3 Surfaces

3.1 Regular Parametrized Surfaces

We approach surfaces in $\mathbb{E}^3$ similarly to how we considered curves; a parametrized surface is a *function* $\mathbf{x} : U \to \mathbb{E}^3$ where U is some open subset of the plane $\mathbb{R}^2$. Our main purpose is to develop and measure the *curvature* of a surface in terms of the parametrizing function $\mathbf{x}$.

Our primary definition should mostly be familiar from elementary multivariable calculus.

Definition 3.1. A *(smooth local) surface* is the range $S = \mathbf{x}(U)$ of a smooth function $\mathbf{x} : U \to \mathbb{E}^3$, where U is a connected open subset of $\mathbb{R}^2$.

Given co-ordinates u, v on U, the *co-ordinate tangent vector fields* are the partial derivatives $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$.

The *exterior derivative* or *differential* of the surface is the vector-valued 1-form $\mathrm{d}\mathbf{x} = \mathbf{x}_u \, \mathrm{d}u + \mathbf{x}_v \, \mathrm{d}v$.

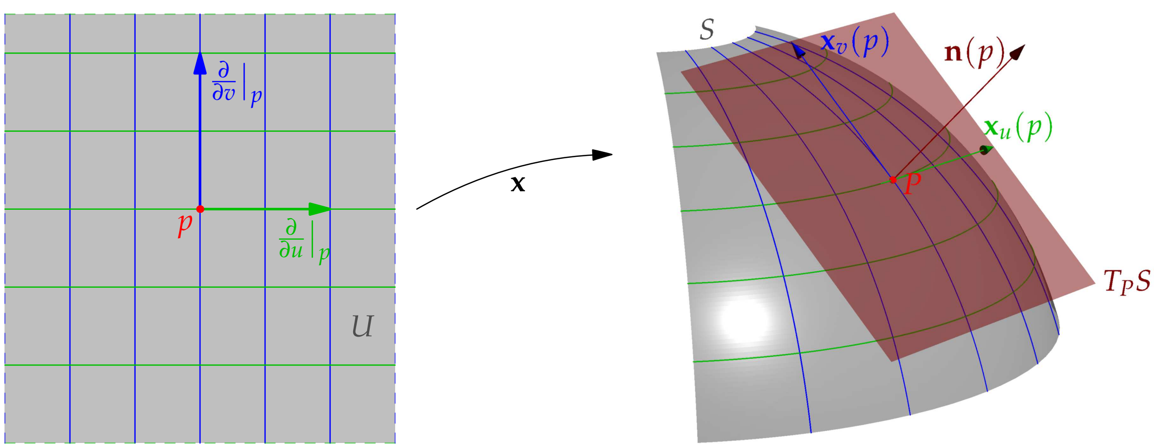
A surface is *regular* at $P = \mathbf{x}(p)$ if the tangent vectors $\mathbf{x}_u(p)$ and $\mathbf{x}_v(p)$ are *linearly independent*: otherwise said, at P, the surface has a well-defined

$$\textit{Tangent plane } \ T_P S = \operatorname{Span}\{\mathbf{x}_u(p), \mathbf{x}_v(p)\} \text{ (a 2-dim subspace of } T_P\mathbb{E}^3\text{), and}$$

$$\textit{Unit normal vector } \ \mathbf{n}(p) = \frac{\mathbf{x}_u(p) \times \mathbf{x}_v(p)}{||\mathbf{x}_u(p) \times \mathbf{x}_v(p)||} \in T_P\mathbb{E}^3$$

S is *regular* if it is regular everywhere. An *orientation* is a smooth choice of *unit normal vector field* $\mathbf{n}$.

The *Möbius strip* (Exercise 9) shows that not every surface is orientable!
For brevity, we will often refer to the parametrizing function $\mathbf{x}$ as the surface, though many different parametrizations will exist! A general surface typically needs to be parametrized by several overlapping functions $\mathbf{x}_1, \mathbf{x}_2, \ldots$. Our definition is *local* since there is only one $\mathbf{x}$.



The partial derivatives $\mathbf{x}_u(p), \mathbf{x}_v(p)$ are *tangent to the surface* at $\mathbf{x}(p)$: if $p = (u_0, v_0)$ then the *curve* $\mathbf{y}(t) := \mathbf{x}(t, v_0)$ lies in the surface and passes through $P = \mathbf{x}(p)$; its tangent vector at P is then

$$\mathbf{y}'(u_0) = \lim_{h \to 0} \frac{\mathbf{y}(u_0 + h) - \mathbf{y}(u_0)}{h} = \lim_{h \to 0} \frac{\mathbf{x}(u_0 + h, v_0) - \mathbf{x}(p)}{h} = \mathbf{x}_u(p)$$

To help distinguish between domain and codomain, we standardize notation.

Domain $U \subseteq \mathbb{R}^2$: *Points* are written *lower case* or as *row vectors*: e.g., $p = (u_0, v_0) \in U$. Typically we'll use u, v as *co-ordinates* unless it is more natural to use angles such as ϕ, θ.

Tangent vectors/fields are written with an arrow in our *new* notation: e.g., $\vec{w}_p = \frac{\partial}{\partial u}\Big|_p \in T_p\mathbb{R}^2$.

Codomain $\mathbb{E}^3$: *Points* are written *upper case* or as *row vectors*, e.g., $P = (3, 4, 8) \in \mathbb{E}^3$. Co-ordinates on $\mathbb{E}^3$ will typically be x, y, z.

Vectors are written *bold-face* as either row or column vectors: e.g., $\mathbf{x}(u, v) = \big(u, v, u^2 + v^2\big)$.

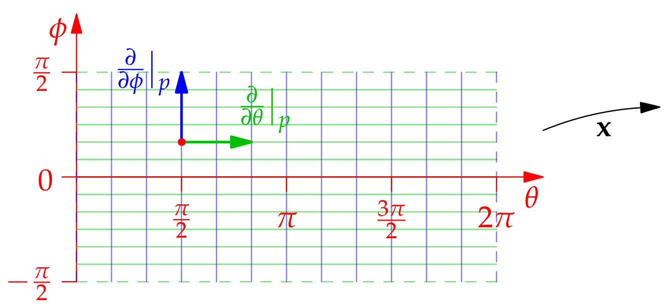
Tangent vectors/fields use the *old* notation:[16] e.g., if $P = \mathbf{x}(p)$, then $\mathbf{x}_u(p) = \frac{\partial \mathbf{x}}{\partial u}\Big|_p \in T_P\mathbb{E}^3$.

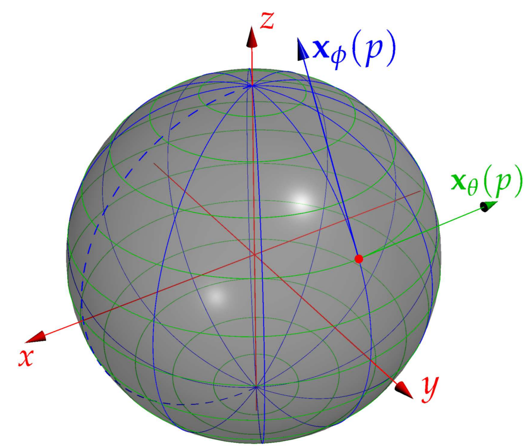
Example 3.2. Consider the sphere of radius a parametrized using spherical polar co-ordinates:

$$\mathbf{x}(\theta, \phi) = a \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix}, \quad \mathrm{d}\mathbf{x} = \mathbf{x}_\theta\,\mathrm{d}\theta + \mathbf{x}_\phi\,\mathrm{d}\phi = a\cos\phi \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \mathrm{d}\theta + a \begin{pmatrix} -\cos\theta\sin\phi \\ -\sin\theta\sin\phi \\ \cos\phi \end{pmatrix} \mathrm{d}\phi$$

The unit normal field is simply $\mathbf{n} = \frac{1}{a}\mathbf{x}$. The domain $U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is an open rectangle whose image $S = \mathbf{x}(U)$ is the sphere minus the (dashed) semicircle $\mathbf{x}(0, \phi)$. While we could extend θ to wrap round the equator, we cannot extend to the north or south poles without sacrificing *regularity*:

$$\mathbf{x}_\theta = a\cos\phi \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \mathbf{0} \text{ when } \phi = \pm\frac{\pi}{2}$$





This illustrates the term *local*: indeed the famous *hairy ball theorem* from topology says that it is impossible to find a regular parametrization of the entire sphere by a single function.
Also observe how the tangent vectors $\frac{\partial}{\partial\phi}\Big|_p, \frac{\partial}{\partial\theta}\Big|_p \in T_p\mathbb{R}^2$ are mapped by $\mathrm{d}\mathbf{x}$ to tangent vectors

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\phi}\bigg|_p = \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\phi}\bigg|_p\right), \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\theta}\bigg|_p = \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\theta}\bigg|_p\right) \in T_{\mathbf{x}(p)}S$$

[16] To use our new notation in $\mathbb{E}^3$ would require a subtle redefinition of $\mathrm{d}\mathbf{x}$: if $\vec{w}$ is a vector field on U, then $\mathrm{d}\mathbf{x}(\vec{w})$ is the vector field on S such that $\big(\mathrm{d}\mathbf{x}(\vec{w})\big)[f] = \vec{w}\big[f \circ \mathbf{x}\big]$ for all $f : S \to \mathbb{R}$. In co-ordinates this benefits from tensor notation:

$$\mathbf{x}(u_1, u_2) = \Big(x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2)\Big) \implies \mathrm{d}\mathbf{x} = \sum_{i,j} \frac{\partial x_j}{\partial u_i}\frac{\partial}{\partial x_j} \otimes \mathrm{d}u_i$$

In more general situations this approach is necessary, but it is overkill for our purposes!

Theorem 3.3. *Let $S = \mathbf{x}(U)$ be a smooth surface containing the point $P = \mathbf{x}(p)$:*

1. *The differential at p is a linear map $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P\mathbb{E}^3$ mapping tangent vectors in $\mathbb{R}^2$ to vectors tangent to S.*

2. *S is regular at P if and only if $\mathrm{d}\mathbf{x}$ is* injective (1–1) *at p. In such a case we can view it as an invertible linear map $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P S$.*

Proof. 1. The differential at p is linear since the co-ordinate 1-forms $\mathrm{d}u, \mathrm{d}v$ are linear: indeed

$$\mathrm{d}\mathbf{x}\left(a\left.\frac{\partial}{\partial u}\right|_p + b\left.\frac{\partial}{\partial v}\right|_p\right) = \mathbf{x}_u(p)\,\mathrm{d}u\left(a\left.\frac{\partial}{\partial u}\right|_p + b\left.\frac{\partial}{\partial v}\right|_p\right) + \mathbf{x}_v(p)\,\mathrm{d}v\left(a\left.\frac{\partial}{\partial u}\right|_p + b\left.\frac{\partial}{\partial v}\right|_p\right)$$
$$= a\mathbf{x}_u(p) + b\mathbf{x}_v(p) = a\,\mathrm{d}\mathbf{x}\left(\left.\frac{\partial}{\partial u}\right|_p\right) + b\,\mathrm{d}\mathbf{x}\left(\left.\frac{\partial}{\partial v}\right|_p\right)$$

This expression is moreover tangent to S at $\mathbf{x}(p)$: if this last assertion is unconvincing, see Exercise 8.

