, but is non-regular since its speed $v(0)$ is zero.



Exercises 1.1. 1. A twice-differentiable curve $\mathbf{x}(t)$ has the property that its second derivative $\mathbf{x}''(t)$ is identically zero. What can be said about $\mathbf{x}$?

2. Find the unique curve such that $\mathbf{x}(0) = (1,0,5)$ and $\mathbf{x}'(t) = (t^2, t, e^t)$.

3. An ellipse in the plane has equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. By modifying the standard parametrization of the circle, find a regular parametrization of this ellipse. What is its speed?

4. Show that $\mathbf{x}(t) = \big(\frac{e^t+e^{-t}}{2}, \frac{e^t-e^{-t}}{2}\big)$ parametrizes half of the hyperbola $x^2 - y^2 = 1$. How would you parametrize the other half?

5. (a) Find the speed of the re-parametrized standard helix $\mathbf{y}(s) = \mathbf{x}(s^3) = \big(\cos s^3, \sin s^3, s^3\big)$.
   (b) More generally, if $\mathbf{x}(t)$ is a regular curve, show that $\mathbf{y}(s) := \mathbf{x}(s^3)$ is non-regular.

6. Verify that our cusp example (above) may instead be parametrized $\mathbf{y}(u) = \big(u, u^{2/3}\big)$. Is the new parametrization still non-regular at the origin? Explain.

7. Show that the tangent vectors to the regular curve $\mathbf{x}(t) = \big(3t, 3t^2, 2t^3\big)$ make a constant angle with the vector $(1,0,1)$.

8. Consider the plane curve $\mathbf{x}(t) = \big(t - 1 + e^{-t}, e^{-t}\big)$. Find the equation of its tangent line at $t = t_0$ and find where the tangent line intersects the x-axis.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Find a parametrization for the graph of $y = f(x)$ and find its tangent line when $x = x_0$.

10. Find a parametrization of the straight line through the points $(1,-3,-1)$ and $(6,2,1)$. Does this line meet the line through the points $(-1,1,0)$ and $(-5,-1,-1)$?

1.2 The Arc-length Parametrization and Curvature

As we've seen, the same 'curve' (viewed as a subset of $\mathbb{E}^3$) may be parametrized in different ways. For instance, in Exercise 1.1.5, the standard helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ was re-parametrized to obtain

$$\mathbf{y}(s) = (\cos s^3, \sin s^3, s^3) \tag{$*$}$$

This new parametrization is non-regular at $s = 0$; it *slows down and stops* before resuming its journey up the helix! Regularity is not, therefore, an intrinsic property of a curve viewed as a *set* (range($\mathbf{x}$)), rather it is a property of the parametrization.

Thankfully it is easy to create new parametrizations that remain regular.

Lemma 1.9. *If $\mathbf{x} : I \to \mathbb{E}^3$ is regular and $\alpha : J \to I$ is smooth with nowhere-zero derivative, then we obtain a new* regular *parametrization*

$$\mathbf{y} : J \to \mathbb{E}^3, \quad \mathbf{y}(s) := \mathbf{x}\big(\alpha(s)\big)$$

Proof. By the chain rule, $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}s} = \alpha'(s)\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$, which is non-zero by assumption. ∎

By contrast, if $\mathbf{x}(t)$ is non-regular, no smooth reparametrization can possible regularize it.

Since $\alpha'(s)$ is continuous and non-zero, there are two distinct cases:[3]

$\alpha(s)$ *increasing* We call this an *orientation-preserving* re-parametrization, since a 'particle' travels along the curve in the same direction.

$\alpha(s)$ *decreasing* The re-parametrization is *orientation-reversing.*

In the language of the Lemma, $(*)$ turned a regular parametrization into a non-regular one because $\alpha(s) = s^3$ has $\alpha'(s) = 3s^2$ which is zero at $s = 0$.

Our next goal is to develop a special parametrization for regular curves. First we recall a concept from multi-variable calculus.

Definition 1.10. The (signed) *arc-length* of a curve $\mathbf{x} : I \to \mathbb{E}^3$ measured from $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$ is the integral of the speed

$$s(t) = \int_{t_0}^{t} \left|\left|\mathbf{x}'(T)\right|\right| \mathrm{d}T = \int_{t_0}^{t} v(T)\,\mathrm{d}T$$

The arc-length is *signed* because it is negative if $t < t_0$: we are measuring length against the orientation of the curve. Of course if $\mathbf{x} : [a, b] \to \mathbb{E}^3$ has domain a closed bounded interval, then it is most sensible to measure arc-length from $t_0 = a$ so that $s(t) \geq 0$ everywhere on the curve.

Example 1.11. The standard helix $\mathbf{x}(t) = \big(\cos t, \sin t, t\big)$ has constant speed $\sqrt{2}$, whence the arc-length measured from $\mathbf{x}(0)$ is simply $s(t) = \sqrt{2}t$.

[3]Observe that $\alpha'(s)$ is *always positive* or *always negative.* In particular, $\alpha(s)$ is 1–1. If, in addition, α is *onto,* then $\mathbf{x}$ and $\mathbf{y}$ parametrize precisely the same subset of $\mathbb{E}^3$.

Recall the Fundamental Theorem of Calculus: is $s(t)$ is the arc-length of a regular curve, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} ||\mathbf{x}'(T)||\,\mathrm{d}T = ||\mathbf{x}'(t)|| = v(t)$$

is the curve's speed, which is *positive* and *continuous*. The same is therefore true for its *inverse function*

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{s'(t)} = \frac{1}{v(t)} > 0$$

Definition 1.12. An *arc-length parameter* for a regular curve $\mathbf{x}(t)$ is the inverse $\alpha(s) = t(s)$ of an arc-length function $s(t)$.

Lemma 1.9 tells us that $\mathbf{y}(s) = \mathbf{x}(\alpha(s))$ is a regular re-parametrization of our original curve. Indeed it is a re-parametrization with a very special property:

$$||\mathbf{y}'(s)|| = \alpha'(s)\,||\mathbf{x}'(\alpha(s))|| = \frac{1}{v(t)}v(t) = 1 \tag{†}$$

The curve $\mathbf{y}(s)$ has *unit-speed*. We have therefore proved a key result.

Theorem 1.13. *Every regular curve has a unit-speed parametrization, namely by an arc-length parameter (measured from wherever you like).*

The usefulness of the Theorem is abstract; by *assuming* that we have a unit-speed parametrization, certain analyses become much simpler. As a practical matter, explicitly finding an arc-length parametrization might be essentially impossible (evaluate an integral then invert a function...).

Examples 1.14. 1. Since the standard helix has arc-length parameter $s(t) = \sqrt{2}t$, it is trivial to observe that the re-parametrization

$$\mathbf{y}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

has unit speed.

2. More generally, if $\mathbf{x}(t)$ has constant speed v, then $s(t) = vt$ is an arc-length parameter and $\mathbf{y}(s) = \mathbf{x}(\frac{s}{v})$ a unit-speed re-parametrization.

3. The graph of $y = \frac{2}{3}x^{3/2}$ $(t \geq 0)$ may be parametrized by $\mathbf{x}(t) = (t, \frac{2}{3}t^{3/2})$. The arc-length measured from the origin is then

$$s(t) = \int_0^t \sqrt{1+T}\,\mathrm{d}T = \tfrac{2}{3}\left[(1+t)^{3/2} - 1\right] \implies \alpha(s) = t(s) = \left(1 + \tfrac{3}{2}s\right)^{2/3} - 1$$

We've obtained an explicit unit-speed parametrization

$$\mathbf{y}(s) = \mathbf{x}(\alpha(s)) = \left(\left(1+\tfrac{3}{2}s\right)^{2/3} - 1, \tfrac{2}{3}\left[\left(1+\tfrac{3}{2}s\right)^{2/3} - 1\right]^{3/2}\right)$$

though is it really something you ever want to compute with?!

Armed with unit-speed curves, we can now define our principal notion of bendiness.

Definition 1.15. The *curvature* of a unit-speed curve $\mathbf{x} : I \to \mathbb{E}^3$ is

$$\kappa(s) = ||\mathbf{x}''(s)||$$

We modify this slightly for curves in the plane: $\kappa(s)$ is positive/negative if the tangent vector rotates *counter-clockwise/clockwise* as we traverse the curve. This corresponds to the usual *right hand rule.*

By Newton's second law, a unit mass travelling along the curve at unit speed experiences a *transverse force* of magnitude $\kappa(s)$.

Examples 1.16. 1. A straight line has curvature zero. For example, the line joining $(1,4)$ and $(-3,1)$ has unit-speed parametrization $\mathbf{x}(s) = \left(-3 + \frac{4}{5}s, 1 + \frac{3}{5}s\right)$, whence $\mathbf{x}''(s) = \mathbf{0} \implies \kappa(s) = 0$.

2. The circle of radius r has unit-speed parametrization $\mathbf{x}(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$, whence

$$\mathbf{x}''(s) = -\frac{1}{r}\begin{pmatrix}\cos\frac{s}{r}\\ \sin\frac{s}{r}\end{pmatrix} \implies \kappa(s) = \frac{1}{r}$$

This is positive since the tangent vector rotates counter-clockwise. Observe that $\kappa = \frac{1}{r}$ is inversely proportional to the radius: smaller circles have larger curvature.

3. The standard helix with unit-speed parametrization $\mathbf{x}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ has

$$\mathbf{x}''(s) = -\frac{1}{2}\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right) \implies \kappa(s) = \frac{1}{2}$$

Since finding a unit-speed parametrization is difficult, there few curves for which this approach is sensible. What we want is a method that works for *arbitrary parametrization.* This is indeed possible, though for spacecurves it will take a while. For curves in the *plane* however, things are fairly easy.

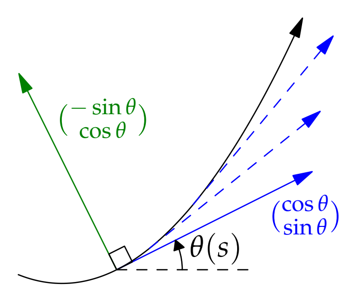
Curvature of Plane Curves If $\mathbf{y} : I \to \mathbb{E}^2$ has unit-speed, we can write

$$\mathbf{y}'(s) = \begin{pmatrix}\cos\theta(s)\\ \sin\theta(s)\end{pmatrix}$$

where $\theta(s)$ is the *angle* between the tangent line and the positive x-axis. Now observe that

$$\mathbf{y}''(s) = \theta'(s)\begin{pmatrix}-\sin\theta(s)\\ \cos\theta(s)\end{pmatrix}$$

Since $(-\sin\theta, \cos\theta)$ points to the *left* of $\mathbf{y}'(s)$, we conclude:



$\kappa(s) = \theta'(s) > 0$

Theorem 1.17. *The curvature of a unit-speed plane curve is the rate of change $\kappa(s) = \theta'(s)$ of the angle of its tangent line.*

This should be intuitive for constant curvature examples such as the straight line and the circle.

Now suppose $\mathbf{x}(t) = \big(x(t), y(t)\big)$ is any regular parametrization of the same curve; its speed satisfies

$$v(t) = \sqrt{x'(t)^2 + y'(t)^2} = s'(t)$$

where $s(t)$ is an arc-length function for $\mathbf{x}(t)$. Moreover, the angle $\theta(s)$ plainly satisfies

$$\theta(s) = \tan^{-1}\frac{y'(t)}{x'(t)}$$

Now differentiate and applying the chain rule:

$$\kappa(s) = \frac{\mathrm{d}}{\mathrm{d}s}\tan^{-1}\frac{y'(t)}{x'(t)} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}\frac{y'(t)}{x'(t)} = \cdots$$

The result is a formula for the curvature as a function of an arbitrary regular parametrization.

Corollary 1.18. *A regular curve $\mathbf{x}(t) = \big(x(t), y(t)\big)$ has curvature*

$$\kappa(t) = \frac{y''x' - x''y'}{[x'^2 + y'^2]^{3/2}} = \frac{y''x' - x''y'}{v^3} = \frac{\mathbf{x}'' \cdot J\mathbf{x}'}{v^3}$$

where $J\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y' \\ x' \end{pmatrix}$. In particular, the graph of a smooth function $y = f(x)$ has curvature

$$\kappa(x) = \frac{f''(x)}{\Big[1 + \big(f'(x)\big)^2\Big]^{3/2}}$$

Examples 1.19. 1. The graph of $y = \frac{2}{3}x^{3/2}$ has curvature

$$\kappa(x) = \frac{\frac{1}{2}x^{-1/2}}{(1+x)^{3/2}} = \frac{1}{2\sqrt{x(1+x)^3}}$$

2. If $f(x) = \sin x$, then $\kappa(x) = \dfrac{-\sin x}{(1+\cos^2 x)^{3/2}}$

3. The spiral $\mathbf{x}(t) = \big(t\cos t, t\sin t\big)$ has

$$\mathbf{x}'(t) = \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{pmatrix}, \quad \mathbf{x}''(t) = \begin{pmatrix} -2\sin t - t\cos t \\ 2\cos t - t\sin t \end{pmatrix}$$

$$\begin{aligned} \implies \kappa(t) &= \frac{(2\cos t - t\sin t)(\cos t - t\sin t) - (-2\sin t - t\cos t)(\sin t + t\cos t)}{[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2]^{3/2}} \\ &= \frac{2+t^2}{[1+t^2]^{3/2}} \end{aligned}$$

Exercises 1.2. 1. Compute the arc-length of the following curves by parametrizing and evaluating an integral:

(a) The straight line between points $(3,1,2)$ and $(1,1,0)$.

(b) The circle centered at $(1,-2)$ with radius 5 measured *clockwise* from $(6,-2)$ to $(1,3)$.

(c) The graph of the function $y=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2}$ for $1\le x\le 9$.

2. Find the curvature of the following plane curves (use Corollary 1.18).

(a) The graph of $y=x^2$.

(b) The catenary: the graph of $y=\frac{1}{2}(e^x+e^{-x})=\cosh x$

(c) The figure-eight curve $\mathbf{x}(t)=\big(\cos t,\sin 2t\big)$

(d) The exponential spiral $\mathbf{x}(t)=\big(e^t\cos t,e^t\sin t\big)$.

3. Find a unit-speed parametrization of the straight line between points with position vectors $\mathbf{a}\neq\mathbf{b}$ in $\mathbb{E}^3$ and hence verify that its curvature is zero.

4. Suppose $\mathbf{x}:\mathbb{R}\to\mathbb{E}^3$ has unit speed. Verify that $\mathbf{x}$ is parametrized by an arc-length parameter.

5. Find the curvature of the spacecurve $\mathbf{x}(s)=\big(\frac{5}{13}\cos s,\sin s,\frac{12}{13}\cos s\big)$. What is this curve?

6. (a) Find the arc-length of the standard helix $\mathbf{x}(t)=(\cos t,\sin t,t)$ between $t=-\pi$ and $t=2\pi$.

(b) Suppose a particle travels *down* the standard helix so that $\mathbf{y}(0)=(1,0,2\pi)$ and such that its speed is $v(t)=2\sqrt{2}t$. Find a parametrization which describes this motion.

(c) Let r,h be positive constants. Find the curvature of the general circular helix

$$\mathbf{x}(t)=\big(r\cos t,r\sin t,ht\big)$$

and interpret how it depends on r and h.

7. Check the evaluation of $\kappa(t)$ and $\kappa(x)$ in the proof of Corollary 1.18.

8. We find the curvature of the exponential spiral $\mathbf{x}(t)=\big(e^t\cos t,e^t\sin t\big)$ the hard way.

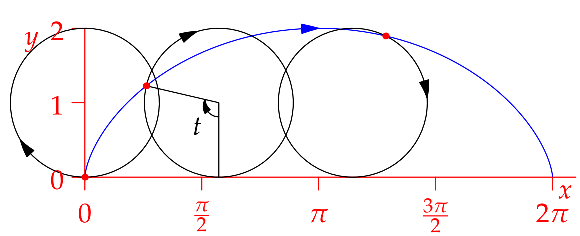
(a) Calculate the arc-length $s(t)$ measured from $\mathbf{x}(0)$.

(b) Find a unit-speed parametrization $\mathbf{y}(s)$ where $\mathbf{y}(0)=(1,0)$.

(c) Hence compute $\kappa(s)$ and show that it equals your answer from Exercise 2d.

9. A circle of radius 1 rolls at constant speed without slipping along the x-axis so that the angle indicated in the picture is t at time t.

The curve described by a point on the circumference of the rolling circle is a *cycloid*.



(a) Find a parametrization $\mathbf{x}:[0,2\pi]\to\mathbb{E}^2$.

(b) Find the curvature of the cycloid as a function of t.

(c) Compute the arc-length of the cycloid over a complete rotation of the circle.

1.3 Orthogonality, Moving Frames & The Structure Equations

Our plan is to analyze a curve with respect to a family of *moving* orthonormal bases. Before embarking on this, we summarize the relevant ideas from linear algebra. Hopefully most of the concepts are familiar. Most proofs are omitted, but will be met in a standard linear algebra class. As usual, definitions and results are stated in 3-dimensions, but are valid in others, particularly 2-dimensions.

In $\mathbb{E}^3$, points are typically denoted with reference to the *standard basis* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. For instance,

$$\mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

The numbers $3, 4, 6$ are the *co-ordinates* of $\mathbf{v}$ with respect to the standard basis. Of course other bases are available...

Definition 1.20. A set $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subseteq \mathbb{E}^3$ is a *basis* if every vector $\mathbf{v} \in \mathbb{E}^3$ can be expressed uniquely[4] as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$: that is

$$\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \tag{$*$}$$

for unique $c_1, c_2, c_3 \in \mathbb{R}$, the *co-ordinates* of $\mathbf{v}$ with respect to β.

A basis is *orthonormal* if $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

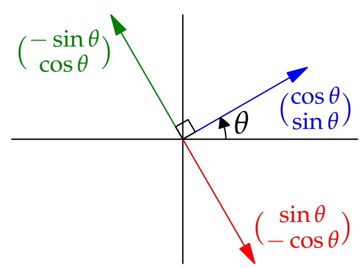
Consider the (invertible) matrix $E = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ whose columns are the elements of β viewed as column vectors (with respect to the standard basis). A basis is *positively oriented* if $\det E > 0$.

Examples 1.21. 1. $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a *negatively* oriented orthonormal basis of $\mathbb{E}^3$ ($\det E = -1 < 0$).

2. Every orthonormal basis of $\mathbb{E}^2$ has the form

$$\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \right\}$$

for some angle θ. The first is positively oriented ($\det = 1 > 0$) and the second negatively ($\det = -1 < 0$).



A positively oriented orthonormal basis in $\mathbb{E}^3$ satisfies the *right-hand rule*: $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. In $\mathbb{E}^2$, positive orientation means that $\mathbf{e}_2$ is obtained by rotating $\mathbf{e}_1$ counter-clockwise by 90°: we can write this as

$$\mathbf{e}_2 = J\mathbf{e}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{e}_1$$

[4]In a linear algebra class this is usually broken into two definitions which imply, respectively, the existence and uniqueness of the linear combination $(*)$.

Spanning Set Every $\mathbf{v} \in \mathbb{E}^3$ can be expressed as a linear combination $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Linear Independence The only linear combination summing to $\mathbf{0}$ is trivial: $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0} \implies c_1 = c_2 = c_3 = 0$.

Finding the co-ordinates of a vector with respect to a basis (∗) is really a matrix problem[5]

$$\mathbf{v} = E\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = E^{-1}\mathbf{v}$$

Inverting a 3×3 matrix is tedious. Thankfully the co-ordinates can be found more easily if the basis is *orthonormal* just by taking dot products!

$$\mathbf{v} \cdot \mathbf{e}_i = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3) \cdot \mathbf{e}_i = c_i$$

Lemma 1.22. *If $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis, then for any vector $\mathbf{v} \in \mathbb{E}^3$,*

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}_3)\mathbf{e}_3$$

Example 1.23. $\beta = \{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}\right\}$ is a positively oriented orthonormal basis of $\mathbb{E}^2$. With respect to β, the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can be written

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = \frac{7}{5}\mathbf{e}_1 + \frac{1}{5}\mathbf{e}_2$$

Orthogonal Matrices Recall Definition 1.5. Given $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and its associated matrix $E = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$, observe that

$$E^TE = \begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{pmatrix} (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) = \begin{pmatrix} ||\mathbf{e}_1||^2 & \mathbf{e}_1 \cdot \mathbf{e}_2 & \mathbf{e}_1 \cdot \mathbf{e}_3 \\ \mathbf{e}_2 \cdot \mathbf{e}_1 & ||\mathbf{e}_2||^2 & \mathbf{e}_2 \cdot \mathbf{e}_3 \\ \mathbf{e}_3 \cdot \mathbf{e}_1 & \mathbf{e}_3 \cdot \mathbf{e}_2 & ||\mathbf{e}_3||^2 \end{pmatrix}$$

When β is an orthonormal basis, this matrix is very simple.

Definition 1.24. A 3×3 matrix A is *orthogonal* if $A^TA = I$ (equivalently $AA^T = I$). The set of all such is denoted $\mathrm{O}_3(\mathbb{R})$. In addition, if $\det A = 1$, we write $A \in \mathrm{SO}_3(\mathbb{R})$ (*special orthogonal matrices*).

Lemma 1.25. 1. *If $A \in \mathrm{O}_3(\mathbb{R})$, then it is invertible with inverse A^T (also orthogonal).*

2. *The product of two orthogonal matrices is orthogonal.*

3. *A is orthogonal if and only if $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{E}^3$.*

4. *Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $E = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) \in M_3(\mathbb{R})$:*

   (a) $E \in \mathrm{O}_3(\mathbb{R}) \iff \beta$ is an orthonormal basis.

   (b) $E \in \mathrm{SO}_3(\mathbb{R}) \iff \beta$ is a positively oriented orthonormal basis.

Parts 1 and 2 together say that $\mathrm{O}_3(\mathbb{R})$ forms a *group* under matrix multiplication; it is therefore known as the *orthogonal group*.[6]

[5] For obvious reasons, this is known as the *change of co-ordinate matrix* from β to the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

[6] Knowledge of group theory is not necessary for these notes. *Lie Groups*, of which $\mathrm{O}_3(\mathbb{R})$ is an example, are critical to more advanced differential geometry.

Examples (1.21 cont). 1. It is no fun to check $E^T E = I$ directly, but since we have an orthonormal basis, the lemma tells us that

$$E = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \in \mathrm{O}_3(\mathbb{R})$$

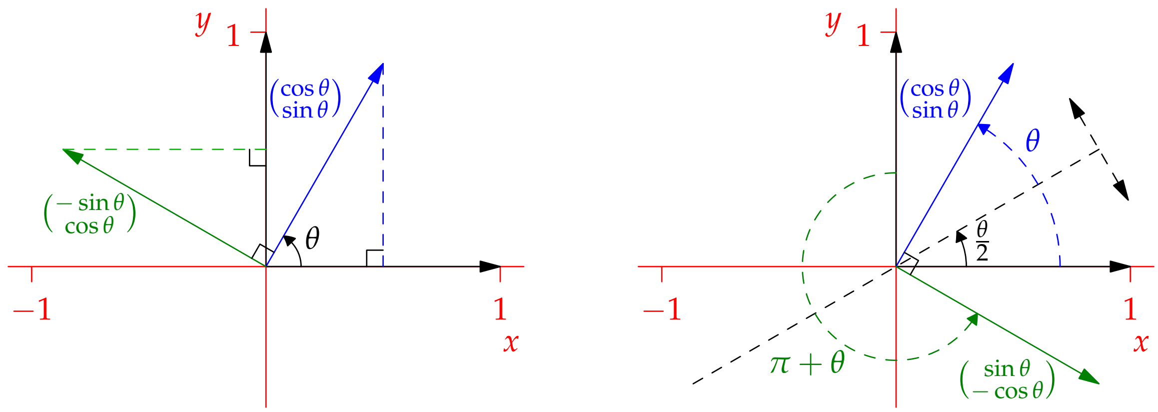
2. Every 2×2 orthogonal matrix has one of two forms:

Rotations $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$

The effect of the map $\mathbf{x} \mapsto A_\theta \mathbf{x}$ is to *rotate* $\mathbf{x}$ counter-clockwise by θ radians.[7]

Reflections $B_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ $(\det B_\theta = -1)$

The effect of $\mathbf{x} \mapsto B_\theta \mathbf{x}$ is to *reflect* $\mathbf{x}$ across the line making angle $\frac{\theta}{2}$ with the positive x-axis.



Motivated by the 2×2 case, it is common to refer to every orthogonal matrix in $\mathrm{O}_3(\mathbb{R})$ as a *rotation* $(\det = 1)$ or a *reflection* $(\det = -1)$.[8]

Part 3 of Lemma 1.25 says that multiplication by an orthogonal matrix preserves the dot product and thus (Definition 1.5) the *lengths* of vectors and the *angles* between them. We use this to define a useful family of transformations of $\mathbb{E}^3$.

Definition 1.26. An *isometry* is a function $S : \mathbb{E}^3 \to \mathbb{E}^3$ acting on points/position vectors by

$$S(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

where $\mathbf{b}$ is a constant vector and $A \in \mathrm{O}_3(\mathbb{R})$. We call S a *direct isometry* or *rigid motion* if $\det A = 1$ $(A \in \mathrm{SO}_3(\mathbb{R}))$, and an *indirect isometry* otherwise.

Isometry literally means *equal length*; it can be seen that every function $S : \mathbb{E}^3 \to \mathbb{E}^3$ which preserves distances between all pairs of points is an isometry. *Congruent* geometric objects (in standard Euclidean geometry) are precisely those which are related by an isometry.

[7] Recall that the matrix of a linear map is found by evaluating the map on the standard basis: the 1st column of A_θ is the column vector $A_\theta \mathbf{i} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. The pictures should verify the remaining columns; for B_θ you might find it helpful to consider how the required reflections of the standard basis vectors $\mathbf{i}, \mathbf{j}$ may be computed using *rotations*.

[8] A full analysis is more complicated. For instance, the map $\mathbf{x} \mapsto E\mathbf{x}$ in the first example is the composition of a reflection across a plane in $\mathbb{E}^3$ followed by a rotation in that plane.

Moving Frames

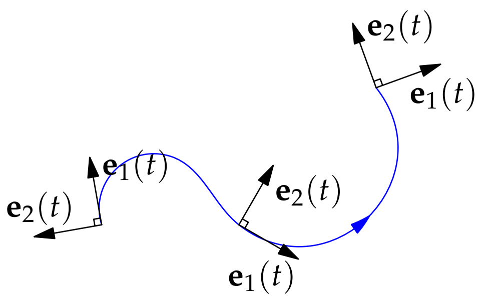
Thus far we have analyzed curves with reference to the standard orthonormal basis $\epsilon = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. We now replace this *static* frame of reference with one that *moves*. The goal is eventually to describe a special moving frame with respect to which the fundamental properties of the curve are clear.

Definition 1.27. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a smooth curve. Suppose that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are smooth functions on I such that, for each $t \in I$,

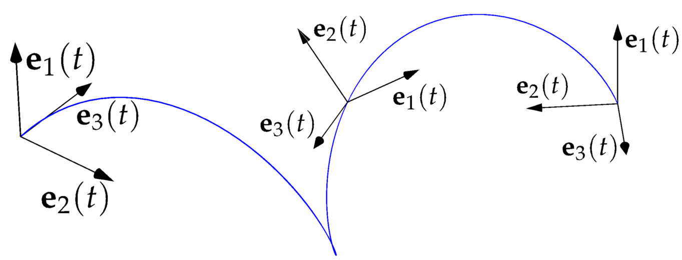
$$\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\} \text{ is a positively oriented orthonormal basis of the tangent space } T_{\mathbf{x}(t)}\mathbb{E}^3$$

We call this family of functions a *moving frame* along $\mathbf{x}$.

Equivalently, $E(t) = \begin{pmatrix} \mathbf{e}_1(t) & \mathbf{e}_2(t) & \mathbf{e}_3(t) \end{pmatrix}$ is a smooth function $E : I \to SO_3(\mathbb{R})$. We will often refer to this matrix-valued function as a moving frame.



A moving frame in $\mathbb{E}^2$



A moving frame in $\mathbb{E}^3$

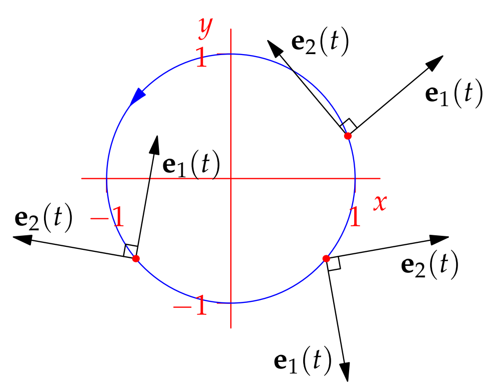
The smoothness criterion needs a little unpacking. At each point on the curve, the tangent space $T_{\mathbf{x}(t)}\mathbb{E}^3$ has a standard basis of tangent vectors $\{\mathbf{i}_{\mathbf{x}(t)}, \mathbf{j}_{\mathbf{x}(t)}, \mathbf{k}_{\mathbf{x}(t)}\}$, with respect to which

$$\mathbf{e}_j(t) = a_j(t)\mathbf{i}_{\mathbf{x}(t)} + b_j(t)\mathbf{j}_{\mathbf{x}(t)} + c_j(t)\mathbf{k}_{\mathbf{x}(t)} = \begin{pmatrix} a_j(t) \\ b_j(t) \\ c_j(t) \end{pmatrix}$$

We require that the functions $a_j, b_j, c_j : I \to \mathbb{R}$ be smooth. Strictly speaking, $\mathbf{e}_j(t)$ is a *smooth vector field* along the curve.

Example 1.28. We define a moving frame along the unit circle $\mathbf{x}(t) = (\cos t, \sin t)$ via

$$\mathbf{e}_1(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \qquad \mathbf{e}_2(t) = \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$



Click on the picture to see how the frame rotates twice as one travels once round the circle!

In accordance with the definition, for each t,

$$E(t) = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \in SO_3(\mathbb{R})$$

The obvious disadvantage of a moving frame is that we have to understand how such a frame moves!

Theorem 1.29 (Structure equations). *Suppose $\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\}$ is a moving frame (orthonormal positive orientation). For each j, express the derivative as a linear combination*

$$\mathbf{e}_j' = \mathbf{e}_1 w_{1j} + \mathbf{e}_2 w_{2j} + \mathbf{e}_3 w_{3j}$$

where each $w_{ij}(t) = \mathbf{e}_i \cdot \mathbf{e}_j'$ is a scalar function. Then the matrix $W = (w_{ij})$ is skew-symmetric:[9]

$$\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3'\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}0 & w_{12} & w_{13}\\ -w_{12} & 0 & w_{23}\\ -w_{13} & -w_{23} & 0\end{pmatrix} \qquad \text{(matrix form } E' = EW\text{)}$$

In $\mathbb{E}^2$, there is only a single function $w_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2'$. As we've done already, we often drop the (t) to make things more readable; just remember that everything is still a *function*!

Proof. Since $\mathbf{e}_i \cdot \mathbf{e}_j$ is constant (equals 0 or 1), the product rule says that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_i \cdot \mathbf{e}_j) = \mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' \implies w_{ji} + w_{ij} = 0$$

∎

Examples 1.30. 1. Example 1.28 described a moving frame in $\mathbb{E}^2$:

$$w_{12}(t) = \mathbf{e}_1(t) \cdot \mathbf{e}_2'(t) = \begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix} \cdot \begin{pmatrix}-2\cos 2t\\ -2\sin 2t\end{pmatrix} = -2$$

The structure equations are therefore

$$\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2'\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix}\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}$$

2. A moving frame can be described without mentioning a specific curve $\mathbf{x}(t)$:

$$\mathbf{e}_1(t) = \begin{pmatrix}\cos^2 t\\ \cos t \sin t\\ \sin t\end{pmatrix} \qquad \mathbf{e}_2(t) = \begin{pmatrix}\sin t\\ -\cos t\\ 0\end{pmatrix} \qquad \mathbf{e}_3(t) = \begin{pmatrix}\sin t\cos t\\ \sin^2 t\\ -\cos t\end{pmatrix}$$

The structure equations are easily computed

$$\begin{aligned}
w_{12}(t) &= \mathbf{e}_1 \cdot \mathbf{e}_2' = \cos^3 t + \cos t \sin^2 t = \cos t,\\
w_{13}(t) &= \mathbf{e}_1 \cdot \mathbf{e}_3' = \cos^2 t(\cos^2 t - \sin^2 t) + 2(\cos t \sin t)^2 + \sin^2 t = 1\\
w_{23}(t) &= \mathbf{e}_2 \cdot \mathbf{e}_3' = \sin t(\cos^2 t - \sin^2 t) - \cos t(2\cos t \sin t) = -\sin t\\
\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3'\end{pmatrix} &= \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}0 & \cos t & 1\\ -\cos t & 0 & -\sin t\\ -1 & \sin t & 0\end{pmatrix}
\end{aligned}$$

[9] The set of skew-symmetric matrices is sometimes denoted $\mathfrak{so}_3(\mathbb{R})$. The structure equations are an example of the relationship between a *Lie group* and its *Lie algebra*, a foundation on which much advanced differential geometry rests.

Exercises 1.3. 1. Express $\mathbf{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ as a linear combination with respect to the orthonormal basis $\beta = \{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix}, \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \right\}$ of $\mathbb{E}^2$.

2. (a) Show that $\beta = \left\{ \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \right\}$ is an orthonormal basis of $\mathbb{E}^3$. Is it positively oriented?
   (b) Find the co-ordinates of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with respect to β.

3. (a) Explain why the product rule $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x}' \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}'$ holds for differentiable curves $\mathbf{x}, \mathbf{y}$.
   (b) Let $\mathbf{x}, \mathbf{y}$ be differentiable on an interval and use the product rule to answer the following:
      i. Suppose $\mathbf{x}(t_0)$ and $\mathbf{x}'(t)$ are orthogonal to a fixed vector $\mathbf{v}$ (the latter for all t). Show that $\mathbf{x}(t)$ is always orthogonal to $\mathbf{v}$.
      ii. If $\mathbf{y}(t_0)$ is a point on $\mathbf{y}$ which is closest to the origin, show that $\mathbf{y}(t_0) \perp \mathbf{y}'(t_0)$.

4. Find the function w_{12} for the moving frame $\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \frac{1}{1+t^2}\begin{pmatrix} 2t \\ 1-t^2 \end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix} t^2-1 \\ 2t \end{pmatrix} \right\}$

5. Find the structure equations for the moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \left\{ \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin t \\ 0 \\ \cos t \end{pmatrix} \right\}$

6. (a) Explain why every moving frame in $\mathbb{E}^2$ has the form $\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}, \begin{pmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{pmatrix} \right\}$ for some function θ.
   (b) Find the structure equations for this frame: how does w_{12} relate to θ?
   (c) If $\mathbf{x}(t)$ is parametrized at unit speed such that $\mathbf{e}_1(t) = \mathbf{x}'(t)$, what is $w_{12}(t)$?