2. The range of $\mathrm{d}\mathbf{x}$ at p is plainly $\operatorname{Span}\{\mathbf{x}_u(p), \mathbf{x}_v(p)\}$. This is 2-dimensional (and thus defines the tangent plane) if and only if $\operatorname{rank}\mathrm{d}\mathbf{x} = 2 \iff \mathrm{d}\mathbf{x}$ is 1–1. ∎

It is worth reiterating two crucially important properties of $\mathrm{d}\mathbf{x}$:

- At a regular point, $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P S$ is an invertible linear map. We shall shortly use this to *pull-back* calculations from S to U.

- The differential is co-ordinate independent and thus does not depend on the parametrization of S. This follows since $\mathrm{d}\mathbf{x}$ is the unique 1-form satisfying $\mathrm{d}\mathbf{x}(\vec{w}) = \vec{w}[\mathbf{x}]$ for all vector fields $\vec{w}$ on U; a description that does not depend on co-ordinates.

Aside: change of co-ordinates To more clearly spell this out, suppose we choose a new parametrization $\mathbf{y}(s,t) = \mathbf{x}\big(F(s,t)\big)$ where $F(s,t) = (u,v)$ is a change of co-ordinates on U. By the chain rule,

$$\begin{pmatrix}\mathbf{y}_s\\ \mathbf{y}_t\end{pmatrix} = \begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} \quad\text{and}\quad (\mathrm{d}u\ \mathrm{d}v) = (\mathrm{d}s\ \mathrm{d}t)\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}$$

from which

$$\mathrm{d}\mathbf{y} = (\mathrm{d}s\ \mathrm{d}t)\begin{pmatrix}\mathbf{y}_s\\ \mathbf{y}_t\end{pmatrix} = (\mathrm{d}u\ \mathrm{d}v)\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} = (\mathrm{d}u\ \mathrm{d}v)\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} = \mathrm{d}\mathbf{x}$$

The matrix of partial derivatives is the *Jacobian* of the co-ordinate change.

To be completely strict, $\mathrm{d}\mathbf{x}$ and $\mathrm{d}\mathbf{y}$ are not identical since they feed on tangent vectors with respect to different co-ordinates. Formally

$$\mathbf{y} = \mathbf{x} \circ F \implies \mathrm{d}\mathbf{y} = \mathrm{d}\mathbf{x} \circ \mathrm{d}F$$

where $\mathrm{d}F$ maps tangent vectors in $\operatorname{Span}\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\}$ to those in $\operatorname{Span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$: in matrix language, $\mathrm{d}F$ is precisely the above Jacobian!

Common Surfaces

You should have met many of these families/examples in multi-variable calculus.

Graphs If $f(x,y)$ is a smooth function, its graph may be parametrized by $\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$. Its differential and unit normal field are

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}1\\0\\f_u\end{pmatrix}\mathrm{d}u + \begin{pmatrix}0\\1\\f_v\end{pmatrix}\mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1+f_u^2+f_v^2}}\begin{pmatrix}-f_u\\-f_v\\1\end{pmatrix}$$

This is regular at all points, regardless of f.

Examples 3.4. 1. The standard circular paraboloid may be parametrized $\mathbf{x}(u,v) = \big(u, v, u^2+v^2\big)$.

2. The upper half of the unit sphere is the graph of $z = f(x,y) = \sqrt{1-x^2-y^2}$ where $x^2+y^2 < 1$.

3. A plane has equation $ax+by+cz = d$ where a,b,c,d are constant. Since at least one of a,b,c must be non-zero, this may be written as a function and graphed. For instance, if $b \neq 0$ we have $y = f(x,z) = \frac{1}{b}(d-ax-cz)$ and $\mathbf{n} = \frac{1}{\sqrt{a^2+b^2+c^2}}\big(a,b,c\big)$.

Surfaces of Revolution If a smooth positive function $x = f(z)$ is rotated around the z-axis, we obtain a parametrization

$$\mathbf{x}(\theta,v) = \big(f(v)\cos\theta, f(v)\sin\theta, v\big), \quad (\theta,v) \in (0,2\pi)\times \mathrm{dom}(f)$$

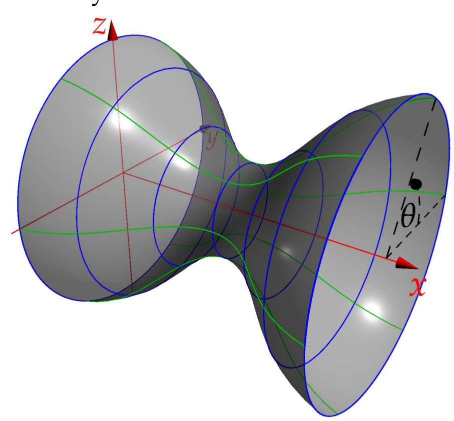
with differential and unit normal field

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}-f(v)\sin\theta\\ f(v)\cos\theta\\ 0\end{pmatrix}\mathrm{d}\theta + \begin{pmatrix}f'(v)\cos\theta\\ f'(v)\sin\theta\\ 1\end{pmatrix}\mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1+f'(v)^2}}\begin{pmatrix}\cos\theta\\ \sin\theta\\ -f'(v)\end{pmatrix}$$

Examples 3.5. 1. The simplest example ($f(z) \equiv 1$) is the right circular cylinder of radius 1.

2. We may rotate around any axis! For instance, if we rotate the curve $z = 2+\cos x$ around the x-axis, the resulting surface may be parametrized

$$\mathbf{x}(\theta,v) = (2+\cos v)\begin{pmatrix}0\\ \cos\theta\\ \sin\theta\end{pmatrix} + \begin{pmatrix}v\\0\\0\end{pmatrix}$$



This time v measures distance along the x-axis and θ the angle of rotation around it.

The differential and unit normal field are

$$\mathrm{d}\mathbf{x} = (2+\cos v)\begin{pmatrix}0\\ -\sin\theta\\ \cos\theta\end{pmatrix}\mathrm{d}\theta + \begin{pmatrix}1\\ -\sin v\cos\theta\\ -\sin v\sin\theta\end{pmatrix}\mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1+\sin^2 v}}\begin{pmatrix}\sin v\\ \cos\theta\\ \sin\theta\end{pmatrix}$$

Note the *orientation* of the surface: the unit normal field points *outward*, away from the x-axis.

Ruled Surfaces Given functions $\mathbf{y}(u), \mathbf{z}(u)$, define

$$\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{z}(u)$$

Through each point $P = \mathbf{x}(u_0, v_0)$ passes a line $t \mapsto \mathbf{x}(u_0, t) = \mathbf{y}(u_0) + t\mathbf{x}(u_0)$ lying in the surface. The surface can be visualized as moving a *ruler* through space. Ruled surfaces are common in engineering applications since they may be constructed using straight beams.

Definition 3.6. The *tangent developable* of a smooth curve $\mathbf{y}$ is the special case when $\mathbf{z} = \mathbf{y}'$.

Examples 3.7. 1. Every plane is a ruled surface! Let $\mathbf{y}$ be a line in the plane and $\mathbf{z}$ any other tangent direction. For instance, the plane passing through $(1,0,9)$ and spanned by $(2,-3,-5)$ ad $(1,2,3)$ may be parametrized

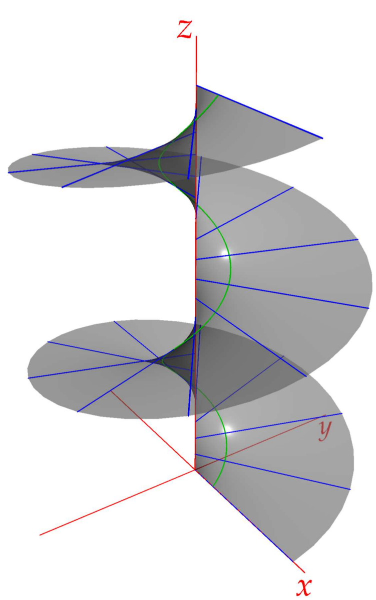
$$\mathbf{x}(u,v) = \underbrace{(1,0,9) + (2,-3,-5)u}_{\mathbf{y}(u)} + \underbrace{(1,2,3)}_{\mathbf{z}(u)}v$$

2. A *helicoid* is built by joining each point of a helix to its axis of rotation. From the standard helix, we obtain the helicoid $\mathbf{x}(u,v) = \big(v\cos u, v\sin u, u\big)$ for $v > 0$.

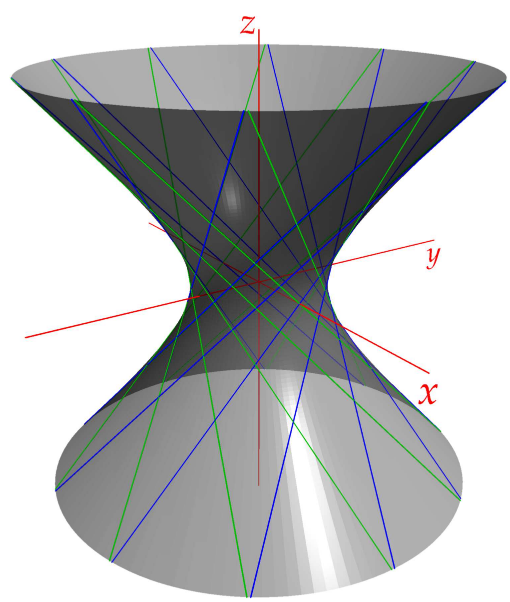
3. The *hyperboloid of one sheet* is a *doubly ruled surface*: through each point there are *two* lines lying on the surface. It may be parametrized as a ruled surface by

$$\mathbf{x}(u,v) = \begin{pmatrix}1\\0\\0\end{pmatrix} + u\begin{pmatrix}0\\1\\1\end{pmatrix} + v\begin{pmatrix}2u\\u^2-1\\u^2+1\end{pmatrix}$$

though convincing yourself there are *two* lines through each point takes a little more work...



Helicoid



Hyperboloid

Implicitly Defined Surfaces

Definition 3.8. A *regular implicitly defined surface* is the zero set of a smooth function $f : \mathbb{E}^3 \to \mathbb{R}$ for which $\mathrm{d}f \neq 0$ (equivalently $\nabla f \neq \mathbf{0}$).

Recall that the directional derivative of f in the direction $\mathbf{v}$ is $D_{\mathbf{v}} f(P) = \mathbf{v} \cdot \nabla f(P)$. This is zero if and only if $\mathbf{v}$ is orthogonal to $\nabla f(P)$. In particular, this says that ∇f provides a *normal field* to an implicitly defined surface.

Examples 3.9. 1. Let a, b, c, d be constants. The function $f(x, y, z) = ax + by + cz - d$ has

$$\mathrm{d}f = a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z$$

which is non-zero provided at least one of a, b, c are non-zero. This defines a plane with unit normal field $\mathbf{n} = \frac{1}{||\nabla f||} \nabla f = \frac{1}{\sqrt{a^2+b^2+c^2}}(a, b, c)$.

2. The sphere of radius a is the zero set of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$. It has unit normal field

$$\mathbf{n} = \frac{1}{||\nabla f||} \nabla f = \frac{1}{a}(x, y, z)$$

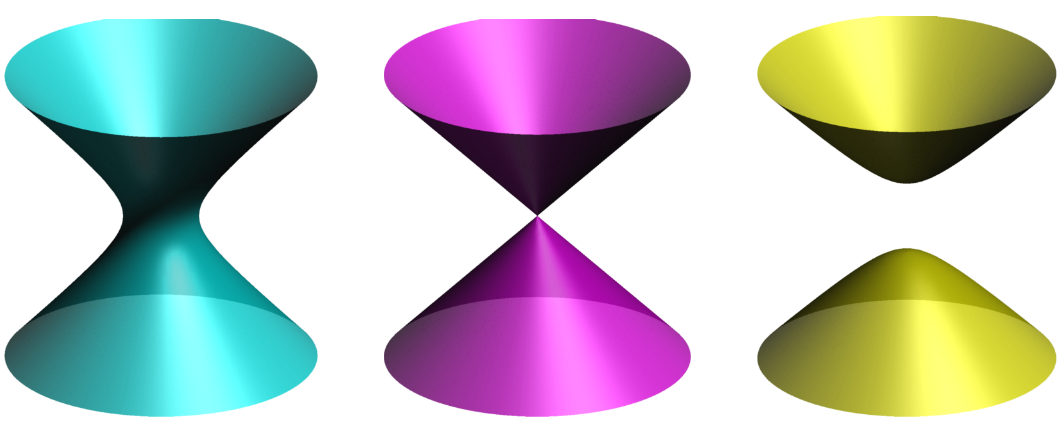
The sphere is everywhere regular since at least one of x, y, z is non-zero at all points of the sphere. Contrast this with our earlier example of the *parametrized* sphere which could not be made regular at the north and south poles. The lack of regularity in this case is an aspect of the parametrization, not the surface itself.

3. The function $f(x, y, z) = x^2 + y^2 - z^2 - c$ has

$$\mathrm{d}f = 2(x\,\mathrm{d}x + y\,\mathrm{d}y - z\,\mathrm{d}z)$$

which is non-zero away from $(x, y, z) = (0, 0, 0)$. Depending on the sign of c, the zero set is a hyperboloid or a cone; visualize the horizontal cross-sectional circles to determine which.

$c > 0$	Hyperboloid of 1-sheet: $x^2 + y^2 = z^2 + c > 0$ for all z
$c = 0$	Cone: $x^2 + y^2 = z^2$ contains a non-regular point $(0, 0, 0)$
$c < 0$	Hyperboloid of 2-sheets: $x^2 + y^2 = z^2 - \lvert c\rvert \geq 0$ only when $\lvert z\rvert \geq \sqrt{\lvert c\rvert}$



Our next result, a corollary of the famous *implicit function theorem*, ties together the notions of *regularity*. In particular, it says that we can always assume the existence of *local* co-ordinates.

Theorem 3.10. *A regular implicitly defined surface $f(x,y,z)=0$ is (locally) the image of a regular local surface.*

Proof. Suppose $P=(x_0,y_0,z_0)$ lies on the surface and $\nabla f(P)\neq \mathbf{0}$. At least one of the partial derivatives of f is non-zero; suppose WLOG that $f_z(P)\neq 0$. By the implicit function theorem, there exists $U\subseteq \mathbb{R}^2$ and a function $g:U\to\mathbb{R}$ for which $g(x_0,y_0)=z_0$ and $f(x,y,g(x,y))=0$. The surface is then (locally) the graph of $z=g(x,y)$.

$$\mathbf{x}:U\to\mathbb{E}^3:(u,v)\mapsto\big(u,v,g(u,v)\big)$$

∎

Example 3.11. The zero set of $f(x,y,z)=x^2+y^2-z^2-6$ is a hyperboloid of one sheet. It has unit normal vector field

$$\mathbf{n}(x,y,z)=\frac{1}{||\nabla f||}\nabla f=\frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix}x\\y\\-z\end{pmatrix}=\frac{1}{\sqrt{6+2z^2}}\begin{pmatrix}x\\y\\-z\end{pmatrix}$$

whenever (x,y,z) is a point on the hyperboloid. For instance, at $P=(3,1,2)$ the unit normal is $\mathbf{n}(P)=\frac{1}{\sqrt{14}}(3,1,2)$ and the tangent plane has equation

$$3x+y-2z=6$$

Alternatively, the hyperboloid can be parametrized in several ways.

(a) In the language of the proof, near $P=(3,1,2)$ it is the graph of $z=g(x,y)=\sqrt{x^2+y^2-6}$. This results in a (local) regular parametrization

$$\mathbf{x}(u,v)=\big(u,v,\sqrt{u^2+v^2-6}\big)$$

(b) The hyperboloid is a surface of revolution around the z-axis:

$$\mathbf{x}(\theta,v)=\begin{pmatrix}\sqrt{6+v^2}\cos\theta\\\sqrt{6+v^2}\sin\theta\\v\end{pmatrix}$$

For this parametrization, the differential and normal field are

$$\mathrm{d}\mathbf{x}=\sqrt{6+v^2}\begin{pmatrix}-\sin\theta\\\cos\theta\\0\end{pmatrix}\mathrm{d}\theta+\frac{1}{\sqrt{6+v^2}}\begin{pmatrix}v\cos\theta\\v\sin\theta\\\sqrt{6+v^2}\end{pmatrix}\mathrm{d}v$$

$$\mathbf{n}=\frac{\mathbf{x}_\theta\times\mathbf{x}_v}{||\mathbf{x}_\theta\times\mathbf{x}_v||}=\frac{1}{\sqrt{6+2v^2}}\begin{pmatrix}\sqrt{6+v^2}\cos\theta\\\sqrt{6+v^2}\sin\theta\\-v\end{pmatrix}$$

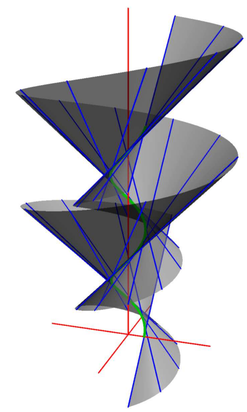
which is precisely what we obtained above.

Yet another expression could be obtained using a parametrization as a ruled surface (e.g., page 51).

Exercises 3.1. 1. Show that parametrization $\mathbf{x}(r,\theta) = \big(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\big)$ of the upper hemisphere is non-regular at $r=0$.

2. Explain why the parametrization in Example 3.11(a) is *local*: what is left out?

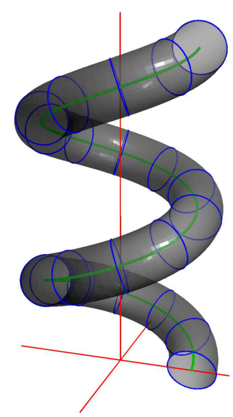
3. (a) Compute $\mathrm{d}\mathbf{x}$ and $\mathbf{n}$ for the paraboloid $\mathbf{x}(u,v) = \big(u, v, u^2+v^2\big)$.
   (b) Repeat for the polar co-ordinate parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$. Is this parametrization everywhere regular?
   (c) Using $x = r\cos\theta$, etc., write $\mathrm{d}\mathbf{x}$ in terms of $r, \theta, \mathrm{d}r, \mathrm{d}\theta$. What do you observe?
   (d) By viewing the paraboloid as the zero set of $f(x,y,z) = z - x^2 - y^2$, find another expression for the unit normal field.



4. (a) Find a parametrization for the tangent developable of the helix $\mathbf{y}(u) = \big(\cos u, \sin u, u\big)$. Compute $\mathrm{d}\mathbf{y}$ and the unit normal field $\mathbf{n}$.
   (*The picture covers $v \in (-3,6)$ with the original curve $\mathbf{y}(u)$ in green*)
   (b) If $\mathbf{y}$ is a unit speed biregular curve, prove that its tangent developable $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ is a regular surface except when $v=0$. Express the differential and unit normal field in terms of the Frenet frame of $\mathbf{y}$.

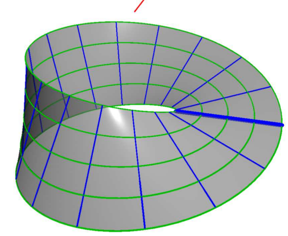
5. Let $f(x,y,z) = z^2$. Show that the zero set of f has a regular parametrization despite the gradient of f vanishing at $z=0$.

6. Let a,b,c be positive constants and define $\mathbf{x}(\theta,\phi) = \begin{pmatrix} a\cos\theta\cos\phi \\ b\sin\theta\cos\phi \\ c\sin\phi \end{pmatrix}$, $(\theta,\phi) \in (0,2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$
   (a) Show that $\mathbf{x}$ parametrizes the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. What part(s) of the ellipsoid are 'missing' from the parametrization?
   (b) Describe geometrically the curves $\theta =$ constant and $\phi =$ constant on the ellipsoid.
   (c) Calculate the differential of $\mathbf{x}$ and show that $\mathrm{d}\mathbf{x}$ is 1–1 for each $p \in U$.



7. The tube of radius $a > 0$ centered on a curve $\mathbf{y}(t)$ may be parametrized in terms of the Frenet frame of $\mathbf{y}$:
$$\mathbf{x}(\phi,t) = \mathbf{y}(t) + a\cos\phi\,\mathbf{N}(t) + a\sin\phi\,\mathbf{B}(t)$$
   (a) *Briefly* explain why the normal field is $\mathbf{n} = \cos\phi\,\mathbf{N}(t) + \sin\phi\,\mathbf{B}(t)$.
   (b) Suppose $\mathbf{y}$ is unit speed. Prove that $\mathbf{x}$ is everywhere regular if and only if $\kappa(t) < \frac{1}{a}$ at all points of the generating curve.