7. (a) Let $E(t)$ be a square matrix-valued function. Show that $\frac{\mathrm{d}}{\mathrm{d}t}(E(t))^{-1} = -E^{-1}E'E^{-1}$.
   (b) Suppose $E : I \to \mathrm{O}_3(\mathbb{R})$ is differentiable and define $W(t) := E^{-1}(t)E'(t)$. Use part (a) to prove that $W(t)$ is skew-symmetric ($W^T = -W$).

8. (a) Verify parts 2 and 3 of Lemma 1.25.
   (b) Suppose f, g are rigid motions. Show that $f \circ g$ and f^{-1} are also rigid motions.

9. Let $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Suppose $\mathbf{p} \in \mathbb{E}^2$ and a unit vector $\mathbf{v}$ are given. Prove that there is a unique rigid motion $S : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ such that

$$S(\mathbf{0}) = \mathbf{p} \quad \text{and} \quad S(\mathbf{i}) = \mathbf{p} + \mathbf{v}$$

   Write $\mathbf{i_0} = (\mathbf{0}, \mathbf{i}) \in T_\mathbf{0}\mathbb{E}^2$ and $\mathbf{v_p} = (\mathbf{p}, \mathbf{v}) \in T_\mathbf{p}\mathbb{E}^2$ as *tangent vectors*, explain why it is reasonable to write $\mathbf{v_p} = S(\mathbf{i_0}) = (A\mathbf{i})_\mathbf{p}$: i.e., only A affects the *directional part* of a tangent vector.

10. (Hard) Suppose that a moving frame has structure equations

$$\mathbf{e}_1' = -\tfrac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3), \qquad \mathbf{e}_2' = \tfrac{1}{\sqrt{2}}\mathbf{e}_1, \qquad \mathbf{e}_3' = \tfrac{1}{\sqrt{2}}\mathbf{e}_1$$

   (a) By considering $\mathbf{e}_1''$, show that the vector $\mathbf{e}_1 \times \mathbf{e}_1'$ is constant.
   (b) Show that $||\mathbf{e}_1'||$ is constant.
   (c) Prove that there exists a constant positively oriented orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ such that $\mathbf{e}_1(t) = \cos t\mathbf{a} + \sin t\mathbf{b}$ and compute $\mathbf{e}_2, \mathbf{e}_3$ in terms of this basis.

1.4 The Frenet Frame for a Spacecurve

In this section we analyze spacecurves with respect to a moving frame *adapted* to the curve. To do this, we need to restrict our class of curves slightly. For this section, we work exclusively in $\mathbb{E}^3$.

Definition 1.31. A regular *spacecurve* $\mathbf{x} : I \to \mathbb{E}^3$ is *biregular* if it has *non-zero curvature* κ.

Every biregular curve is necessarily regular, but the converse is false. For instance, a straight line is regular but *not biregular.* Indeed for a biregular curve, the vectors $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$ must be linearly independent.

Definition 1.32. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a biregular unit-speed curve. The *Frenet frame* $E(t) = (\mathbf{T}\ \mathbf{N}\ \mathbf{B})$ is the moving frame defined as follows:

$\mathbf{T} := \mathbf{x}'$ is the *unit tangent* vector field

$\mathbf{N} := \frac{1}{\kappa}\mathbf{T}'$ is the *principal normal* vector field

$\mathbf{B} := \mathbf{T} \times \mathbf{N}$ is the *binormal* vector field

We verify that the Frenet frame is indeed a moving frame:

1. Since $\mathbf{x}$ is unit-speed, $\mathbf{T}$ has unit length.
2. By the product rule, $\mathbf{T} \cdot \mathbf{T} = 1 \implies 2\mathbf{T}' \cdot \mathbf{T} = 0 \implies \mathbf{N} \cdot \mathbf{T} = 0$. Moreover, the definition of curvature tells us tht
$$||\mathbf{N}|| = \frac{1}{\kappa}\,||\mathbf{T}'|| = \frac{1}{\kappa}\,||\mathbf{x}''|| = \frac{\kappa}{\kappa} = 1$$
so that $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$.
3. Standard properties of the cross product finish things off:
   - $\mathbf{B}$ has unit length since $||\mathbf{B}|| = ||\mathbf{T}||\,||\mathbf{N}|| \sin\theta = 1$ ($\theta = 90°$ is the angle between $\mathbf{T}, \mathbf{N}$).
   - $(\mathbf{T} \times \mathbf{N}) \cdot \mathbf{z} = \det(\mathbf{T}\ \mathbf{N}\ \mathbf{z})$ with $\mathbf{z} = \mathbf{T}$ or $\mathbf{N}$ says that $\mathbf{B}$ is perpendicular to $\mathbf{T}, \mathbf{N}$. Finally, let $\mathbf{z} = \mathbf{B}$ to see that the Frenet frame is positively oriented.

Theorem 1.33. *The Frenet frame is a moving frame. Its structure equations are*

$$\begin{pmatrix}\mathbf{T}' & \mathbf{N}' & \mathbf{B}'\end{pmatrix} = \begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix}\begin{pmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix} \qquad \begin{cases}\mathbf{T}' = \kappa\mathbf{N}\\ \mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}\\ \mathbf{B}' = -\tau\mathbf{N}\end{cases}$$

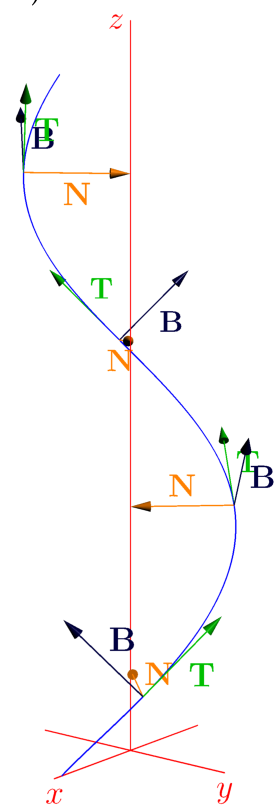
where $\kappa > 0$ is the curvature and $\tau = \mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot \mathbf{B}'$ is called the torsion.

The structure equations for the Frenet frame are also known as the *Frenet–Serret equations.* The moving planes spanned by pairs of these vectors have special names:

Span$\{\mathbf{T}, \mathbf{N}\}$, Span$\{\mathbf{T}, \mathbf{B}\}$ and Span$\{\mathbf{N}, \mathbf{B}\}$ are the *osculating, rectifying* and *normal* planes.

At any point, the tangent line lies in the osculating plane.

Examples 1.34. 1. We compute the Frenet frame and its structure equations for the standard helix $\mathbf{x}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ parametrized by arc-length (3D pic)(animation)



$$\mathbf{T}(s) = \mathbf{x}'(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\frac{s}{\sqrt{2}} \\ \cos\frac{s}{\sqrt{2}} \\ 1 \end{pmatrix} \implies \mathbf{T}'(s) = -\frac{1}{2}\begin{pmatrix} \cos\frac{t}{\sqrt{2}} \\ \sin\frac{t}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\implies \mathbf{N}(s) = -\begin{pmatrix} \cos\frac{t}{\sqrt{2}} \\ \sin\frac{t}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \kappa(s) = \frac{1}{2}$$

$$\implies \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sin\frac{s}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} \\ 1 \end{pmatrix}$$

$$\tau(s) = \mathbf{N}'(s) \cdot \mathbf{B}(s) = \frac{1}{2}\begin{pmatrix} \sin\frac{s}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sin\frac{s}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} \\ 1 \end{pmatrix} = \frac{1}{2}$$

The Frenet–Serret equations for the helix are therefore

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' & \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

2. Let $\mathbf{x}(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{\sqrt{2}}s, \frac{1}{3}(1-s)^{3/2}\right)$ for $s \in (-1,1)$. First we verify this is unit-speed

$$\mathbf{x}'(s) = \frac{1}{2}\begin{pmatrix} \sqrt{1+s} \\ \sqrt{2} \\ -\sqrt{1-s} \end{pmatrix} \implies v(s) = \left|\left|\mathbf{x}'(s)\right|\right| = \frac{1}{2}\sqrt{1+s+2+1-s} = 1$$

It follows that $\mathbf{T} = \mathbf{x}'$. Now compute the rest of the Frenet apparatus:

$$\mathbf{T}' = \frac{1}{4}\begin{pmatrix} (1+s)^{-1/2} \\ 0 \\ (1-s)^{-1/2} \end{pmatrix} \implies \kappa = \left|\left|\mathbf{T}'\right|\right| = \frac{1}{4}\sqrt{\frac{1}{1+s} + \frac{1}{1-s}} = \frac{1}{2\sqrt{2}\sqrt{1-s^2}}$$

$$\mathbf{N} = \frac{1}{\kappa}\mathbf{T}' = \frac{2\sqrt{2}\sqrt{1-s^2}}{4}\begin{pmatrix} (1+s)^{-1/2} \\ 0 \\ (1-s)^{-1/2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-s} \\ 0 \\ \sqrt{1+s} \end{pmatrix}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{2}\begin{pmatrix} \sqrt{1+s} \\ -\sqrt{2} \\ -\sqrt{1-s} \end{pmatrix} \implies \tau = \mathbf{N}' \cdot \mathbf{B} = \frac{-1}{2\sqrt{2}\sqrt{1-s^2}}$$

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' & \mathbf{B}' \end{pmatrix} = \frac{\begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix}}{2\sqrt{2}\sqrt{1-s^2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The Frenet Frame in arbitrary parametrization

Since there are relatively few curves for which an *explicit* unit-speed parametrization can be found, we want to be able to compute the Frenet frame for any biregular curve, regardless of parametrization. This requires nothing more than the careful application of the chain rule…

Example 1.35. We compute for the exponential spiral $\mathbf{x}(t) = \left(e^t \cos t, e^t \sin t, e^t\right)$.

$$\mathbf{x}'(t) = e^t \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \\ 1 \end{pmatrix} \implies v(t) = \sqrt{3}e^t \implies \mathbf{T}(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \\ 1 \end{pmatrix}$$

Since $\mathbf{T}(t)$ has unit length, $\mathbf{T}' \perp \mathbf{T}$. But then

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix} \implies \mathbf{N}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix} \qquad (\text{unit length} \parallel \mathbf{T}')$$

$$\implies \mathbf{B}(t) = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{6}} \begin{pmatrix} -\cos t + \sin t \\ -\sin t - \cos t \\ 2 \end{pmatrix}$$

It is tempting to think that the curvature should be $||\mathbf{T}'(t)|| = \sqrt{\frac{2}{3}}$, but this is not so. Since $\mathbf{x}$ is not unit-speed, we need to use the chain rule:

$$\kappa = \left|\left|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}(t)\right|\right| = \left|\left|\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t)\right|\right| = \frac{1}{v(t)}\,||\mathbf{T}'(t)|| = \frac{\sqrt{2}}{3}e^{-t}$$

The torsion may be computed similarly

$$\tau = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} \cdot \mathbf{B} = \frac{1}{v(t)}\mathbf{N}'(t) \cdot \mathbf{B}(t) = \frac{1}{3}e^{-t}$$

For the general result, simply(!) repeat the example in the abstract.

Corollary 1.36. *Let $\mathbf{x}(t)$ be a biregular spacecurve with arbitrary parametrization. The speed, curvature, torsion, Frenet frame, and structure equations are as follows.*

$$v(t) = ||\mathbf{x}'(t)|| \qquad \kappa(t) = \frac{||\mathbf{x}' \times \mathbf{x}''||}{v^3} \qquad \tau(t) = \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{v^6\kappa^2}$$

$$\mathbf{T}(t) = \frac{1}{v}\mathbf{x}' \qquad \mathbf{N}(t) = \frac{v\mathbf{x}'' - v'\mathbf{x}'}{v^3\kappa} \qquad \mathbf{B}(t) = \frac{\mathbf{x}' \times \mathbf{x}''}{v^3\kappa}$$

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' & \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix} \begin{pmatrix} 0 & -v\kappa & 0 \\ v\kappa & 0 & -v\tau \\ 0 & v\tau & 0 \end{pmatrix}$$

The curvature formula also holds if $\mathbf{x}(t)$ is merely regular.

Exercises 1.4. 1. Compute the curvature and torsion of the spiral $\mathbf{x}(t) = \big(e^t \cos t, e^t \sin t, e^t\big)$ directly using the expressions in Corollary 1.36.

2. A circular helix has the form $\mathbf{x}(t) = \big(r\cos t, r\sin t, ht\big)$, where $r > 0$ and h are constants. Find its Frenet frame and show that its curvature and torsion are given by

$$\kappa = \frac{r}{r^2+h^2}, \qquad \tau = \frac{h}{r^2+h^2}$$

3. Find the curvature and torsion of the curve $\mathbf{x}(t) = \big(t, t^2, t^3\big)$.

4. Given $\mathbf{x}(t) = \frac{1}{\sqrt{5}}\Big(\sqrt{1+t^2}, 2t, \ln\big(t+\sqrt{1+t^2}\big)\Big)$, find the Frenet frame, curvature and torsion.

5. Let $f(t) = \sqrt{2}\int_0^t \sqrt{1-e^{-2u}}\,\mathrm{d}u$, and define the curve $\mathbf{x}(t) = \frac{1}{\sqrt{2}}\big(e^{-t}\cos t, e^{-t}\sin t, f(t)\big)$, $t > 0$.

   (a) Verify that $\mathbf{x}(t)$ has unit speed.

   (b) Calculate the curvature of $\mathbf{x}$ and show that $\lim_{t\to\infty} \kappa(t) = 0$.

6. Let a, b be positive constants and $\mathbf{x}(t) = \big(4a\cos^3 t, 4a\sin^3 t, 3b\cos 2t\big)$ where $0 < t < \frac{\pi}{2}$. Find the Frenet frame, curvature and torsion of $\mathbf{x}$.

7. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a twice-differentiable regular curve.

   (a) Prove the formula for κ in Corollary 1.36:

   $$\kappa(t) = \frac{||\mathbf{x}' \times \mathbf{x}''||}{v^3}$$

   Hence conclude that $\kappa(t_0) = 0 \iff \mathbf{x}'(t_0)$ and $\mathbf{x}''(t_0)$ are parallel.
   (*Hint: let* $\mathbf{x}(t) = \mathbf{y}\big(s(t)\big)$ *where* $\mathbf{y}(s)$ *has unit speed*)

   (b) Prove as much else as you can tolerate of Corollary 1.36.

8. Suppose $\mathbf{x} : I \to \mathbb{E}^3$ is a curve lying on the surface of the unit sphere ($||\mathbf{x}|| = 1$).

   (a) If $\mathbf{x}$ has unit speed, show that $\mathbf{x}'' \cdot \mathbf{x} = -1$.

   (b) Hence or otherwise, prove that the curvature of $\mathbf{x}$ is at least 1 everywhere.
   (*Hint:* $\mathbf{x}$ *and* $\mathbf{x}'$ *are orthonormal...*)

   (c) What happens if $\mathbf{x}$ lies on the surface of the sphere $||\mathbf{x}|| = r$ of radius $r > 0$?

   (d) (Hard) If a *unit-speed* curve lies on a sphere of radius r, show that

   $$\kappa^2\tau^2(r^2\kappa^2 - 1) = (\kappa')^2$$

   (*Hint: compute the coefficients of* $\mathbf{x}$ *with respect to the Frenet frame*)

9. (Hard) Let $d(t) > 0$. Suppose $\mathbf{x}(t)$ and $\mathbf{y}(t) = \mathbf{x}(t) + d\mathbf{N}(t)$ are unit-speed curves such that the principal normal vector field $\mathbf{N}$ of $\mathbf{x}$ is the translate[a] of the binormal vector field $\hat{\mathbf{B}}$ of $\mathbf{y}$.

   Prove that the distance d between corresponding points of the curves is *constant*. Prove also that the curvature and torsion of $\mathbf{x}$ satisfy $2\kappa = d(\kappa^2 + \tau^2)$.

   (*Hint: Compute* $\hat{\mathbf{T}}$ *and take dot products with something useful...*)

[a] That is, the directional parts of $\mathbf{N}, \hat{\mathbf{B}}$ are identical: of course these are members of different tangent spaces.

1.5 The Fundamental Theorem of Biregular Spacecurves

Our goal for this section is to see that curvature and torsion determine a spacecurve uniquely up to rigid motions. We do this by recognizing the Frenet–Serret equations satisfied by the Frenet frame as a system of ordinary differential equations; provided sufficient initial conditions (starting point and orientation), the usual existence and uniqueness theorem for initial value problems shows that there is a unique curve with this data.

As a precursor, we consider how to interpret curvature and torsion, and how they change (or don't!) under rigid motions of a curve.

Theorem 1.37. 1. *A regular spacecurve has $\kappa \equiv 0$ if and only if it is a straight line.*

2. *A biregular spacecurve has $\tau \equiv 0$ if and only if it is contained in a fixed plane (the unmoving osculating plane of the curve).*

Proof. In both cases, we assume, without loss of generality, that $\mathbf{x}(s)$ is a unit-speed parametrization of our spacecurve.

1. $\kappa(s) = ||\mathbf{x}''(s)|| = 0 \iff \mathbf{x}''(s) = \mathbf{0}$. Thus $\mathbf{x}$ is a straight line.

2. ($\Leftarrow$) Suppose the curve lies in a fixed plane. Then $\mathbf{x}'$ and $\mathbf{x}''$ are parallel to this plane, whence $\mathbf{T}$ and $\mathbf{N}$ are also. But then $\mathbf{B}$ is a continuous unit vector orthogonal to the plane and is therefore *constant*. From the Frenet equations, $-\tau\mathbf{N} = \mathbf{B}' = \mathbf{0} \implies \tau \equiv 0$.

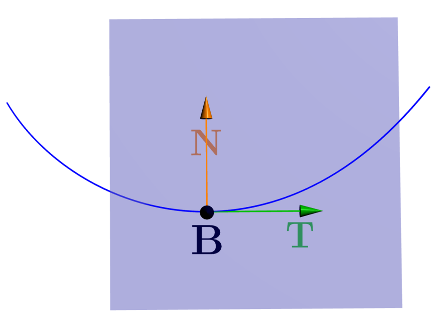
   ($\Rightarrow$) As above, if $\tau \equiv 0$, then $\mathbf{B}$ is constant. But then

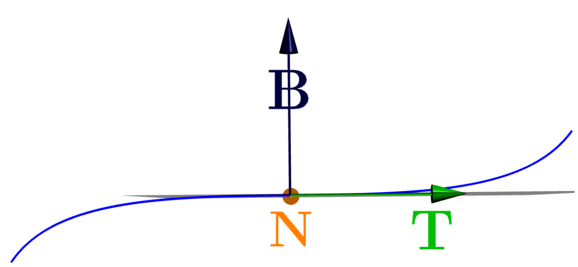
   $$(\mathbf{x} \cdot \mathbf{B})' = \mathbf{x}' \cdot \mathbf{B} + \mathbf{x} \cdot \mathbf{B}' = \mathbf{T} \cdot \mathbf{B} = 0$$

   from which $\mathbf{x} \cdot \mathbf{B}$ is constant. The curve therefore lies in a fixed plane perpendicular to $\mathbf{B}$. ∎

Curvature measures the deviation of a curve from a straight line; its *bending*. Torsion measures how badly a curve fails to be planar; its *twisting*.

To visualize the difference, the pictures below show a segment of a standard helix. In the first we look down the binormal onto the osculating plane; the non-zero curvature is clearly visible. In the second we look along the principal normal vector $\mathbf{N}$ and across the osculating plane; the positive torsion ($\tau = \frac{1}{2}$) indicates that the curve crosses the plane similarly to how the cubic function $y = x^3$ crosses the x-axis. The full 3D curve is linked via either picture.





Theorem 1.38. *Under an isometry $\hat{\mathbf{x}} := A\mathbf{x} + \mathbf{b}$ (recall Definition 1.26), the curvature and torsion of a biregular spacecurve transform as follows:*

Direct isometry/rigid motion: $\hat{\kappa} = \kappa, \quad \hat{\tau} = \tau.$

Indirect isometry: $\hat{\kappa} = \kappa, \quad \hat{\tau} = -\tau.$

Proof. Suppose $\mathbf{x}(s)$ has unit-speed. We relate the Frenet frame $(\hat{\mathbf{T}}\ \hat{\mathbf{N}}\ \hat{\mathbf{B}})$ of $\hat{\mathbf{x}}$ to the original.[10] Since orthogonal matrices preserve the dot product (Lemma 1.25), $\hat{\mathbf{x}}$ has unit-speed also:

$$\hat{\mathbf{x}}'(s) = A\mathbf{x}'(s) \implies \hat{v}(s) = \left|\left|\hat{\mathbf{x}}'(s)\right|\right| = \left|\left|\mathbf{x}'(s)\right|\right| = 1 \implies \hat{\mathbf{T}} = A\mathbf{T}$$

Moreover, since A is constant and both $\hat{\mathbf{N}}$ and $\mathbf{N}$ have unit length,

$$\frac{1}{\hat{\kappa}}\hat{\mathbf{N}} = \hat{\mathbf{T}}' = A\mathbf{T}' = \frac{1}{\kappa}A\mathbf{N} \implies \hat{\kappa} = \kappa \quad \text{and} \quad \hat{\mathbf{N}} = A\mathbf{N}$$

Curvature is therefore invariant under any isometry. Since A preserves angles, $A\mathbf{B}$ is perpendicular to both $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$, and so $A\mathbf{B} = \pm\hat{\mathbf{B}}$. Since the Frenet frame $E = (\mathbf{T}\ \mathbf{N}\ \mathbf{B})$ is a special orthogonal matrix, AE is also orthogonal, and moreover

$$\det AE = \det A \det E = \det A$$

We conclude that $\det A = 1 \iff AE = (A\mathbf{T}\ A\mathbf{N}\ A\mathbf{B}) = (\hat{\mathbf{T}}\ \hat{\mathbf{N}}\ A\mathbf{B})$ is *positively oriented*, whence

$$\hat{\mathbf{B}} = (\det A)A\mathbf{B} = \begin{cases} A\mathbf{B} & \text{if the isometry is direct,} \\ -A\mathbf{B} & \text{if the isometry is indirect.} \end{cases}$$

Finally, we compute the torsion:

$$\hat{\tau} = \hat{\mathbf{N}}' \cdot \hat{\mathbf{B}} = A\mathbf{N}' \cdot \big((\det A)(A\mathbf{B})\big) = (\det A)(A\mathbf{N}') \cdot (A\mathbf{B}) = (\det A)\mathbf{N}' \cdot \mathbf{B} = (\det A)\tau$$

■

Existence and Uniqueness of Solutions to ODEs

Our classification of spacecurves depends on the 'usual' existence and uniqueness result for ODEs. Here is a version suitable for our needs.

Theorem 1.39 (Existence/Uniqueness for Linear ODE (Picard, Lindelöf, etc.)).
Let $t_0 \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^n$ be given, and let $M(t) \in M_n(\mathbb{R})$ be a continuous matrix-valued function defined on an interval $|t - t_0| \le T$. Then the initial value problem

$$\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} = M(t)\mathbf{E}, \qquad \mathbf{E}(t_0) = \mathbf{c}$$

has a unique solution $\mathbf{E} : [t_0 - T, t_0 + T] \to \mathbb{R}^n$.

[10] Recall Exercise 1.3.9: when we write $\hat{\mathbf{T}} = A\mathbf{T}$ we mean that the *directional parts* of the tangent vectors are thus related.

The rough idea of the proof is to define a sequence of functions

$$\mathbf{E}_0(t) := \mathbf{c}, \qquad \mathbf{E}_1(t) := \mathbf{c} + \int_{t_0}^{t} M(u)\mathbf{E}_0(u)\,\mathrm{d}u, \qquad \mathbf{E}_2(t) := \mathbf{c} + \int_{t_0}^{t} M(u)\mathbf{E}_1(u)\,\mathrm{d}u, \ \ldots$$

which are seen to converge to the required solution; this last step requires advanced ideas from topology/analysis. A simple example should convince you of the approach.

Example 1.40. Given the initial value problem $\frac{\mathrm{d}E}{\mathrm{d}t} = 2tE$, $E(0) = 1$, we obtain

$$E_0(t) = 1, \quad E_1(t) = 1 + \int_0^t 2u\,\mathrm{d}u = 1 + t^2, \quad E_2(t) = 1 + \int_0^t 2u(1+u^2)\,\mathrm{d}u = 1 + t^2 + \frac{1}{2}t^4, \ldots$$

The *Picard iteration* builds up the correct solution as a power series

$$E(t) = e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots$$

Corollary 1.41. *Let $\mathcal{O}$ be an orthogonal matrix, $I = [t_0 - T, t_0 + T]$ an interval, and $W : I \to M_3(\mathbb{R})$ a matrix-valued function such that each $W(t)$ is* skew-symmetric*. Then:*

1. *There exists a unique solution $E : I \to \mathrm{O}_3(\mathbb{R})$ to the initial value problem*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = EW, \qquad E(t_0) = \mathcal{O}$$

2. *If $\det \mathcal{O} = 1$, then $E : I \to \mathrm{SO}_3(\mathbb{R})$.*

Proof.

1. The initial value problem is a system of nine linear first order ODEs in the entries of the 3×3 matrix E. We are therefore in the case of Picard's theorem where $\mathbf{E} : I \to \mathbb{R}^9$. There therefore exists a unique solution $E : I \to M_3(\mathbb{R})$. Now differentiate:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(EE^T) &= E'E^T + E(E')^T = EWE^T + E(EW)^T = EWE^T + EW^TE^T \\ &= EWE^T + E(-W)E^T = 0 \qquad\qquad (W^T = -W!) \end{aligned}$$

Thus EE^T is constant. However $E(t_0)E(t_0)^T = \mathrm{I}$ since $E(t_0) = \mathcal{O}$ is orthogonal. We conclude that $E(t)$ is always orthogonal.

2. Determinant is continuous (it is a polynomial!); E is differentiable, and so $\det E : I \to \mathbb{R}$ is continuous on an interval. But $\det E = \pm 1$ since E is orthogonal. It follows that $\det E$ is the constant 1.

■

For simple W, we might be able to state the solution using the matrix exponential; for instance

$$W \text{ constant} \implies E(t) = \mathcal{O}e^{tW}$$

This is of limited utility: the matrix exponential is rarely computable except as an infinite series, and the approach fails for general $W(t)$.

Corollary 1.42 (Fundamental theorem of biregular spacecurves).
Suppose we are given the following data:

- *Smooth functions $\kappa > 0$ and τ on an interval $I = [t_0 - T, t_0 + T]$.*
- *A position vector $\mathbf{c} \in \mathbb{E}^3$ and a positively oriented orthonormal basis $(\mathbf{T}_0\ \mathbf{N}_0\ \mathbf{B}_0)$ of $T_{\mathbf{c}}\mathbb{E}^3$.*

Then there exists a unique unit-speed biregular spacecurve $\mathbf{x} : I \to \mathbb{E}^3$ with curvature κ, torsion τ, initial position $\mathbf{x}(t_0) = \mathbf{c}$ and Frenet frame $E(t_0) = (\mathbf{T}_0\ \mathbf{N}_0\ \mathbf{B}_0)$ at $\mathbf{x}(t_0)$.

Proof. The structure equations $E' = EW$ put us in the situation of Corollary 1.41; there exists a unique solution $E = (\mathbf{T}\ \mathbf{N}\ \mathbf{B}) : I \to SO_3(\mathbb{R})$. Integrate the unit tangent vector field to finish:

$$\mathbf{x}(t) = \mathbf{c} + \int_{t_0}^{t} \mathbf{T}(u)\,\mathrm{d}u$$

This is plainly the unique curve with the required initial conditions, curvature and torsion. ∎

Alternatively, a biregular curve is determined up to rigid motions by its curvature and torsion.

Corollary 1.43. *Given two biregular spacecurves with the same curvature and torsion functions, there exists a unique direct isometry transforming one to the other.*

Proof. Suppose $\mathbf{x}_1 : I \to \mathbb{E}^3$ and $\mathbf{x}_2 : I \to \mathbb{E}^3$ have Frenet frames E_1, E_2, and the same curvature and torsion functions. Choose some (any!) $t_0 \in I$. The required rigid motion $S : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ must satisfy the conditions at t_0, whence[11]

$$S\big(\mathbf{x}_1(t_0)\big) = \mathbf{x}_2(t_0) \quad \text{and} \quad AE_1(t_0) = E_2(t_0)$$

Plainly $A = E_2(t_0)\big(E_1(t_0)\big)^{-1}$ and $\mathbf{b} = \mathbf{x}_2(t_0) - A\mathbf{x}_1(t_0)$ provide the unique isometry S. Moreover $\det A = 1$ since both E_1 and E_2 do so also.

By Theorem 1.38, $\mathbf{x}_3 := S(\mathbf{x}_1)$ is a spacecurve with the *same* initial conditions (at t_0), curvature and torsion as $\mathbf{x}_2$. The Fundamental Theorem says that $\mathbf{x}_2 = \mathbf{x}_3 = S(\mathbf{x}_1)$. ∎

Compare what we've done to the standard acceleration/position kinematics problem, where *three scalar functions* $\mathbf{x}''(t) = \big(x''(t), y''(t), z''(t)\big)$ and *six scalar initial conditions* $\mathbf{x}(t_0)$ and $\mathbf{x}'(t_0)$ recover the motion by twice integrating.

The Fundamental Theorem says that a spacecurve is determined uniquely by *three scalar functions* $v(t), \kappa(t), \tau(t)$ and the *initial conditions* $\mathbf{x}(t_0), \mathbf{T}(t_0), \mathbf{N}(t_0)$, which also amount to *six* scalar constants.[12]

One benefit of our result is that, by standardizing $v(t) \equiv 1$ and ignoring rigid motions, we see that the *physical shape* of a curve depends only on *two* scalar functions $\kappa(t)$ and $\tau(t)$.

[11] As in Theorem 1.38, S acts on *position vectors* but Frenet frames consist of *tangent vectors* and thus only see A.

[12] You don't need explicitly to specify $\mathbf{B}(t_0) = \mathbf{T}(t_0) \times \mathbf{N}(t_0)$! The position $\mathbf{x}(t_0)$ requires three constants; $\mathbf{T}(t_0)$ needs two angles (spherical polar co-ordinates), and $\mathbf{N}(t_0)$ a single angle in the plane $(\mathbf{T}(t_0))^{\perp}$.

We finish this discussion with a quick application of the Fundamental Theorem.

Corollary 1.44. *Every biregular curve with κ and τ constant is a circular helix (circle if $\tau \equiv 0$).*

Proof. By (Exercise 1.4.2), the circular helix $\mathbf{x}(t) = \big(r\cos t, r\sin t, ht\big)$ has constant curvature $\kappa = \frac{r}{r^2+h^2}$ and torsion $\tau = \frac{h}{r^2+h^2}$.
Given constant κ, τ, it is a simple exercise to find suitable r, h. By Corollary 1.43, this is the *only* such curve up to direct isometry (and constant speed re-parametrization). ∎

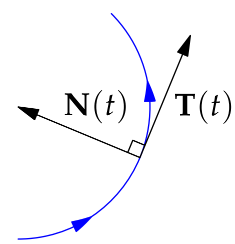
What changes in other dimensions?

For plane curves things are a little simpler. Here is a summary.

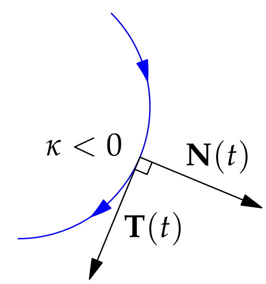
Assume: $\mathbf{x} : I \to \mathbb{E}^2$ is regular; we don't need biregularity.

Frenet frame: Define $\mathbf{T} := \frac{1}{v}\mathbf{x}'$ and $\mathbf{N} := J\mathbf{T}$, where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is rotation by 90° counter-clockwise; no differentiation is required to compute $\mathbf{N}$!

Curvature: $\kappa = \frac{1}{v}\mathbf{T}' \cdot \mathbf{N}$ is *signed* as we saw in Section 1.2:



$\kappa > 0$: $\mathbf{x}$ bends towards $\mathbf{N}$



$\kappa < 0$: $\mathbf{x}$ bends away from $\mathbf{N}$

Frenet–Serret equations: In arbitrary parametrization

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' \end{pmatrix} = \begin{pmatrix} \mathbf{T} & \mathbf{N} \end{pmatrix} \begin{pmatrix} 0 & -v\kappa \\ v\kappa & 0 \end{pmatrix}$$

Isometries: Direct preserves κ, indirect changes its sign.

Fundamental Theorem: Given $\kappa(s)$, $\mathbf{x}(s_0) \in \mathbb{E}^2$ and $\mathbf{T}_0 \in T_{\mathbf{x}(s_0)}\mathbb{E}^2$, there exists a unique unit-speed curve with curvature $\kappa(s)$ and these initial data. Exercise 7 gives an elementary proof.

Example 1.45. Constant κ curves are circles, as we can see explicitly in a couple of ways. The unit-speed structure equations $E' = E\left(\begin{smallmatrix} 0 & -\kappa \\ \kappa & 0 \end{smallmatrix}\right)$ become $\mathbf{T}'' = -\kappa^2\mathbf{T}$ which may be explicitly integrated. Alternatively, $\theta'(t) = \kappa \implies \theta(t) = \kappa t + c$, yields an explicit circle of radius $\frac{1}{\kappa}$:

$$\mathbf{T}(t) = \begin{pmatrix} \cos(\kappa t + c) \\ \sin(\kappa t + c) \end{pmatrix} \implies \mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{T}(u)\,\mathrm{d}u = \mathbf{x}(t_0) + \frac{1}{\kappa}\begin{pmatrix} -\sin(\kappa t + c) \\ \cos(\kappa t + c) \end{pmatrix}$$

We can also play this game in higher dimensions. Given a unit-speed curve $\mathbf{x} : I \to \mathbb{E}^n$ whose first $n-1$ derivatives at each point are linearly independent, apply Gram-Schmidt orthogonalization to obtain a moving frame $E = \begin{pmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{pmatrix}$ and functions $\kappa_1, \ldots, \kappa_{n-1}$ (the generalized curvatures) satisfying the structure equations shown.

$$E' = E \begin{pmatrix} 0 & -\kappa_1 & 0 & \cdots & 0 & 0 \\ \kappa_1 & 0 & -\kappa_2 & & 0 & 0 \\ 0 & \kappa_2 & 0 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & -\kappa_{n-1} \\ 0 & 0 & 0 & \cdots & \kappa_{n-1} & 0 \end{pmatrix}$$

Conversely, the $n-1$ generalized curvatures determine the curve up to rigid motions.

Exercises 1.5. 1. Find an explicitly parametrized curve with constant curvature κ and torsion τ.

2. Reflection in the xy-plane $S(\mathbf{x}) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)\mathbf{x}$ is an indirect isometry. Explicitly compare the curvature and torsion of the standard helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ with those of $S(\mathbf{x})$.