8. Let $c(t) : (-\epsilon,\epsilon) \to U$ be a curve and $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$ the corresponding curve in the surface $\mathbf{x} : U \to \mathbb{E}^3$. Prove that $\mathrm{d}\mathbf{x}\big(c'(0)\big) = \mathbf{y}'(0)$.
(*Hint: Recall how to write $c'(t)$ as a* vector field)



9. (Möbius strip) Show that $\mathbf{x}(u,v) = \begin{pmatrix} (2+v\cos\frac{u}{2})\cos u \\ (2+v\cos\frac{u}{2})\sin u \\ v\sin\frac{u}{2} \end{pmatrix}$ is regular and orientable whenever $0 < u < 2\pi$ and $-1 < v < 1$. By computing $\mathbf{n}(0,0)$ and $\mathbf{n}(2\pi,0)$, explain what happens if we try to extend u to $[0,2\pi]$.

3.2 The Fundamental Forms

Our immediate goal is to use differentials to describe the shape of a surface. Before making the main definition, we need another product of 1-forms.

Definition 3.12. Given 1-forms α, β on U, define the *symmetric 2-form* $\alpha\beta$ by

$$\alpha\beta(\vec{v},\vec{w}) = \frac{1}{2}\big(\alpha(\vec{v})\beta(\vec{w}) + \alpha(\vec{w})\beta(\vec{v})\big)$$

where $\vec{v},\vec{w}$ are vector fields on U. Note that $\alpha^2(\vec{v},\vec{w}) := \alpha\alpha(\vec{v},\vec{w}) = \alpha(\vec{v})\alpha(\vec{w})$.

Symmetric 2-forms behave the way you (hopefully!) think they should.

Lemma 3.13. *On each tangent space, $\alpha\beta : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ is a symmetric and bilinear. Moreover $\alpha\beta = \beta\alpha$, and the product is linear in each slot:*

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma \quad \text{and} \quad (\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma \tag{$*$}$$

Take care when using co-ordinate 1-forms; convention dictates that $\mathrm{d}x^2 = (\mathrm{d}x)^2$ is a symmetric 2-form, *not* the exterior derivative (1-form) $\mathrm{d}(x^2) = 2x\,\mathrm{d}x$.

Example 3.14. Let $\vec{v} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $\vec{w} = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$. Then

$$\mathrm{d}x^2(\vec{v},\vec{w}) = ac, \qquad \mathrm{d}y^2(\vec{v},\vec{w}) = bd, \qquad \mathrm{d}x\,\mathrm{d}y(\vec{v},\vec{w}) = \frac{1}{2}(ad+bc)$$

In particular, $\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)(\vec{v},\vec{w}) = ac + bd$ is the *dot product* in disguise.
To evaluate symmetric 2-forms with respect to co-ordinates, linearity and distributivity $(*)$ are all you need. For instance, if $\alpha = x\,\mathrm{d}x - \mathrm{d}y$ and $\beta = xy\,\mathrm{d}y$, then $\alpha\beta = x^2y\,\mathrm{d}x\mathrm{d}y - xy\,\mathrm{d}y^2$.

If α, β take values in $\mathbb{E}^n$, we use the dot product for multiplication of the resulting *vectors* $\alpha(\vec{v})$, etc.

$$(\alpha\cdot\beta)(\vec{v},\vec{w}) := \frac{1}{2}\big(\alpha(\vec{v})\cdot\beta(\vec{w}) + \alpha(\vec{w})\cdot\beta(\vec{v})\big)$$

Definition 3.15. The *first and second fundamental forms* of a regular local surface $\mathbf{x} : U \to \mathbb{E}^3$ are

$$\mathrm{I} = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}, \qquad \mathbb{I} = -\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{n}$$

where $\mathrm{d}\mathbf{n}$ is the differential of the unit normal field ($\mathbb{I}$ requires that the surface be *oriented*). The first fundamental form is also commonly denoted $\mathrm{d}s^2$ (see Example 3.17 and Theorem 3.20 for why).

Example 3.16. If $\mathbf{x}(u,v) = \big(u, uv, 1+u\big)$, then

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}1\\v\\1\end{pmatrix}\mathrm{d}u + \begin{pmatrix}0\\u\\0\end{pmatrix}\mathrm{d}v, \qquad \mathbf{n} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad \mathrm{d}\mathbf{n} = \mathbf{0}$$

from which $\mathrm{I} = (2+v^2)\,\mathrm{d}u^2 + 2uv\,\mathrm{d}u\,\mathrm{d}v + u^2\,\mathrm{d}v^2$ and $\mathbb{I} = 0$.

Why should we care about I & $\mathbb{I}$?

Basic interpretation $\mathrm{I}(\vec{v},\vec{w}) = \mathrm{d}\mathbf{x}(\vec{v})\cdot\mathrm{d}\mathbf{x}(\vec{w})$ pulls back the dot product from T_PS to $T_p\mathbb{R}^2$. The length of and angle between tangent vectors to the surface S at P may now be computed in $T_p\mathbb{R}^2$.

$\mathbb{I}(\vec{v},\vec{w}) = -\frac{1}{2}\big(\mathrm{d}\mathbf{x}(\vec{v})\cdot\mathrm{d}\mathbf{n}(\vec{w}) + \mathrm{d}\mathbf{x}(\vec{w})\cdot\mathrm{d}\mathbf{n}(\vec{v})\big)$ describes how the normal field $\mathbf{n}$ changes over the surface. In the example, $\mathbb{I}\equiv 0$ encapsulates the *constancy* of the normal field: the surface is (part of) the plane $\mathbf{x}\cdot(1,0,-1) = -1$.

Co-ordinate invariance Since $\mathrm{d}\mathbf{x}$ is independent of co-ordinates, so also is I. The unit normal field is independent of *oriented* co-ordinate changes. More formally, if $\mathbf{y}(s,t) = \mathbf{x}(u,v)$ parametrize the same surface, then[17]

$$\mathrm{I}_{\mathbf{y}} = \mathrm{I}_{\mathbf{x}} \quad \text{and} \quad \mathbb{I}_{\mathbf{y}} = \begin{cases} \mathbb{I}_{\mathbf{x}} & \text{if the orientations are identical} \\ -\mathbb{I}_{\mathbf{x}} & \text{if the orientations are reversed} \end{cases}$$

The upshot is that the fundamental forms provide a co-ordinate independent way to compute information about a surface from within the *parametrization space* U.

Example 3.17. For the sphere of radius a in spherical polar co-ordinates, recall Example 3.2:

$$\mathbf{x}(\theta,\phi) = a\begin{pmatrix}\cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi\end{pmatrix} \implies \mathrm{d}\mathbf{x} = a\cos\phi\begin{pmatrix}-\sin\theta \\ \cos\theta \\ 0\end{pmatrix}\mathrm{d}\theta + a\begin{pmatrix}-\cos\theta\sin\phi \\ -\sin\theta\sin\phi \\ \cos\phi\end{pmatrix}\mathrm{d}\phi$$
$$\implies \mathrm{I} = a^2\big(\cos^2\!\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$$

If you revisit the pictures in Example 3.2, the effect of I is easy to visualize:

- $\mathrm{I}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) = \lvert\lvert\mathbf{x}_\theta\rvert\rvert^2 = a^2\cos^2\!\phi$: the tangent vector $\mathbf{x}_\theta$ is *shorter* near the poles, where $\cos\phi\to 0$.
- $\mathrm{I}(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}) = \lvert\lvert\mathbf{x}_\phi\rvert\rvert^2 = a^2$: the tangent vector $\mathbf{x}_\phi$ always has the *same length.*
- $\mathrm{I}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}) = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0$: the co-ordinate tangent vectors are always *orthogonal.*

At a point $P = \mathbf{x}(p)$ on the sphere, if we increase the co-ordinates by tiny quantities $\Delta p = \big(\Delta\theta,\Delta\phi\big)$, then the distance Δs travelled along the surface approximately satisfies

$$(\Delta s)^2 \approx \lvert\lvert\mathbf{x}(p+\Delta p)-\mathbf{x}(p)\rvert\rvert \approx \big\lvert\big\lvert\mathbf{x}_\theta\,\Delta\theta + \mathbf{x}_\phi\,\Delta\phi\big\rvert\big\rvert^2 = a^2\cos^2\!\phi\,(\Delta\theta)^2 + a^2(\Delta\phi)^2$$

with equality in the limit $\Delta\theta,\Delta\phi\to 0$. Near the poles, a change in *longitude* $\Delta\theta$ corresponds to a smaller distance on the sphere. This is analogous to how a standard *map* of the Earth works, with distances appearing distorted near the poles. We'll return to this idea shortly…

Computing $\mathbb{I}$ is very easy for the sphere, since $\mathbf{n} = \frac{1}{a}\mathbf{x}$ is merely the scaled position vector:

$$\mathbb{I} = -\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{n} = -\frac{1}{a}\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = -\frac{1}{a}\mathrm{I} = -a\big(\cos^2\!\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$$

[17] As in the Aside on page 49, we strictly have $\mathrm{I}_{\mathbf{y}} = \mathrm{I}_{\mathbf{x}}\circ\mathrm{d}F$, etc., where $\mathbf{y}(s,t) = \mathbf{x}\big(F(s,t)\big) = \mathbf{x}(u,v)$. The $\pm$-sign in the expressions for $\mathbb{I}$ is that of the *determinant* of the Jacobian $\mathrm{d}F$.

The fundamental forms I, II may be computed directly in terms of co-ordinates u, v.

Theorem 3.18. *If $\mathbf{x} : U \to \mathbb{E}^3$ is a regular (oriented) surface, then*

$$\mathrm{I} = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \quad \textit{and} \quad \mathrm{II} = l\,\mathrm{d}u^2 + 2m\,\mathrm{d}u\,\mathrm{d}v + n\,\mathrm{d}v^2$$

where the smooth functions $E, F, G, l, m, n : U \to \mathbb{R}$ are defined by

$$\begin{array}{lll} E = \mathbf{x}_u \cdot \mathbf{x}_u & F = \mathbf{x}_u \cdot \mathbf{x}_v & G = \mathbf{x}_v \cdot \mathbf{x}_v \\ l = \mathbf{x}_{uu} \cdot \mathbf{n} = -\mathbf{x}_u \cdot \mathbf{n}_u & m = \mathbf{x}_{uv} \cdot \mathbf{n} = -\mathbf{x}_u \cdot \mathbf{n}_v = -\mathbf{x}_v \cdot \mathbf{n}_u & n = \mathbf{x}_{vv} \cdot \mathbf{n} = -\mathbf{x}_v \cdot \mathbf{n}_v \end{array}$$

The expressions for II come from differentiating $\mathbf{x}_u \cdot \mathbf{n} = 0 = \mathbf{x}_v \cdot \mathbf{n}$, and are particularly helpful because they avoid computing derivatives of $\mathbf{n}$ (which likely contains a square-root).