3. In the manner of Example 1.40, compute the Picard iteration process up to $\mathbf{E}_3(t)$ for the initial value problem

$$\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{E}, \quad \mathbf{E}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Verify that this comports with the correct solution $\mathbf{E}(t) = \left(\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix}\right)$ to this system of ODEs.

4. Suppose f is a function such that $\mathbf{x}(t) = (\cos t, \sin t, f(t))$ lies in a fixed plane. Show that f satisfies the 3rd-order linear ODE $f'''(t) + f'(t) = 0$ and thus find all possible functions f.

   (*Hints: What is the torsion of a plane curve?*)

5. Assume that all principal normals of a biregular curve in $\mathbb{E}^3$ pass through a fixed point: $\exists\alpha(t)$ and a constant $\mathbf{n}$ such that $\mathbf{x}(t) + \alpha(t)\mathbf{N}(t) = \mathbf{n}$. Show that the curve is (part of) a *circle*.

6. Let $\mathbf{x} : I \to \mathbb{E}^2$ be a regular curve and let $\mathbf{y} = S(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ be a new curve resulting from a rigid motion. Prove that the curvatures of $\mathbf{x}$ and $\mathbf{y}$ are identical.

7. For regular curves in $\mathbb{E}^2$, the Fundamental Theorem is relatively simple to prove.

   (a) Suppose you are given a smooth function $\kappa : I \to \mathbb{R}$ on an interval I containing t_0, an initial position $\mathbf{x}(t_0) = (a, b)$ and an initial direction $\theta(t_0) = \theta_0$ (angle with positive x-axis).

   Use the Fundamental Theorem of Calculus to describe the unique unit-speed curve $\mathbf{x} : I \to \mathbb{E}^2$ with curvature κ and given initial data.

   (*Hints: use* $\theta(t) := \theta_0 + \int_{t_0}^{t} \kappa(u)\,\mathrm{d}u$ *to define* $\mathbf{T}(t)$ *and integrate! Your answer will contain definite integrals.*)

   (b) Suppose $\mathbf{x} : \mathbb{R} \to \mathbb{E}^2$ is unit-speed with $\kappa(t) = \frac{1}{1+t^2}$, $\mathbf{x}(0) = (0, 0)$, and $\mathbf{x}'(0) = (1, 0)$. Find $\mathbf{x}(t)$.

8. (Hard) A *cylindrical helix* is a curve $\mathbf{x}(t)$ whose unit tangent field $\mathbf{T}(t)$ makes a constant angle $\theta \in (0, \frac{\pi}{2})$ with a fixed vector $\mathbf{n}$.

   (a) If $\mathbf{x}(t) = (\cos t, \sin t, t)$ is the standard circular helix, describe a suitable vector $\mathbf{n}$.

   (b) Use the Frenet–Serret formulas to prove that a (unit-speed) non-planar curve is a cylindrical helix if and only if κ/τ is constant.

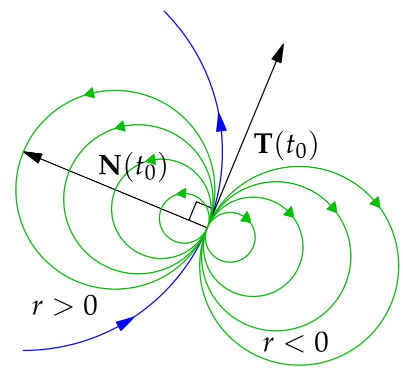
9. (Very hard) Suppose a moving frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ has structure equations where all three functions w_{12}, w_{13}, w_{23} are constant. Find the moving frame $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ where $\mathbf{f}_1 = \mathbf{e}_1$ such that $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ is the Frenet frame of a unit-speed circular helix. Calculate the curvature κ and torsion τ of this helix in terms of w_{12}, w_{13}, w_{23}. Can you find an orthogonal matrix A such that

$$A^{-1}\begin{pmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}?$$

1.6 Radii of curvature

We have seen how curvature measures the deviation of a curve from a straight line and that the only planar curves with constant curvature κ are circles of radius $\frac{1}{\kappa}$. We could have started with this as our definition; at a given point, a curve has curvature κ if the circle which best approximates the curve has radius $\frac{1}{\kappa}$. Of course, we have to define what is meant by *best approximation.*

Definition 1.46. Unit-speed curves $\mathbf{x}, \mathbf{y}$ have n^{th} *order contact* at an intersection point $\mathbf{x}(t_0) = \mathbf{y}(s_0)$, if their first n derivatives agree there: $\mathbf{x}^{(j)}(t_0) = \mathbf{y}^{(j)}(s_0)$ for all $1 \le j \le n$.



Let $\mathbf{x}(t)$ be a unit-speed curve in $\mathbb{E}^2$, fix $r \neq 0$ and consider the unit-speed circle $\mathbf{c}(s)$ with (signed) radius r for which

$$\mathbf{c}(0) = \mathbf{x}(t_0) \quad \text{and} \quad \mathbf{c}'(0) = \mathbf{x}'(t_0)$$

We take $r > 0 \iff$ the circle lies on the same side of the curve as the principal normal vector $\mathbf{N}$.

The circle is straightforward to parametrize:

$$\mathbf{c}(s) = \underbrace{\mathbf{x}(t_0) + r\mathbf{N}(t_0)}_{\text{center}} + \underbrace{r\sin(s/r)\mathbf{T}(t_0) - r\cos(s/r)\mathbf{N}(t_0)}_{\text{rotation}}$$

Certainly this circle has 1st-order contact with the curve: $\mathbf{c}(0) = \mathbf{x}(t_0)$ and

$$\mathbf{c}'(s) = \cos(s/r)\mathbf{T}(t_0) + \sin(s/r)\mathbf{N}(t_0) \implies \mathbf{c}'(0) = \mathbf{T}(t_0) = \mathbf{x}'(t_0)$$

Moreover,

$$\mathbf{c}''(s) = -\frac{1}{r}\sin(s/r)\mathbf{T}(t_0) + \frac{1}{r}\cos(s/r)\mathbf{N}(t_0) \implies \mathbf{c}''(0) = \frac{1}{r}\mathbf{N}(t_0)$$

The circle has second-order contact with the curve if and only if

$$\mathbf{c}''(0) = \mathbf{x}''(t_0) \iff \frac{1}{r} = \kappa(t_0)$$

There is nothing stopping us from finding this circle for an arbitrary speed regular curve, since all we need is the curvature and the Frenet frame at the relevant point.

Definition 1.47. Let $\mathbf{x}(t)$ be a regular curve. At a point $\mathbf{x}(t_0)$ with non-zero curvature:

- The *radius of curvature* is $r = \frac{1}{\kappa(t_0)}$.
- The *center of curvature* is the point with position vector $\mathbf{x}(t_0) + r\mathbf{N}(t_0)$.
- The *osculating circle* is the radius r circle centered at the center of curvature. It has unit-speed parametrization

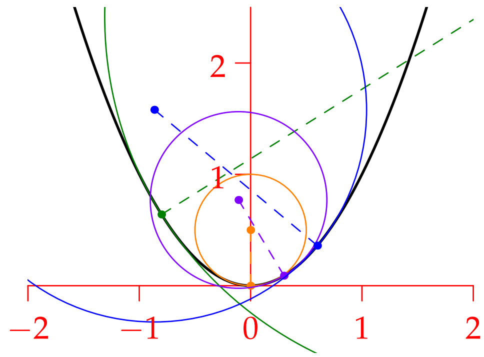
$$\mathbf{c}(s) = \mathbf{x}(t_0) + \frac{1}{\kappa(t_0)}\big(\sin(s/r)\mathbf{T}(t_0) + (1 - \cos(s/r))\mathbf{N}(t_0)\big)$$

Osculating means 'kissing.' If $\kappa(t_0) = 0$, some consider the tangent line to be an osculating circle with infinite radius!

Example 1.48. We find the osculating circles for the parabola $y = x^2$ parametrized in the obvious manner $\mathbf{x}(t) = (t, t^2)$. The relevant ingredients are

$$\mathbf{x}'(t) = \begin{pmatrix}1\\2t\end{pmatrix} \implies \mathbf{T}(t) = \frac{1}{\sqrt{1+4t^2}}\begin{pmatrix}1\\2t\end{pmatrix} \qquad \mathbf{N}(t) = \frac{1}{\sqrt{1+4t^2}}\begin{pmatrix}-2t\\1\end{pmatrix}$$

$$\mathbf{x}''(t) = \begin{pmatrix}0\\2\end{pmatrix}, \qquad \kappa(t) = \frac{2}{(1+4t^2)^{3/2}}$$



The center of curvature when $t = t_0$ has position vector

$$\mathbf{x}(t_0) + \frac{1}{\kappa(t_0)}\mathbf{N}(t_0) = \begin{pmatrix}-4t_0^3\\ \frac{1}{2}+3t_0^2\end{pmatrix}$$

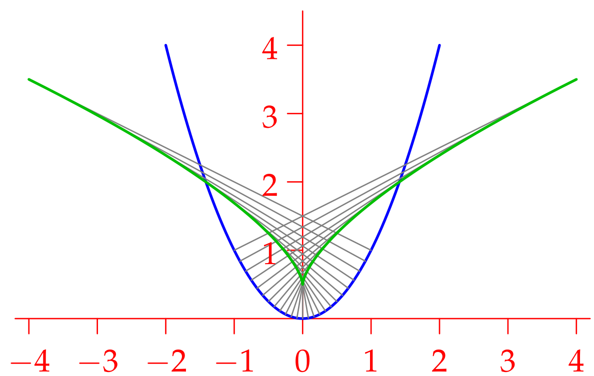
Several osculating circles are drawn and their centers of curvature indicated.

The centers of curvature describe a curve that is interesting in its own right.

Definition 1.49. Let $\mathbf{x}(t)$ be a regular plane curve with non-zero curvature. The curve $\mathbf{e}(t)$ defined by the centers of curvature is the *evolute* of $\mathbf{x}(t)$:

$$\mathbf{e}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$$

Example (1.48 cont). The evolute of the parabola $\mathbf{x}(t) = (t, t^2)$ was found above:



$$\mathbf{e}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = \begin{pmatrix}-4t^3\\ \frac{1}{2}+3t^2\end{pmatrix}$$

Alternatively, this is the graph $y = \frac{1}{2} + 3\left(\frac{x}{4}\right)^{2/3}$: notice that this isn't regular at $x = 0$.

The picture now animates to show the osculating circles and the construction of the evolute.

The gray lines are the *normal lines* to the parabola, and are also *tangent* to the evolute.

$$\mathbf{e}' = \mathbf{x}' - \frac{\kappa'}{\kappa^2}\mathbf{N} + \frac{1}{\kappa}(-v\kappa\mathbf{T}) = -\frac{\kappa'}{\kappa^2}\mathbf{N}$$

This last means that the evolute is a *focal curve* for the family of normal lines. The same equation shows that the evolute is regular precisely when $\kappa'(t) \neq 0$.

A related notion is the *involute*, which may be imagined by rolling a line along a curve and seeing what curve the end of the line traces out.

Definition 1.50. Suppose $\mathbf{x}(t)$ has unit speed. Its *involute* is the curve

$$\mathbf{i}(t) := \mathbf{x}(t) - t\mathbf{x}'(t) = \mathbf{x}(t) - t\mathbf{T}(t)$$

An involute depends crucially on its parametrization: it intersects its source curve when $t = 0$.

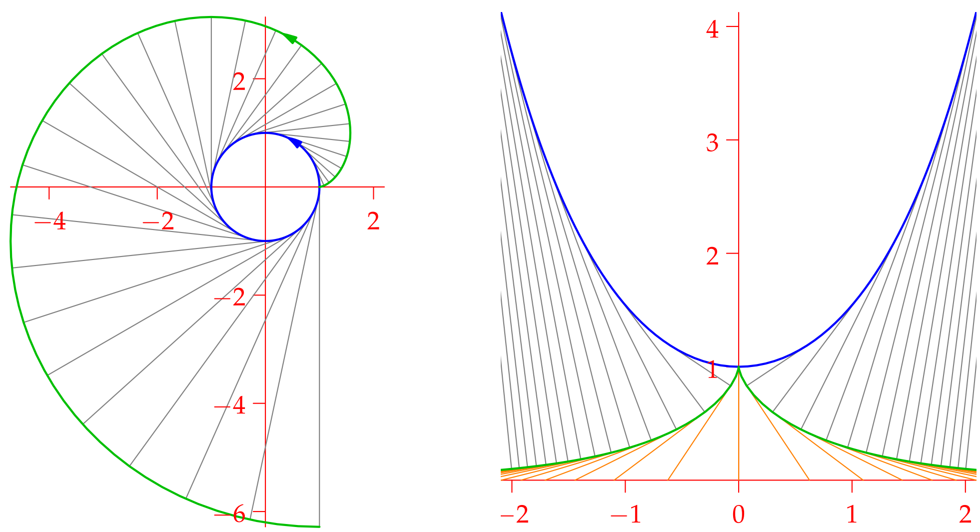
Examples 1.51. 1. The unit speed unit circle $\mathbf{x}(t) = \big(\cos t, \sin t\big)$. Its involute is therefore

$$\mathbf{i}(t) = \mathbf{x}(t) - t\mathbf{T}(t) = \begin{pmatrix} \cos t + t \sin t \\ \sin t - t \cos t \end{pmatrix}$$

2. The involute of the unit speed *catenary* $\mathbf{x}(t) = \big(\sinh^{-1} t, \sqrt{1+t^2}\big)$ is the *tractrix*:

$$\mathbf{i}(t) = \begin{pmatrix} \sinh^{-1} t - t(1+t^2)^{-1/2} \\ (1+t^2)^{-1/2} \end{pmatrix}$$

This is the curve obtained when an object starting at the point $(0,1)$ is dragged (subjected to *traction*) by attaching a rope of length 1 to a vehicle moving along the x-axis.



Circle and involute (spiral) Catenary and involute (tractrix)

Another way to visualize the involute of the catenary is to imagine attaching a weight at $(0,1)$ to a long string wrapped tightly along the catenary and then releasing the weight. Similarly, imagine a string is wound tightly around the circle and then uncoiled; the result is the involute.

Theorem 1.52. *The evolute of any involute is the original curve, except where $t = 0$ or $\kappa = 0$.*

We leave the argument as an exercise. The reverse process fails, as an observation of the parabola example should convince you: remember that an involute intersects its source curve at $t = 0$...

Exercises 1.6. 1. Find the center of curvature for the curve $\mathbf{x}(t) = (1 - t^{-1}, 1 + t)$ at $t = 1$.

2. Consider the ellipse $\mathbf{x}(t) = \big(a\cos t, b\sin t\big)$ where $a > b > 0$.
   (a) Compute the curvature of the ellipse.
   (b) Show that its evolute is the *astroid* $\mathbf{e}(t) = (a^2 - b^2)\left(\begin{smallmatrix} a^{-1}\cos^3 t \\ -b^{-1}\sin^3 t\end{smallmatrix}\right)$
   (c) The four-vertex theorem states that a simple closed plane curve with differentiable curvature has at least four points where $\kappa' = 0$. Show that the ellipse has precisely four.

3. Describe the involutes of a straight line.
   (*Hint: this is a trick question!*)

4. In Example 1.51.2 we constructed the tractrix as the involute of the catenary.
   (a) Use $\sinh^{-1} t = \ln\big(t + \sqrt{1+t^2}\big)$ to verify that $\mathbf{x}(t)$ has unit speed and thus confirm the derivation of $\mathbf{i}(t)$.
   (b) Compute the tangent line to the tractrix when $t > 0$ and show that this line cuts the x-axis a distance 1 from the curve, thus justifying the *traction* claim.

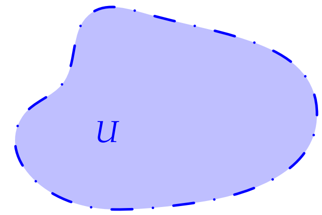
5. Suppose that the graph of a smooth function $y = f(x)$ passes horizontally through the origin: $f(0) = 0 = f'(0)$. Show that its Maclaurin series is
   $$f(x) \approx \frac{1}{2}\kappa(0)x^2 + \text{higher order terms}$$
   Use this to *quickly* state the curvature at $x = 0$ of the graph of $y = x^2(7x^2 - 29)$.

6. Let $\mathbf{x}(t)$ be unit speed with non-zero curvature κ and Frenet frame $\{\mathbf{T}, \mathbf{N}\}$. Moreover, let $\mathbf{i}(t) = \mathbf{x}(t) - t\mathbf{T}(t)$ be an involute and denote the speed and corresponding data for the involute $\hat{v}, \hat{\kappa}, \hat{\mathbf{T}}, \hat{\mathbf{N}}$. For simplicity, suppose $\kappa, t > 0$.
   (a) Compute the Frenet frame of $\mathbf{i}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$.
   (b) Show that $\hat{\kappa}(t) = \frac{1}{t}$.
   (c) Show that the evolute of $\mathbf{i}(t)$ is the original curve $\mathbf{x}(t)$.

7. We see how an involute of the evolute fails to recover the original curve.
   Let $\mathbf{x}(t)$ be regular with non-zero curvature, $\kappa'(t) \neq 0$, and evolute $\mathbf{e}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$. Since $\mathbf{e}(t)$ is regular, we may assume it is parametrized by arc-length.
   (a) If $\kappa' > 0$, explain why $\kappa' = \kappa^2$.
   (b) Show that the natural involute of the evolute is
   $$\mathbf{e}(t) - t\mathbf{e}'(t) = \mathbf{x}(t) + \frac{1}{\kappa(0)}\mathbf{N}(t)$$
   that is, the original curve shifted a constant distance $\frac{1}{\kappa(0)}$ in its normal direction.
   (*Hint: the ODE in part (a) is separable*)
   (c) Find the involute of the evolute of the parabola $\mathbf{x}(t) = (t, t^2)$.

2 Vector Fields & Differential Forms

In preparation for our study of surfaces, we further develop the notion of a tangent vector. To permit easy differentiation, throughout this section all functions are assumed to be *smooth* (infinitely differentiable) and $U \subseteq \mathbb{R}^n$ will denote a *connected open set*: (informally) a region consisting of a single piece without edge points. As previously, n will always be 1, 2 or 3: when $n = 1$, $U = (a, b)$ is an open interval; the picture illustrates $n = 2$.



2.1 Directional Derivatives, Tangent Vectors & Vector Fields

First recall some basic objects and facts from elementary multivariable calculus.

Definition 2.1. The *gradient* of $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is the function $\nabla f : U \to \mathbb{R}^n$ defined by

$$\nabla f(x_1, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Given a point $p \in U$, a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, and a function $f : U \to \mathbb{R}$, the *directional derivative* of f at p in the direction $\mathbf{v}$ is the *scalar*

$$D_{\mathbf{v}} f(p) := \sum_{k=1}^{n} v_k \left. \frac{\partial f}{\partial x_k} \right|_p = \mathbf{v} \cdot (\nabla f(p))$$

Example 2.2. Suppose $f(x, y, z) = x^2 - z \cos y$, $p = (1, \pi, 0)$, and $\mathbf{v} = (3, 5, 1)$. Then

$$\nabla f = \begin{pmatrix} 2x \\ z \sin y \\ -\cos z \end{pmatrix} \implies D_{\mathbf{v}} f(p) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 7$$

The directional derivative describes the rate of change of the value of f in a given direction.

Lemma 2.3. 1. *By the chain rule, if $\mathbf{x}(t)$ is a curve such that $\mathbf{x}(0) = p$ and $\mathbf{x}'(0) = \mathbf{v}$, then*

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} f(\mathbf{x}(t)) = \sum_{k=1}^{n} \left. \frac{\partial f}{\partial x_k} \right|_p x_k'(0) = D_{\mathbf{v}} f(p)$$

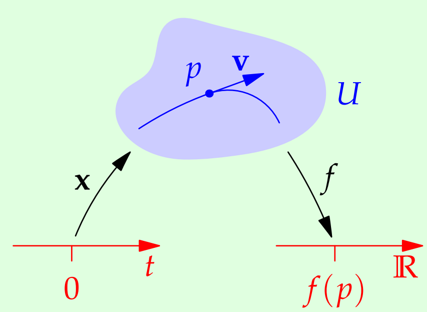
is the rate of change of f at p as one travels along the curve.

2. *If t is small, then $f(p + t\mathbf{v}) \approx f(p) + D_{\mathbf{v}} f(p)\, t$.*

3. *If $\mathbf{v}$ is a unit vector making angle θ with $\nabla f(p)$, then*

$$D_{\mathbf{v}} f(p) = \mathbf{v} \cdot \nabla f(p) = ||\nabla f(p)|| \cos \theta$$

is maximal when $\mathbf{v}$ points in the same direction as $\nabla f(p)$. Otherwise said, $\nabla f(p)$ points in the direction of greatest increase of f at p; its magnitude measures the rate of change.



By placing the function f at the end of the directional derivative, we are tempted to create an *operator*

$$D_{\mathbf{v}}\big|_p = \sum_{k=1}^n v_k \left.\frac{\partial}{\partial x_k}\right|_p$$

which takes a *function* $f : U \to \mathbb{R}$ and returns the *scalar* $D_{\mathbf{v}} f(p)$. This operator is a map (function) from the set of smooth functions $f : U \to \mathbb{R}$ to the real numbers. It is even more tempting to drop the point p and allow the components of $\mathbf{v}$ to be *smooth functions.* This yields a new definition of an old concept.

Definition 2.4. The set of directional derivative operators $D_{\mathbf{v}}\big|_p$ is the *tangent space* $T_p\mathbb{R}^n$ at $p \in \mathbb{R}^n$. A *vector field* v on $U \subseteq \mathbb{R}^n$ is a smooth choice for each $p \in U$ of an element of $T_p\mathbb{R}^n$: that is

$$v = \sum_{k=1}^n v_k \frac{\partial}{\partial x_k} \quad \text{where each } v_k : U \to \mathbb{R} \text{ is smooth}$$

Each operator $\frac{\partial}{\partial x_k}$ is termed a *co-ordinate vector field.*
If $f : U \to \mathbb{R}$ is smooth, we write $v[f] = \sum v_k \frac{\partial f}{\partial x_k}$ for the result of applying the vector field v to f; this is itself a smooth function $v[f] : U \to \mathbb{R}$.

Each tangent space $T_p\mathbb{R}^n$ is a vector space, with natural basis $\frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_n}\big|_p$. In this brave new world, a tangent vector $v_p = \sum v_k \frac{\partial}{\partial x_k}\big|_p$ corresponds to our previous notion $\mathbf{v}_p = (v_1, \ldots, v_n)$. While this might seem artificially complicated, the rational is simple: the purpose of tangent vectors is to measure how functions change in given directions (Lemma 2.3!).

Examples 2.5. 1. The vector field $v = 3x\frac{\partial}{\partial x} + 2xz\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ on $\mathbb{R}^3$ corresponds to the vector-valued function $\mathbf{v}(x,y,z) = (3x, 2xz, -x)$. Given $f(x,y,z) = xy^2 + z$, we have

$$v[f] = 3x\frac{\partial f}{\partial x} + 2xz\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial z} = 3xy^2 + 4x^2yz - x$$

which, as expected, is a smooth function $v[f] : \mathbb{R}^3 \to \mathbb{R}$.

2. Suppose, in $\mathbb{R}^2$, that we are given a *vector field* $v = y^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, a *function* $f(x,y) = x^2y$, and a *point* $p = (2, -1)$. These may be combined in various ways, for instance:

Vector field on $\mathbb{R}^2$	$fv = x^2y\left(y^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = x^2y^3\frac{\partial}{\partial x} - x^3y\frac{\partial}{\partial y}$
Tangent vector	$(fv)(p) = f(p)v_p = -4\left.\frac{\partial}{\partial x}\right\vert_p + 8\left.\frac{\partial}{\partial y}\right\vert_p \in T_p\mathbb{R}^2$
Function $\mathbb{R}^2 \to \mathbb{R}$	$v[f] = y^2\frac{\partial}{\partial x}(x^2y) - x\frac{\partial}{\partial y}(x^2y) = 2xy^3 - x^3$
Number	$\big(v[f]\big)(p) = -4 - 8 = -12$

Note the use of different brackets! Note also that fv denotes the vector field obtained by *multiplying* v by the *value* of f at each point. It does *not* mean *apply the function f to the vector field v*, which makes no sense!

Here are the basic rules of computation for vector fields. These are all essentially trivial if you take $v = \sum v_k \frac{\partial}{\partial x_k}$, etc., as in Definition 2.4. Just be careful with notation!

Lemma 2.6. *Let v, w be vector fields on U, let $f, g : U \to \mathbb{R}$ be smooth, and $a, b \in \mathbb{R}$ constant. Then,*

1. *$fv + gw$ is a vector field: at each $p \in U$, $(fv + gw)(p) := f(p)v_p + g(p)w_p$*
2. *Vector fields act linearly on smooth functions: $v[af + bg] = av[f] + bv[g]$*
3. *(Leibniz rule) Vector fields obey a product rule: $v[fg] = fv[g] + gv[f]$*

Examples 2.7. 1. We verify the Leibniz rule for the vector field $v = \frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}$ and functions $f(x,y) = x$ and $g(x,y) = ye^x$.

$$v[fg] = \left(\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\right)[xye^x] = ye^x + xye^x - x^2ye^x$$

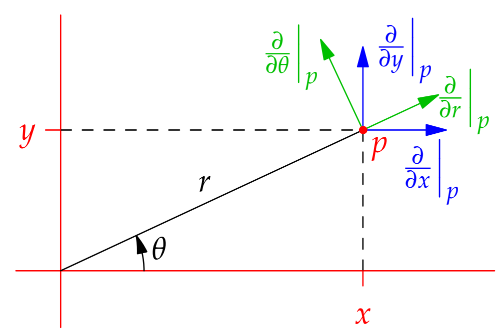
$$fv[g] + gv[f] = x\left(\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\right)[ye^x] + ye^x\left(\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\right)[x] = x(ye^x - xye^x) + ye^x$$

2. (Polar co-ordinates) Let U be the plane without the non-positive x-axis. On U, the standard *rectangular co-ordinates* (x, y) are related to the *polar co-ordinates* (r, θ) via

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \leftrightsquigarrow \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\frac{y}{x} \quad (\text{or } \pm\frac{\pi}{2} \text{ if } x = 0) \end{cases}$$

The chain rule tells us that the co-ordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ are related via

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ &= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned}$$



At p, these point in the direction of maximal increase for the corresponding co-ordinate.

We could similarly compute $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by differentiating. For variety, we instead use linear algebra:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \implies \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$

$$\implies \begin{cases} \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{cases}$$

The first matrix is the familiar *Jacobian* $\frac{\partial(x,y)}{\partial(r,\theta)}$ from multivariable calculus. Strictly, we are viewing U as subsets of *two different versions of* $\mathbb{R}^2$:

- In rectangular co-ordinates, $U = \mathbb{R}^2 \setminus \{(x,0) : x \le 0\}$ is a cut plane.
- In polar co-ordinates, $U = (0,\infty) \times (-\pi,\pi)$ is an infinite open rectangle.

In practice, particularly since we are so familiar with polar co-ordinates, it is easier to stick to the first interpretation and draw all four co-ordinate tangent vectors on the same picture.

Exercises 2.1. 1. You are given the following vector fields and functions

$$u = 7\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} \qquad v = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \qquad w = \sin x\frac{\partial}{\partial x} - 2\cos x\frac{\partial}{\partial y}$$
$$f(x,y) = xy^2 \qquad g(x,y) = -y$$

Compute the *functions*:

(a) $u[f]$ (b) $v[f]$ (c) $w[f]$

(d) $v[fg]$ (e) $fu[g]$ (f) $v\big[w[g]\big]$

2. Revisit Example 2.7.2 on polar co-ordinates.

   (a) Use the chain rule to compute $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ directly in terms of r,θ, $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ and verify that you obtain the same expressions as the linear algebra approach.

   (b) Suppose $T_p\mathbb{R}^2$ is equipped with the standard dot product so that $\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$ are considered orthonormal.

      i. Show that $\frac{\partial}{\partial r}\Big|_p$ and $\frac{\partial}{\partial \theta}\Big|_p$ are perpendicular.

      ii. What are the lengths of $\frac{\partial}{\partial r}\Big|_p$ and $\frac{\partial}{\partial \theta}\Big|_p$?

3. Consider the spherical polar co-ordinate system

$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\sin\theta\cos\phi \\ z = r\sin\phi \end{cases} \qquad \text{where } r > 0,\ 0 < \theta < 2\pi \text{ and } -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

   Show that

$$\frac{\partial}{\partial r} = \frac{1}{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

4. Prove the Leibniz rule (Lemma 2.6 part 3).

5. If f,g,h are smooth functions and v is a vector field, expand $v[fgh]$ using the Leibniz rule.

6. Let $s = x^2 - y^2$ and $t = 2xy$. Compute $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

   (*Hint: use the chain rule to find* $\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial y}$*, then invert the Jacobian*)

2.2 Differential 1-forms

Make sure you are comfortable with vector fields *before* you tackle this section and the next! There is a lot of new notation to get used to here, but with a little practice it is very easy to use.

Definition 2.8. Let $(x_1, \ldots, x_n)$ be co-ordinates on $U \subseteq \mathbb{R}^n$ and $p \in U$. The *(co-ordinate) 1-form* $\mathrm{d}x_k$ *at* p is the *linear map*[13] $\mathrm{d}x_k : T_p\mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathrm{d}x_k \left(\left. \frac{\partial}{\partial x_j} \right|_p \right) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

A *1-form* $\alpha = \sum_{k=1}^{n} a_k \, \mathrm{d}x_k$ on U is a smooth assignment ($a_k : U \to \mathbb{R}$ *smooth*) of 1-forms.
If v is a vector field on U, we write $\alpha(v)$ for the *function* $U \to \mathbb{R}$ obtained by mapping $p \mapsto \alpha(v_p)$.

Examples 2.9. 1. Consider the vector field $v = xy\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. At each $p \in \mathbb{R}^2$, the components xy and -2 are *scalars* and thus ignored by the linear map $\mathrm{d}x : T_p\mathbb{R}^2 \to \mathbb{R}$. We therefore obtain a *function* $\mathrm{d}x(v) : \mathbb{R}^2 \to \mathbb{R}$

$$\mathrm{d}x(v) = \mathrm{d}x \left(xy\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} \right) = xy \, \mathrm{d}x \left(\frac{\partial}{\partial x} \right) - 2 \, \mathrm{d}x \left(\frac{\partial}{\partial y} \right) = xy$$

2. Again on $\mathbb{R}^2$, let $\alpha = 2x \, \mathrm{d}x + \mathrm{d}y$ and $v = x^2y\frac{\partial}{\partial x} - e^{xy}\frac{\partial}{\partial y}$. Then

$$\alpha(v) = (2x \, \mathrm{d}x + \mathrm{d}y) \left(x^2y\frac{\partial}{\partial x} - e^{xy}\frac{\partial}{\partial y} \right) = 2x^3y - e^{xy}$$

Remember that a 1-form α is linear *when restricted to each tangent space* $T_p\mathbb{R}^n$: if $v_p \in T_p\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$, we obtain a *real number*

$$\alpha_p(f(p)v_p) = f(p)\alpha_p(v_p) \in \mathbb{R}$$

by pointwise multiplication by the value of f. Taken over all points p, this means that scalar *functions* come straight through a 1-form: if v is a vector field on U, then

$$\alpha(fv) = f\alpha(v)$$

Definition 2.10. Let $f : U \to \mathbb{R}$ be *smooth*. The *exterior derivative* of f is the 1-form

$$\mathrm{d}f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \mathrm{d}x_k = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n$$

If a 1-form is the exterior derivative of a function, we say that it is *exact*.

[13] For those who've met dual vector spaces in linear algebra, the set of 1-forms at p is the *cotangent space* $T_p^*\mathbb{R}^n$, or the space of *covectors*. At each p, $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ is the *dual basis* to $\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \ldots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$.

Our approach essentially splits a derivative into two pieces: for each k, we have $\mathrm{d}f\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial f}{\partial x_k}$. Moreover, since a linear map $(\mathrm{d}f_p : T_p\mathbb{R}^n \to \mathbb{R})$ is determined by what it does to a basis, the exterior derivative $\mathrm{d}f$ is the *unique* 1-form with the property that $\mathrm{d}f(v) = v[f]$ for all vector fields v on U. This says that the definition is *co-ordinate independent* (does not depend on $x_1, \ldots, x_n$).

Examples 2.11. 1. Let $f(x,y) = x^2y$, then $\mathrm{d}f = \alpha = 2xy\,\mathrm{d}x + x^2\mathrm{d}y$. As a sanity check, consider a general vector field $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ (remember that a, b are smooth functions!) and compute

$$\mathrm{d}f(v) = 2axy + bx^2 = v[x^2y]$$

2. If $\alpha = 4xy^2\,\mathrm{d}x + (4x^2y+1)\,\mathrm{d}y = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$ is exact, then 'partial integration' forces

$$f(x,y) = \int 4xy^2\,\mathrm{d}x = 2x^2y^2 + g(y) = \int 4x^2y + 1\,\mathrm{d}y = 2x^2y^2 + y + h(x)$$

for some functions g, h. Plainly g, h must be constant and $\alpha = \mathrm{d}(2x^2y^2 + y)$.

3. We could a similar game to see that $\alpha = 3x^2y\,\mathrm{d}x + 2\,\mathrm{d}y$ is *not* exact on $\mathbb{R}^2$. Alternatively, note that if $\alpha = \mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$, we obtain a contradiction by observing that the mixed partial derivative is simultaneously

$$3x^2 = \frac{\partial f_x}{\partial y} = f_{xy} = f_{yx} = \frac{\partial f_y}{\partial x} = 0$$

See Exercise 6 for the general result.

Lemma 2.12. *If f, g are smooth functions, then*

1. $\mathrm{d}(f+g) = \mathrm{d}f + \mathrm{d}g$
2. $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$
3. $\mathrm{d}f = 0 \iff f$ *is a constant function*

Proof. These follow straight from the definition of $\mathrm{d}f$. For instance

$$\mathrm{d}f = 0 \iff \frac{\partial f}{\partial x_j} = \mathrm{d}f\left(\frac{\partial}{\partial x_j}\right) = 0 \text{ for all } j = 1, \ldots, n \iff f \text{ is constant}$$

■

Example (2.7.2 cont). The exterior derivative and part 2 of the Lemma make it easy to compute the relationship between the 1-forms $\mathrm{d}x, \mathrm{d}y, \mathrm{d}r, \mathrm{d}\theta$:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} \mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta \end{cases} \implies \begin{cases} \mathrm{d}r = \frac{1}{r}(x\,\mathrm{d}x + y\,\mathrm{d}y) \\ \mathrm{d}\theta = \frac{1}{r^2}(-y\,\mathrm{d}x + x\,\mathrm{d}y) \end{cases}$$

We may also verify directly that the dual basis relations hold; for instance,

$$\begin{aligned}\mathrm{d}r\left(\frac{\partial}{\partial r}\right) &= \frac{1}{r}(x\,\mathrm{d}x + y\,\mathrm{d}y)\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) = \frac{1}{r}(x\cos\theta + y\sin\theta) \\ &= \cos^2\theta + \sin^2\theta = 1\end{aligned}$$

Elementary Calculus & Line Integrals

It is worth reviewing some staples from basic calculus in our new language.