Example 3.19. Parametrize the graph of $z = f(x, y)$ by $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$ to obtain,

$$\mathbf{x}_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}, \quad \mathbf{x}_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix} \implies E = 1 + f_u^2, \quad F = f_u f_v, \quad G = 1 + f_v^2$$

$$\implies \mathrm{I} = (1 + f_u^2)\,\mathrm{d}u^2 + 2f_u f_v\,\mathrm{d}u\,\mathrm{d}v + (1 + f_v^2)\,\mathrm{d}v^2$$

$$\mathbf{x}_{uu} = \begin{pmatrix} 0 \\ 0 \\ f_{uu} \end{pmatrix}, \quad \mathbf{x}_{uv} = \begin{pmatrix} 0 \\ 0 \\ f_{uv} \end{pmatrix}, \quad \mathbf{x}_{vv} = \begin{pmatrix} 0 \\ 0 \\ f_{vv} \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$\implies l = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad m = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad n = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}$$

$$\implies \mathrm{II} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \left(f_{uu}\,\mathrm{d}u^2 + 2f_{uv}\,\mathrm{d}u\,\mathrm{d}v + f_{vv}\,\mathrm{d}v^2 \right)$$

As a particular example, the circular paraboloid $z = x^2 + y^2$ has fundamental forms

$$\mathrm{I} = (1 + 4u^2)\,\mathrm{d}u^2 + 8uv\,\mathrm{d}u\,\mathrm{d}v + (1 + 4v^2)\,\mathrm{d}v^2 = \mathrm{d}u^2 + \mathrm{d}v^2 + 4\big(u\,\mathrm{d}u + v\,\mathrm{d}v\big)^2$$

$$\mathrm{II} = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}} (\mathrm{d}u^2 + \mathrm{d}v^2)$$

As a sanity check, compare with the parametrization of the same paraboloid in polar co-ordinates $\mathbf{y}(r, \theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$ (Exercise 3.1.3). By computing the partial derivatives $\mathbf{y}_r, \mathbf{y}_\theta, \mathbf{y}_{rr}, \mathbf{y}_{r\theta}, \mathbf{y}_{\theta\theta}$ directly, one easily verifies that

$$\mathrm{I} = (1 + 4r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \qquad \mathrm{II} = \frac{2}{\sqrt{1 + 4r^2}} (\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$$

These expressions are identical to the originals (same orientation!) since

$$\begin{cases} \mathrm{d}u = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}v = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta \end{cases} \implies \begin{cases} \mathrm{d}u^2 + \mathrm{d}v^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \\ \big(u\,\mathrm{d}u + v\,\mathrm{d}v\big)^2 = r^2\,\mathrm{d}r^2 \end{cases}$$

Curves in Surfaces: interpreting I and $\mathrm{I\!I}$

Given a regular (oriented) surface $\mathbf{x} : U \to \mathbb{E}^3$ and a curve $c(t)$ in U, we may transfer this curve to the surface $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$. Its tangent vector (Exercise 3.1.8) and speed are then

$$\mathbf{y}'(t) = \mathrm{d}\mathbf{x}\big(c'(t)\big) \implies ||\mathbf{y}'(t)|| = \sqrt{\mathrm{d}\mathbf{x}\big(c'(t)\big) \cdot \mathrm{d}\mathbf{x}\big(c'(t)\big)} = \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}$$

We can do something similar for the second fundamental form.

Theorem 3.20. *Let $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$ parametrize a curve in a surface $\mathbf{x}$ with unit normal field $\mathbf{n}$.*

1. *If $a < b$, then the* arc-length *of $\mathbf{y}$ between $\mathbf{y}(a)$ and $\mathbf{y}(b)$ is $\int_a^b \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}\,\mathrm{d}t$.*
2. *The* normal acceleration *of the curve is $\mathbf{y}''(t) \cdot \mathbf{n} = \mathrm{I\!I}(c', c')$.*

This puts some flesh on our earlier observations (page 56). I measures *infinitesimal squared-distance* on the surface, while $\mathrm{I\!I}$ measures how the surface bends away from the normal field: recall how force/acceleration motivated the curvature κ of a curve (Definition 1.15).

Proof. 1. Arc-length is the integral of the speed $||\mathbf{y}'(t)|| = \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}$.

2. Since $\mathbf{y}'$ lies in the tangent plane, we have $\mathbf{y}' \cdot \mathbf{n} \equiv 0$. Differentiate to obtain

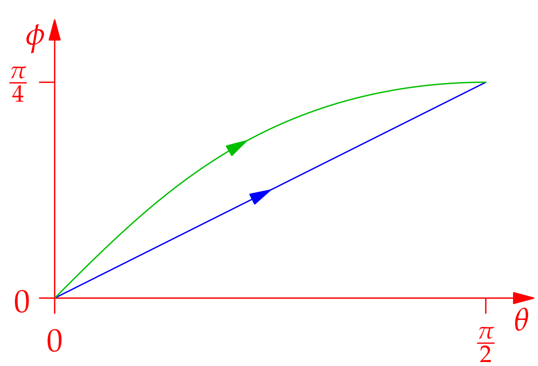
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{y}' \cdot \mathbf{n}) = \mathbf{y}'' \cdot \mathbf{n} + \mathbf{y}' \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}\big(c(t)\big) = \mathbf{y}'' \cdot \mathbf{n} + \mathrm{d}\mathbf{x}(c') \cdot \mathrm{d}\mathbf{n}(c') = \mathbf{y}'' \cdot \mathbf{n} - \mathrm{I\!I}(c', c')$$

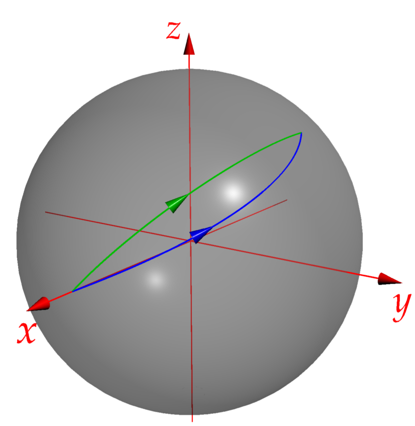
∎

Example (3.17, cont). Consider the curve $c(t) = \big(\theta(t), \phi(t)\big) = \big(2t, t\big)$ where $0 \le t \le \frac{\pi}{4}$. This has tangent field $c'(t) = 2\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}$. Translated to the unit sphere, the resulting curve has arc-length

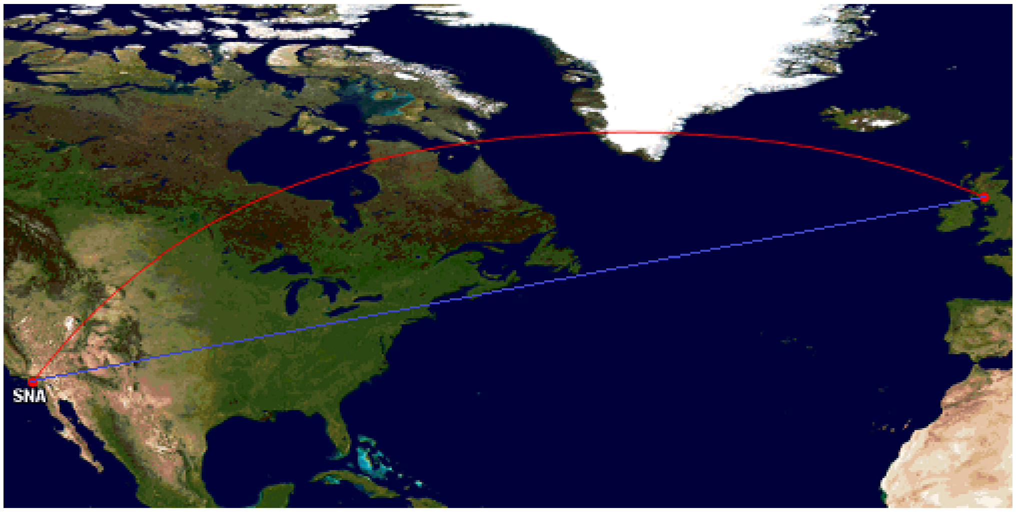
$$\int_0^{\frac{\pi}{4}} \sqrt{\mathrm{I}(c', c')}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} \sqrt{4\cos^2 t + 1}\,\mathrm{d}t \approx 1.619$$

In the parametrization space U, $c(t)$ is a straight line. The shortest path between the endpoints of the curve *on the sphere* is the great circle arc with length $\frac{2\pi}{4} = \frac{\pi}{2} \approx 1.571$; its pre-image in U *appears* longer but isn't due to the $\cos^2\phi$ factor in the first fundamental form. By spending more time at northerly latitudes, I is smaller for more of the great circle arc and the resulting arc-length is shorter.





If a map of the Earth covers a small latitude range (almost constant $\phi \approx \phi_0$), the first fundamental form is almost similar to a standard dot product $\mathrm{I} \approx (a \cos \phi_0 \, \mathrm{d}\theta)^2 + (a \, \mathrm{d}\phi)^2$. If not, say when we travel by plane, the distortion becomes much more apparent.



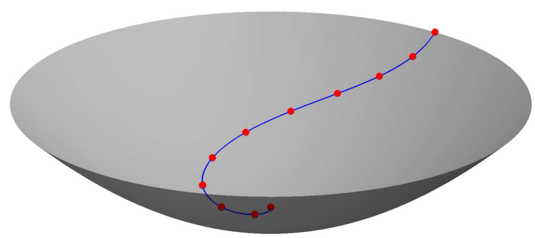
The picture shows the shortest path from Irvine (California) to Irvine (Scotland), as flown by an aircraft in ideal conditions. The straight line on the map corresponds to a *longer real-world path.*

If we travel at constant speed, it can be checked that great circles are precisely those curves whose acceleration is entirely normal to the surface. This observation, and its relation to *geodesics* (paths minimizing distance), is a matter for another course.

Example 3.21. A skater descends into a paraboloidal bowl $z = \frac{1}{2}r^2$ following the path described by $c(t) = (r(t), \theta(t)) = (1-t, 4t^2)$ in polar co-ordinates. If we parametrize the bowl in polar co-ordinates $\mathbf{x}(r,\theta) = \left(r\cos\theta, r\sin\theta, \frac{1}{2}r^2\right)$, the fundamental forms are seen to be

$$\mathrm{I} = (1+r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$$

$$\mathbb{I} = \frac{1}{\sqrt{1+r^2}}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$$

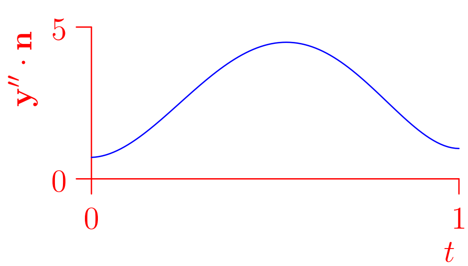


For the skater's path, $c'(t) = -\frac{\partial}{\partial r} + 8t\frac{\partial}{\partial \theta}$, whence

$$\mathrm{I}(c', c') = (1 + (1-t)^2) + 64t^2(1-t)^2$$

The path therefore has arc-length

$$\int_0^1 \sqrt{\mathrm{I}(z', z')}\,\mathrm{d}t = \int_0^1 \sqrt{1 + (64t^2+1)(1-t)^2}\,\mathrm{d}t \approx 1.82$$



and normal acceleration

$$\mathbf{y}'' \cdot \mathbf{n} = \mathbb{I}(c', c') = \frac{1}{\sqrt{1+(1-t)^2}}\left(1 + 64t^2(1-t)^2\right)$$

By Newton's second law, this is proportional to the component of the force experienced by the skater pushing perpendicularly out from the surface.