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then its exterior derivative $\mathrm{d}f = f'(x)\,\mathrm{d}x$ feels familiar.[14] To make sense of this as a relation between 1-forms we need *vector fields*: the derivative of f isn't the ratio of two 1-forms, rather it is the application of the 1-form $\mathrm{d}f$ to the vector field $\frac{\mathrm{d}}{\mathrm{d}x}$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[f] = \mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

Vector fields in $\mathbb{R}$ are written with a straight d rather than partial ∂ since there is only one direction in which to differentiate!

You've seen 1-forms before when integrating: we integrate 1-forms over oriented curves.

Definition 2.13. Let α be a 1-form on $U \subseteq \mathbb{R}^n$ and suppose $\mathbf{x} : [a, b] \to U$ parametrizes a smooth curve C. Our usual identification (Definition 2.4) produces the *tangent vector field*

$$\mathbf{x}'(t) = x_1'(t)\frac{\partial}{\partial x_1} + \cdots + x_n'(t)\frac{\partial}{\partial x_n}$$

along the curve. Now define the integral of α along C by

$$\int_C \alpha := \int_a^b \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t = \int_a^b \alpha\left(x_1'(t)\frac{\partial}{\partial x_1} + \cdots + x_n'(t)\frac{\partial}{\partial x_n}\right)\mathrm{d}t$$

Examples 2.14. 1. We integrate $\alpha = x\,\mathrm{d}y$ over the unit-circle $\mathbf{x}(t) = (\cos t, \sin t)$ counter-clockwise. Differentiate to obtain the tangent vector field $\mathbf{x}'(t) = -\sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y}$, then

$$\int_C \alpha = \int_0^{2\pi} \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t \int_0^{2\pi} \cos^2 t\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} 1 + \cos 2t\,\mathrm{d}t = \pi$$

2. Integrate $\alpha = y^2\,\mathrm{d}x - x^2\,\mathrm{d}y$ over the curve $\mathbf{x}(t) = (t, t^2)$ between $(0, 0)$ and $(1, 1)$:

$$\begin{aligned}\int_C \alpha &= \int_0^1 \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t = \int_0^1 \alpha\left(\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\right)\mathrm{d}t = \int_0^1 \Big(\big(y(t)\big)^2 - 2t\big(x(t)\big)^2\Big)\,\mathrm{d}t \\ &= \int_0^1 t^4 - 2t^3\,\mathrm{d}t = \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}\end{aligned}$$

Lemma 2.15. *The integral of a 1-form along a curve is independent of the choice of (orientation-preserving) parametrization.*

Otherwise said, if $\mathbf{x}(t) = \mathbf{y}\big(s(t)\big)$ parametrizes the same curve where $s'(t) > 0$, then

$$\int_a^b \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t = \int_{s(a)}^{s(b)} \alpha\big(\mathbf{y}'(s)\big)\,\mathrm{d}s$$

The proof is an easy exercise in interpreting old material (the chain rule/substitution).

[14] Consider the equivalence of notations $\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x)$, linear approximations (differentials) & integration by substitution.

Our final result from elementary calculus shows that integrals of exact forms are independent of path. This is essentially the fundamental theorem of calculus for curves.

Theorem 2.16 (Fundamental Theorem of Line Integrals). *If f is a function on $U \subseteq \mathbb{R}^2$ and C is a curve in U, then the integral of $\mathrm{d}f$ depends only on the values of f at the endpoints of C:*

$$\int_C \mathrm{d}f = f(\text{end of } C) - f(\text{start of } C)$$

The converse also holds: if $\int_C \alpha$ is independent of path, then α is exact.

Proof. Suppose $\mathbf{x} : [a, b] \to U$ parametrizes C, then

$$\begin{aligned}\int_C \mathrm{d}f &= \int_a^b \mathrm{d}f(\mathbf{x}')\,\mathrm{d}t = \int_a^b \mathbf{x}'[f]\,\mathrm{d}t = \int_a^b \left(x_1'(t)\frac{\partial f}{\partial x_1} + \cdots + x_n'(t)\frac{\partial f}{\partial x_n}\right)\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\big(f(\mathbf{x}(t))\big)\,\mathrm{d}t = f\big(\mathbf{x}(b)\big) - f\big(\mathbf{x}(a)\big)\end{aligned}$$

The converse is sketched in an exercise. ∎

In elementary multivariable calculus this result was written $\int_C \nabla f \cdot \mathrm{d}\mathbf{x} = f\big(\mathbf{x}(b)\big) - f\big(\mathbf{x}(b)\big)$ which comports with our new notation when we view $\mathrm{d}\mathbf{x}$ as a vector of 1-forms:

$$\nabla f \cdot \mathrm{d}\mathbf{x} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix} = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n = \mathrm{d}f$$

The exterior derivative $\mathrm{d}f$ is just the gradient in disguise!

Example 2.17. If $\alpha = \cos(xy)(y\,\mathrm{d}x + x\,\mathrm{d}y)$, find the integral of α over any curve C joining the points $\left(\pi, \frac{1}{3}\right)$ and $\left(\frac{1}{2}, \pi\right)$. Since $\alpha = \mathrm{d}\sin(xy)$ is exact on $\mathbb{R}^2$, we see that

$$\int_C \alpha = \sin(xy)\Big|_{\left(\pi,\frac{1}{3}\right)}^{\left(\frac{1}{2},\pi\right)} = \sin\frac{\pi}{2} - \sin\frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2}$$

Summary

- Tangent vectors & vector fields encode *directional derivatives*, measuring how functions change in given directions.
- Vector fields and 1-forms break standard derivatives into two pieces: the result is a more flexible and extensible language for describing familiar results from multi-variable calculus.

The real pay-off comes once our new language is applied to surfaces and higher-dimensional objects. Here is a précis. A parametrized surface is a function $\mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{E}^3$; its exterior derivative $\mathrm{d}\mathbf{x}$ is a vector-valued 1-form which, at each point $p \in U$, describes a *linear map* between tangent spaces

$$\mathrm{d}\mathbf{x}_p : T_p\mathbb{R}^2 \to T_{\mathbf{x}(p)}\mathbb{E}^3$$

which maps the co-ordinate fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ on U to corresponding vector fields *tangent to the surface.*

Exercises 2.2. 1. In $\mathbb{R}^2$, let $\alpha = 2y\,dx - 3\,dy$ and $v = 3x^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Compute $\alpha(v)$, and $v\big[\alpha(v)\big]$.

2. On $\mathbb{R}^3$, suppose $f(x,y,z) = x^2\cos(yz)$ and $v = e^x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}$. Verify that $df(v) = v[f]$.

3. Find dr directly by taking the exterior derivative of the equation $r^2 = x^2 + y^2$.

4. Prove parts 1 and 2 of Lemma 2.12.

5. Continuing Example 2.7.2, verify that $d\theta\left(\frac{\partial}{\partial\theta}\right) = 1$, and $dr\left(\frac{\partial}{\partial\theta}\right) = 0 = d\theta\left(\frac{\partial}{\partial r}\right)$.

6. Suppose that $\alpha = \sum a_k\,dx_k$ is exact. Prove that $\frac{\partial a_k}{\partial x_j} = \frac{\partial a_j}{\partial x_k}$ for all j,k.

7. Decide whether the 1-forms α are exact on $\mathbb{R}^2$. If yes, find a function f such that $\alpha = df$.
   (a) $\alpha = 2x\,dx + dy$
   (b) $\alpha = dx + 2x\,dy$
   (c) $\alpha = \cos(x^2y)(2y\,dx + x\,dy)$
   (d) $\alpha = x\cos(x^2y)(2y\,dx + x\,dy)$

8. We consider a partial converse to Exercise 6.
   (a) Suppose $\alpha = a\,dx + b\,dy$ is a 1-form on a *rectangle* $[p,q]\times[r,s]$, where $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$. Define
   $$f(x,y) := \int_p^x a(s,y)\,ds + \int_r^y b(p,t)\,dt$$
   Prove that $df = \alpha$ is exact.
   (b) Let $\alpha = \frac{-y\,dx + x\,dy}{x^2+y^2} = a\,dx + b\,dy$ be defined on the *punctured plane* $\mathbb{R}^2\setminus\{(0,0)\}$.
   Show that $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$ but that α is *not exact*: the full converse to Exercise 6 is therefore *false*. (*Hint: $\alpha = d\theta$ except on the non-positive real axis; why is this a problem?*)

9. Evaluate the integral $\int_C \alpha$ given C and α.
   (a) $\alpha = dx - x^{-1}\,dy$, where C is parametrized by $\mathbf{x}(t) = (t^2, t^3)$, $0 \le t \le 1$.
   (b) $\alpha = 2x\tan^{-1}y\,dx + \frac{x^2}{1+y^2}\,dy$, where C is parametrized by $\mathbf{x}(t) = \left(\frac{1}{t+1}, 1\right)$, $0 \le t \le 2$.
   (c) $\alpha = \cos x\,dx + dy$, with C the graph of $y = \cos x$ over one period of the curve.

10. Which of the integrals in the previous question are path-independent?

11. Prove Lemma 2.15. Moreover, show that if we reverse the orientation of the curve ($s'(t) < 0$) then the order of the limits is reversed and $\int \alpha$ becomes $-\int \alpha$.

12. Let $p \in U \subseteq \mathbb{R}^2$ and let $\alpha = a\,dx + b\,dy$ be a 1-form on U. For each q define $f(q) := \int_C \alpha$ where we additionally assume this value is *independent of the path* C joining p to q.
    Let h be small and C_h the straight line from q to $q + h\mathbf{i}$. Integrate over C_h to show that
    $$\left.\frac{\partial f}{\partial x}\right|_q = \lim_{h\to 0}\frac{f(q+h\mathbf{i}) - f(q)}{h} = a(q)$$
    Make a similar argument to conclude that $\alpha = df$ is exact.

13. (If you've done complex analysis) Let $f(x,y) = u(x,y) + iv(x,y)$ be a complex-valued function $f : \mathbb{R}^2 \to \mathbb{C}$ where u,v are real-valued. Viewing $z = x+iy$ and $\overline{z} = x - iy$ as co-ordinates on $\mathbb{R}^2$, prove that $df\left(\frac{\partial}{\partial\overline{z}}\right) = 0$ if and only if u,v satisfy the *Cauchy-Riemann equations*:
    $$u_x = v_y, \quad v_x = -u_y$$

2.3 Higher-degree Forms

We introduce a new operation on forms which generalizes the cross product of vectors.

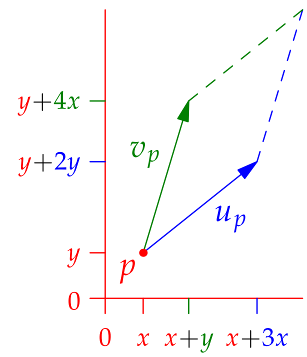
Definition 2.18. Given 1-forms α, β on U, their *wedge product* $\alpha \wedge \beta$ is the function which takes two vector fields and returns the *smooth function*

$$\alpha \wedge \beta(u, v) = \det \begin{pmatrix} \alpha(u) & \alpha(v) \\ \beta(u) & \beta(v) \end{pmatrix} : U \to \mathbb{R}$$

We call $\alpha \wedge \beta$ a *2-form.*

Example 2.19. Let x, y be the usual co-ordinates on $\mathbb{R}^2$. The *standard area form* is the object $\mathrm{d}x \wedge \mathrm{d}y$ which takes two vector fields $u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y}$ and $v = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$ and returns the determinant

$$\mathrm{d}x \wedge \mathrm{d}y(u, v) = \begin{vmatrix} \mathrm{d}x(u) & \mathrm{d}x(v) \\ \mathrm{d}y(u) & \mathrm{d}y(v) \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$



This gets its name since, at each point p, it returns the (signed) area of the parallelogram spanned by the tangent vectors u_p, v_p.
For instance, if $u = 3x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$ and $v = y \frac{\partial}{\partial x} + 4x \frac{\partial}{\partial y}$, then

$$\mathrm{d}x \wedge \mathrm{d}y(u, v) = \begin{vmatrix} 3x & y \\ 2y & 4x \end{vmatrix} = 12x^2 - 2y^2$$

Recall that determinants change sign if you switch its rows or columns, and that they are linear functions of both their rows and columns. This has two consequences for $\alpha \wedge \beta$.

Lemma 2.20. 1. *(Columns) At each $p \in U$, a wedge product of 1-forms is an* alternating, bilinear *function $\alpha \wedge \beta : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$: given vector fields u, v, w and functions $f, g : U \to \mathbb{R}$,*

$$\alpha \wedge \beta(v, u) = -\alpha \wedge \beta(u, v) \qquad \text{(alternating)}$$

$$\alpha \wedge \beta(fu + gv, w) = f\,\alpha \wedge \beta(u, w) + g\,\alpha \wedge \beta(v, w) \qquad \text{(linear in 1}^{\text{st}}\text{ slot)}$$

2. *(Rows) Wedge products are alternating and addition distributes over $\wedge$*

$$\beta \wedge \alpha = -\alpha \wedge \beta \quad \text{and} \quad \alpha \wedge \alpha = 0 \qquad \text{(alternating)}$$

$$(\alpha + \gamma) \wedge \beta = \alpha \wedge \beta + \gamma \wedge \beta \qquad \text{(distributivity in 1}^{\text{st}}\text{ slot)}$$

Linearity/distributivity in the second slot is similar in both cases.

The linearity and alternating properties tell us that every wedge product of 1-forms on $\mathbb{R}^2$ may be written

$$\alpha \wedge \beta = (a_1\,\mathrm{d}x + a_2\,\mathrm{d}y) \wedge (b_1\,\mathrm{d}x + b_2\,\mathrm{d}y) = (a_1b_2 - a_2b_1)\,\mathrm{d}x \wedge \mathrm{d}y$$

Notice the determinant again!

For higher order forms, we extend the same approach.

Definition 2.21. The *wedge product* of 1-forms $\alpha_1, \dots, \alpha_k$ on $U \subseteq \mathbb{R}^n$ takes k *vector fields* and returns a *smooth function*:

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, \dots, v_k) = \begin{vmatrix} \alpha_1(v_1) & \cdots & \alpha_1(v_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \cdots & \alpha_k(v_k) \end{vmatrix} : U \to \mathbb{R}$$

Let $x_1, \dots, x_n$ be co-ordinates on U. A *k-form* on U (alternating form of degree k) is an expression

$$\alpha = \sum a_I \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}, \quad a_I : U \to \mathbb{R} \text{ smooth}$$

where we sum over all *increasing multi-indices* $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, 2, \dots, n\}$ of length k. The *wedge product* of a k-form α and an l-form β is the $(k+l)$-form

$$\alpha \wedge \beta = \sum_{I,J} a_I b_J \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}$$

where the 1-forms $\mathrm{d}x$ may be rearranged/cancelled using the alternating property (Lemma 2.20.2).

By convention, a 0-form is a smooth function $f : U \to \mathbb{R}$, whose wedge product with anything is pointwise multiplication. At each point $p \in U$, the k-forms comprise the vector space of alternating multilinear maps with basis $\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : i_1 < \cdots < i_k\}$ and dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
In this course we'll never have reason to work in more then three dimensions!

The table describes all k-forms in 2 and 3 dimensions written in standard co-ordinates.

Analogous to Example 2.19, $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the *standard volume form* on $\mathbb{R}^3$.

k	$\mathbb{R}^2$	$\mathbb{R}^3$
0	function f	f
1	$f\,\mathrm{d}x + g\,\mathrm{d}y$	$f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$
2	$f\,\mathrm{d}x \wedge \mathrm{d}y$	$f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}x \wedge \mathrm{d}z + h\,\mathrm{d}y \wedge \mathrm{d}z$
3	None	$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
4+	None	None

Examples 2.22. 1. Given 1-forms $\alpha = 2\,\mathrm{d}x - 3x\,\mathrm{d}y$ and $\beta = y^2\,\mathrm{d}x + y\,\mathrm{d}y$ on $\mathbb{R}^2$,

$$\begin{aligned} \alpha \wedge \beta &= (2\,\mathrm{d}x - 3x\,\mathrm{d}y) \wedge (y^2\,\mathrm{d}x + y\,\mathrm{d}y) \\ &= 2y^2\,\mathrm{d}x \wedge \mathrm{d}x + 2y\,\mathrm{d}x \wedge \mathrm{d}y - 3xy^2\,\mathrm{d}y \wedge \mathrm{d}x - 3xy\,\mathrm{d}y \wedge \mathrm{d}y \\ &= (2y + 3xy^2)\,\mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

2. Given the 1-forms $\alpha = \mathrm{d}x + 2\,\mathrm{d}y + x\,\mathrm{d}z$ and 2-form $\beta = 3z\,\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{R}^3$, the wedge product $\alpha \wedge \beta$ is the 3-form

$$\begin{aligned} \alpha \wedge \beta &= \mathrm{d}x \wedge (-\mathrm{d}y \wedge \mathrm{d}z) + 3xz\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= (3xz - 1)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned}$$

Note how $\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ requires *two* swaps, so the sign is ultimately unchanged!

Lemma 2.23. *For any forms α, β,*

$$\beta \wedge \alpha = (-1)^{\deg \alpha \deg \beta} \alpha \wedge \beta$$

where $\deg \alpha = k$ means that α is a k-form.

This is true by definition when α, β are 1-forms, and trivially true when α is a 0-form. Check the previous examples to make sure they agree.

Example 2.24 (Polar co-ordinates). Changing to polar co-ordinates, the standard area form on $\mathbb{R}^2$ becomes

$$\mathrm{d}x \wedge \mathrm{d}y = (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta) \wedge (\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta) = r\,\mathrm{d}r \wedge \mathrm{d}\theta$$

This should remind you of change of variables in integration: if $f(x,y) = g(r,\theta)$, then

$$\int f(x,y)\mathrm{d}x\mathrm{d}y = \int g(r,\theta) r\,\mathrm{d}r\mathrm{d}\theta$$

The example illustrates one of the advantages of forms: change of variables (Jacobians) are built in!

The Exterior Derivative Just as with functions, we can apply 'd' to forms.

Definition 2.25. The *exterior derivative* of a k-form $\alpha = \sum a_I\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ is the $(k+1)$-form

$$\mathrm{d}\alpha = \sum \mathrm{d}a_I \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

where $\mathrm{d}a_I = \sum_j \frac{\partial a}{\partial x_j}\,\mathrm{d}x_j$ is the usual exterior derivative of a function (Definition 2.10).

Example 2.26. In $\mathbb{R}^3$, let $\alpha = xy^2z\,\mathrm{d}x - xz\,\mathrm{d}z$. Then

$$\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}(xy^2z) \wedge \mathrm{d}x - \mathrm{d}(xz) \wedge \mathrm{d}z \\
&= (y^2z\,\mathrm{d}x + 2xyz\,\mathrm{d}y + xy^2\,\mathrm{d}z) \wedge \mathrm{d}x - (z\,\mathrm{d}x + x\,\mathrm{d}z) \wedge \mathrm{d}z \\
&= -2xyz\,\mathrm{d}x \wedge \mathrm{d}y - (xy^2 + z)\,\mathrm{d}x \wedge \mathrm{d}z
\end{aligned}$$

Since $\mathrm{d}x \wedge \mathrm{d}x = 0 = \mathrm{d}z \wedge \mathrm{d}z$, there was no need to write the blue terms.

Theorem 2.27. *Let α, β be forms:*

1. $\mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$ *(α, β must have the same degree)*
2. $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$
3. $\mathrm{d}(\mathrm{d}\alpha) = 0$. *This is often written*[15] $\mathrm{d}^2\alpha = 0$, *or just* $\mathrm{d}^2 = 0$.

[15] A k-form α is *closed* if $\mathrm{d}\alpha = 0$, and *exact* if $\exists\beta$ such that $\alpha = \mathrm{d}\beta$. The result says that every exact form is closed. *Poincare's Lemma* gives a partial converse: every closed form on an open ball/hypercube is exact. Exercise 2.2.8 is a simple version.

Example (2.26 cont). We verify that $\mathrm{d}^2\alpha = 0$:

$$\begin{aligned}\mathrm{d}(\mathrm{d}\alpha) &= \mathrm{d}(-2xyz)\wedge \mathrm{d}x\wedge \mathrm{d}y - \mathrm{d}(xy^2+z)\wedge \mathrm{d}x\wedge \mathrm{d}z \\ &= -2xy\,\mathrm{d}z\wedge \mathrm{d}x\wedge \mathrm{d}y - 2xy\,\mathrm{d}y\wedge \mathrm{d}x\wedge \mathrm{d}z = 0\end{aligned}$$

Proof. This is very easy to prove explicitly for the only forms we'll ever see (up to 3-forms in $\mathbb{R}^3$). Here are general arguments that work in any dimension.

For simplicity of notation, write $\mathrm{d}x_I = \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}x_{i_k}$, whenever $I = \{i_1 < \cdots < i_k\}$. Then

$$\mathrm{d}(\alpha+\beta) = \sum_I \mathrm{d}a_I\wedge \mathrm{d}x_I + \mathrm{d}b_I\wedge \mathrm{d}x_I = \sum_I (\mathrm{d}a_I + \mathrm{d}b_I)\wedge \mathrm{d}x_I = \mathrm{d}\alpha + \mathrm{d}\beta$$

Part 2 is an exercise. For part 3, we extend Exercise 2.2.6 which in fact shows that $\mathrm{d}^2 f = 0$ for any function (0-form)

$$\begin{aligned}\mathrm{d}(\mathrm{d}\alpha) &= \mathrm{d}\sum_I \mathrm{d}a_I\wedge \mathrm{d}x_I = \mathrm{d}\sum_{j\notin I}\frac{\partial a_I}{\partial x_j}\,\mathrm{d}x_j\wedge \mathrm{d}x_I = \sum_{i,j\notin I}\frac{\partial^2 a_I}{\partial x_i x_j}\,\mathrm{d}x_i\wedge \mathrm{d}x_j\wedge \mathrm{d}x_I \\ &= \sum_{i<j\notin I}\left[\frac{\partial^2 a_I}{\partial x_i x_j} - \frac{\partial^2 a_I}{\partial x_j x_i}\right]\mathrm{d}x_i\wedge \mathrm{d}x_j\wedge \mathrm{d}x_I = 0\end{aligned}$$

since mixed partial derivatives commute. ∎

A New Take on Vector Calculus

The standard vector calculus operations of div, grad and curl in $\mathbb{E}^3$ are closely related to the exterior derivative. For instance, compare the curl of a vector field $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ with the exterior derivative of the 1-form $\alpha = a_1\,\mathrm{d}x + a_2\,\mathrm{d}y + a_3\,\mathrm{d}z$:

$$\nabla\times\mathbf{v} = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\times\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\mathbf{k}$$

$$\mathrm{d}\alpha = \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\mathrm{d}y\wedge \mathrm{d}z + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\mathrm{d}z\wedge \mathrm{d}x + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\mathrm{d}x\wedge \mathrm{d}y$$

Comparing coefficients gives part of the dictionary for comparing forms and traditional vector fields.

$$\begin{array}{ccc}
\text{function } f & \longleftrightarrow & \text{function } f \\
\Big\downarrow \mathrm{d} & & \text{grad}\Big\downarrow \nabla \\
a_1\,\mathrm{d}x + a_2\,\mathrm{d}y + a_3\,\mathrm{d}z & \longleftrightarrow & a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\
\Big\downarrow \mathrm{d} & & \text{curl}\Big\downarrow \nabla\times \\
b_1\,\mathrm{d}y\wedge \mathrm{d}z + b_2\,\mathrm{d}z\wedge \mathrm{d}x + b_3\,\mathrm{d}x\wedge \mathrm{d}y & \longleftrightarrow & b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \\
\Big\downarrow \mathrm{d} & & \text{div}\Big\downarrow \nabla\cdot \\
c\,\mathrm{d}x\wedge \mathrm{d}y\wedge \mathrm{d}z & \longleftrightarrow & \text{function } c
\end{array}$$

The exterior derivative d is div, grad and curl all in one tidy package! Moreover:

- The identity $\mathrm{d}^2 = 0$ translates to two familiar results from vector calculus:

$$\nabla \times (\nabla f) = \mathbf{0} \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{v}) = 0$$

- Under the above identification, the wedge product of 1-forms corresponds to the cross product, and the wedge product of a 1-form and a 2-form to the dot product. Various identities may be obtained this way: for instance, if α is a 1-form, then

$$\mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha \quad \longleftrightarrow \quad \nabla \times f\mathbf{v} = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v}$$

- Changes of co-ordinates are built into forms (e.g. Example 2.24).
- The exterior derivative and wedge product apply in any dimension, thus extending standard vector calculus and the cross product to arbitrary dimensions.

None of what we've done in this chapter is strictly necessary for the analysis of surfaces in $\mathbb{E}^3$. However, forms are the language of modern differential geometry (and other things besides) and it is easier to meet them first in a familiar setting. And if you want to do higher-dimensional geometry (e.g., general relativity), this new language becomes almost essential.

Exercises 2.3. 1. Compute $\alpha(u, v)$, given $\alpha = \mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$, $u = \frac{\partial}{\partial x} - \frac{\partial}{\partial z}$ and $v = y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

2. Let $\alpha = y^2\,\mathrm{d}x \wedge \mathrm{d}z - \mathrm{d}y \wedge \mathrm{d}z$ and $u = x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$ and $v = -y\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y}$.
   (a) Compute $\alpha(u, v)$.
   (b) Find the 3-form $\mathrm{d}\alpha$.
3. Given $s = x^2 - y^2$ and $t = 2xy$, compute $\mathrm{d}s \wedge \mathrm{d}t$ in terms of $\mathrm{d}x \wedge \mathrm{d}y$
4. Revisit Lemma 2.20. State what it means for a wedge product of 1-forms $\alpha \wedge \beta$ to be linear in the second slot.
5. Let f, g be functions and consider the 1-form $\alpha = g\,\mathrm{d}f$. Show that $\alpha \wedge \mathrm{d}\alpha = 0$. Can the 1-form $\mathrm{d}x + y\,\mathrm{d}z$ be written in the form $g\,\mathrm{d}f$?
6. (a) Check the claim that the wedge product of 1-forms on $\mathbb{R}^3$ corresponds to the cross product.
   (b) Suppose α is a 2-form on $\mathbb{R}^3$. To what vector calculus identity does $\mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha$ correspond?
   (c) State an expression using forms, d and $\wedge$ which corresponds to the vector calculus identity

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

7. Let r, θ, ϕ be the spherical polar co-ordinate system in Exercise 2.1.3. Show that

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \cos\phi\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$$

8. A 2-form is *decomposable* if it can be written as a wedge product $\alpha \wedge \beta$ for some 1-forms α, β.
   (a) Show that every 2-form on $\mathbb{R}^3$ is decomposable.
   (b) If w, x, y, z are co-ordinates on $\mathbb{R}^4$, show that the 2-form $\mathrm{d}w \wedge \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}z$ is *not* decomposable.
   (*Hint: if a 2-form γ is decomposable, what is $\gamma \wedge \gamma$?*)

9. (Hard) Suppose α, β are forms, sketch an argument for why
$$\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha$$
Now prove that
$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$$

10. (Hard) Given vector fields u, v, their *Lie bracket* $[u, v]$ is the vector field such that
$$[u, v][f] := u\big[v[f]\big] - v\big[u[f]\big]$$
for all functions f.
   (a) Compute $[u, v][f]$ where $u = 3x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $v = \frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $f(x, y) = x^2 y$.
   (b) If $u = \sum u_j \frac{\partial}{\partial x_j}$ and $v = \sum v_k \frac{\partial}{\partial x_k}$, show that $[u, v]$ really is a vector field by explicitly computing $[u, v][f]$ in the form $\sum c_j \frac{\partial f}{\partial x_j}$: how do the coefficients c_j of the vector field $[u, v]$ depend on those of u, v? Find the *field* $[u, v]$ when u, v are as in part (a).
   (c) If α is a 1-form and u, v are vector fields, prove that
$$\mathrm{d}\alpha\big(u, v\big) = u\big[\alpha(v)\big] - v\big[\alpha(u)\big] - \alpha\big([u, v]\big)$$
   This provides a co-ordinate-free definition of $\mathrm{d}\alpha$*; similar expressions exist for k-forms*
   (*Hint: Write everything out as sums over j, k so that all differentiations of scalars are with respect to the single variable x_k; now compare!*)

3 Surfaces

3.1 Regular Parametrized Surfaces

We approach surfaces in $\mathbb{E}^3$ similarly to how we considered curves; a parametrized surface is a *function* $\mathbf{x} : U \to \mathbb{E}^3$ where U is some open subset of the plane $\mathbb{R}^2$. Our main purpose is to develop and measure the *curvature* of a surface in terms of the parametrizing function $\mathbf{x}$.

Our primary definition should mostly be familiar from elementary multivariable calculus.

Definition 3.1. A *(smooth local) surface* is the range $S = \mathbf{x}(U)$ of a smooth function $\mathbf{x} : U \to \mathbb{E}^3$, where U is a connected open subset of $\mathbb{R}^2$.

Given co-ordinates u, v on U, the *co-ordinate tangent vector fields* are the partial derivatives $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$.

The *exterior derivative* or *differential* of the surface is the vector-valued 1-form $\mathrm{d}\mathbf{x} = \mathbf{x}_u \, \mathrm{d}u + \mathbf{x}_v \, \mathrm{d}v$.

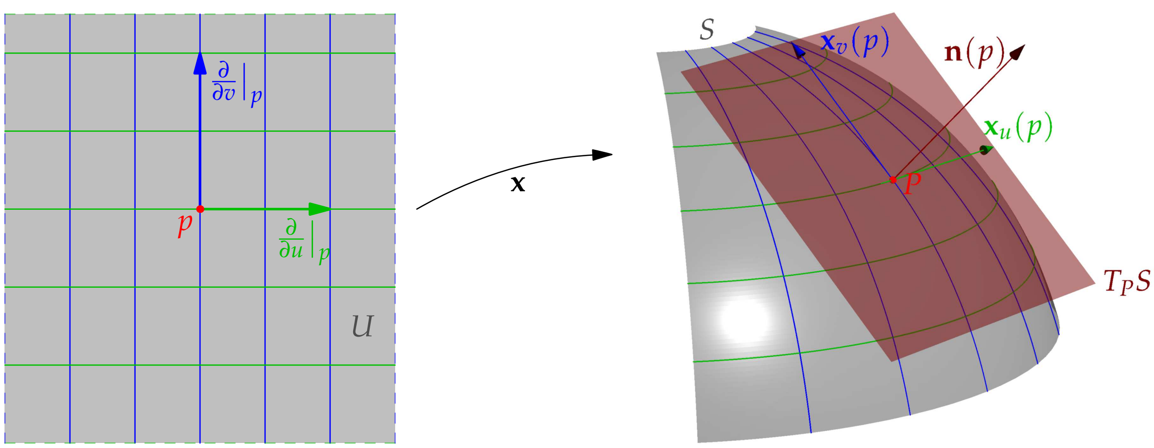
A surface is *regular* at $P = \mathbf{x}(p)$ if the tangent vectors $\mathbf{x}_u(p)$ and $\mathbf{x}_v(p)$ are *linearly independent*: otherwise said, at P, the surface has a well-defined

$$\textit{Tangent plane } \; T_P S = \operatorname{Span}\{\mathbf{x}_u(p), \mathbf{x}_v(p)\} \text{ (a 2-dim subspace of } T_P\mathbb{E}^3\text{), and}$$

$$\textit{Unit normal vector } \; \mathbf{n}(p) = \frac{\mathbf{x}_u(p) \times \mathbf{x}_v(p)}{||\mathbf{x}_u(p) \times \mathbf{x}_v(p)||} \in T_P\mathbb{E}^3$$

S is *regular* if it is regular everywhere. An *orientation* is a smooth choice of *unit normal vector field* $\mathbf{n}$.

The *Möbius strip* (Exercise 9) shows that not every surface is orientable!
For brevity, we will often refer to the parametrizing function $\mathbf{x}$ as the surface, though many different parametrizations will exist! A general surface typically needs to be parametrized by several overlapping functions $\mathbf{x}_1, \mathbf{x}_2, \ldots$. Our definition is *local* since there is only one $\mathbf{x}$.



The partial derivatives $\mathbf{x}_u(p), \mathbf{x}_v(p)$ are *tangent to the surface* at $\mathbf{x}(p)$: if $p = (u_0, v_0)$ then the *curve* $\mathbf{y}(t) := \mathbf{x}(t, v_0)$ lies in the surface and passes through $P = \mathbf{x}(p)$; its tangent vector at P is then

$$\mathbf{y}'(u_0) = \lim_{h \to 0} \frac{\mathbf{y}(u_0 + h) - \mathbf{y}(u_0)}{h} = \lim_{h \to 0} \frac{\mathbf{x}(u_0 + h, v_0) - \mathbf{x}(p)}{h} = \mathbf{x}_u(p)$$

To help distinguish between domain and codomain, we standardize notation.

Domain $U \subseteq \mathbb{R}^2$: *Points* are written *lower case* or as *row vectors*: e.g., $p = (u_0, v_0) \in U$. Typically we'll use u, v as *co-ordinates* unless it is more natural to use angles such as ϕ, θ.

Tangent vectors/fields are written with an arrow in our *new* notation: e.g., $\vec{w}_p = \frac{\partial}{\partial u}\Big|_p \in T_p\mathbb{R}^2$.

Codomain $\mathbb{E}^3$: *Points* are written *upper case* or as *row vectors*, e.g., $P = (3, 4, 8) \in \mathbb{E}^3$. Co-ordinates on $\mathbb{E}^3$ will typically be x, y, z.

Vectors are written *bold-face* as either row or column vectors: e.g., $\mathbf{x}(u, v) = \big(u, v, u^2 + v^2\big)$.

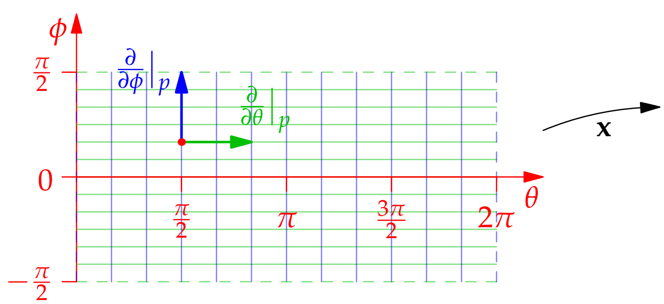
Tangent vectors/fields use the *old* notation:[16] e.g., if $P = \mathbf{x}(p)$, then $\mathbf{x}_u(p) = \frac{\partial \mathbf{x}}{\partial u}\Big|_p \in T_P\mathbb{E}^3$.

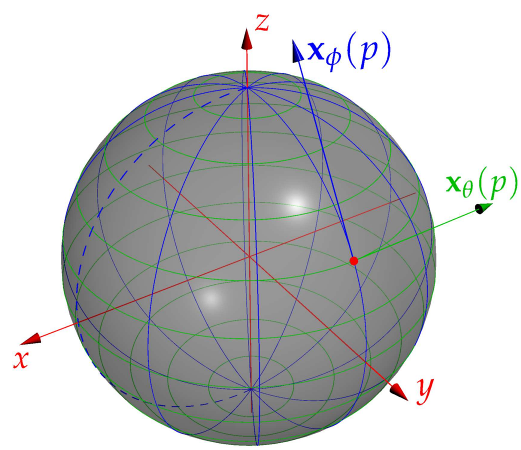
Example 3.2. Consider the sphere of radius a parametrized using spherical polar co-ordinates:

$$\mathbf{x}(\theta, \phi) = a\begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix}, \quad \mathrm{d}\mathbf{x} = \mathbf{x}_\theta\,\mathrm{d}\theta + \mathbf{x}_\phi\,\mathrm{d}\phi = a\cos\phi\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\mathrm{d}\theta + a\begin{pmatrix} -\cos\theta\sin\phi \\ -\sin\theta\sin\phi \\ \cos\phi \end{pmatrix}\mathrm{d}\phi$$

The unit normal field is simply $\mathbf{n} = \frac{1}{a}\mathbf{x}$. The domain $U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is an open rectangle whose image $S = \mathbf{x}(U)$ is the sphere minus the (dashed) semicircle $\mathbf{x}(0, \phi)$. While we could extend θ to wrap round the equator, we cannot extend to the north or south poles without sacrificing *regularity*:

$$\mathbf{x}_\theta = a\cos\phi\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \mathbf{0} \text{ when } \phi = \pm\frac{\pi}{2}$$





This illustrates the term *local*: indeed the famous *hairy ball theorem* from topology says that it is impossible to find a regular parametrization of the entire sphere by a single function.
Also observe how the tangent vectors $\frac{\partial}{\partial\phi}\Big|_p, \frac{\partial}{\partial\theta}\Big|_p \in T_p\mathbb{R}^2$ are mapped by $\mathrm{d}\mathbf{x}$ to tangent vectors

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\phi}\bigg|_p = \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\phi}\bigg|_p\right), \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\theta}\bigg|_p = \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\theta}\bigg|_p\right) \in T_{\mathbf{x}(p)}S$$

[16] To use our new notation in $\mathbb{E}^3$ would require a subtle redefinition of $\mathrm{d}\mathbf{x}$: if $\vec{w}$ is a vector field on U, then $\mathrm{d}\mathbf{x}(\vec{w})$ is the vector field on S such that $\big(\mathrm{d}\mathbf{x}(\vec{w})\big)[f] = \vec{w}\big[f \circ \mathbf{x}\big]$ for all $f : S \to \mathbb{R}$. In co-ordinates this benefits from tensor notation:

$$\mathbf{x}(u_1, u_2) = \Big(x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2)\Big) \implies \mathrm{d}\mathbf{x} = \sum_{i,j}\frac{\partial x_j}{\partial u_i}\frac{\partial}{\partial x_j} \otimes \mathrm{d}u_i$$

In more general situations this approach is necessary, but it is overkill for our purposes!

Theorem 3.3. *Let $S = \mathbf{x}(U)$ be a smooth surface containing the point $P = \mathbf{x}(p)$:*

1. *The differential at p is a linear map $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P\mathbb{E}^3$ mapping tangent vectors in $\mathbb{R}^2$ to vectors tangent to S.*

2. *S is regular at P if and only if $\mathrm{d}\mathbf{x}$ is* injective (1–1) *at p. In such a case we can view it as an invertible linear map $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_PS$.*

Proof. 1. The differential at p is linear since the co-ordinate 1-forms $\mathrm{d}u, \mathrm{d}v$ are linear: indeed

$$\mathrm{d}\mathbf{x}\left(a\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p\right) = \mathbf{x}_u(p)\,\mathrm{d}u\left(a\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p\right) + \mathbf{x}_v(p)\,\mathrm{d}v\left(a\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p\right)$$
$$= a\mathbf{x}_u(p) + b\mathbf{x}_v(p) = a\,\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial u}\bigg|_p\right) + b\,\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial v}\bigg|_p\right)$$

This expression is moreover tangent to S at $\mathbf{x}(p)$: if this last assertion is unconvincing, see Exercise 8.

2. The range of $\mathrm{d}\mathbf{x}$ at p is plainly $\mathrm{Span}\{\mathbf{x}_u(p), \mathbf{x}_v(p)\}$. This is 2-dimensional (and thus defines the tangent plane) if and only if $\mathrm{rank}\,\mathrm{d}\mathbf{x} = 2 \iff \mathrm{d}\mathbf{x}$ is 1–1. ∎

It is worth reiterating two crucially important properties of $\mathrm{d}\mathbf{x}$:

- At a regular point, $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_PS$ is an invertible linear map. We shall shortly use this to *pull-back* calculations from S to U.

- The differential is co-ordinate independent and thus does not depend on the parametrization of S. This follows since $\mathrm{d}\mathbf{x}$ is the unique 1-form satisfying $\mathrm{d}\mathbf{x}(\vec{w}) = \vec{w}[\mathbf{x}]$ for all vector fields $\vec{w}$ on U; a description that does not depend on co-ordinates.

Aside: change of co-ordinates To more clearly spell this out, suppose we choose a new parametrization $\mathbf{y}(s,t) = \mathbf{x}\big(F(s,t)\big)$ where $F(s,t) = (u,v)$ is a change of co-ordinates on U. By the chain rule,

$$\begin{pmatrix}\mathbf{y}_s\\ \mathbf{y}_t\end{pmatrix} = \begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} \quad \text{and} \quad (\mathrm{d}u\ \mathrm{d}v) = (\mathrm{d}s\ \mathrm{d}t)\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}$$

from which

$$\mathrm{d}\mathbf{y} = (\mathrm{d}s\ \mathrm{d}t)\begin{pmatrix}\mathbf{y}_s\\ \mathbf{y}_t\end{pmatrix} = (\mathrm{d}u\ \mathrm{d}v)\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} = (\mathrm{d}u\ \mathrm{d}v)\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} = \mathrm{d}\mathbf{x}$$

The matrix of partial derivatives is the *Jacobian* of the co-ordinate change.

To be completely strict, $\mathrm{d}\mathbf{x}$ and $\mathrm{d}\mathbf{y}$ are not identical since they feed on tangent vectors with respect to different co-ordinates. Formally

$$\mathbf{y} = \mathbf{x}\circ F \implies \mathrm{d}\mathbf{y} = \mathrm{d}\mathbf{x}\circ \mathrm{d}F$$

where $\mathrm{d}F$ maps tangent vectors in $\mathrm{Span}\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\}$ to those in $\mathrm{Span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$: in matrix language, $\mathrm{d}F$ is precisely the above Jacobian!

Common Surfaces

You should have met many of these families/examples in multi-variable calculus.

Graphs If $f(x,y)$ is a smooth function, its graph may be parametrized by $\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$. Its differential and unit normal field are

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix} \mathrm{d}u + \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix} \mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

This is regular at all points, regardless of f.

Examples 3.4. 1. The standard circular paraboloid may be parametrized $\mathbf{x}(u,v) = \big(u, v, u^2 + v^2\big)$.

2. The upper half of the unit sphere is the graph of $z = f(x,y) = \sqrt{1 - x^2 - y^2}$ where $x^2 + y^2 < 1$.

3. A plane has equation $ax + by + cz = d$ where a, b, c, d are constant. Since at least one of a, b, c must be non-zero, this may be written as a function and graphed. For instance, if $b \neq 0$ we have $y = f(x,z) = \frac{1}{b}(d - ax - cz)$ and $\mathbf{n} = \frac{1}{\sqrt{a^2+b^2+c^2}}\big(a, b, c\big)$.

Surfaces of Revolution If a smooth positive function $x = f(z)$ is rotated around the z-axis, we obtain a parametrization

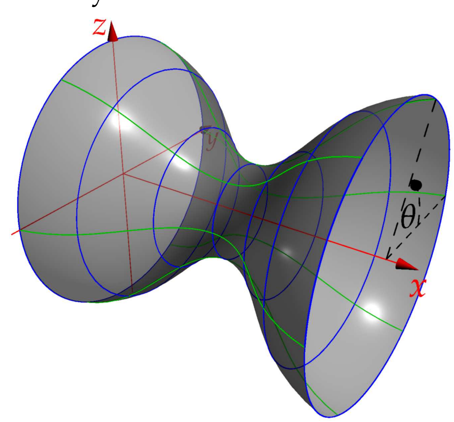
$$\mathbf{x}(\theta, v) = \big(f(v)\cos\theta, f(v)\sin\theta, v\big), \quad (\theta, v) \in (0, 2\pi) \times \mathrm{dom}(f)$$

with differential and unit normal field

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} -f(v)\sin\theta \\ f(v)\cos\theta \\ 0 \end{pmatrix} \mathrm{d}\theta + \begin{pmatrix} f'(v)\cos\theta \\ f'(v)\sin\theta \\ 1 \end{pmatrix} \mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1 + f'(v)^2}} \begin{pmatrix} \cos\theta \\ \sin\theta \\ -f'(v) \end{pmatrix}$$

Examples 3.5. 1. The simplest example ($f(z) \equiv 1$) is the right circular cylinder of radius 1.

2. We may rotate around any axis! For instance, if we rotate the curve $z = 2 + \cos x$ around the x-axis, the resulting surface may be parametrized



$$\mathbf{x}(\theta, v) = (2 + \cos v) \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$$

This time v measures distance along the x-axis and θ the angle of rotation around it.

The differential and unit normal field are

$$\mathrm{d}\mathbf{x} = (2 + \cos v) \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} \mathrm{d}\theta + \begin{pmatrix} 1 \\ -\sin v\cos\theta \\ -\sin v \sin\theta \end{pmatrix} \mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1 + \sin^2 v}} \begin{pmatrix} \sin v \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

Note the *orientation* of the surface: the unit normal field points *outward*, away from the x-axis.

Ruled Surfaces Given functions $\mathbf{y}(u), \mathbf{z}(u)$, define

$$\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{z}(u)$$

Through each point $P = \mathbf{x}(u_0, v_0)$ passes a line $t \mapsto \mathbf{x}(u_0, t) = \mathbf{y}(u_0) + t\mathbf{x}(u_0)$ lying in the surface. The surface can be visualized as moving a *ruler* through space. Ruled surfaces are common in engineering applications since they may be constructed using straight beams.

Definition 3.6. The *tangent developable* of a smooth curve $\mathbf{y}$ is the special case when $\mathbf{z} = \mathbf{y}'$.

Examples 3.7. 1. Every plane is a ruled surface! Let $\mathbf{y}$ be a line in the plane and $\mathbf{z}$ any other tangent direction. For instance, the plane passing through $(1,0,9)$ and spanned by $(2,-3,-5)$ ad $(1,2,3)$ may be parametrized

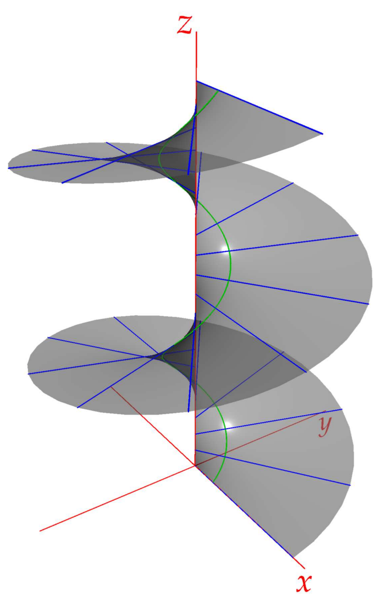
$$\mathbf{x}(u,v) = \underbrace{(1,0,9) + (2,-3,-5)u}_{\mathbf{y}(u)} + \underbrace{(1,2,3)}_{\mathbf{z}(u)}v$$

2. A *helicoid* is built by joining each point of a helix to its axis of rotation. From the standard helix, we obtain the helicoid $\mathbf{x}(u,v) = \big(v\cos u, v\sin u, u\big)$ for $v > 0$.

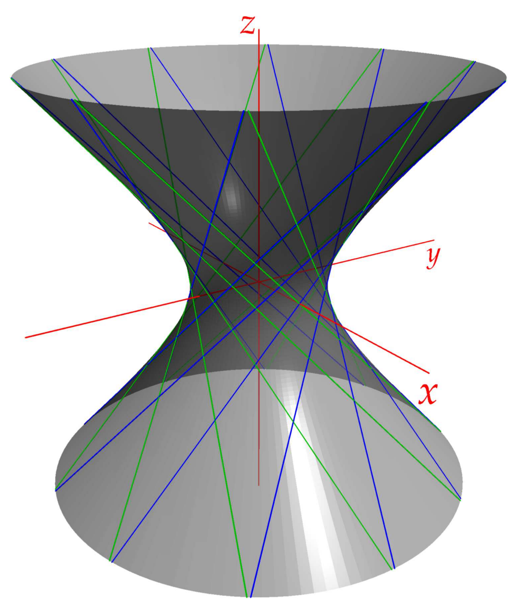
3. The *hyperboloid of one sheet* is a *doubly ruled surface*: through each point there are *two* lines lying on the surface. It may be parametrized as a ruled surface by

$$\mathbf{x}(u,v) = \begin{pmatrix}1\\0\\0\end{pmatrix} + u\begin{pmatrix}0\\1\\1\end{pmatrix} + v\begin{pmatrix}2u\\u^2-1\\u^2+1\end{pmatrix}$$

though convincing yourself there are *two* lines through each point takes a little more work…



Helicoid



Hyperboloid

Implicitly Defined Surfaces

Definition 3.8. A *regular implicitly defined surface* is the zero set of a smooth function $f : \mathbb{E}^3 \to \mathbb{R}$ for which $\mathrm{d}f \neq 0$ (equivalently $\nabla f \neq \mathbf{0}$).

Recall that the directional derivative of f in the direction $\mathbf{v}$ is $D_{\mathbf{v}} f(P) = \mathbf{v} \cdot \nabla f(P)$. This is zero if and only if $\mathbf{v}$ is orthogonal to $\nabla f(P)$. In particular, this says that ∇f provides a *normal field* to an implicitly defined surface.

Examples 3.9. 1. Let a, b, c, d be constants. The function $f(x, y, z) = ax + by + cz - d$ has

$$\mathrm{d}f = a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z$$

which is non-zero provided at least one of a, b, c are non-zero. This defines a plane with unit normal field $\mathbf{n} = \frac{1}{||\nabla f||} \nabla f = \frac{1}{\sqrt{a^2+b^2+c^2}}(a, b, c)$.

2. The sphere of radius a is the zero set of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$. It has unit normal field

$$\mathbf{n} = \frac{1}{||\nabla f||} \nabla f = \frac{1}{a}(x, y, z)$$

The sphere is everywhere regular since at least one of x, y, z is non-zero at all points of the sphere. Contrast this with our earlier example of the *parametrized* sphere which could not be made regular at the north and south poles. The lack of regularity in this case is an aspect of the parametrization, not the surface itself.

3. The function $f(x, y, z) = x^2 + y^2 - z^2 - c$ has

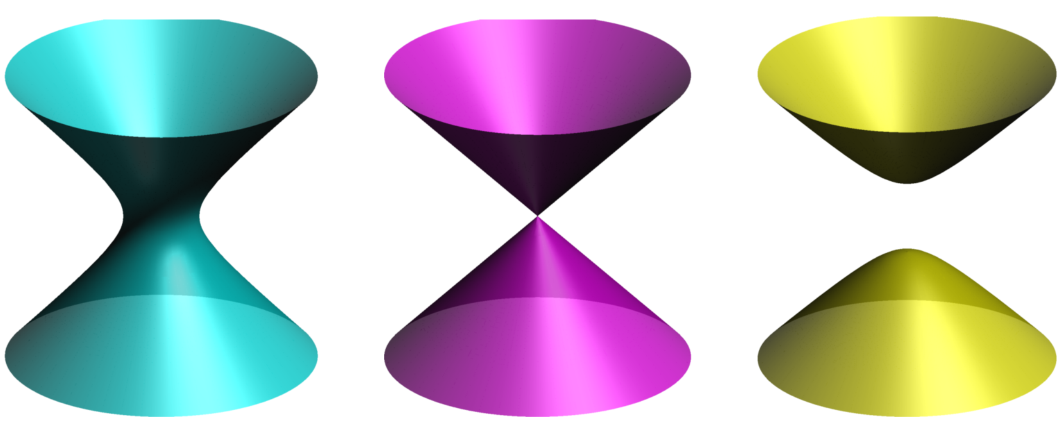
$$\mathrm{d}f = 2(x\,\mathrm{d}x + y\,\mathrm{d}y - z\,\mathrm{d}z)$$

which is non-zero away from $(x, y, z) = (0, 0, 0)$. Depending on the sign of c, the zero set is a hyperboloid or a cone; visualize the horizontal cross-sectional circles to determine which.

$c > 0$ Hyperboloid of 1-sheet: $x^2 + y^2 = z^2 + c > 0$ for all z

$c = 0$ Cone: $x^2 + y^2 = z^2$ contains a non-regular point $(0, 0, 0)$

$c < 0$ Hyperboloid of 2-sheets: $x^2 + y^2 = z^2 - |c| \geq 0$ only when $|z| \geq \sqrt{|c|}$



Our next result, a corollary of the famous *implicit function theorem*, ties together the notions of *regularity*. In particular, it says that we can always assume the existence of *local* co-ordinates.

Theorem 3.10. *A regular implicitly defined surface $f(x,y,z) = 0$ is (locally) the image of a regular local surface.*

Proof. Suppose $P = (x_0, y_0, z_0)$ lies on the surface and $\nabla f(P) \neq \mathbf{0}$. At least one of the partial derivatives of f is non-zero; suppose WLOG that $f_z(P) \neq 0$. By the implicit function theorem, there exists $U \subseteq \mathbb{R}^2$ and a function $g : U \to \mathbb{R}$ for which $g(x_0, y_0) = z_0$ and $f(x, y, g(x,y)) = 0$. The surface is then (locally) the graph of $z = g(x,y)$.

$$\mathbf{x} : U \to \mathbb{E}^3 : (u,v) \mapsto (u, v, g(u,v))$$

∎

Example 3.11. The zero set of $f(x,y,z) = x^2 + y^2 - z^2 - 6$ is a hyperboloid of one sheet. It has unit normal vector field

$$\mathbf{n}(x,y,z) = \frac{1}{||\nabla f||}\nabla f = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ -z \end{pmatrix} = \frac{1}{\sqrt{6+2z^2}}\begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

whenever (x,y,z) is a point on the hyperboloid. For instance, at $P = (3,1,2)$ the unit normal is $\mathbf{n}(P) = \frac{1}{\sqrt{14}}(3,1,2)$ and the tangent plane has equation

$$3x + y - 2z = 6$$

Alternatively, the hyperboloid can be parametrized in several ways.

(a) In the language of the proof, near $P = (3,1,2)$ it is the graph of $z = g(x,y) = \sqrt{x^2+y^2-6}$. This results in a (local) regular parametrization

$$\mathbf{x}(u,v) = \left(u, v, \sqrt{u^2+v^2-6}\right)$$

(b) The hyperboloid is a surface of revolution around the z-axis:

$$\mathbf{x}(\theta, v) = \begin{pmatrix} \sqrt{6+v^2}\cos\theta \\ \sqrt{6+v^2}\sin\theta \\ v \end{pmatrix}$$

For this parametrization, the differential and normal field are

$$\mathrm{d}\mathbf{x} = \sqrt{6+v^2}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\mathrm{d}\theta + \frac{1}{\sqrt{6+v^2}}\begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \sqrt{6+v^2} \end{pmatrix}\mathrm{d}v$$

$$\mathbf{n} = \frac{\mathbf{x}_\theta \times \mathbf{x}_v}{||\mathbf{x}_\theta \times \mathbf{x}_v||} = \frac{1}{\sqrt{6+2v^2}}\begin{pmatrix} \sqrt{6+v^2}\cos\theta \\ \sqrt{6+v^2}\sin\theta \\ -v \end{pmatrix}$$

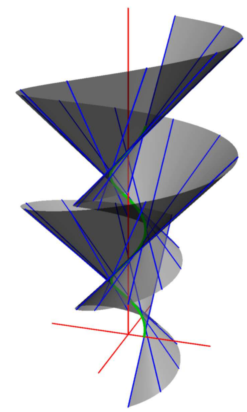
which is precisely what we obtained above.

Yet another expression could be obtained using a parametrization as a ruled surface (e.g., page 51).

Exercises 3.1. 1. Show that parametrization $\mathbf{x}(r,\theta) = \big(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\big)$ of the upper hemisphere is non-regular at $r=0$.

2. Explain why the parametrization in Example 3.11(a) is *local*: what is left out?

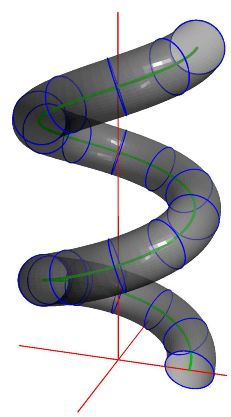
3. (a) Compute $\mathrm{d}\mathbf{x}$ and $\mathbf{n}$ for the paraboloid $\mathbf{x}(u,v) = \big(u, v, u^2+v^2\big)$.
   (b) Repeat for the polar co-ordinate parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$. Is this parametrization everywhere regular?
   (c) Using $x = r\cos\theta$, etc., write $\mathrm{d}\mathbf{x}$ in terms of $r, \theta, \mathrm{d}r, \mathrm{d}\theta$. What do you observe?
   (d) By viewing the paraboloid as the zero set of $f(x,y,z) = z - x^2 - y^2$, find another expression for the unit normal field.



4. (a) Find a parametrization for the tangent developable of the helix $\mathbf{y}(u) = \big(\cos u, \sin u, u\big)$. Compute $\mathrm{d}\mathbf{y}$ and the unit normal field $\mathbf{n}$.
   (*The picture covers $v \in (-3,6)$ with the original curve $\mathbf{y}(u)$ in green*)
   (b) If $\mathbf{y}$ is a unit speed biregular curve, prove that its tangent developable $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ is a regular surface except when $v=0$. Express the differential and unit normal field in terms of the Frenet frame of $\mathbf{y}$.

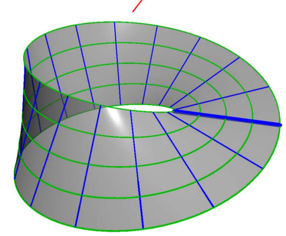
5. Let $f(x,y,z) = z^2$. Show that the zero set of f has a regular parametrization despite the gradient of f vanishing at $z=0$.

6. Let a,b,c be positive constants and define $\mathbf{x}(\theta,\phi) = \begin{pmatrix} a\cos\theta\cos\phi \\ b\sin\theta\cos\phi \\ c\sin\phi \end{pmatrix}$, $(\theta,\phi) \in (0,2\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})$
   (a) Show that $\mathbf{x}$ parametrizes the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$. What part(s) of the ellipsoid are 'missing' from the parametrization?
   (b) Describe geometrically the curves $\theta =$ constant and $\phi =$ constant on the ellipsoid.
   (c) Calculate the differential of $\mathbf{x}$ and show that $\mathrm{d}\mathbf{x}$ is 1–1 for each $p \in U$.



7. The tube of radius $a > 0$ centered on a curve $\mathbf{y}(t)$ may be parametrized in terms of the Frenet frame of $\mathbf{y}$:
   $$\mathbf{x}(\phi,t) = \mathbf{y}(t) + a\cos\phi\,\mathbf{N}(t) + a\sin\phi\,\mathbf{B}(t)$$
   (a) *Briefly* explain why the normal field is $\mathbf{n} = \cos\phi\,\mathbf{N}(t) + \sin\phi\,\mathbf{B}(t)$.
   (b) Suppose $\mathbf{y}$ is unit speed. Prove that $\mathbf{x}$ is everywhere regular if and only if $\kappa(t) < \frac{1}{a}$ at all points of the generating curve.

8. Let $c(t) : (-\epsilon,\epsilon) \to U$ be a curve and $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$ the corresponding curve in the surface $\mathbf{x} : U \to \mathbb{E}^3$. Prove that $\mathrm{d}\mathbf{x}\big(c'(0)\big) = \mathbf{y}'(0)$.
   (*Hint: Recall how to write $c'(t)$ as a* vector field)



9. (Möbius strip) Show that $\mathbf{x}(u,v) = \begin{pmatrix} (2+v\cos\frac{u}{2})\cos u \\ (2+v\cos\frac{u}{2})\sin u \\ v\sin\frac{u}{2} \end{pmatrix}$ is regular and orientable whenever $0 < u < 2\pi$ and $-1 < v < 1$. By computing $\mathbf{n}(0,0)$ and $\mathbf{n}(2\pi,0)$, explain what happens if we try to extend u to $[0,2\pi]$.

3.2 The Fundamental Forms

Our immediate goal is to use differentials to describe the shape of a surface. Before making the main definition, we need another product of 1-forms.

Definition 3.12. Given 1-forms α, β on U, define the *symmetric 2-form* $\alpha\beta$ by

$$\alpha\beta(\vec{v}, \vec{w}) = \frac{1}{2}\big(\alpha(\vec{v})\beta(\vec{w}) + \alpha(\vec{w})\beta(\vec{v})\big)$$

where $\vec{v}, \vec{w}$ are vector fields on U. Note that $\alpha^2(\vec{v}, \vec{w}) := \alpha\alpha(\vec{v}, \vec{w}) = \alpha(\vec{v})\alpha(\vec{w})$.

Symmetric 2-forms behave the way you (hopefully!) think they should.

Lemma 3.13. *On each tangent space, $\alpha\beta : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ is a symmetric and bilinear. Moreover $\alpha\beta = \beta\alpha$, and the product is linear in each slot:*

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad \text{and} \quad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma \tag{$*$}$$

Take care when using co-ordinate 1-forms; convention dictates that $\mathrm{d}x^2 = (\mathrm{d}x)^2$ is a symmetric 2-form, *not* the exterior derivative (1-form) $\mathrm{d}(x^2) = 2x\,\mathrm{d}x$.

Example 3.14. Let $\vec{v} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $\vec{w} = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$. Then

$$\mathrm{d}x^2(\vec{v}, \vec{w}) = ac, \qquad \mathrm{d}y^2(\vec{v}, \vec{w}) = bd, \qquad \mathrm{d}x\,\mathrm{d}y(\vec{v}, \vec{w}) = \frac{1}{2}(ad + bc)$$

In particular, $\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)(\vec{v}, \vec{w}) = ac + bd$ is the *dot product* in disguise.
To evaluate symmetric 2-forms with respect to co-ordinates, linearity and distributivity $(*)$ are all you need. For instance, if $\alpha = x\,\mathrm{d}x - \mathrm{d}y$ and $\beta = xy\,\mathrm{d}y$, then $\alpha\beta = x^2y\,\mathrm{d}x\mathrm{d}y - xy\,\mathrm{d}y^2$.

If α, β take values in $\mathbb{E}^n$, we use the dot product for multiplication of the resulting *vectors* $\alpha(\vec{v})$, etc.

$$(\alpha \cdot \beta)(\vec{v}, \vec{w}) := \frac{1}{2}\big(\alpha(\vec{v}) \cdot \beta(\vec{w}) + \alpha(\vec{w}) \cdot \beta(\vec{v})\big)$$

Definition 3.15. The *first and second fundamental forms* of a regular local surface $\mathbf{x} : U \to \mathbb{E}^3$ are

$$\mathrm{I} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x}, \qquad \mathrm{I\!I} = -\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{n}$$

where $\mathrm{d}\mathbf{n}$ is the differential of the unit normal field ($\mathrm{I\!I}$ requires that the surface be *oriented*). The first fundamental form is also commonly denoted $\mathrm{d}s^2$ (see Example 3.17 and Theorem 3.20 for why).

Example 3.16. If $\mathbf{x}(u, v) = \big(u, uv, 1 + u\big)$, then

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 1 \\ v \\ 1 \end{pmatrix} \mathrm{d}u + \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} \mathrm{d}v, \qquad \mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathrm{d}\mathbf{n} = \mathbf{0}$$

from which $\mathrm{I} = (2 + v^2)\,\mathrm{d}u^2 + 2uv\,\mathrm{d}u\,\mathrm{d}v + u^2\,\mathrm{d}v^2$ and $\mathrm{I\!I} = 0$.

Why should we care about I & $\mathrm{I\!I}$?

Basic interpretation $\mathrm{I}(\vec{v},\vec{w}) = \mathrm{d}\mathbf{x}(\vec{v})\cdot\mathrm{d}\mathbf{x}(\vec{w})$ pulls back the dot product from $T_P S$ to $T_p\mathbb{R}^2$. The length of and angle between tangent vectors to the surface S at P may now be computed in $T_p\mathbb{R}^2$.

$\mathrm{I\!I}(\vec{v},\vec{w}) = -\frac{1}{2}\big(\mathrm{d}\mathbf{x}(\vec{v})\cdot\mathrm{d}\mathbf{n}(\vec{w}) + \mathrm{d}\mathbf{x}(\vec{w})\cdot\mathrm{d}\mathbf{n}(\vec{v})\big)$ describes how the normal field $\mathbf{n}$ changes over the surface. In the example, $\mathrm{I\!I}\equiv 0$ encapsulates the *constancy* of the normal field: the surface is (part of) the plane $\mathbf{x}\cdot(1,0,-1) = -1$.

Co-ordinate invariance Since $\mathrm{d}\mathbf{x}$ is independent of co-ordinates, so also is I. The unit normal field is independent of *oriented* co-ordinate changes. More formally, if $\mathbf{y}(s,t) = \mathbf{x}(u,v)$ parametrize the same surface, then[17]

$$\mathrm{I}_{\mathbf{y}} = \mathrm{I}_{\mathbf{x}} \quad\text{and}\quad \mathrm{I\!I}_{\mathbf{y}} = \begin{cases}\mathrm{I\!I}_{\mathbf{x}} & \text{if the orientations are identical}\\ -\mathrm{I\!I}_{\mathbf{x}} & \text{if the orientations are reversed}\end{cases}$$

The upshot is that the fundamental forms provide a co-ordinate independent way to compute information about a surface from within the *parametrization space* U.

Example 3.17. For the sphere of radius a in spherical polar co-ordinates, recall Example 3.2:

$$\mathbf{x}(\theta,\phi) = a\begin{pmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ \sin\phi\end{pmatrix} \implies \mathrm{d}\mathbf{x} = a\cos\phi\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}\mathrm{d}\theta + a\begin{pmatrix}-\cos\theta\sin\phi\\ -\sin\theta\sin\phi\\ \cos\phi\end{pmatrix}\mathrm{d}\phi$$
$$\implies \mathrm{I} = a^2\big(\cos^2\!\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$$

If you revisit the pictures in Example 3.2, the effect of I is easy to visualize:

- $\mathrm{I}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) = \lvert\lvert\mathbf{x}_\theta\rvert\rvert^2 = a^2\cos^2\!\phi$: the tangent vector $\mathbf{x}_\theta$ is *shorter* near the poles, where $\cos\phi\to 0$.
- $\mathrm{I}(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}) = \big\lvert\big\lvert\mathbf{x}_\phi\big\rvert\big\rvert^2 = a^2$: the tangent vector $\mathbf{x}_\phi$ always has the *same length.*
- $\mathrm{I}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}) = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0$: the co-ordinate tangent vectors are always *orthogonal.*

At a point $P = \mathbf{x}(p)$ on the sphere, if we increase the co-ordinates by tiny quantities $\Delta p = \big(\Delta\theta,\Delta\phi\big)$, then the distance Δs travelled along the surface approximately satisfies

$$(\Delta s)^2 \approx \lvert\lvert\mathbf{x}(p+\Delta p) - \mathbf{x}(p)\rvert\rvert \approx \big\lvert\big\lvert\mathbf{x}_\theta\,\Delta\theta + \mathbf{x}_\phi\,\Delta\phi\big\rvert\big\rvert^2 = a^2\cos^2\!\phi\,(\Delta\theta)^2 + a^2(\Delta\phi)^2$$

with equality in the limit $\Delta\theta,\Delta\phi\to 0$. Near the poles, a change in *longitude* $\Delta\theta$ corresponds to a smaller distance on the sphere. This is analogous to how a standard *map* of the Earth works, with distances appearing distorted near the poles. We'll return to this idea shortly…

Computing $\mathrm{I\!I}$ is very easy for the sphere, since $\mathbf{n} = \frac{1}{a}\mathbf{x}$ is merely the scaled position vector:

$$\mathrm{I\!I} = -\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{n} = -\frac{1}{a}\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = -\frac{1}{a}\mathrm{I} = -a\big(\cos^2\!\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$$

[17] As in the Aside on page 49, we strictly have $\mathrm{I}_{\mathbf{y}} = \mathrm{I}_{\mathbf{x}}\circ\mathrm{d}F$, etc., where $\mathbf{y}(s,t) = \mathbf{x}\big(F(s,t)\big) = \mathbf{x}(u,v)$. The $\pm$-sign in the expressions for $\mathrm{I\!I}$ is that of the *determinant* of the Jacobian $\mathrm{d}F$.

The fundamental forms $\mathrm{I}, \mathbb{I}$ may be computed directly in terms of co-ordinates u, v.

Theorem 3.18. *If $\mathbf{x} : U \to \mathbb{E}^3$ is a regular (oriented) surface, then*

$$\mathrm{I} = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \quad \textit{and} \quad \mathbb{I} = l\,\mathrm{d}u^2 + 2m\,\mathrm{d}u\,\mathrm{d}v + n\,\mathrm{d}v^2$$

where the smooth functions $E, F, G, l, m, n : U \to \mathbb{R}$ are defined by

$$\begin{aligned} &E = \mathbf{x}_u \cdot \mathbf{x}_u && F = \mathbf{x}_u \cdot \mathbf{x}_v && G = \mathbf{x}_v \cdot \mathbf{x}_v \\ &l = \mathbf{x}_{uu} \cdot \mathbf{n} = -\mathbf{x}_u \cdot \mathbf{n}_u && m = \mathbf{x}_{uv} \cdot \mathbf{n} = -\mathbf{x}_u \cdot \mathbf{n}_v = -\mathbf{x}_v \cdot \mathbf{n}_u && n = \mathbf{x}_{vv} \cdot \mathbf{n} = -\mathbf{x}_v \cdot \mathbf{n}_v \end{aligned}$$

The expressions for $\mathbb{I}$ come from differentiating $\mathbf{x}_u \cdot \mathbf{n} = 0 = \mathbf{x}_v \cdot \mathbf{n}$, and are particularly helpful because they avoid computing derivatives of $\mathbf{n}$ (which likely contains a square-root).