Exercises 3.2. 1. Verify the final details of Example 3.19: that is, compute I, $\mathbb{I}$ directly using the polar co-ordinate parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$.

2. Find the fundamental forms for the surface of revolution $\mathbf{x}(\theta, v) = \big(f(v)\cos\theta, f(v)\sin\theta, v\big)$.

3. Compute the first fundamental forms of each parametrized surface wherever they are regular (a, b, c are non-zero constants). Where does each parametrization fail to be regular?

   (a) Ellipsoid $\mathbf{x}(\theta,\phi) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi, c\sin\phi)$

   (b) Elliptic paraboloid $\mathbf{x}(r,\theta) = (ar\cos\theta, br\sin\theta, r^2)$

   (c) Hyperboloid of two sheets $\mathbf{x}(u,v) = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u)$

4. Calculate the fundamental forms of Enneper's surface

$$\mathbf{x}(u,v) = \big(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + vu^2, u^2 - v^2\big)$$

5. Compute $\mathrm{d}\mathbf{y}$ for the parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\big)$ of the upper unit hemisphere. Verify that the first fundamental form is the same as in Example 3.17.

6. Let $\mathbf{x}$ be the tangent developable of a unit speed biregular curve $\mathbf{y}$ (Exercise 3.1.4).

   (a) Compute the fundamental forms of $\mathbf{x}$ in terms of the curvature and torsion of $\mathbf{y}$.

   (b) If $\mathbf{y}(u) = \big(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\big)$ is the unit speed helix, show that

$$\mathrm{I} = \big(1 + \tfrac{v^2}{4}\big)\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2, \qquad \mathbb{I} = -\tfrac{v}{4}\,\mathrm{d}u^2$$

7. Prove that $\mathbb{I} \equiv 0$ if and only if $\mathbf{x}$ is (part of) a plane.

8. Parametrize the great circle in Example 3.17 (cont) by $\mathbf{z}(t) = \big(\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\sin t\big), 0 \le t \le \frac{\pi}{2}$. Verify that the arc has length $\frac{\pi}{2}$ and that the acceleration of $\mathbf{z}$ is entirely *normal*; $\mathbf{z}'' = (\mathbf{z}'' \cdot \mathbf{n})\mathbf{n}$.

9. Equip the upper half plane $y > 0$ with the abstract first fundamental form $\mathrm{I} = \frac{1}{y^2}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)$. Compare the arc-length between the points $(1,1)$ and $(-1,1)$:

   (a) Over the circular arc $c(t) = \sqrt{2}\big(\cos t, \sin t\big)$ centered at the origin.

   (b) Over the 'straight' line $y = 1$.

   This is the Poincaré half-plane model of hyperbolic space. There is neither a surface $\mathbf{x} : U \to \mathbb{E}^3$ nor a second fundamental form $\mathbb{I}$!

10. (Hard) The *torus* obtained by rotating the unit circle in the x,z-plane centered at $(2,0,0)$ around the z-axis may be parametrized

$$\mathbf{x}(u,v) = \big((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi\big), \quad (\theta,\phi) \in \mathbb{R}^2$$

   Let $k \neq 0$ be constant and consider the curve $\mathbf{y}(t) = \mathbf{x}(kt, t)$ on the torus.

   (a) Prove that $\mathbf{y}(t)$ has a self-intersection ($\exists s \neq t$ such that $\mathbf{y}(t) = \mathbf{y}(s)$) if and only if $k \in \mathbb{Q}$.

   (b) If $k \in \mathbb{Q}$, show that the curve is *periodic* in that there exists a minimum positive T for which $\mathbf{y}(t+T) = \mathbf{y}(t)$ for all t. Find T in terms of k and write down (don't evaluate!) the integral for the arc-length of the curve over one period.

3.3 Principal, Gauss & Mean Curvatures

Since I and $\mathrm{I\!I}$ are symmetric bilinear forms on each tangent space $T_p\mathbb{R}^2$, they may be expressed in matrix form: their matrices with respect to linearly independent vector fields $\vec{s}, \vec{t}$ are

$$[\mathrm{I}] := \begin{pmatrix} \mathrm{I}(\vec{s},\vec{s}) & \mathrm{I}(\vec{s},\vec{t}) \\ \mathrm{I}(\vec{s},\vec{t}) & \mathrm{I}(\vec{t},\vec{t}) \end{pmatrix} \quad \text{and} \quad [\mathrm{I\!I}] := \begin{pmatrix} \mathrm{I\!I}(\vec{s},\vec{s}) & \mathrm{I\!I}(\vec{s},\vec{t}) \\ \mathrm{I\!I}(\vec{s},\vec{t}) & \mathrm{I\!I}(\vec{t},\vec{t}) \end{pmatrix}$$

Otherwise said

$$\mathrm{I}\big(f\vec{s}+g\vec{t}, h\vec{s}+k\vec{t}\big) = \big(f\ g\big)\,[\mathrm{I}]\begin{pmatrix} h \\ k \end{pmatrix}$$

and similarly for $\mathrm{I\!I}$. Matters are simplest when these matrices are *diagonal*...

Definition 3.22. Linearly independent vector fields $\vec{s}, \vec{t}$ are said to be *orthogonal* if $\mathrm{I}(\vec{s},\vec{t}) = 0$. They additionally describe *curvature directions* if $\mathrm{I\!I}(\vec{s},\vec{t}) = 0$.
Co-ordinates u, v are *orthogonal/curvature-line* if the above apply to the the co-ordinate fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$.

In the language of Theorem 3.18, the matrices of the fundamental forms with respect to $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ are

$$A := \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} l & m \\ m & n \end{pmatrix} \tag{$*$}$$

Co-ordinates are orthogonal iff $F = \mathbf{x}_u \cdot \mathbf{x}_v \equiv 0$ (I has no $\mathrm{d}u\,\mathrm{d}v$ term), and are curvature-line iff $\mathrm{I\!I}$ is also diagonal:

$$\mathrm{I} = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2 \quad \text{and} \quad \mathrm{I\!I} = l\,\mathrm{d}u^2 + n\,\mathrm{d}v^2$$

While the meaning of *orthogonal* is clear, the reason for the term *curvature-line* will take a little work.

Examples 3.23.
1. Since the sphere of radius a has $\mathrm{I\!I} = -\frac{1}{a}\mathrm{I}$, any orthogonal co-ordinates on the sphere are curvature-line! E.g., spherical polar co-ordinates: $\mathrm{I} = a^2\big(\cos^2\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$.
2. (Example 3.2.3.19) Standard polar co-ordinates are curvature-line for the paraboloid $z = r^2$:

$$\mathrm{I} = (1+4r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \qquad \mathrm{I\!I} = \frac{2}{\sqrt{1+4r^2}}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\big)$$

3. In curvature-line co-ordinates $\mathbf{n}_u = -\frac{l}{E}\mathbf{x}_u$ and $\mathbf{n}_v = -\frac{n}{G}\mathbf{x}_v$ (see Exercise 11).

A Little Linear Algebra The existence of curvature directions is equivalent to the *simultaneous diagonalization* of both matrices ($*$). This requires an extension of the concepts of eigenvalues/vectors.

Definition 3.24. Let A, B be square matrices of the same dimension. A non-zero vector $\vec{v}$ is an *eigenvector* of B with respect to A with *eigenvalue* λ if

$$(B - \lambda A)\vec{v} = \vec{0}$$

If $A = I$ is the identity matrix, these are standard eigenvalues/vectors. We compute in the usual manner: solve the characteristic polynomial and find $\vec{v} \in \mathcal{N}(B - \lambda A)$ in the nullspace...

Example 3.25. Let $A = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 5 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 3 \end{smallmatrix}\right)$.

$$\det(B - \lambda A) = \begin{vmatrix} -2\lambda & 1-3\lambda \\ 1-3\lambda & 3-5\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1$$
$$\vec{v}_1 \in \mathcal{N}(B - A) = \mathcal{N}\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$\vec{v}_2 \in \mathcal{N}(B + A) = \mathcal{N} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Note that $\{\vec{v}_1, \vec{v}_2\} = \left\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)\right\}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of B with respect to A.

Theorem 3.26. *Let A, B be symmetric matrices of the same dimension, with A* positive-definite.[18]

1. *There exists a basis of eigenvectors of B with respect to A. Moreover, all eigenvalues are* real.
2. *If $\vec{s}, \vec{t}$ are eigenvectors corresponding to distinct eigenvalues, then $\vec{s}^T A\vec{t} = 0 = \vec{s}^T B\vec{t}$.*

Proof. 1. This follows from the famous *spectral theorem* in linear algebra.[19]

2. Assume $B\vec{s} = k_1 A\vec{s}$ and $B\vec{t} = k_2 A\vec{t}$ where $k_1 \neq k_2$, and apply the symmetry of A and B,

$$\left.\begin{array}{l} \vec{s}^T B\vec{t} = \vec{s}^T(k_2 A\vec{t}) = k_2 \vec{s}^T A\vec{t} \\ \quad \| \\ \vec{t}^T B\vec{s} = \vec{t}^T(k_1 A\vec{s}) = k_1 \vec{t}^T A\vec{s} = k_1 \vec{s}^T A\vec{t} \end{array}\right\} \implies (k_2 - k_1)\vec{s}^T A\vec{t} = 0 \implies \vec{s}^T A\vec{t} = 0$$

■

Application to Regular Surfaces With respect to independent vector fields, the matrices A, B of I, II are symmetric. Moreover, the regularity of $\mathbf{x}$ guarantees the positive-definiteness of A:

$$\forall \vec{w} \neq \vec{0} \implies \mathrm{I}(\vec{w}, \vec{w}) = d\mathbf{x}(\vec{w}) \cdot d\mathbf{x}(\vec{w}) = ||d\mathbf{x}(\vec{w})||^2 > 0$$

We may therefore apply Theorem 3.26 to the matrices of the fundamental forms.

Definition 3.27. The *principal curvatures* $k_1, k_2 : U \to \mathbb{R}$ of an oriented surface $\mathbf{x} : U \to \mathbb{E}^3$ are the eigenvalues of II with respect to I. Corresponding eigenvectors are *curvature directions.*
The *Gauss* and *mean curvatures* are, respectively, $K := k_1 k_2$ and $H := \frac{1}{2}(k_1 + k_2)$.
A point $\mathbf{x}(p)$ is *umbilic* if $k_1(p) = k_2(p)$.

[18] For all non-zero vectors, $\vec{v}^T A\vec{v} > 0$. Equivalently, all eigenvalues of A are positive. This means that $\langle \vec{v}, \vec{w} \rangle := \vec{v}^T A\vec{w}$ defines an *inner product* on $\mathbb{R}^n$. In Example 3.25, A has eigenvalues $\frac{1}{2}(7 \pm \sqrt{45}) > 0$.

[19] In case you're interested: A has an orthogonal eigenbasis $\{\vec{x}_1, \ldots, \vec{x}_n\}$ by the spectral theorem. Since its eigenvalues μ_i are positive, we may scale such that $||\vec{x}_i||^2 = \frac{1}{\mu_i}$. Let $X = (\vec{x}_1 \ \cdots \ \vec{x}_n)$ so that $X^T A X = I$ is the identity matrix. But then,

$$\det(B - \lambda A) = \det(X^T)^{-1} \det\big(X^T B X - \lambda I\big) \det(X^{-1}) = 0 \iff \det(X^T B X - \lambda I) = 0$$

Since $X^T B X$ is symmetric (spectral theorem again), it has an orthogonal eigenbasis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ and real eigenvalues λ_k. Each $\vec{v}_k := X\vec{w}_k$ is an eigenvector of B with respect to A with eigenvalue λ_k. Since X is invertible, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis.

The curvatures are independent of *oriented* co-ordinate changes. If we reverse orientation, then k_1, k_2 and H change sign, while $K = k_1 k_2$ is unchanged.

At non-umbilic points, Theorem 3.26 says that curvature directions diagonalize both fundamental forms, in line with Definition 3.22.

At umbilic points, $\mathrm{II} = k\mathrm{I}$ and all directions are curvature directions; any orthogonal directions necessarily diagonalize both fundamental forms.

Example 3.28. Here are two *totally umbilic* surfaces where the curvatures are constant.

1. A plane: $\mathrm{II} \equiv 0 \implies$ all curvatures are zero.
2. A sphere of radius a: $\mathrm{II} = -\frac{1}{a}\mathrm{I} \implies k_1 = k_2 = -\frac{1}{a}$, $K = \frac{1}{a^2}$ and $H = -\frac{1}{a}$.

In fact these comprise all totally umbilic surfaces (see Exercise 12).

Theorem 3.29. 1. *In co-ordinates, the Gauss and mean curvatures are given by*

$$K = \frac{ln - m^2}{EG - F^2} = \frac{\det B}{\det A} = \det(A^{-1}B) \quad \textit{and} \quad H = \frac{lG + nE - 2mF}{2(EG - F^2)} = \frac{1}{2}\operatorname{tr} A^{-1}B$$

2. *At non-umbilic points, the curvatures k_1, k_2, K, H are smooth functions and the curvature directions may be described locally by (smooth) vector fields.*

Proof. 1. The principal curvatures are the solutions to the quadratic equation

$$\det\left(\begin{pmatrix} l & m \\ m & n \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = (EG - F^2)\lambda^2 - (lG + nE - 2mF)\lambda + (ln - m^2)$$

of which K and H are the product and half the sum of the roots.

2. The roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of a quadratic are smooth functions of the coefficients unless $b^2 - 4ac = 0$, in which case we have a repeated root ($k_1 = k_2$). At non-umbilic points, each eigenspace is one-dimensional, so there is no obstruction to choosing smooth eigenvectors.[20]

∎

Examples 3.30. 1. (Example 3.19) For the paraboloid $\mathbf{x}(r, \theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$, standard polar co-ordinates are curvature-line:

$$A = [\mathrm{I}] = \begin{pmatrix} 1 + 4r^2 & 0 \\ 0 & r^2 \end{pmatrix} \qquad B = [\mathrm{II}] = \begin{pmatrix} \frac{2}{\sqrt{1+4r^2}} & 0 \\ 0 & \frac{2r^2}{\sqrt{1+4r^2}} \end{pmatrix}$$

The curvatures are therefore

$$k_1 = \frac{2}{(1 + 4r^2)^{3/2}}, \quad k_2 = \frac{2}{\sqrt{1 + 4r^2}}, \quad K = \frac{4}{(1 + 4r^2)^2}, \quad H = \frac{2 + 4r^2}{(1 + 4r^2)^{3/2}}$$

The curvatures make sense at the single umbilic point ($r = 0$), but the co-ordinates are not curvature-line there since the parametrization fails to be regular ($\mathbf{x}_\theta(0, \theta) = \mathbf{0}$).

[20] At an umbilic point $\mathbf{x}(p)$, the eigenspace is 2-dimensional so $\lim_{q \to p} \vec{v}(q)$ need not exist and $\vec{v}$ need not be continuous.

2. Parametrize a graph $z = f(x,y)$ in the usual manner $\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$. Then

$$A = [\mathrm{I}] = \begin{pmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{pmatrix} \qquad B = [\mathrm{II}] = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

Theorem 3.29 tells us that

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} \qquad H = \frac{f_{vv}(1+f_u^2) + f_{uu}(1+f_v^2) - 2f_u f_v f_{uv}}{2(1+f_u^2+f_v^2)^{3/2}}$$

In the abstract, solving for the curvatures and directions is disgusting. As a sanity check, you should verify that $f(u,v) = u^2 + v^2$ recovers exactly the curvatures in the previous example!

3. (Exercise 3.2.6) The tangent developable of the unit-speed helix has

$$A = [\mathrm{I}] = \begin{pmatrix} 1+\frac{v^2}{4} & 1 \\ 1 & 1 \end{pmatrix} \qquad B = [\mathrm{II}] = \begin{pmatrix} -\frac{v}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

Now solve for the curvatures:

$$\begin{vmatrix} -\frac{v}{4} - \lambda\left(1+\frac{v^2}{4}\right) & -\lambda \\ -\lambda & -\lambda \end{vmatrix} = \frac{v^2}{4}\lambda^2 + \frac{v}{4}\lambda = 0 \implies k_1 = 0,\ k_2 = -\frac{1}{v}, \quad K = 0,\ H = -\frac{1}{2v}$$

In this case an explicit computation of the curvature directions is not difficult:

$$k_1 = 0 \implies \mathcal{N}(B - k_1 A) = \mathcal{N}\begin{pmatrix} -\frac{v}{4} & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \vec{s} = \frac{\partial}{\partial v}$$
$$k_2 = -\frac{1}{v} \implies \mathcal{N}(B - k_2 A) = \mathcal{N}\begin{pmatrix} \frac{1}{v} & \frac{1}{v} \\ \frac{1}{v} & \frac{1}{v} \end{pmatrix} = \mathrm{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightsquigarrow \vec{t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$

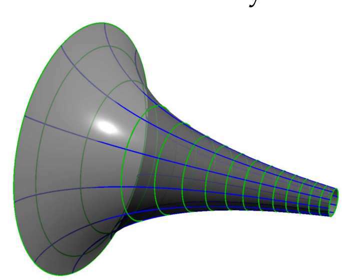
where we made the natural choice of vector fields $\vec{s}, \vec{t}$. As a sanity check, here are the matrices of the fundamental forms with respect to $\vec{s}, \vec{t}$:

$$\mathrm{I}(\vec{s},\vec{s}) = (0\ 1)A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \ \ldots \implies [\mathrm{I}] = \begin{pmatrix} \mathrm{I}(\vec{s},\vec{s}) & \mathrm{I}(\vec{s},\vec{t}) \\ \mathrm{I}(\vec{s},\vec{t}) & \mathrm{I}(\vec{t},\vec{t}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{v^2}{4} \end{pmatrix} \qquad [\mathrm{II}] = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{v}{4} \end{pmatrix}$$

in which the principal curvatures are clearly visible: $1k_1 = 0$, $\frac{v^2}{4}k_2 = -\frac{v}{4}$

Constant Gauss & Mean Curvature Surfaces

Minimal Surfaces $H \equiv 0$: Among all surfaces whose boundary is a given closed curve, a surface with minimal surface area has $H \equiv 0$. This is the shape made by a soap film whose boundary is the curve: it minimizes the 'total tension' of the soap film. More generally, *constant mean curvature* (CMC) surfaces model soap *bubbles.*



Constant Gauss Curvature Surfaces: We've see that planes, cones and cylinders have $K = 0$, and that spheres have constant positive Gauss curvature. A *pseudosphere* with constant $K = -1$ is shown in the picture.

Existence of (Curvature-Line) Co-ordinates

At non-umbilic points, Theorems 3.26 and 3.29 tell us how to find curvature *directions* as vector fields $\vec{s}, \vec{t}$. Unfortunately, being able to compute *explicit* curvature *co-ordinates* is exceptionally unlikely.

Example (3.30.3 cont). Recall that we chose curvature direction fields $\vec{s} = \frac{\partial}{\partial v}$ and $\vec{t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$. By inspection, the functions $s = u + v$ and $t = u$ satisfy the required equations:

$$\vec{s}[s] = 1 = \vec{t}[t], \qquad \vec{s}[t] = 0 = \vec{t}[s] \tag{$*$}$$

It follows that $\vec{s} = \frac{\partial}{\partial s}$ and $\vec{t} = \frac{\partial}{\partial t}$ for curvature-line co-ordinates s, t, as you can easily verify using the chain rule. Indeed

$$\mathrm{I} = \frac{v^2}{4}\,\mathrm{d}u^2 + \mathrm{d}(u+v)^2 = \mathrm{d}s^2 + \frac{v^2}{4}\,\mathrm{d}t^2, \qquad \mathrm{I\!I} = 0\,\mathrm{d}s^2 - \frac{v}{4}\,\mathrm{d}t^2$$

so that the co-ordinates really do diagonalize both fundamental forms.

The simple reason the example is so unlikely is that *mixed partial derivatives must commute*: if $\vec{s} = \frac{\partial}{\partial s}$ and $\vec{t} = \frac{\partial}{\partial t}$ are co-ordinate fields ($\exists s, t : U \to \mathbb{R}$), then their *Lie bracket* (Exercise 2.3.10) vanishes:

$$[\vec{s}, \vec{t}] = \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = \frac{\partial}{\partial s} \circ \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \circ \frac{\partial}{\partial s} = 0$$

The astonishing fact is that this simple condition is *locally sufficient*.