Example 3.19. Parametrize the graph of $z = f(x, y)$ by $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$ to obtain,

$$\mathbf{x}_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}, \quad \mathbf{x}_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix} \implies E = 1 + f_u^2, \quad F = f_u f_v, \quad G = 1 + f_v^2$$

$$\implies \mathrm{I} = (1 + f_u^2)\,\mathrm{d}u^2 + 2 f_u f_v\,\mathrm{d}u\,\mathrm{d}v + (1 + f_v^2)\,\mathrm{d}v^2$$

$$\mathbf{x}_{uu} = \begin{pmatrix} 0 \\ 0 \\ f_{uu} \end{pmatrix}, \quad \mathbf{x}_{uv} = \begin{pmatrix} 0 \\ 0 \\ f_{uv} \end{pmatrix}, \quad \mathbf{x}_{vv} = \begin{pmatrix} 0 \\ 0 \\ f_{vv} \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$\implies l = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad m = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad n = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}$$

$$\implies \mathbb{I} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \left(f_{uu}\,\mathrm{d}u^2 + 2 f_{uv}\,\mathrm{d}u\,\mathrm{d}v + f_{vv}\,\mathrm{d}v^2 \right)$$

As a particular example, the circular paraboloid $z = x^2 + y^2$ has fundamental forms

$$\begin{aligned} \mathrm{I} &= (1 + 4u^2)\,\mathrm{d}u^2 + 8uv\,\mathrm{d}u\,\mathrm{d}v + (1 + 4v^2)\,\mathrm{d}v^2 = \mathrm{d}u^2 + \mathrm{d}v^2 + 4\big(u\,\mathrm{d}u + v\,\mathrm{d}v\big)^2 \\ \mathbb{I} &= \frac{2}{\sqrt{1 + 4u^2 + 4v^2}} (\mathrm{d}u^2 + \mathrm{d}v^2) \end{aligned}$$

As a sanity check, compare with the parametrization of the same paraboloid in polar co-ordinates $\mathbf{y}(r, \theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$ (Exercise 3.1.3). By computing the partial derivatives $\mathbf{y}_r, \mathbf{y}_\theta, \mathbf{y}_{rr}, \mathbf{y}_{r\theta}, \mathbf{y}_{\theta\theta}$ directly, one easily verifies that

$$\mathrm{I} = (1 + 4r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \qquad \mathbb{I} = \frac{2}{\sqrt{1 + 4r^2}} (\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$$

These expressions are identical to the originals (same orientation!) since

$$\begin{cases} \mathrm{d}u = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}v = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta \end{cases} \implies \begin{cases} \mathrm{d}u^2 + \mathrm{d}v^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \\ \big(u\,\mathrm{d}u + v\,\mathrm{d}v\big)^2 = r^2\,\mathrm{d}r^2 \end{cases}$$

Curves in Surfaces: interpreting I and $\mathrm{I\!I}$

Given a regular (oriented) surface $\mathbf{x} : U \to \mathbb{E}^3$ and a curve $c(t)$ in U, we may transfer this curve to the surface $\mathbf{y}(t) = \mathbf{x}(c(t))$. Its tangent vector (Exercise 3.1.8) and speed are then

$$\mathbf{y}'(t) = \mathrm{d}\mathbf{x}\big(c'(t)\big) \implies ||\mathbf{y}'(t)|| = \sqrt{\mathrm{d}\mathbf{x}\big(c'(t)\big)\cdot\mathrm{d}\mathbf{x}\big(c'(t)\big)} = \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}$$

We can do something similar for the second fundamental form.

Theorem 3.20. *Let $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$ parametrize a curve in a surface $\mathbf{x}$ with unit normal field $\mathbf{n}$.*

1. *If $a < b$, then the* arc-length *of $\mathbf{y}$ between $\mathbf{y}(a)$ and $\mathbf{y}(b)$ is $\int_a^b \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}\,\mathrm{d}t$.*
2. *The* normal acceleration *of the curve is $\mathbf{y}''(t)\cdot\mathbf{n} = \mathrm{I\!I}(c', c')$.*

This puts some flesh on our earlier observations (page 56). I measures *infinitesimal squared-distance* on the surface, while $\mathrm{I\!I}$ measures how the surface bends away from the normal field: recall how force/acceleration motivated the curvature κ of a curve (Definition 1.15).

Proof. 1. Arc-length is the integral of the speed $||\mathbf{y}'(t)|| = \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}$.

2. Since $\mathbf{y}'$ lies in the tangent plane, we have $\mathbf{y}'\cdot\mathbf{n} \equiv 0$. Differentiate to obtain

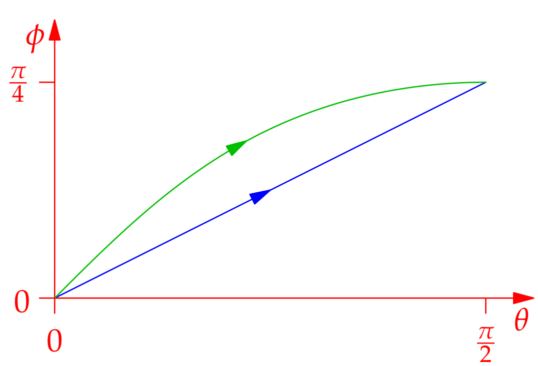
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{y}'\cdot\mathbf{n}) = \mathbf{y}''\cdot\mathbf{n} + \mathbf{y}'\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}\big(c(t)\big) = \mathbf{y}''\cdot\mathbf{n} + \mathrm{d}\mathbf{x}(c')\cdot\mathrm{d}\mathbf{n}(c') = \mathbf{y}''\cdot\mathbf{n} - \mathrm{I\!I}(c', c')$$

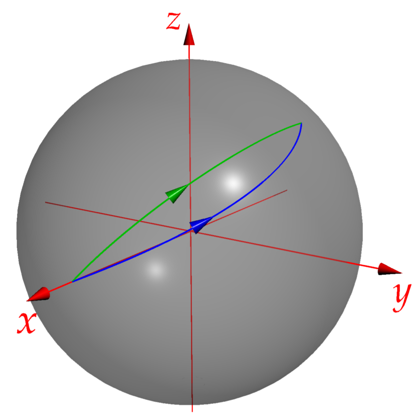
∎

Example (3.17, cont). Consider the curve $c(t) = \big(\theta(t), \phi(t)\big) = \big(2t, t\big)$ where $0 \le t \le \frac{\pi}{4}$. This has tangent field $c'(t) = 2\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}$. Translated to the unit sphere, the resulting curve has arc-length

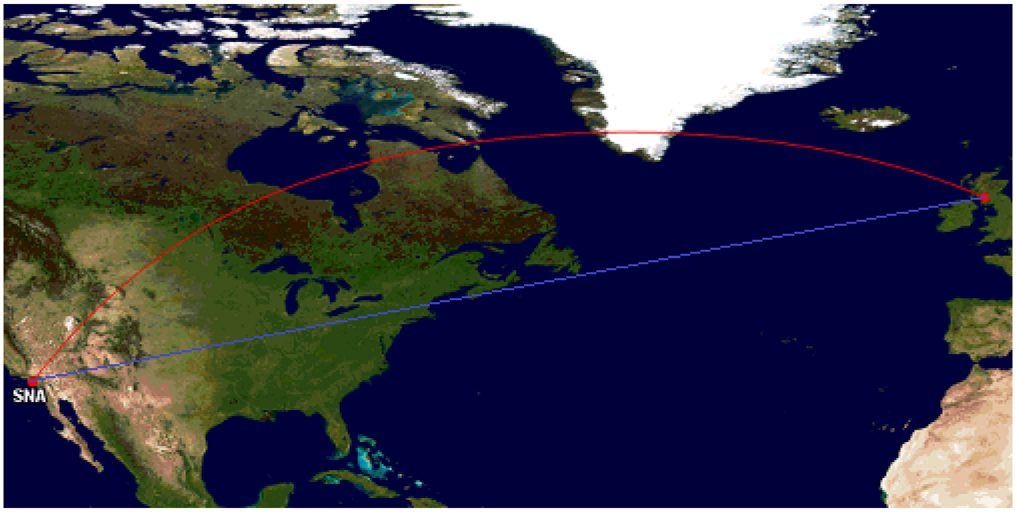
$$\int_0^{\frac{\pi}{4}} \sqrt{\mathrm{I}(c', c')}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} \sqrt{4\cos^2 t + 1}\,\mathrm{d}t \approx 1.619$$

In the parametrization space U, $c(t)$ is a straight line. The shortest path between the endpoints of the curve *on the sphere* is the great circle arc with length $\frac{2\pi}{4} = \frac{\pi}{2} \approx 1.571$; its pre-image in U *appears* longer but isn't due to the $\cos^2\phi$ factor in the first fundamental form. By spending more time at northerly latitudes, I is smaller for more of the great circle arc and the resulting arc-length is shorter.





If a map of the Earth covers a small latitude range (almost constant $\phi \approx \phi_0$), the first fundamental form is almost similar to a standard dot product $\mathrm{I} \approx (a\cos\phi_0\, \mathrm{d}\theta)^2 + (a\,\mathrm{d}\phi)^2$. If not, say when we travel by plane, the distortion becomes much more apparent.



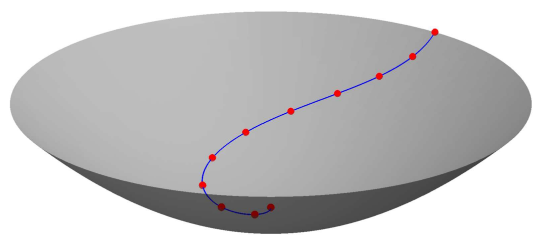
The picture shows the shortest path from Irvine (California) to Irvine (Scotland), as flown by an aircraft in ideal conditions. The straight line on the map corresponds to a *longer real-world path.*

If we travel at constant speed, it can be checked that great circles are precisely those curves whose acceleration is entirely normal to the surface. This observation, and its relation to *geodesics* (paths minimizing distance), is a matter for another course.

Example 3.21. A skater descends into a paraboloidal bowl $z = \frac{1}{2}r^2$ following the path described by $c(t) = (r(t), \theta(t)) = (1-t, 4t^2)$ in polar co-ordinates. If we parametrize the bowl in polar co-ordinates $\mathbf{x}(r,\theta) = \left(r\cos\theta, r\sin\theta, \frac{1}{2}r^2\right)$, the fundamental forms are seen to be

$$\mathrm{I} = (1+r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$$

$$\mathrm{I\!I} = \frac{1}{\sqrt{1+r^2}}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$$

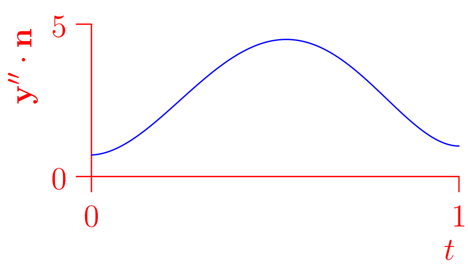


For the skater's path, $c'(t) = -\frac{\partial}{\partial r} + 8t\frac{\partial}{\partial\theta}$, whence

$$\mathrm{I}(c',c') = (1+(1-t)^2) + 64t^2(1-t)^2$$

The path therefore has arc-length

$$\int_0^1 \sqrt{\mathrm{I}(z',z')}\,\mathrm{d}t = \int_0^1 \sqrt{1+(64t^2+1)(1-t)^2}\,\mathrm{d}t \approx 1.82$$



and normal acceleration

$$\mathbf{y}'' \cdot \mathbf{n} = \mathrm{I\!I}(c',c') = \frac{1}{\sqrt{1+(1-t)^2}}\left(1+64t^2(1-t)^2\right)$$

By Newton's second law, this is proportional to the component of the force experienced by the skater pushing perpendicularly out from the surface.

Exercises 3.2. 1. Verify the final details of Example 3.19: that is, compute $\mathrm{I}, \mathbb{I}$ directly using the polar co-ordinate parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$.

2. Find the fundamental forms for the surface of revolution $\mathbf{x}(\theta, v) = \big(f(v)\cos\theta, f(v)\sin\theta, v\big)$.

3. Compute the first fundamental forms of each parametrized surface wherever they are regular (a, b, c are non-zero constants). Where does each parametrization fail to be regular?

   (a) Ellipsoid $\mathbf{x}(\theta,\phi) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi, c\sin\phi)$

   (b) Elliptic paraboloid $\mathbf{x}(r,\theta) = (ar\cos\theta, br\sin\theta, r^2)$

   (c) Hyperboloid of two sheets $\mathbf{x}(u,v) = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u)$

4. Calculate the fundamental forms of Enneper's surface

$$\mathbf{x}(u,v) = \big(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + vu^2, u^2 - v^2\big)$$

5. Compute $\mathrm{d}\mathbf{y}$ for the parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\big)$ of the upper unit hemisphere. Verify that the first fundamental form is the same as in Example 3.17.

6. Let $\mathbf{x}$ be the tangent developable of a unit speed biregular curve $\mathbf{y}$ (Exercise 3.1.4).

   (a) Compute the fundamental forms of $\mathbf{x}$ in terms of the curvature and torsion of $\mathbf{y}$.

   (b) If $\mathbf{y}(u) = \big(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\big)$ is the unit speed helix, show that

$$\mathrm{I} = \big(1 + \tfrac{v^2}{4}\big)\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2, \qquad \mathbb{I} = -\tfrac{v}{4}\,\mathrm{d}u^2$$

7. Prove that $\mathbb{I} \equiv 0$ if and only if $\mathbf{x}$ is (part of) a plane.

8. Parametrize the great circle in Example 3.17 (cont) by $\mathbf{z}(t) = \big(\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\sin t\big), 0 \le t \le \frac{\pi}{2}$. Verify that the arc has length $\frac{\pi}{2}$ and that the acceleration of $\mathbf{z}$ is entirely *normal*; $\mathbf{z}'' = (\mathbf{z}'' \cdot \mathbf{n})\mathbf{n}$.

9. Equip the upper half plane $y > 0$ with the abstract first fundamental form $\mathrm{I} = \frac{1}{y^2}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)$. Compare the arc-length between the points $(1,1)$ and $(-1,1)$:

   (a) Over the circular arc $c(t) = \sqrt{2}\big(\cos t, \sin t\big)$ centered at the origin.

   (b) Over the 'straight' line $y = 1$.

   This is the Poincaré half-plane model of hyperbolic space. There is neither a surface $\mathbf{x} : U \to \mathbb{E}^3$ nor a second fundamental form $\mathbb{I}$!

10. (Hard) The *torus* obtained by rotating the unit circle in the x,z-plane centered at $(2,0,0)$ around the z-axis may be parametrized

$$\mathbf{x}(u,v) = \big((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi\big), \quad (\theta,\phi) \in \mathbb{R}^2$$

   Let $k \neq 0$ be constant and consider the curve $\mathbf{y}(t) = \mathbf{x}(kt, t)$ on the torus.

   (a) Prove that $\mathbf{y}(t)$ has a self-intersection ($\exists s \neq t$ such that $\mathbf{y}(t) = \mathbf{y}(s)$) if and only if $k \in \mathbb{Q}$.

   (b) If $k \in \mathbb{Q}$, show that the curve is *periodic* in that there exists a minimum positive T for which $\mathbf{y}(t+T) = \mathbf{y}(t)$ for all t. Find T in terms of k and write down (don't evaluate!) the integral for the arc-length of the curve over one period.

3.3 Principal, Gauss & Mean Curvatures

Since I and $\mathrm{I\!I}$ are symmetric bilinear forms on each tangent space $T_p\mathbb{R}^2$, they may be expressed in matrix form: their matrices with respect to linearly independent vector fields $\vec{s},\vec{t}$ are

$$[\mathrm{I}] := \begin{pmatrix} \mathrm{I}(\vec{s},\vec{s}) & \mathrm{I}(\vec{s},\vec{t}) \\ \mathrm{I}(\vec{s},\vec{t}) & \mathrm{I}(\vec{t},\vec{t}) \end{pmatrix} \quad \text{and} \quad [\mathrm{I\!I}] := \begin{pmatrix} \mathrm{I\!I}(\vec{s},\vec{s}) & \mathrm{I\!I}(\vec{s},\vec{t}) \\ \mathrm{I\!I}(\vec{s},\vec{t}) & \mathrm{I\!I}(\vec{t},\vec{t}) \end{pmatrix}$$

Otherwise said

$$\mathrm{I}\big(f\vec{s}+g\vec{t}, h\vec{s}+k\vec{t}\big) = \begin{pmatrix} f & g \end{pmatrix} [\mathrm{I}] \begin{pmatrix} h \\ k \end{pmatrix}$$

and similarly for $\mathrm{I\!I}$. Matters are simplest when these matrices are *diagonal*...

Definition 3.22. Linearly independent vector fields $\vec{s},\vec{t}$ are said to be *orthogonal* if $\mathrm{I}(\vec{s},\vec{t}) = 0$. They additionally describe *curvature directions* if $\mathrm{I\!I}(\vec{s},\vec{t}) = 0$.
Co-ordinates u,v are *orthogonal*/*curvature-line* if the above apply to the the co-ordinate fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$.

In the language of Theorem 3.18, the matrices of the fundamental forms with respect to $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ are

$$A := \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} l & m \\ m & n \end{pmatrix} \tag{$*$}$$

Co-ordinates are orthogonal iff $F = \mathbf{x}_u \cdot \mathbf{x}_v \equiv 0$ (I has no $\mathrm{d}u\,\mathrm{d}v$ term), and are curvature-line iff $\mathrm{I\!I}$ is also diagonal:

$$\mathrm{I} = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2 \quad \text{and} \quad \mathrm{I\!I} = l\,\mathrm{d}u^2 + n\,\mathrm{d}v^2$$

While the meaning of *orthogonal* is clear, the reason for the term *curvature-line* will take a little work.

Examples 3.23. 1. Since the sphere of radius a has $\mathrm{I\!I} = -\frac{1}{a}\mathrm{I}$, any orthogonal co-ordinates on the sphere are curvature-line! E.g., spherical polar co-ordinates: $\mathrm{I} = a^2\big(\cos^2\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$.

2. (Example 3.2.3.19) Standard polar co-ordinates are curvature-line for the paraboloid $z = r^2$:

$$\mathrm{I} = (1+4r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \qquad \mathrm{I\!I} = \frac{2}{\sqrt{1+4r^2}}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\big)$$

3. In curvature-line co-ordinates $\mathbf{n}_u = -\frac{l}{E}\mathbf{x}_u$ and $\mathbf{n}_v = -\frac{n}{G}\mathbf{x}_v$ (see Exercise 11).

A Little Linear Algebra The existence of curvature directions is equivalent to the *simultaneous diagonalization* of both matrices ($*$). This requires an extension of the concepts of eigenvalues/vectors.

Definition 3.24. Let A, B be square matrices of the same dimension. A non-zero vector $\vec{v}$ is an *eigenvector* of B with respect to A with *eigenvalue* λ if

$$(B - \lambda A)\vec{v} = \vec{0}$$

If $A = I$ is the identity matrix, these are standard eigenvalues/vectors. We compute in the usual manner: solve the characteristic polynomial and find $\vec{v} \in \mathcal{N}(B - \lambda A)$ in the nullspace...

Example 3.25. Let $A = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 5 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 3 \end{smallmatrix}\right)$.

$$\det(B - \lambda A) = \begin{vmatrix} -2\lambda & 1-3\lambda \\ 1-3\lambda & 3-5\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1$$

$$\vec{v}_1 \in \mathcal{N}(B - A) = \mathcal{N}\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} = \text{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\vec{v}_2 \in \mathcal{N}(B + A) = \mathcal{N} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = \text{Span}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Note that $\{\vec{v}_1, \vec{v}_2\} = \left\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)\right\}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of B with respect to A.

Theorem 3.26. *Let A, B be symmetric matrices of the same dimension, with A* positive-definite.[18]

1. *There exists a basis of eigenvectors of B with respect to A. Moreover, all eigenvalues are* real.
2. *If $\vec{s}, \vec{t}$ are eigenvectors corresponding to distinct eigenvalues, then $\vec{s}^T A\vec{t} = 0 = \vec{s}^T B\vec{t}$.*

Proof.

1. This follows from the famous *spectral theorem* in linear algebra.[19]
2. Assume $B\vec{s} = k_1 A\vec{s}$ and $B\vec{t} = k_2 A\vec{t}$ where $k_1 \neq k_2$, and apply the symmetry of A and B,

$$\left.\begin{array}{l} \vec{s}^T B\vec{t} = \vec{s}^T(k_2 A\vec{t}) = k_2\vec{s}^T A\vec{t} \\ \quad \| \\ \vec{t}^T B\vec{s} = \vec{t}^T(k_1 A\vec{s}) = k_1\vec{t}^T A\vec{s} = k_1\vec{s}^T A\vec{t} \end{array}\right\} \implies (k_2 - k_1)\vec{s}^T A\vec{t} = 0 \implies \vec{s}^T A\vec{t} = 0$$

■

Application to Regular Surfaces With respect to independent vector fields, the matrices A, B of $\mathrm{I}, \mathrm{I\!I}$ are symmetric. Moreover, the regularity of $\mathbf{x}$ guarantees the positive-definiteness of A:

$$\forall \vec{w} \neq \vec{0} \implies \mathrm{I}(\vec{w}, \vec{w}) = \mathbf{dx}(\vec{w}) \cdot \mathbf{dx}(\vec{w}) = ||\mathbf{dx}(\vec{w})||^2 > 0$$

We may therefore apply Theorem 3.26 to the matrices of the fundamental forms.

Definition 3.27. The *principal curvatures* $k_1, k_2 : U \to \mathbb{R}$ of an oriented surface $\mathbf{x} : U \to \mathbb{E}^3$ are the eigenvalues of $\mathrm{I\!I}$ with respect to I. Corresponding eigenvectors are *curvature directions.*
The *Gauss* and *mean curvatures* are, respectively, $K := k_1 k_2$ and $H := \frac{1}{2}(k_1 + k_2)$.
A point $\mathbf{x}(p)$ is *umbilic* if $k_1(p) = k_2(p)$.

[18] For all non-zero vectors, $\vec{v}^T A\vec{v} > 0$. Equivalently, all eigenvalues of A are positive. This means that $\langle \vec{v}, \vec{w} \rangle := \vec{v}^T A\vec{w}$ defines an *inner product* on $\mathbb{R}^n$. In Example 3.25, A has eigenvalues $\frac{1}{2}(7 \pm \sqrt{45}) > 0$.

[19] In case you're interested: A has an orthogonal eigenbasis $\{\vec{x}_1, \ldots, \vec{x}_n\}$ by the spectral theorem. Since its eigenvalues μ_i are positive, we may scale such that $||\vec{x}_i||^2 = \frac{1}{\mu_i}$. Let $X = (\vec{x}_1 \ \cdots \ \vec{x}_n)$ so that $X^T AX = I$ is the identity matrix. But then,

$$\det(B - \lambda A) = \det(X^T)^{-1}\det\big(X^T BX - \lambda I\big)\det(X^{-1}) = 0 \iff \det(X^T BX - \lambda I) = 0$$

Since $X^T BX$ is symmetric (spectral theorem again), it has an orthogonal eigenbasis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ and real eigenvalues λ_k. Each $\vec{v}_k := X\vec{w}_k$ is an eigenvector of B with respect to A with eigenvalue λ_k. Since X is invertible, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis.

The curvatures are independent of *oriented* co-ordinate changes. If we reverse orientation, then k_1, k_2 and H change sign, while $K = k_1k_2$ is unchanged.

At non-umbilic points, Theorem 3.26 says that curvature directions diagonalize both fundamental forms, in line with Definition 3.22.

At umbilic points, $\mathrm{I\!I} = k\mathrm{I}$ and all directions are curvature directions; any orthogonal directions necessarily diagonalize both fundamental forms.

Example 3.28. Here are two *totally umbilic* surfaces where the curvatures are constant.

1. A plane: $\mathrm{I\!I} \equiv 0 \implies$ all curvatures are zero.
2. A sphere of radius a: $\mathrm{I\!I} = -\frac{1}{a}\mathrm{I} \implies k_1 = k_2 = -\frac{1}{a}$, $K = \frac{1}{a^2}$ and $H = -\frac{1}{a}$.

In fact these comprise all totally umbilic surfaces (see Exercise 12).

Theorem 3.29. 1. *In co-ordinates, the Gauss and mean curvatures are given by*

$$K = \frac{ln - m^2}{EG - F^2} = \frac{\det B}{\det A} = \det(A^{-1}B) \quad \text{and} \quad H = \frac{lG + nE - 2mF}{2(EG - F^2)} = \frac{1}{2}\operatorname{tr} A^{-1}B$$

2. *At non-umbilic points, the curvatures k_1, k_2, K, H are smooth functions and the curvature directions may be described locally by (smooth) vector fields.*

Proof. 1. The principal curvatures are the solutions to the quadratic equation

$$\det\left(\begin{pmatrix} l & m \\ m & n \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = (EG - F^2)\lambda^2 - (lG + nE - 2mF)\lambda + (ln - m^2)$$

of which K and H are the product and half the sum of the roots.

2. The roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of a quadratic are smooth functions of the coefficients unless $b^2 - 4ac = 0$, in which case we have a repeated root ($k_1 = k_2$). At non-umbilic points, each eigenspace is one-dimensional, so there is no obstruction to choosing smooth eigenvectors.[20] ∎

Examples 3.30. 1. (Example 3.19) For the paraboloid $\mathbf{x}(r, \theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$, standard polar co-ordinates are curvature-line:

$$A = [\mathrm{I}] = \begin{pmatrix} 1 + 4r^2 & 0 \\ 0 & r^2 \end{pmatrix} \qquad B = [\mathrm{I\!I}] = \begin{pmatrix} \frac{2}{\sqrt{1+4r^2}} & 0 \\ 0 & \frac{2r^2}{\sqrt{1+4r^2}} \end{pmatrix}$$

The curvatures are therefore

$$k_1 = \frac{2}{(1 + 4r^2)^{3/2}}, \quad k_2 = \frac{2}{\sqrt{1 + 4r^2}}, \quad K = \frac{4}{(1 + 4r^2)^2}, \quad H = \frac{2 + 4r^2}{(1 + 4r^2)^{3/2}}$$

The curvatures make sense at the single umbilic point ($r = 0$), but the co-ordinates are not curvature-line there since the parametrization fails to be regular ($\mathbf{x}_\theta(0, \theta) = \mathbf{0}$).

[20] At an umbilic point $\mathbf{x}(p)$, the eigenspace is 2-dimensional so $\lim_{q \to p} \vec{v}(q)$ need not exist and $\vec{v}$ need not be continuous.

2. Parametrize a graph $z = f(x,y)$ in the usual manner $\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$. Then

$$A = [\mathrm{I}] = \begin{pmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{pmatrix} \qquad B = [\mathrm{I\!I}] = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

Theorem 3.29 tells us that

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} \qquad H = \frac{f_{vv}(1+f_u^2) + f_{uu}(1+f_v^2) - 2 f_u f_v f_{uv}}{2(1+f_u^2+f_v^2)^{3/2}}$$

In the abstract, solving for the curvatures and directions is disgusting. As a sanity check, you should verify that $f(u,v) = u^2 + v^2$ recovers exactly the curvatures in the previous example!

3. (Exercise 3.2.6) The tangent developable of the unit-speed helix has

$$A = [\mathrm{I}] = \begin{pmatrix} 1+\frac{v^2}{4} & 1 \\ 1 & 1 \end{pmatrix} \qquad B = [\mathrm{I\!I}] = \begin{pmatrix} -\frac{v}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

Now solve for the curvatures:

$$\begin{vmatrix} -\frac{v}{4} - \lambda\left(1+\frac{v^2}{4}\right) & -\lambda \\ -\lambda & -\lambda \end{vmatrix} = \frac{v^2}{4}\lambda^2 + \frac{v}{4}\lambda = 0 \implies k_1 = 0,\ k_2 = -\frac{1}{v}, \quad K = 0,\ H = -\frac{1}{2v}$$

In this case an explicit computation of the curvature directions is not difficult:

$$\begin{aligned} k_1 = 0 &\implies \mathcal{N}(B - k_1 A) = \mathcal{N}\begin{pmatrix} -\frac{v}{4} & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \vec{s} = \frac{\partial}{\partial v} \\ k_2 = -\frac{1}{v} &\implies \mathcal{N}(B - k_2 A) = \mathcal{N}\begin{pmatrix} \frac{1}{v} & \frac{1}{v} \\ \frac{1}{v} & \frac{1}{v} \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightsquigarrow \vec{t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \end{aligned}$$

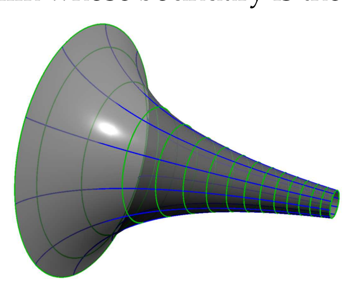
where we made the natural choice of vector fields $\vec{s}, \vec{t}$. As a sanity check, here are the matrices of the fundamental forms with respect to $\vec{s}, \vec{t}$:

$$\mathrm{I}(\vec{s},\vec{s}) = (0\ 1)\, A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \ \ldots \implies [\mathrm{I}] = \begin{pmatrix} \mathrm{I}(\vec{s},\vec{s}) & \mathrm{I}(\vec{s},\vec{t}) \\ \mathrm{I}(\vec{s},\vec{t}) & \mathrm{I}(\vec{t},\vec{t}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{v^2}{4} \end{pmatrix} \qquad [\mathrm{I\!I}] = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{v}{4} \end{pmatrix}$$

in which the principal curvatures are clearly visible: $1k_1 = 0$, $\frac{v^2}{4} k_2 = -\frac{v}{4}$

Constant Gauss & Mean Curvature Surfaces

Minimal Surfaces $H \equiv 0$: Among all surfaces whose boundary is a given closed curve, a surface with minimal surface area has $H \equiv 0$. This is the shape made by a soap film whose boundary is the curve: it minimizes the 'total tension' of the soap film. More generally, *constant mean curvature* (CMC) surfaces model soap *bubbles.*



Constant Gauss Curvature Surfaces: We've see that planes, cones and cylinders have $K = 0$, and that spheres have constant positive Gauss curvature. A *pseudosphere* with constant $K = -1$ is shown in the picture.

Existence of (Curvature-Line) Co-ordinates

At non-umbilic points, Theorems 3.26 and 3.29 tell us how to find curvature *directions* as vector fields $\vec{s}, \vec{t}$. Unfortunately, being able to compute *explicit* curvature *co-ordinates* is exceptionally unlikely.

Example (3.30.3 cont). Recall that we chose curvature direction fields $\vec{s} = \frac{\partial}{\partial v}$ and $\vec{t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$. By inspection, the functions $s = u + v$ and $t = u$ satisfy the required equations:

$$\vec{s}[s] = 1 = \vec{t}[t], \qquad \vec{s}[t] = 0 = \vec{t}[s] \tag{$*$}$$

It follows that $\vec{s} = \frac{\partial}{\partial s}$ and $\vec{t} = \frac{\partial}{\partial t}$ for curvature-line co-ordinates s, t, as you can easily verify using the chain rule. Indeed

$$\mathrm{I} = \frac{v^2}{4}\,\mathrm{d}u^2 + \mathrm{d}(u+v)^2 = \mathrm{d}s^2 + \frac{v^2}{4}\,\mathrm{d}t^2, \qquad \mathrm{I\!I} = 0\,\mathrm{d}s^2 - \frac{v}{4}\,\mathrm{d}t^2$$

so that the co-ordinates really do diagonalize both fundamental forms.

The simple reason the example is so unlikely is that *mixed partial derivatives must commute*: if $\vec{s} = \frac{\partial}{\partial s}$ and $\vec{t} = \frac{\partial}{\partial t}$ are co-ordinate fields ($\exists s, t : U \to \mathbb{R}$), then their *Lie bracket* (Exercise 2.3.10) vanishes:

$$[\vec{s}, \vec{t}] = \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = \frac{\partial}{\partial s} \circ \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \circ \frac{\partial}{\partial s} = 0$$

The astonishing fact is that this simple condition is *locally sufficient.*

Theorem 3.31 (Co-ordinate fields). *Let $\vec{s}, \vec{t}$ be linearly independent vector fields on $U \subseteq \mathbb{R}^2$.*

1. *If there exist functions $s, t : U \to \mathbb{R}$ such that $\vec{s} = \frac{\partial}{\partial s}, \vec{t} = \frac{\partial}{\partial t}$, then $[\vec{s}, \vec{t}] = 0$.*
2. *Suppose $[\vec{s}, \vec{t}] = 0$ and let $p \in U$. Then there exists a neighborhood V of p and co-ordinate functions $s, t : V \to \mathbb{R}$ for which $\vec{s} = \frac{\partial}{\partial s}, \vec{t} = \frac{\partial}{\partial t}$.*

Examples 3.32. 1. The fields $\vec{s} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\vec{t} = \frac{\partial}{\partial y}$ do not arise simultaneously from co-ordinates:

$$[\vec{s}, \vec{t}] = \frac{\partial^2}{\partial x \partial y} + y\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y \partial x} - \frac{\partial}{\partial y} - y\frac{\partial^2}{\partial y^2} = -\frac{\partial}{\partial y} \neq 0$$

2. The cylindrical paraboloid $\mathbf{x}(u, v) = (u, v, \frac{1}{2}u^2 + v)$ has curvatures and curvature directions

$$k_1 = 0,\ \vec{s} = \frac{\partial}{\partial v}, \qquad k_2 = \frac{2}{[2+u^2]^{3/2}},\ \vec{t} = 2\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$$

The Lie bracket condition $[\vec{s}, \vec{t}] = 0$ is satisfied, so co-ordinates s, t corresponding to these fields must exist. You can try to find such by inspection, though simultaneously solving $(*)$ is messy. Alternatively, following the proof of part 2 (Exercise 13), observe that the dual 1-forms are

$$\alpha = \frac{1}{2}u\,\mathrm{d}u + \mathrm{d}v, \qquad \beta = \frac{1}{2}\,\mathrm{d}u \qquad (\alpha(\vec{s}) = \beta(\vec{t}) = 1, \alpha(\vec{t}) = \beta(\vec{s}) = 0)$$

These forms are *exact*: $\alpha = \mathrm{d}(\frac{1}{4}u^2 + v)$ and $\beta = \mathrm{d}(\frac{1}{2}u)$. We therefore conclude that $s = \frac{1}{4}u^2 + v$ and $t = \frac{1}{2}u$ are suitable curvature-line co-ordinates.

The Lie bracket condition says that explicit co-ordinates corresponding to given vector fields are very unlikely to exist. This is no matter: we typically only require co-ordinates s, t whose fields $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ are *parallel* to $\vec{s}, \vec{t}$: that is

$$\frac{\partial}{\partial s} = f\vec{s} \quad \text{and} \quad \frac{\partial}{\partial t} = g\vec{t} \quad \text{for some functions } f, g \qquad (\text{equivalently } \vec{s}[t] = 0 = \vec{t}[s])$$

Such co-ordinates indeed exist, though only *locally*, as shown by one of the most important foundational results in differential geometry.

Theorem 3.33 (Frobenius). *Let $\vec{s}, \vec{t}$ be linearly independent vector fields on a domain U. Then there exist local co-ordinates s, t whose co-ordinate fields are parallel to $\vec{s}, \vec{t}$.*

In particular, if $\mathbf{x}(p)$ is a non-umbilic point on a surface $\mathbf{x} : U \to \mathbb{E}^3$, then there exists a neighborhood V of p and curvature-line co-ordinates s, t on V.

Frobenius' theorem comes in many guises and generalizes to higher dimensions, taking the place of Picard's ODE existence/uniqueness theorem (1.39) for particular classes of PDE. Its proof is too involved for us, though the informal idea is to search for functions f, g such that $[f\vec{s}, g\vec{t}] = 0$, a lengthy process that indeed depends on Picard's theorem.

Exercises 3.3. 1. Find the eigenvalues of $B = \left(\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ with respect to $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. If $\vec{s}, \vec{t}$ are corresponding eigenvectors, verify that $\vec{s}^T A\vec{t} = 0 = \vec{s}^T B\vec{t}$.

2. Parametrize the graph of $x = z^2$; compute I, II and the principal, Gauss and mean curvatures.

3. Use Theorem 3.29 to find the Gauss and mean curvatures of the graph of $y = x^2 - z^2$.

4. Show that Enneper's surface (Exercise 3.2.4) is minimal.

5. Let $\mathbf{x}(u, v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ be the tangent developable of a unit speed biregular curve $\mathbf{y}$.

   (a) Find the principal curvatures, Gauss and mean curvatures of $\mathbf{x}$.

   (b) Compute the curvature directions and find curvature line co-ordinates.

   (*This is very similar to Example 3.30.3 - keep track of the changes!*)

6. With respect to some co-ordinates u, v, suppose that a surface has fundamental forms

$$\mathrm{I} = u^2\,\mathrm{d}u^2 + v^2\,\mathrm{d}v^2, \qquad \mathrm{II} = u^2\,\mathrm{d}u^2 + 2uv\,\mathrm{d}u\,\mathrm{d}v + v^2\,\mathrm{d}v^2$$

   (a) Show that the principal curvatures are constant: $k_1 = 0$ and $k_2 = 2$.

   (b) Show that $\vec{s} = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$ and $\vec{t} = v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$ are curvature directions.

   (c) Compute the Lie bracket $[\vec{s}, \vec{t}]$ to show that these are *not* vector fields with respect to some curvature-line co-ordinates s, t.

   (d) Find explicit curvature-line co-ordinates for the surface; functions s, t such that $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ are *parallel* to $\vec{s}, \vec{t}$ and express I, II with respect to s, t.
   (*Hint: try to guess solutions to* $\vec{s}[t] = 0 = \vec{t}[s]$)

7. Rotate $y = f(x)$ around the x-axis and parametrize the surface via

$$\mathbf{x}(\phi, v) = \big(v, f(v)\cos\phi, f(v)\sin\phi\big)$$

   (a) Verify that the co-ordinates ϕ, v are curvature-line, compute the principal curvatures, and show that the Gauss and mean curvatures are

$$K = -\frac{f''(v)}{f(v)\big(1+f'(v)^2\big)^2} \qquad H = \frac{f(v)f''(v) - 1 - f'(v)^2}{2f(v)\big(1+f'(v)^2\big)^{3/2}}$$

   (b) Demonstrate the following by choosing suitable $f(v)$:
      i. A cylinder has $K = 0$;
      ii. A cone has $K = 0$;
      iii. A sphere of radius a has $K = \frac{1}{a^2}$.
      iv. A *catenoid* $f(v) = a^{-1}\cosh(av - c)$ is a minimal surface.

   (c) Suppose $\mathbf{x}$ is a minimal surface $H \equiv 0$. By writing $g(v) = 1 + \big(f'(v)\big)^2$, show that

$$1 + f'^2 = g = a^2 f^2 \quad \text{for some constant } a$$

   By substituting $f(v) = a^{-1}\cosh\big(ah(v)\big)$, show that the surface is a *catenoid.*

   (d) Plainly $K \equiv 0$ if and only if $f''(v) \equiv 0$. What are these surfaces? More generally, if the surface has constant non-zero Gauss curvature K, show that f satisfies a non-linear ODE

$$Kf^2 = (1 + f'^2)^{-1} + c \quad \text{for some constant } c$$

   (*Solving for f requires an elliptic integral when $c \neq 0$, so don't try!*)

8. The *tractrix* is parametrized by $\mathbf{y}(t) = \big(\sinh^{-1} t - t(1+t^2)^{-1/2}, (1+t^2)^{-1/2}\big)$. By revolving this curve around the x-axis, show that the resulting surface is a pseudosphere with $K \equiv -1$.

9. We know that the Gauss and mean curvature are defined in terms of the principal curvatures. By writing down a suitable quadratic polynomial, prove that knowing of H, K is sufficient to recover the principal curvatures.

10. The graph of a function $z = f(x, y)$ is parametrized by $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$. What can you say about the surface if (u, v) are curvature-line co-ordinates?

    (*Hint: recall Example 3.19*)

11. Suppose u, v are curvature-line co-ordinates for a surface $\mathbf{x}$. Explain why $\mathbf{n}_u = -k_1\mathbf{x}_u$ and $\mathbf{n}_v = -k_2\mathbf{x}_v$.

12. Suppose that a surface $\mathbf{x} : U \to \mathbb{E}^3$ is totally umbilic $\mathrm{I\!I} = k\mathrm{I}$ for some function $k : U \to \mathbb{R}$.

    (a) Use Exercise 11 and $\mathbf{n}_{uv} = \mathbf{n}_{vu}$ to prove that k is constant.
    (b) Prove that $\mathbf{x}$ is (part of) a plane or a sphere (recall Example 3.28).
    (*Hint: If $k \neq 0$ consider $\mathbf{c} := \mathbf{x} + \frac{1}{k}\mathbf{n}\ldots$*)

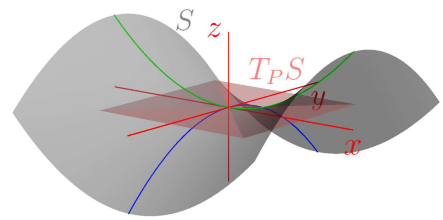
13. We prove part 2 of Theorem 3.31. Given the assumptions, define the *dual 1-forms* to $\vec{s}, \vec{t}$:

$$\alpha(\vec{s}) = 1 = \beta(\vec{t}) \quad \text{and} \quad \alpha(\vec{t}) = 0 = \beta(\vec{s})$$

    Use Exercise 2.3.10 to prove that $\mathrm{d}\alpha = 0 = \mathrm{d}\beta$. Hence conclude (footnote, page 43) that (locally) $\alpha = \mathrm{d}s$ and $\beta = \mathrm{d}t$ for some functions s, t.

3.4 Power Series Expansions and Euler's Theorem

In this section we intersect a surface with certain planes and consider the resulting curves. The curvatures provide data about these curves and thus tell us something about the local shape of the surface. The key is to see how curvatures describe a quadratic approximation to a surface.



At a regular point P on a surface S, choose axes such that P is the origin and the (x,y)-plane is tangent[21] to S. By Theorem 3.10, S is locally the graph of a function $z = f(x,y)$, which we may parametrize in the usual manner

$$\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$$

The unit normal vector $\mathbf{n}_P = \mathbf{k}$ is therefore the standard vertical basis vector. Since the tangent plane at P is the (x,y)-plane, we see that $f_u(0,0) = 0 = f_v(0,0)$; substituting into Example 3.19 yields the fundamental forms *at* P:

$$\begin{aligned} \mathrm{I}_P &= \mathrm{d}u^2 + \mathrm{d}v^2 \\ \mathrm{I\!I}_P &= f_{uu}\,\mathrm{d}u^2 + 2f_{uv}\,\mathrm{d}u\mathrm{d}v + f_{uv}\,\mathrm{d}v^2 \end{aligned} \qquad [\mathrm{I}]_P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\mathrm{I\!I}]_P = \operatorname{Hess} f = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

The last matrix is the *Hessian* of f, and the Gauss and mean curvatures at P are

$$K(P) = \det \operatorname{Hess} f(0,0) \quad \text{and} \quad H(P) = \frac{1}{2} \operatorname{tr} \operatorname{Hess} f(0,0)$$

It bears repeating that these expressions are only valid *at the origin* $O \in U$ (equivalently $P \in S$). Although the co-ordinates u, v will extend nearby on the surface, the first fundamental form need not be diagonal anywhere except at the origin.

Now suppose we rotate the (x,y)-plane so that the axes point in the principal directions. Then the Hessian is also diagonal ($f_{uv}(0,0) = 0$) and the principal curvatures at P are

$$k_1 = f_{uu}(0,0) \quad \text{and} \quad k_2 = f_{vv}(0,0)$$

Theorem 3.34. *If the graph of $z = f(x,y)$ is tangent to the (x,y)-plane at the origin O so that the axes are the curvature directions, then the Maclaurin approximation of the function $f(x,y)$ is*

$$\begin{aligned} f(x,y) &\approx f(O) + (x\ y)\ \nabla f\big|_O + \frac{1}{2}(x\ y) \operatorname{Hess} f(O) \begin{pmatrix} x \\ y \end{pmatrix} + \textit{ higher order terms} \\ &= \frac{1}{2}k_1(O)x^2 + \frac{1}{2}k_2(O)y^2 + \textit{ higher order terms} \end{aligned}$$

Example 3.35. Let $f(x,y) = x^2 - y^2$ (above picture). *At the origin*, $\mathbf{x}(u,v) = \big(u, v, u^2 - v^2\big)$ has

$$\mathrm{I} = \mathrm{d}u^2 + \mathrm{d}v^2, \quad \mathrm{I\!I} = 2(\mathrm{d}u^2 - \mathrm{d}v^2), \quad k_1 = 2, \quad k_2 = -2, \quad K = -4, \quad H = 0$$

In this case the Maclaurin approximation is exact!

$$\frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2 = x^2 - y^2 = f(x,y)$$

[21] This amounts to applying a rigid motion (direct isometry) to the surface, which does nothing to the fundamental forms.

Level Curves: intersections with planes parallel to the tangent plane

If c is small, then the intersection of S with a plane $c\mathbf{n}_P + T_PS$ parallel to the tangent plane is a *level curve*; in our analysis, they correspond to level curves $f(x,y) =$ constant. Theorem 3.34 tells us how level curves depend on the curvatures. For instance, if k_1, k_2 have opposite signs, then for small c,

$$k_1x^2 + k_2y^2 \approx 2c$$

is approximately a *hyperbola.*

Definition 3.36. Suppose k_1, k_2, K, H are the curvatures of a surface S at a point P. We say that P is:

Elliptic $\iff K > 0 \iff k_1, k_2 \neq 0$ and have the same sign.
Level curves near P are approximately *ellipses.*

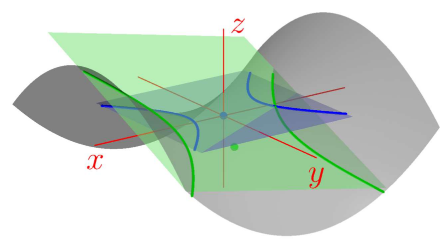
Hyperbolic $\iff K < 0 \iff k_1, k_2 \neq 0$ and have opposite signs.
Level curves near P are approximately *hyperbolæ.*

Parabolic $\iff K = 0$ and $H \neq 0 \iff$ exactly one of k_1, k_2 is zero.
Level curves near P are approximately a pair of *parallel lines*, e.g. $x = \pm c$.

Planar $\iff K = H = 0 \iff k_1 = k_2 = 0$.
The curvatures provide no data as to the level curves near P.

Example (3.35, mk. II). For the graph of $z = x^2 - y^2$, the level curve $x^2 - y^2 = c \neq 0$ is a hyperbola. In fact this is true everywhere on this surface: under the usual parametrization $\mathbf{x}(u,v) = \big(u, v, u^2 - v^2\big)$, we have

$$K = -\frac{4}{(1+4u^2+4v^2)^2} \quad \text{and} \quad H = \frac{4(v^2-u^2)}{(1+4u^2+4v^2)^{3/2}}$$



Since $K < 0$ everywhere, all points are hyperbolic.

In the picture, shifted tangent planes $c\mathbf{n}_P + T_PS$ and their intersections with the surface are drawn for two points. In both cases the level curves are genuine hyperbolæ.

Normal Curvature: intersections with planes containing the normal vector

Theorem 3.34 is the surface analogy of Exercise 1.6.5: a regular curve in $\mathbb{E}^2$ passing through the origin horizontally at $t = 0$ has its graph given locally by

$$y = \frac{1}{2}\kappa(0)x^2 + \text{higher order terms} \tag{$*$}$$

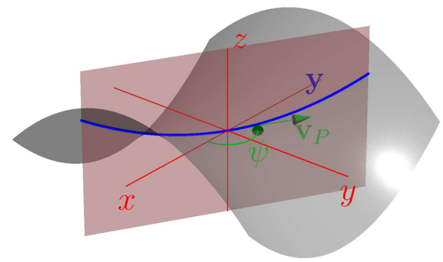
We put this to work by considering the curvature of curves passing through a point on a surface.

Definition 3.37. Let S be a surface and $\mathbf{v}_P \in T_PS$ a non-zero tangent vector.
The *normal curvature* $\nu(\mathbf{v}_P)$ is the curvature at P of the curve[22] defined by the intersection of the surface S and the normal plane $\operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$.
We say that $\mathbf{v}_P$ is *asymptotic* if $\nu(\mathbf{v}_P) = 0$.

[22] Strictly, the curve is the *connected component* of $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$ containing P.

Example (3.35, mk. III). Consider the hyperbolic paraboloid $z = x^2 - y^2$ at the origin $P = O$. Fix an angle ψ and let $\mathbf{v}_P = (\cos\psi, \sin\psi)$. The intersection curve $\mathbf{y} \subseteq S \cap \operatorname{Span}\{\mathbf{v}_O, \mathbf{n}_O\}$ may be parametrized using polar co-ordinates:

$$\mathbf{y}(r) = \left(r\cos\psi, r\sin\psi, r^2(\cos^2\psi - \sin^2\psi)\right)$$



which amounts to the graph of the function $g(r) = r^2\cos 2\psi$. The normal curvature is the curvature at $r = 0$ of this curve:

$$\nu(\mathbf{v}_O) = \kappa(0) = \frac{g''(0)}{[1 + g'(0)^2]^{3/2}} = 2\cos 2\psi$$

Think about how the this corresponds to the picture and observe that

$$\mathbf{v}_O \text{ is asymptotic} \iff \cos 2\psi = 0 \iff \psi = \pm\tfrac{\pi}{4}$$

Our next result generalizes the method in the example.

Theorem 3.38 (Euler). *Suppose $\mathbf{v}_P$ makes angle ψ with the first principal curvature direction. Then*

$$\nu(\mathbf{v}_P) = k_1\cos^2\psi + k_2\sin^2\psi$$

In particular, the principal curvatures are the extremes of normal curvature: if $k_1 \le k_2$, then

$$k_1 \le \nu(\mathbf{v}_P) \le k_2$$

where the bounds are realized precisely when $\mathbf{v}_P$ points in a curvature direction.

Proof. Choose axes so the curvature directions at P are $\mathbf{i}, \mathbf{j}$, and $\mathbf{n}_P = \mathbf{k}$. The surface is locally a graph $z = f(x, y)$. If (r, ψ) are polar co-ordinates in the (x, y)-plane, Theorem 3.34 says that

$$z = f(x, y) \approx \tfrac{1}{2}k_1(r\cos\psi)^2 + \tfrac{1}{2}k_2(r\sin\psi)^2 + \cdots = \frac{1}{2}(k_1\cos^2\psi + k_2\sin^2\psi)r^2 + \cdots$$

Fix ψ and let $\mathbf{v}_P = \left(\begin{smallmatrix}\cos\psi \\ \sin\psi\end{smallmatrix}\right)$ (assume unit length since only the direction matters). Our curve of interest $\mathbf{y} \subseteq S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$ may be parametrized

$$\mathbf{y}(r) = r\mathbf{v}_P + f(r\cos\psi, r\sin\psi)\,\mathbf{n}_P = \begin{pmatrix} r\cos\psi \\ r\sin\psi \\ f(r\cos\psi, r\sin\psi) \end{pmatrix} = \begin{pmatrix} r\cos\psi \\ r\sin\psi \\ \frac{1}{2}\nu r^2 + \cdots \end{pmatrix}$$

The last equality used observation ($*$), where ν is the normal curvature. For the first result, simply compare the z-expressions in the displayed equations. For the final observation, note that

$$\nu(\mathbf{v}_P) = k_1(1 - \sin^2\psi) + k_2\sin^2\psi = k_1 + (k_2 - k_1)\sin^2\psi \in [k_1, k_2]$$

and that the bounds are achieved precisely when $\psi = 0, \frac{\pi}{2}$, when $\mathbf{v}_P$ is a curvature direction. ■

Examples 3.39. 1. If P is a planar point ($k_1 = k_2 = 0$), all normal curvatures are zero and all directions are asymptotic.

2. (Example 3.30.1) All points of the paraboloid $z = r^2$ are elliptic (everywhere $k_1, k_2 > 0$). The surface has no asymptotic directions at any point, indeed the normal curvature in the direction $\mathbf{v}_P = (\cos\psi, \sin\psi)$ at $P = (r\cos\theta, r\sin\theta, r^2)$ is

$$\nu(\mathbf{v}_P) = k_1 \cos^2\psi + k_2 \sin^2\psi = \frac{2}{(1+4r^2)^{3/2}}\left[\cos^2\psi + (1+4r^2)\sin^2\psi\right] > 0$$

3. If $k_2 \neq 0$, then $\mathbf{v}_P = \left(\begin{smallmatrix}\cos\psi \\ \sin\psi\end{smallmatrix}\right)$ is asymptotic $\iff$ $\tan\psi = \pm\sqrt{-\frac{k_1}{k_2}}$.

The Second Fundamental Form and the Local Shape of a Surface

Our standard approach is to transfer calculations about surfaces back to the parametrization space. With this in mind, we consider special tangent vectors with respect to the second fundamental form.

Definition 3.40. Let $\mathbf{x} : U \to \mathbb{E}^3$ be an oriented surface and $\vec{w}_p \in T_p\mathbb{R}^2$.

1. A tangent vector $\vec{w}_p \neq \vec{0}$ is *asymptotic* if $\mathbb{I}(\vec{w}_p, \vec{w}_p) = 0$.

2. The *Dupin indicatrix* at $p \in U$ is the set of tangent vectors $\vec{w}_p$ such that $\mathbb{I}(\vec{w}_p, \vec{w}_p) = \pm 1$.

Theorem 3.41. *The notions of* asymptotic *in Definitions 3.37 & 3.40 coincide:*

$$\mathbf{v}_P = \mathrm{d}\mathbf{x}(\vec{w}_p) \in T_P S \text{ is asymptotic} \iff \vec{w}_p \in T_p\mathbb{R}^2 \text{ is asymptotic}$$

The proof is an exercise. Recalling Theorem 3.20, a direction is asymptotic if and only if the *normal acceleration* in said direction is zero.

The Dupin indicatrix turns out to precisely describe level curves near a point. To see this, write $\vec{w}_p = a\vec{s}_p + b\vec{t}_p$ where $\vec{s}_p, \vec{t}_p$ are orthonormal curvature directions;[23] the indicatrix at p has equation

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = k_1 a^2 + k_2 b^2 = \pm 1$$

This defines a conic in the tangent space $T_p\mathbb{R}^2$ whose type depends on the signs of the principal curvatures. In essence, the Dupin indicatrix indicates the level curve obtained by taking the intersection $S \cap (c\mathbf{n}_P + T_P S)$ for *infinitesimal* c. We summarize all possibilities in a table using the point-types introduced in Definition 3.36:

type of point	# asymptotic directions	Dupin indicatrix
elliptic	0	ellipse
hyperbolic	2	two hyperbolæ
parabolic	1	two parallel lines
planar	∞	empty

[23] With respect to $\vec{s}_p, \vec{t}_p$, the matrices of the fundamental forms at p are $[\mathrm{I}_p] = \left(\begin{smallmatrix}1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ and $[\mathbb{I}_p] = \left(\begin{smallmatrix}k_1 & 0 \\ 0 & k_2\end{smallmatrix}\right)$

Examples 3.42. For a parametrized surface $\mathbf{x}$ at a given point $p = (u_0, v_0)$, write $\vec{w}_p = a\,\frac{\partial}{\partial u}\Big|_p + b\,\frac{\partial}{\partial v}\Big|_p$.

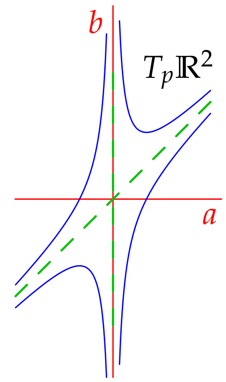
1. (Exercise 3.2.6) The tangent developable of the unit-speed helix has

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = -\frac{v_0}{4}\,\mathrm{d}u^2(\vec{w}_p, \vec{w}_p) = -\frac{v_0}{4}a^2$$

The single asymptotic direction is $\vec{w}_p = \frac{\partial}{\partial v}\Big|_p$. The Dupin indicatrix is a pair of parallel lines

$$-\frac{v_0}{4}a^2 = \pm 1 \implies \vec{w}_p = \pm\frac{2}{\sqrt{|v_0|}}\,\frac{\partial}{\partial u}\Big|_p + b\,\frac{\partial}{\partial v}\Big|_p \qquad (b \text{ is arbitrary!})$$

2. In its usual parametrization, the surface $z = x^2y$ has



$$\mathbb{I} = \frac{2}{\sqrt{1+4u^2v^2+u^4}}\big(v\,\mathrm{d}u^2 + 2u\,\mathrm{d}u\mathrm{d}v\big)$$

At $p = (-1, 2)$ (i.e., $\mathbf{x}(p) = (-1, 2, 2)$) we see that

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = \frac{2}{\sqrt{18}}\big(2a^2 - 2ab\big) = \frac{2\sqrt{2}}{3}a(a-b)$$

The point is hyperbolic with asymptotic directions $\frac{\partial}{\partial v}\Big|_p$ and $\frac{\partial}{\partial u}\Big|_p - \frac{\partial}{\partial v}\Big|_p$ ($a = 0$ and $a - b = 0$). The indicatrix comprises two hyperbolæ $a(a-b) = \pm\frac{3}{2\sqrt{2}}$.

Exercises 3.4. 1. Consider the graph of the function $z = x^2 - 3y^2 + 7xy^3 + 9y^4$.

   (a) Find the Gauss and mean curvatures *at the origin.*
   (*Hint: use Theorem 3.34*)

   (b) Find the normal curvature at the origin for the curve in the surface described by $x = y$.

2. As in Example 3.35, mk. III (page 70), find the asymptotic directions at the origin for the surface $z = y^2 - 3x^2$.

3. For the elliptic paraboloid $z = x^2 + y^2$, let $P = (1, 2, 5)$ be a fixed point.

   (a) Find the maximum and minimum values for the normal curvature at P.

   (b) Find the Dupin indicatrix at P.

4. For the hyperbolic paraboloid $z = x^2 - y^2$, let $p = (u_0, v_0)$ and $P = \big(u_0, v_0, u_0^2 - v_0^2\big)$. If $c \neq 0$, prove that the intersection of the parallel plane $c\mathbf{n}_P + T_PS$ and the paraboloid may be expressed

$$(x-u_0)^2 - (y-v_0)^2 = \text{constant}, \quad z = x^2 - y^2$$

   That is, the level curves really are hyperbolæ.

5. Consider the graph of the surface $z = x^2 + y^4$.

   (a) Compute the Gauss curvature and classify all points according to Definition 3.36.

   (b) Sketch the level curves $z = 1, \frac{1}{100}$ and $\frac{1}{10000}$ and compare to the Dupin indicatrix at $(0, 0)$.

6. Prove Theorem 3.41 by considering the normal acceleration of the curve $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$.

3.5 Adaptive Frames & Gauss' Remarkable Theorem

In this section we repurpose the idea of a moving frame first encountered when studying curves.

Definition 3.43. Let $\mathbf{x} : U \to \mathbb{E}^3$ parametrize a surface S. A *moving frame* for S is a triple of smooth functions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ on U such that, for each $p \in U$,

$$\{\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p)\} \text{ is a positively oriented orthonormal basis of } T_{\mathbf{x}(p)}\mathbb{E}^3$$

When S is oriented, we say that a moving frame is *adaptive* if $\mathbf{e}_3 = \mathbf{n}$ is the unit normal field.

For an adaptive frame, the tangent plane at each point is $T_{\mathbf{x}(p)}S = \operatorname{Span}\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$.
We will often refer to the matrix-valued function $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) : U \to SO_3(\mathbb{R})$ as the frame.

Examples 3.44. We'll repeatedly analyze three examples through this section.

1. The parabolic cylinder $\mathbf{x}(u,v) = \left(u, v, \frac{1}{2}u^2\right)$ has an adaptive frame

$$\mathbf{e}_1 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}1\\0\\u\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}-u\\0\\1\end{pmatrix}$$

2. The sphere of radius R in spherical polar co-ordinates $\mathbf{x}(\psi, \phi)$ has an adaptive frame

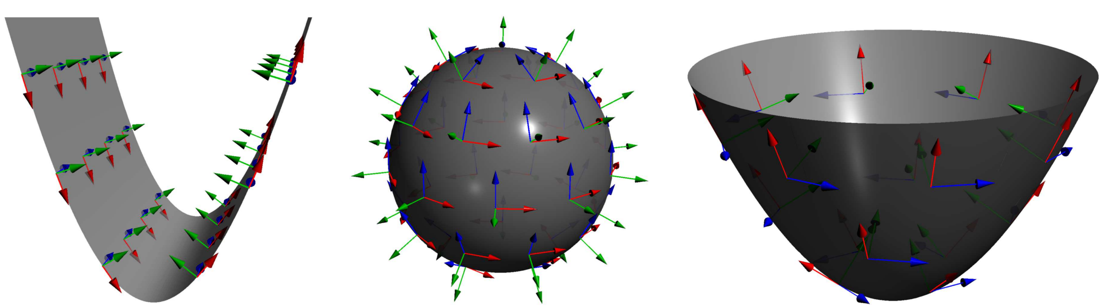
$$\mathbf{e}_1 = \begin{pmatrix}-\sin\psi\\ \cos\psi\\ 0\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}-\cos\psi\sin\phi\\ -\sin\psi\sin\phi\\ \cos\phi\end{pmatrix} \qquad \mathbf{e}_3 = \mathbf{x} = \begin{pmatrix}\cos\psi\cos\phi\\ \sin\psi\cos\phi\\ \sin\phi\end{pmatrix}$$

   We use ψ instead of θ since we'll need the latter for something else momentarily…

3. The paraboloid $\mathbf{x}(r,\psi) = \left(r\cos\psi, r\sin\psi, \frac{1}{2}r^2\right)$ has an adaptive frame

$$\mathbf{e}_1 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}\cos\psi\\ \sin\psi\\ r\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}-\sin\psi\\ \cos\psi\\ 0\end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}-r\cos\psi\\ -r\sin\psi\\ 1\end{pmatrix}$$

In the pictures we've reduced the lengths of the frame vectors for clarity.



In each case $\mathbf{e}_1, \mathbf{e}_2$ were obtained by differentiating with respect to the co-ordinates (and normalizing if necessary). This works because the co-ordinate systems for all three examples are *orthogonal.*

As with the Frenet frame approach to curves, our strategy is to analyse a surface $\mathbf{x} : U \to \mathbb{E}^3$ two stages:

1. Describe how $\mathbf{x}$ moves with respect to the frame $\mathcal{E}$.
2. Describe how the frame $\mathcal{E}$ moves (with respect to itself).

We describe infinitesimal changes using 1-forms, following an approach pioneered by Élie Cartan around 1899.

Definition 3.45. Let $\mathbf{x} : U \to \mathbb{E}^3$ be a smooth map and $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ a moving frame. The *metric forms* θ_j and *connection forms* ω_{jk} are the 1-forms on U defined by

$$\theta_j := \mathbf{e}_j \cdot \mathrm{d}\mathbf{x}, \quad \omega_{jk} = \mathbf{e}_j \cdot \mathrm{d}\mathbf{e}_k$$

where $j, k \in \{1, 2, 3\}$.

Since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthonormal, these forms are nothing more than the co-ordinates of $\mathrm{d}\mathbf{x}$, $\mathrm{d}\mathbf{e}_1$, $\mathrm{d}\mathbf{e}_2$ and $\mathrm{d}\mathbf{e}_3$ with respect to the moving frame:

$$\mathrm{d}\mathbf{x} = \sum_{j=1}^{3} (\mathbf{e}_j \cdot \mathrm{d}\mathbf{x})\mathbf{e}_j = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2 + \mathbf{e}_3\theta_3, \qquad \mathrm{d}\mathbf{e}_k = \sum_{j=1}^{3} \mathbf{e}_j\omega_{jk} \tag{$*$}$$

The frame is adaptive if and only if $\theta_3 = 0$. Moreover, as the next result shows, for any frame there are only three independent connection forms (compare this with Theorem 1.29).

Lemma 3.46. *For all j, k, we have $\omega_{jk} = -\omega_{kj}$. In particular $\omega_{jj} = 0$.*

Proof. Take the exterior derivative of the identity $\mathbf{e}_j \cdot \mathbf{e}_k = 0$ or 1, to obtain

$$0 = \mathrm{d}\mathbf{e}_j \cdot \mathbf{e}_k + \mathbf{e}_j \cdot \mathrm{d}\mathbf{e}_k = \omega_{kj} + \omega_{jk}$$

■

If $(*)$ are arranged in matrix format, the subscripts follow the usual row/column convention:

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} \begin{pmatrix}\theta_1 \\ \theta_2 \\ \theta_3\end{pmatrix} = \mathcal{E}\Theta, \qquad \mathrm{d}\mathcal{E} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} = \mathcal{E}\omega$$

The second expression should remind you of the Frenet–Serret equations for a curve! The metric forms get their name because they *measure* small changes on the surface. The connection forms tell us how nearby frames are related (*connected*): abusing notation a little, if $\vec{s}_p \in T_p\mathbb{R}^2$, then

$$\mathcal{E}(p + \vec{s}_p) - \mathcal{E}(p) \approx \mathrm{d}\mathcal{E}(\vec{s}_p) = \mathcal{E}(p)\omega(\vec{s}_p)$$

The fundamental forms of $\mathbf{x}$ can be written in terms of Θ and ω; in an adaptive frame this is particularly simple.

Lemma 3.47. *In an adaptive frame*

$$\mathrm{I} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \theta_1^2 + \theta_2^2 \quad \textit{and} \quad \mathbb{I} = -\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{e}_3 = -\theta_1\omega_{13} - \theta_2\omega_{23}$$

Examples (3.44, mk. II). You needn't compute all exterior derivatives $\mathrm{d}\mathbf{e}_k$: use the skew-symmetry of ω to help; also consider which frame fields are easier to differentiate! The expressions for the fundamental forms should be a sanity check since we know how to compute them already.

1. The parabolic cylinder has

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}1\\0\\u\end{pmatrix}\mathrm{d}u + \begin{pmatrix}0\\1\\0\end{pmatrix}\mathrm{d}v = \sqrt{1+u^2}\mathbf{e}_1\,\mathrm{d}u + \mathbf{e}_2\,\mathrm{d}v \implies \theta_1 = \sqrt{1+u^2}\,\mathrm{d}u,\ \ \theta_2 = \mathrm{d}v$$
$$\implies \mathrm{I} = (1+u^2)\,\mathrm{d}u^2 + \mathrm{d}v^2$$

   Since $\mathbf{e}_2$ is constant, we have $\mathrm{d}\mathbf{e}_2 = \mathbf{0}$ from which

$$\omega_{12} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_2 = 0, \quad \omega_{23} = -\omega_{32} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_2 = 0$$

   The final connection form requires a derivative:

$$\omega_{13} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_3 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}1\\0\\u\end{pmatrix}\cdot\left[\frac{-u}{(1+u^2)^{3/2}}\begin{pmatrix}-u\\0\\1\end{pmatrix} + \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}-1\\0\\0\end{pmatrix}\right] = \frac{-1}{1+u^2}\mathrm{d}u$$

   Putting it together, we have

$$\omega = \frac{1}{1+u^2}\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\mathrm{d}u \quad\text{and}\quad \mathbb{I} = -\theta_1\omega_{13} - \theta_2\omega_{23} = \mathrm{d}u^2$$

2. For the sphere of radius R, $\mathrm{d}\mathbf{x} = R\cos\phi\,\mathbf{e}_1\,\mathrm{d}\psi + R\mathbf{e}_2\,\mathrm{d}\phi$, whence

$$\theta_1 = R\cos\phi\,\mathrm{d}\psi,\ \ \theta_2 = R\mathrm{d}\phi \implies \mathrm{I} = R^2\big(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2\big)$$
$$\mathrm{d}\mathbf{e}_1 = \begin{pmatrix}-\cos\psi\\-\sin\psi\\0\end{pmatrix}\mathrm{d}\psi \implies \begin{cases}\omega_{12} = -\mathbf{e}_2\cdot\mathrm{d}\mathbf{e}_1 = -\sin\phi\,\mathrm{d}\psi\\ \omega_{13} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_1 = \cos\phi\,\mathrm{d}\psi\end{cases}$$
$$\omega_{23} = \mathbf{e}_2\cdot\mathrm{d}\mathbf{e}_3 = \begin{pmatrix}-\cos\psi\sin\phi\\-\sin\psi\sin\phi\\\cos\phi\end{pmatrix}\cdot\left[\begin{pmatrix}-\sin\psi\cos\phi\\\cos\psi\cos\phi\\0\end{pmatrix}\mathrm{d}\psi + \begin{pmatrix}-\cos\psi\sin\phi\\-\sin\psi\sin\phi\\\cos\phi\end{pmatrix}\mathrm{d}\phi\right] = \mathrm{d}\phi$$
$$\implies \mathbb{I} = -\theta_1\omega_{13} - \theta_2\omega_{23} = -R\big(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2\big)$$

3. For the paraboloid,

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}\cos\psi\\\sin\psi\\r\end{pmatrix}\mathrm{d}r + r\begin{pmatrix}-\sin\psi\\\cos\phi\\0\end{pmatrix}\mathrm{d}\psi = \sqrt{1+r^2}\mathbf{e}_1\,\mathrm{d}r + r\mathbf{e}_2\,\mathrm{d}\psi$$
$$\implies \theta_1 = \sqrt{1+r^2}\,\mathrm{d}r, \quad \theta_2 = r\,\mathrm{d}\psi \implies \mathrm{I} = (1+r^2)\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2$$

   The connection forms are comparatively ugly. The low-hanging fruit is $\mathrm{d}\mathbf{e}_2 = \left(\begin{smallmatrix}-\cos\psi\\-\sin\psi\\0\end{smallmatrix}\right)\mathrm{d}\psi$, which quickly yields two of them:

$$\omega_{12} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_2 = -\frac{\mathrm{d}\psi}{\sqrt{1+r^2}}, \quad \omega_{23} = -\omega_{32} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_2 = \frac{-r\,\mathrm{d}\psi}{\sqrt{1+r^2}}$$

The last connection form requires a nastier differentiation, though only one of the three terms in $\mathrm{d}\mathbf{e}_3$ provides a non-zero result when dotted with $\mathbf{e}_1$:

$$\omega_{13} = \mathbf{e}_1 \cdot \mathrm{d}\mathbf{e}_3 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}\cos\psi\\ \sin\psi\\ r\end{pmatrix} \cdot \left[\cdots + \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}-\cos\psi\\ -\sin\psi\\ 0\end{pmatrix}\mathrm{d}r\right] = \frac{-\mathrm{d}r}{1+r^2}$$

We therefore obtain the connection form matrix

$$\omega = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix} 0 & -\mathrm{d}\psi & \frac{-1}{\sqrt{1+r^2}}\,\mathrm{d}r \\ \mathrm{d}\psi & 0 & -r\,\mathrm{d}\psi \\ \frac{1}{\sqrt{1+r^2}}\,\mathrm{d}r & r\,\mathrm{d}\psi & 0 \end{pmatrix}$$

and second fundamental form

$$\mathbb{I} = -\sqrt{1+r^2}\,\mathrm{d}r\frac{-\mathrm{d}r}{1+r^2} - r\,\mathrm{d}\psi\frac{-r\,\mathrm{d}\psi}{\sqrt{1+r^2}} = \frac{1}{\sqrt{1+r^2}}(\mathrm{d}r^2 + r^2\mathrm{d}\psi^2)$$

The Structure Equations for a Moving Frame

The metric and connection forms satisfy matrix equations $\mathrm{d}\mathbf{x} = \mathcal{E}\Theta$ and $\mathrm{d}\mathcal{E} = \mathcal{E}\omega$. Since $\mathrm{d}^2 = 0$, something nice happens when we take the exterior derivatives of these expressions:

$$\mathbf{0} = \mathrm{d}^2\mathbf{x} = \mathrm{d}(\mathrm{d}\mathbf{x}) = \mathrm{d}(\mathcal{E}\,\Theta) = \mathrm{d}\mathcal{E}\wedge\Theta + \mathcal{E}\,\mathrm{d}\Theta = \mathcal{E}(\omega\wedge\Theta + \mathrm{d}\Theta)$$
$$0 = \mathrm{d}^2\mathcal{E} = \mathrm{d}(\mathrm{d}\mathcal{E}) = \mathrm{d}(\mathcal{E}\omega) = \mathrm{d}\mathcal{E}\wedge\omega + \mathcal{E}\mathrm{d}\omega = \mathcal{E}(\omega\wedge\omega + \mathrm{d}\omega)$$

The notation $\omega\wedge\Theta$ means matrix multiplication using the wedge product of forms to evaluate each entry.[24] Since each $\mathcal{E}(p)$ is an invertible matrix, we conclude two identities.

Theorem 3.48. *The metric and connection forms satisfy the* structure equations*; each amounts to three separate equations after multiplying out the matrix expressions.*

1. $\mathrm{d}\Theta + \omega\wedge\Theta = \mathbf{0}$, *equivalently* $\mathrm{d}\theta_j + \sum_{k\neq j}\omega_{jk}\wedge\theta_k = 0$ *for each* $j = 1, 2, 3$
2. $\mathrm{d}\omega + \omega\wedge\omega = 0$, *equivalently* $\mathrm{d}\omega_{jk} + \omega_{ji}\wedge\omega_{ik} = 0$ *where* i, j, k *are distinct.*

These are easy to remember if you pay attention to the indices! In an adaptive frame ($\theta_3 = 0$), things are a little simpler and some of the equations get special names:

First structure equations	$\begin{cases}\mathrm{d}\theta_1 + \omega_{12}\wedge\theta_2 = 0\\ \mathrm{d}\theta_2 + \omega_{21}\wedge\theta_1 = 0\end{cases}$
Symmetry equation	$\omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = 0$
Gauss equation	$\mathrm{d}\omega_{12} + \omega_{13}\wedge\omega_{32} = 0$
Codazzi equations	$\begin{cases}\mathrm{d}\omega_{13} + \omega_{12}\wedge\omega_{23} = 0\\ \mathrm{d}\omega_{23} + \omega_{21}\wedge\omega_{13} = 0\end{cases}$

[24] Be careful not to reverse the order: $\Theta\wedge\omega$ makes no sense since the dimensions of the matrices are incompatible! Similarly, $\omega\wedge\omega$ is unlikely to be zero…

Examples (3.44, mk. III). 1. For the parabolic cylinder, $\Theta = \begin{pmatrix} \sqrt{1+u^2}\,\mathrm{d}u \\ \mathrm{d}v \\ 0 \end{pmatrix}$ and $\omega = \frac{1}{1+u^2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\mathrm{d}u$, so all the structure equations are trivial:

$$\mathrm{d}\Theta = \mathbf{0} = -\omega \wedge \Theta, \quad \mathrm{d}\omega = 0 = -\omega \wedge \omega$$

2. For the sphere, $\Theta = R\begin{pmatrix} \cos\phi\,\mathrm{d}\psi \\ \mathrm{d}\phi \\ 0 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & \sin\phi\,\mathrm{d}\psi & -\cos\phi\,\mathrm{d}\psi \\ -\sin\phi\,\mathrm{d}\psi & 0 & \mathrm{d}\phi \\ \cos\phi\,\mathrm{d}\psi & -\mathrm{d}\phi & 0 \end{pmatrix}$, from which

$$\mathrm{d}\Theta = R\begin{pmatrix} -\sin\phi \\ 0 \\ 0 \end{pmatrix}\mathrm{d}\phi \wedge \mathrm{d}\psi = -\omega \wedge \Theta$$

$$\mathrm{d}\omega = \begin{pmatrix} 0 & \cos\phi & \sin\phi \\ -\cos\phi & 0 & 0 \\ -\sin\phi & 0 & 0 \end{pmatrix}\mathrm{d}\phi \wedge \mathrm{d}\psi = -\omega \wedge \omega$$

3. For the paraboloid, $\Theta = \begin{pmatrix} \sqrt{1+r^2}\mathrm{d}r \\ r\,\mathrm{d}\psi \\ 0 \end{pmatrix}$ and $\omega = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix} 0 & -\mathrm{d}\psi & -\frac{\mathrm{d}r}{\sqrt{1+r^2}} \\ \mathrm{d}\psi & 0 & -r\,\mathrm{d}\psi \\ \frac{\mathrm{d}r}{\sqrt{1+r^2}} & r\,\mathrm{d}\psi & 0 \end{pmatrix}$.

The first equations aren't too bad to check:

$$\mathrm{d}\Theta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}r \wedge \mathrm{d}\psi = -\omega \wedge \Theta$$

The second are a little nastier: you should check that

$$\mathrm{d}\omega = \frac{1}{(1+r^2)^{3/2}}\begin{pmatrix} 0 & r & 0 \\ -r & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\mathrm{d}r \wedge \mathrm{d}\psi = -\omega \wedge \omega$$

Gauss' Remarkable Theorem

Suppose we have an adaptive frame for an oriented local surface $\mathbf{x}$. If θ_1, θ_2 were linearly dependent at p, then the differential $\mathrm{d}\mathbf{x} = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2 : T_p\mathbb{R}^2 \to T_{\mathbf{x}(p)}S = \operatorname{Span}\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$ would have rank ≤ 1 and thus not be a bijection. We conclude that $\{\theta_1, \theta_2\}$ forms a basis of the space of 1-forms at p, and that *any other 1-form may be written as a linear combination thereof…*

Lemma 3.49. *There exist unique functions a, b, c such that*

$$\omega_{13} = a\theta_1 + b\theta_2, \quad \omega_{23} = b\theta_1 + c\theta_2$$

With respect to these functions, the second fundamental form, Gauss and mean curvatures are

$$\mathrm{I\!I} = -a\theta_1^2 - 2b\theta_1\theta_2 - c\theta_1^2, \qquad K = ac - b^2, \qquad H = -\frac{1}{2}(a+c)$$

Proof. That $\omega_{13} = a\theta_1 + b\theta_2$ and $\omega_{23} = \hat{b}\theta_1 + c\theta_2$ are linear combinations of θ_1, θ_2 is the above discussion. By the symmetry equation and the fact that $\theta_1 \wedge \theta_2 \neq 0$,

$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2 = (-b + \hat{b})\,\theta_1 \wedge \theta_2 \implies \hat{b} = b$$

The formula for $\mathbb{I}$ follows from Lemma 3.47.

Moreover, if $\vec{w}_1$ and $\vec{w}_2$ are the dual vector fields to θ_1, θ_2, then the matrices of $\mathrm{I}, \mathbb{I}$ with respect to these fields are the identity matrix[25] and $B = \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}$. The Gauss and mean curvatures are the determinant and half the trace of B (Theorem 3.29). ■

Now consider the final connection form ω_{12}. Since θ_1, θ_2 form a basis at each point, we may write

$$\omega_{12} = f\theta_1 + g\theta_2$$

for some functions $f, g : U \to \mathbb{R}$. Applying the 1st structure equations,

$$\begin{aligned} d\theta_1 &= -\omega_{12} \wedge \theta_2 = -f\theta_1 \wedge \theta_2 \\ d\theta_2 &= -\omega_{21} \wedge \theta_1 = -\theta_1 \wedge \omega_{12} = -g\theta_1 \wedge \theta_2 \end{aligned}$$

whence f, g (and ω_{12}) are determined by θ_1, θ_2. This brings us to the capstone result of these notes.

Theorem 3.50 (Gauss' Theorem Egregium). *The Gauss curvature depends only on the first fundamental form.*

Proof. By the above discussion, ω_{12} (and thus $d\omega_{12}$) depends only on θ_1, θ_2, which may be recovered from $\mathrm{I} = \theta_1^2 + \theta_2$ by writing it as a sum of squares. But now the Gauss equation reads

$$d\omega_{12} = \omega_{13} \wedge \omega_{23} = (a\theta_1 + b\theta_2) \wedge (b\theta_1 + c\theta_2) = (ac - b^2)\theta_1 \wedge \theta_2 = K\theta_1 \wedge \theta_2$$ ■

An explicit formula for K as a function of the coefficients E, F, G of I can be found; see Exercise 9.

Egregium (Latin for *remarkable/outstanding*) is the (modest!) term Gauss applied after proving his result in 1827. Why did he consider it so remarkable? The original definition of K relied on the normal field; an object *outside* the surface which helps describe its position/orientation in $\mathbb{E}^3$. Gauss' Theorem, however, says that K is *intrinsic* to the surface: it depends only on the *metric* (first fundamental form) which may be understood by an occupant of the surface with no ability to escape (travel outside the surface) in to view its shape. By contrast, the second fundamental form and the mean curvature depend on how a surface is embedded; these are *extrinsic* quantities.

As a nice side-effect, the result provides what is often a faster method for calculating K.

1. Compute the first fundamental form $\mathrm{I} = d\mathbf{x} \cdot d\mathbf{x}$ and express it as a sum of squares $\mathrm{I} = \theta_1^2 + \theta_2^2$.
2. Write $\omega_{12} = f\theta_1 + g\theta_2$ and compute f, g using the 1st structure equations.
3. Use the Gauss equation to find K.

We need only calculate 1-forms $\theta_1, \theta_2, \omega_{12}$ that are related to the *tangent* part of the moving frame. The unit normal $\mathbf{e}_3$ needn't be considered or calculated.

---

[25] $\theta_j(\vec{w}_k) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ implies that $d\mathbf{x}(\vec{w}_1) = \mathbf{e}_1$ and $d\mathbf{x}(\vec{w}_2) = \mathbf{e}_2$ are orthonormal.

Examples (3.44, mk. IV). We return to our examples one last time. Even though we've already calculated the connection forms, the goal is to see that $\omega_{12} = f\theta_1 + g\theta_2$ and thus K may be found directly from I.

1. The parabolic cylinder has $\mathrm{I} = (1+u^2)\,\mathrm{d}u^2 + \mathrm{d}v^2$ so the natural choice is
$$\theta_1 = \sqrt{1+u^2}\,\mathrm{d}u \quad \text{and} \quad \theta_2 = \mathrm{d}v$$
Since $\mathrm{d}\theta_1 = 0 = \mathrm{d}\theta_2$ we see that $f = g = 0$. We conclude that
$$\omega_{12} = 0 \implies \mathrm{d}\omega_{12} = 0 \implies K = 0$$

2. For the sphere $\mathrm{I} = R^2(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2)$ so we choose $\theta_1 = R\cos\phi\,\mathrm{d}\psi$ and $\theta_2 = R\,\mathrm{d}\phi$. Certainly $0 = \mathrm{d}\theta_2 = -g\theta_1\wedge\theta_2 \implies g = 0$. Moreover,
$$\mathrm{d}\theta_1 = -f\theta_1\wedge\theta_2 \implies R\sin\phi\,\mathrm{d}\psi\wedge\mathrm{d}\phi = -fR^2\cos\phi\,\mathrm{d}\psi\wedge\mathrm{d}\phi \implies f = -R^{-1}\tan\phi$$
We conclude that $\omega_{12} = -R^{-1}\tan\phi\,\theta_1 = -\sin\phi\,\mathrm{d}\psi$, from which
$$\mathrm{d}\omega_{12} = \cos\phi\,\mathrm{d}\psi\wedge\mathrm{d}\phi = \frac{1}{R^2}\theta_1\wedge\theta_2 \implies K = \frac{1}{R^2}$$

3. For the paraboloid, $\mathrm{I} = (1+r^2)\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2$ so we choose $\theta_1 = \sqrt{1+r^2}\,\mathrm{d}r$ and $\theta_2 = r\,\mathrm{d}\psi$. This time $\mathrm{d}\theta_1 = 0 \implies f = 0$ and
$$\mathrm{d}\theta_2 = -g\theta_1\wedge\theta_2 \implies \mathrm{d}r\wedge\mathrm{d}\psi = -gr\sqrt{1+r^2}\,\mathrm{d}r\wedge\mathrm{d}\psi \implies g = -\frac{1}{r\sqrt{1+r^2}}$$
We conclude that $\omega_{12} = -\frac{1}{r\sqrt{1+r^2}}\,\theta_2 = -\frac{1}{\sqrt{1+r^2}}\,\mathrm{d}\psi$, from which
$$\mathrm{d}\omega_{12} = \frac{r}{(1+r^2)^{3/2}}\,\mathrm{d}r\wedge\mathrm{d}\psi = \frac{1}{(1+r^2)^2}\theta_1\wedge\theta_2 \implies K = \frac{1}{(1+r^2)^2}$$

Since K depends only on the metric, it is invariant under *isometric* transformations of the surface. This helps explain why the Gauss curvature of a cylinder and a cone are both zero: both may constructed by rolling up a flat plane without other distortion.

The contrapositive of Gauss' Theorem is also important: surfaces with distinct Gauss curvatures cannot be isometric. Since the metric I determines how we measure angle and length, this explains why a perfect flat map ($K = 0$) of any part of the Earth ($K = \frac{1}{R^2}$) is impossible to achieve. The holy grail of map-making would be a map free of direction, angle and length/area distortion:

1. Straight lines on the map should correspond to paths of shortest distance on the Earth.

2. Angles on the map should equal corresponding angles on the Earth's surface.

3. Areas on the map and the Earth should be in constant ratio.

Gauss' Theorem implies that you cannot have all these properties in one map. In fact, at most one of these properties is possible in a single map.

Riemannian Geometry

We can even employ the method when there is no surface! The idea is to equip a domain with an abstract first fundamental form and use it to compute lengths, angles, area, geodesics, curvature, etc.

Example 3.51. The Poincaré disk model of hyperbolic space is the disk $\{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ equipped with the metric (first fundamental form)

$$\mathrm{I} = \frac{4(\mathrm{d}x^2+\mathrm{d}y^2)}{(1-x^2-y^2)^2} = \frac{4(\mathrm{d}r^2+r^2\,\mathrm{d}\psi^2)}{(1-r^2)^2}$$

As one approaches the boundary of the disk, the idea is that measured distance gets larger: the boundary circle is in fact *infinitely far* from any point inside the disk. To express I as a sum of squares, a natural choice is $\theta_1 = \frac{2\mathrm{d}r}{1-r^2}$ and $\theta_2 = \frac{2r\,\mathrm{d}\psi}{1-r^2}$, from which $\mathrm{d}\theta_1 = 0 \implies f = 0$ and

$$\mathrm{d}\theta_2 = -g\theta_1\wedge\theta_2 \implies \frac{2(1+r^2)}{(1-r^2)^2}\mathrm{d}r\wedge\mathrm{d}\psi = -\frac{4gr}{(1-r^2)^2}\,\mathrm{d}r\wedge\mathrm{d}\psi \implies g = -\frac{1+r^2}{2r}$$

from which

$$\mathrm{d}\omega_{12} = \mathrm{d}(g\theta_2) = -\mathrm{d}\left(\frac{1+r^2}{1-r^2}\right)\wedge\mathrm{d}\psi = \frac{-4r}{(1-r^2)^2}\,\mathrm{d}r\wedge\mathrm{d}\psi = -\theta_1\wedge\theta_2 \implies K=-1$$

Hyperbolic space is the canonical example of a *negatively curved geometry*. There is no surface here, no second fundamental form, and no mean curvature! Since there is no surface, it is harder to visualize what K means in this context (e.g. Section 3.4).[26]

The Gauss curvature of a surface is the simplest avatar of a more general object called the *Riemann curvature tensor.* As an example of how this is applied, in general relativity[27] mass is construed as changing the metric of spacetime (i.e. I); it can be seen that this metric is compatible with unique connection (essentially ω) from which the curvature ($\mathrm{d}\omega+\omega\wedge\omega$) may be computed. When a physicist says *spacetime is curved,* this is what they mean: there is no *exterior* to spacetime from which we can measure curvature, so everything is computed intrinsically.

The Fundamental Theorem of Surfaces

Recall the equivalence of spacecurves up to rigid motions (Theorem 1.38) and the Fundamental Theorem of Biregular Spacecurves (Corollary 1.42). A similar discussion is available for surfaces once we replace curvature and torsion with the fundamental forms $\mathrm{I}, \mathbb{I}$.

The equivalence problem is almost identical. Suppose $\mathbf{x}: U \to \mathbb{E}^3$ is an oriented surface, and that $A \in O_3(\mathbb{R})$ and $\mathbf{b} = \mathbb{E}^3$ are constants. Then $\mathbf{y} := A\mathbf{x}+\mathbf{b}$ is a new surface, the result of applying an isometry to $\mathbf{x}$. A moving frame for $\mathbf{x}$ is transformed to a frame for $\mathbf{y}$ via

$$\mathcal{E}_{\mathbf{y}} = \begin{pmatrix} A\mathbf{e}_1 & A\mathbf{e}_2 & \pm A\mathbf{e}_3\end{pmatrix} \quad \text{where } \pm 1 = \det A$$

[26] The most famous consequence concerns angle-sums of geodesic triangles: $A+B+C = \pi + \int_\triangle K$. If $K<0$, the angle-sum of a geodesic triangle is *less than* 180°. When $K>0$ (e.g., a sphere), the angle sum is greater than 180°. This topic, the related *Gauss–Bonnet Theorem,* and other consequences, are a matter for another course.

[27] Really this is *pseudo*-Riemannian geometry, since I is not positive-definite.

The upshot is that $\mathbf{n_y} = (\det A)A\mathbf{n_x}$, and $\mathrm{I}, \mathbb{I}$ transform exactly as κ, τ:

$$\mathrm{I_y} = \mathrm{d}\mathbf{y}\cdot\mathrm{d}\mathbf{y} = (A\mathrm{d}\mathbf{x})\cdot(A\mathrm{d}\mathbf{x}) = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{I_x}$$
$$\mathbb{I}_\mathbf{y} = -\mathrm{d}\mathbf{y}\cdot\mathrm{d}\mathbf{n_y} = -(\det A)(A\mathrm{d}\mathbf{x})\cdot(A\mathrm{d}\mathbf{n_x}) = (\det A)\mathbb{I}_\mathbf{x}$$

As with curves, we may ask the question in reverse. If we know the fundamental forms, can we also recover the surface up to a rigid motion? The answer is yes, though with a caveat: unlike κ, τ for spacecurves, the fundamental forms cannot be chosen independently.

Theorem 3.52 (Bonnet). *Suppose* I *and* $\mathbb{I}$ *are symmetric bilinear forms where* I *is positive-definite. Provided the Gauss–Codazzi equations are satisfied, there exists a local parametrized surface with these fundamental forms, which is moreover unique up to rigid motions.*

Everything ultimately depends on a generalization of the existence/uniqueness theorem for ODE (another version of the *Frobenius Theorem* (3.33)). Here is a rough sketch of how the process works.

1. Suppose we are given $\mathrm{I}, \mathbb{I}$ on U, and initial conditions at some $p \in U$ (for the surface $\mathbf{x}(p) = \mathbf{x}_0$ and frame $\mathcal{E}(p) = \mathcal{E}_0$).
2. Since I is positive-definite, it may be written $\mathrm{I} = \theta_1^2 + \theta_2^2$.
3. The first structure equations determine ω_{12} and $\mathbb{I}$ determines ω_{13} and ω_{23} (Lemma 3.49).
4. The Frobenius Theorem shows that the initial value problem
$$\mathrm{d}\mathcal{E} = \mathcal{E}\omega \qquad \mathcal{E}(p) = \mathcal{E}_0 \tag{$*$}$$
has a unique local solution provided the Gauss–Codazzi equations $\mathrm{d}\omega + \omega\wedge\omega = 0$ are satisfied. The solution $\mathcal{E}$ is $SO_3(\mathbb{R})$-valued and supplies an adapted frame (compare Corollary 1.41).
5. To find the surface, solve a second initial value problem
$$\mathrm{d}\mathbf{x} = \mathcal{E}\Theta \qquad \mathbf{x}(p) = \mathbf{x}_0$$
Frobenius says this has a unique solution provided $\mathrm{d}\Theta + \omega\wedge\Theta = 0$. Since this is precisely what we used to determine ω in step 2, we don't need to check this condition.
6. Any different choice of metric forms in step 2 merely rotates $\mathcal{E}$ around $\mathbf{n} = \mathbf{e}_3$ and does not affect the resulting surface.

It is a little easier to understand the integrability condition when written in co-ordinates: $(*)$ is a linear system of eighteen PDE in nine unknowns

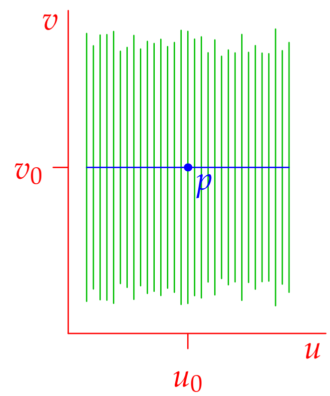
$$\begin{cases} \frac{\partial\mathcal{E}}{\partial u} = \mathcal{E}P \\ \frac{\partial\mathcal{E}}{\partial v} = \mathcal{E}Q \end{cases} \quad \text{where } P = \omega\big(\tfrac{\partial}{\partial u}\big),\ Q = \omega\big(\tfrac{\partial}{\partial v}\big) \text{ are skew-symmetric matrix functions}$$

The Gauss–Codazzi equations are essentially the fact that mixed partial derivatives commute:[28]

$$0 = \mathcal{E}_{uv} - \mathcal{E}_{vu} = \mathcal{E}_v P + \mathcal{E}P_v - \mathcal{E}_u Q + \mathcal{E}Q_u = \mathcal{E}\big(P_v - Q_u - [P,Q]\big)$$
$$P_v - Q_u - [P,Q] = \tfrac{\partial}{\partial v}\omega\big(\tfrac{\partial}{\partial u}\big) - \tfrac{\partial}{\partial u}\omega\big(\tfrac{\partial}{\partial v}\big) - \big[\omega\big(\tfrac{\partial}{\partial u}\big), \omega\big(\tfrac{\partial}{\partial v}\big)\big] = (\mathrm{d}\omega + \omega\wedge\omega)\left(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial u}\right)$$

[28] $[P,Q] = PQ - QP$ and $\mathrm{d}\omega$ is evaluated as in Exercise 2.3.10.

The part that requires some proof is that the integrability condition ($P_v - Q_u = [P, Q]$) is *sufficient* for a solution. This is not as hard as it sounds; here is another sketch:



1. If $p = (u_0, v_0)$, use Picard's ODE existence/uniqueness theorem to solve an initial value problem on the horizontal line $v = v_0$:

$$\frac{d\widetilde{\mathcal{E}}}{du} = \widetilde{\mathcal{E}}P(u, v_0), \qquad \widetilde{\mathcal{E}}(u_0, v_0) = \mathcal{E}_O$$

2. For *each* u_1, apply the ODE theorem to solve another IVP on the vertical line $u = u_1$:

$$\frac{d\mathcal{E}}{dv} = \mathcal{E}Q(u_1, v), \qquad \mathcal{E}(u_1, v_0) = \widetilde{\mathcal{E}}(u_1, v_0)$$

3. Finally, one shows that the resulting $\mathcal{E}$ is differentiable with respect to u, and uses the integrability condition to check that $\mathcal{E}_u = \mathcal{E}P$ as required.

The first two steps may be accomplished approximately using a numerical method to desired accuracy, so this amounts to an algorithm for the approximation of $\mathcal{E}$. The same approach can then be followed to approximate the surface.

The Gauss–Codazzi equations in curvature-line co-ordinates Suppose (u, v) are curvature-line co-ordinates. Then the fundamental forms are

$$\mathrm{I} = E\,du^2 + G\,dv^2, \qquad \mathbb{I} = k_1 E\,du^2 + k_2 G dv^2 \tag{†}$$

where E, G are positive functions and k_1, k_2 are the principal curvatures. We therefore choose metric forms $\theta_1 = \sqrt{E}\,du$ and $\theta_2 = \sqrt{G}\,dv$. In the language of Lemma 3.49,

$$a = -k_1, \quad b = 0, \quad c = -k_2, \quad \omega_{13} = -k_1\sqrt{E}\,du, \quad \omega_{23} = -k_2\sqrt{G}\,dv$$

The first structure equations determine

$$\omega_{12} = \frac{1}{2\sqrt{EG}}(E_v\,du - G_u\,dv)$$

(see Exercise 9). Moreover, the Gauss–Codazzi equations are equivalent to

$$\begin{aligned}
d\omega_{12} + \omega_{21} \wedge \omega_{13} = 0 &\iff \left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v = -2k_1k_2\sqrt{EG} \\
d\omega_{13} + \omega_{12} \wedge \omega_{23} = 0 &\iff 2(k_1)_v E = (k_2 - k_1)E_v \\
d\omega_{23} + \omega_{21} \wedge \omega_{13} = 0 &\iff 2(k_2)_u G = (k_1 - k_2)G_u
\end{aligned}$$

These equations show the relationship between I and $\mathbb{I}$: we cannot independently choose the metric (E, G) and the curvatures (k_1, k_2). However, if E, G, k_1, k_2 satisfy these equations, Bonnet's theorem guarantees the existence of a surface with fundamental forms (†), unique up to rigid motions.

While I, $\mathbb{I}$ cannot be chosen independently, Bonnet's result is considered the best description of the *minimal data* for a surface. You might suspect/hope that knowledge of K, H would be enough to determine a surface up to rigid motions, but Exercise 10 shows such to be vain!

Exercises 3.5. 1. The unit cylinder $\mathbf{x}(\phi, v) = \big(\cos\phi, \sin\phi, v\big)$ has adaptive frame

$$\mathbf{e}_1 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

(a) Directly compute the metric forms θ_j and connection forms ω_{jk}.

(b) That the six structure equations are satisfied should be obvious from your answers to (a): why?

(c) Why is it completely obvious from your answer to (a) that $K \equiv 0$?

2. For a general regular surface, explain why we cannot, in general, find co-ordinates u, v for which $\mathrm{I} = \mathrm{d}u^2 + \mathrm{d}v^2$.

3. For the paraboloid example (3.44.3) verify the Gauss–Codazzi equations $\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0$.

(*Hint: this is easier if you treat the three equations separately!*)

4. Verify that the metric $\mathrm{I} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ on the upper half-plane $y > 0$ has curvature $K = -1$.

(*Hint: Recall Example 3.51 and Exercise 3.2.9*)

5. Consider the catenoid $\mathbf{x}(u, v) = \big(\cos u \cosh v, \sin u \cosh v, v\big)$ obtained by revolving the catenary $x = \cosh z$ around the z-axis.

(a) Show that there exists a moving frame for which the metric forms are

$$\theta_1 = \cosh v \, \mathrm{d}u, \quad \theta_2 = \cosh v \, \mathrm{d}v$$

(b) Show that $\omega_{12} = \tanh v \, \mathrm{d}u = \frac{\sinh v}{\cosh v} \mathrm{d}u$ and use it to prove that the Gaussian curvature of the catenoid is

$$K = -\frac{1}{\cosh^4 v}$$

6. We re-prove Exercise 3.3.12 using our new language.

(a) Suppose a surface $\mathbf{x}$ is totally umbilic: $\mathbb{I} = \lambda \mathrm{I}$, where λ is some function. Explain why $\omega_{13} = -\lambda\theta_1$ and $\omega_{23} = -\lambda\theta_2$.

(b) Use the 1st structure equations and the Codazzi equations to prove that $\mathrm{d}\lambda = 0$.

(c) If $a = 0$, what is $\mathbf{x}$?

(d) If $a \neq 0$, define $\mathbf{c} := \mathbf{x} - \frac{1}{a}\mathbf{e}_3$. Prove that $\mathrm{d}\mathbf{c} = \mathbf{0}$ and hence conclude that the surface is (part of a) round sphere.

7. Suppose $\mathcal{E} = \big(\mathbf{e}_1 \; \mathbf{e}_2 \; \mathbf{e}_3\big)$ is an adaptive frame for a surface. Any other adaptive frame (with the same orientation) is obtained by *rotating* around $\mathbf{e}_3$: that is $\hat{\mathcal{E}} = \big(\hat{\mathbf{e}}_1 \; \hat{\mathbf{e}}_2 \; \mathbf{e}_3\big)$ where

$$\hat{\mathbf{e}}_1 = \cos\varphi \, \mathbf{e}_1 + \sin\varphi \, \mathbf{e}_2, \quad \hat{\mathbf{e}}_2 = -\sin\varphi \, \mathbf{e}_1 + \cos\varphi \, \mathbf{e}_2$$

for some smooth function $\varphi : U \to \mathbb{R}$.

(a) Compute θ_1, θ_2 in terms of $\hat{\theta}_1, \hat{\theta}_2$ and conclude that $\hat{\theta}_1 \wedge \hat{\theta}_2 = \theta_1 \wedge \theta_2$.

(b) Use Definition 3.45 to compute $\hat{\omega}_{12}$ in terms of ω_{12} and φ. Verify that $\mathrm{d}\hat{\omega}_{12} = \mathrm{d}\omega_{12}$ so that the Gauss equation is identical for the new moving frame.

8. Suppose I is the 1st fundamental form of a surface. Suppose $\mathrm{I} = \theta_1^2 + \theta_2^2$ for some 1-forms θ_1, θ_2. Prove that there exists a moving frame $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ for which $\mathrm{d}\mathbf{x} = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2$.

   (*Hint: consider the dual vector fields to* θ_1, θ_2)

9. Suppose u, v are orthogonal co-ordinates so that $\theta_1 = \sqrt{E}\,\mathrm{d}u$ and $\theta_2 = \sqrt{G}\,\mathrm{d}v$.

   (a) Use the structure equations to prove that

   $$\omega_{12} = \frac{1}{2\sqrt{EG}}\,(E_v\,\mathrm{d}u - G_u\,\mathrm{d}v)$$

   (b) Hence deduce an explicit formula for the Gauss curvature in terms of the coefficients of the 1st fundamental form:

   $$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right)$$

   This can be multiplied out to remove the square roots, though you'll get more terms. A nastier expression (the Brioshi formula) may be found for general co-ordinates with $F \neq 0$.

10. In Exercise 3.3.5 we saw that the tangent developable $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ of a unit-speed curve has curvatures $K = 0$, $H = -\frac{\tau}{2v\kappa}$. Use this to describe *two* surfaces with the same curvature functions which are not related by a direct isometry.

11. Show that the surfaces parametrized by

   $$\mathbf{x}(u,v) = \big(u\cos\phi, u\sin\phi, \ln u\big), \quad \mathbf{y}(u,v) = \big(u\cos\phi, u\sin\phi, \phi\big)$$

   have the same Gauss curvature but distinct first fundamental forms $\mathrm{I}_\mathbf{x} \neq \mathrm{I}_\mathbf{y}$. To do this properly, you should argue that there is no reparametrization of $\mathbf{y}$ so that $K_\mathbf{x} = K_\mathbf{y}$ and $\mathrm{I}_\mathbf{x} = \mathrm{I}_\mathbf{y}$.

   (*Gauss' Theorem isn't biconditional: surfaces can have the same K without being locally isometric*)

12. Consider the family of surfaces

   $$\mathbf{x}^t(u,v) = \cos t\begin{pmatrix}\sin u\sinh v\\ -\cos u\sinh v\\ u\end{pmatrix} + \sin t\begin{pmatrix}\cos u\cosh v\\ \sin u\cosh v\\ v\end{pmatrix}, \qquad t \in [0, \tfrac{\pi}{2}]$$

   When $t = 0$ this is a *helicoid*. When $t = \frac{\pi}{2}$ this is the *catenoid* from Exercise 5.

   (a) Compute the first fundamental form of $\mathbf{x}^t$ and show that it is independent of t (the family $\mathbf{x}^t$ is therefore *isometric*).

   (b) Show that the unit normal of $\mathbf{x}^t$ is also independent of t:

   $$\mathbf{n}^t = \frac{1}{\cosh v}\begin{pmatrix}\cos u\\ \sin u\\ -\sinh v\end{pmatrix}$$

   Hence compute the second fundamental form of $\mathbf{x}^t$ for each t.

   (c) Find the Gauss and mean curvatures of all surfaces $\mathbf{x}^t$. What is special about this family? Relate this to Gauss' Theorem.