Theorem 3.31 (Co-ordinate fields). *Let $\vec{s}, \vec{t}$ be linearly independent vector fields on $U \subseteq \mathbb{R}^2$.*

1. *If there exist functions $s, t : U \to \mathbb{R}$ such that $\vec{s} = \frac{\partial}{\partial s}, \vec{t} = \frac{\partial}{\partial t}$, then $[\vec{s}, \vec{t}] = 0$.*
2. *Suppose $[\vec{s}, \vec{t}] = 0$ and let $p \in U$. Then there exists a neighborhood V of p and co-ordinate functions $s, t : V \to \mathbb{R}$ for which $\vec{s} = \frac{\partial}{\partial s}, \vec{t} = \frac{\partial}{\partial t}$.*

Examples 3.32. 1. The fields $\vec{s} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\vec{t} = \frac{\partial}{\partial y}$ do not arise simultaneously from co-ordinates:

$$[\vec{s}, \vec{t}] = \frac{\partial^2}{\partial x \partial y} + y\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y \partial x} - \frac{\partial}{\partial y} - y\frac{\partial^2}{\partial y^2} = -\frac{\partial}{\partial y} \neq 0$$

2. The cylindrical paraboloid $\mathbf{x}(u, v) = (u, v, \frac{1}{2}u^2 + v)$ has curvatures and curvature directions

$$k_1 = 0,\ \vec{s} = \frac{\partial}{\partial v}, \qquad k_2 = \frac{2}{[2+u^2]^{3/2}},\ \vec{t} = 2\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$$

The Lie bracket condition $[\vec{s}, \vec{t}] = 0$ is satisfied, so co-ordinates s, t corresponding to these fields must exist. You can try to find such by inspection, though simultaneously solving ($*$) is messy. Alternatively, following the proof of part 2 (Exercise 13), observe that the dual 1-forms are

$$\alpha = \frac{1}{2}u\,\mathrm{d}u + \mathrm{d}v, \qquad \beta = \frac{1}{2}\,\mathrm{d}u \qquad (\alpha(\vec{s}) = \beta(\vec{t}) = 1,\ \alpha(\vec{t}) = \beta(\vec{s}) = 0)$$

These forms are *exact*: $\alpha = \mathrm{d}(\frac{1}{4}u^2 + v)$ and $\beta = \mathrm{d}(\frac{1}{2}u)$. We therefore conclude that $s = \frac{1}{4}u^2 + v$ and $t = \frac{1}{2}u$ are suitable curvature-line co-ordinates.

The Lie bracket condition says that explicit co-ordinates corresponding to given vector fields are very unlikely to exist. This is no matter: we typically only require co-ordinates s, t whose fields $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ are *parallel* to $\vec{s}, \vec{t}$: that is

$$\frac{\partial}{\partial s} = f\vec{s} \quad \text{and} \quad \frac{\partial}{\partial t} = g\vec{t} \quad \text{for some functions } f, g \qquad\qquad (\text{equivalently } \vec{s}[t] = 0 = \vec{t}[s])$$

Such co-ordinates indeed exist, though only *locally*, as shown by one of the most important foundational results in differential geometry.

Theorem 3.33 (Frobenius). *Let $\vec{s}, \vec{t}$ be linearly independent vector fields on a domain U. Then there exist local co-ordinates s, t whose co-ordinate fields are parallel to $\vec{s}, \vec{t}$.*

In particular, if $\mathbf{x}(p)$ is a non-umbilic point on a surface $\mathbf{x} : U \to \mathbb{E}^3$, then there exists a neighborhood V of p and curvature-line co-ordinates s, t on V.

Frobenius' theorem comes in many guises and generalizes to higher dimensions, taking the place of Picard's ODE existence/uniqueness theorem (1.39) for particular classes of PDE. Its proof is too involved for us, though the informal idea is to search for functions f, g such that $[f\vec{s}, g\vec{t}] = 0$, a lengthy process that indeed depends on Picard's theorem.

Exercises 3.3.

1. Find the eigenvalues of $B = \left(\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ with respect to $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. If $\vec{s}, \vec{t}$ are corresponding eigenvectors, verify that $\vec{s}^T A\vec{t} = 0 = \vec{s}^T B\vec{t}$.

2. Parametrize the graph of $x = z^2$; compute I, II and the principal, Gauss and mean curvatures.

3. Use Theorem 3.29 to find the Gauss and mean curvatures of the graph of $y = x^2 - z^2$.

4. Show that Enneper's surface (Exercise 3.2.4) is minimal.

5. Let $\mathbf{x}(u, v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ be the tangent developable of a unit speed biregular curve $\mathbf{y}$.

   (a) Find the principal curvatures, Gauss and mean curvatures of $\mathbf{x}$.

   (b) Compute the curvature directions and find curvature line co-ordinates.

   (*This is very similar to Example 3.30.3 - keep track of the changes!*)

6. With respect to some co-ordinates u, v, suppose that a surface has fundamental forms

   $$\mathrm{I} = u^2\,\mathrm{d}u^2 + v^2\,\mathrm{d}v^2, \qquad \mathrm{II} = u^2\,\mathrm{d}u^2 + 2uv\,\mathrm{d}u\,\mathrm{d}v + v^2\,\mathrm{d}v^2$$

   (a) Show that the principal curvatures are constant: $k_1 = 0$ and $k_2 = 2$.

   (b) Show that $\vec{s} = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$ and $\vec{t} = v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$ are curvature directions.

   (c) Compute the Lie bracket $[\vec{s}, \vec{t}]$ to show that these are *not* vector fields with respect to some curvature-line co-ordinates s, t.

   (d) Find explicit curvature-line co-ordinates for the surface; functions s, t such that $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ are *parallel* to $\vec{s}, \vec{t}$ and express I, II with respect to s, t.
   (*Hint: try to guess solutions to* $\vec{s}[t] = 0 = \vec{t}[s]$)

7. Rotate $y = f(x)$ around the x-axis and parametrize the surface via

$$\mathbf{x}(\phi, v) = \big(v, f(v)\cos\phi, f(v)\sin\phi\big)$$

   (a) Verify that the co-ordinates ϕ, v are curvature-line, compute the principal curvatures, and show that the Gauss and mean curvatures are

$$K = -\frac{f''(v)}{f(v)\big(1+f'(v)^2\big)^2} \qquad H = \frac{f(v)f''(v) - 1 - f'(v)^2}{2f(v)\big(1+f'(v)^2\big)^{3/2}}$$

   (b) Demonstrate the following by choosing suitable $f(v)$:
      i. A cylinder has $K = 0$;
      ii. A cone has $K = 0$;
      iii. A sphere of radius a has $K = \frac{1}{a^2}$.
      iv. A *catenoid* $f(v) = a^{-1}\cosh(av - c)$ is a minimal surface.

   (c) Suppose $\mathbf{x}$ is a minimal surface $H \equiv 0$. By writing $g(v) = 1 + \big(f'(v)\big)^2$, show that

$$1 + f'^2 = g = a^2 f^2 \quad \text{for some constant } a$$

   By substituting $f(v) = a^{-1}\cosh\big(ah(v)\big)$, show that the surface is a *catenoid.*

   (d) Plainly $K \equiv 0$ if and only if $f''(v) \equiv 0$. What are these surfaces? More generally, if the surface has constant non-zero Gauss curvature K, show that f satisfies a non-linear ODE

$$Kf^2 = (1 + f'^2)^{-1} + c \quad \text{for some constant } c$$

   (*Solving for f requires an elliptic integral when $c \neq 0$, so don't try!*)

8. The *tractrix* is parametrized by $\mathbf{y}(t) = \big(\sinh^{-1} t - t(1+t^2)^{-1/2}, (1+t^2)^{-1/2}\big)$. By revolving this curve around the x-axis, show that the resulting surface is a pseudosphere with $K \equiv -1$.

9. We know that the Gauss and mean curvature are defined in terms of the principal curvatures. By writing down a suitable quadratic polynomial, prove that knowing of H, K is sufficient to recover the principal curvatures.

10. The graph of a function $z = f(x, y)$ is parametrized by $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$. What can you say about the surface if (u, v) are curvature-line co-ordinates?

    (*Hint: recall Example 3.19*)

11. Suppose u, v are curvature-line co-ordinates for a surface $\mathbf{x}$. Explain why $\mathbf{n}_u = -k_1\mathbf{x}_u$ and $\mathbf{n}_v = -k_2\mathbf{x}_v$.

12. Suppose that a surface $\mathbf{x} : U \to \mathbb{E}^3$ is totally umbilic $\mathrm{I\!I} = k\mathrm{I}$ for some function $k : U \to \mathbb{R}$.

    (a) Use Exercise 11 and $\mathbf{n}_{uv} = \mathbf{n}_{vu}$ to prove that k is constant.
    (b) Prove that $\mathbf{x}$ is (part of) a plane or a sphere (recall Example 3.28).
    (*Hint: If $k \neq 0$ consider $\mathbf{c} := \mathbf{x} + \frac{1}{k}\mathbf{n} \ldots$*)

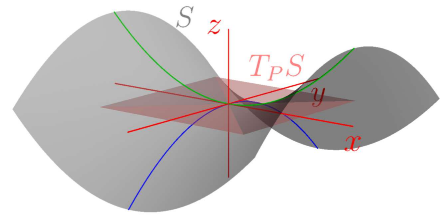
13. We prove part 2 of Theorem 3.31. Given the assumptions, define the *dual 1-forms* to $\vec{s}, \vec{t}$:

$$\alpha(\vec{s}) = 1 = \beta(\vec{t}) \quad \text{and} \quad \alpha(\vec{t}) = 0 = \beta(\vec{s})$$

    Use Exercise 2.3.10 to prove that $\mathrm{d}\alpha = 0 = \mathrm{d}\beta$. Hence conclude (footnote, page 43) that (locally) $\alpha = \mathrm{d}s$ and $\beta = \mathrm{d}t$ for some functions s, t.

3.4 Power Series Expansions and Euler's Theorem

In this section we intersect a surface with certain planes and consider the resulting curves. The curvatures provide data about these curves and thus tell us something about the local shape of the surface. The key is to see how curvatures describe a quadratic approximation to a surface.



At a regular point P on a surface S, choose axes such that P is the origin and the (x,y)-plane is tangent[21] to S. By Theorem 3.10, S is locally the graph of a function $z = f(x,y)$, which we may parametrize in the usual manner

$$\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$$

The unit normal vector $\mathbf{n}_P = \mathbf{k}$ is therefore the standard vertical basis vector. Since the tangent plane at P is the (x,y)-plane, we see that $f_u(0,0) = 0 = f_v(0,0)$; substituting into Example 3.19 yields the fundamental forms *at* P:

$$\begin{aligned} \mathrm{I}_P &= \mathrm{d}u^2 + \mathrm{d}v^2 \\ \mathbb{I}_P &= f_{uu}\,\mathrm{d}u^2 + 2f_{uv}\,\mathrm{d}u\mathrm{d}v + f_{uv}\,\mathrm{d}v^2 \end{aligned} \qquad [\mathrm{I}]_P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\mathbb{I}]_P = \operatorname{Hess} f = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

The last matrix is the *Hessian* of f, and the Gauss and mean curvatures at P are

$$K(P) = \det \operatorname{Hess} f(0,0) \quad \text{and} \quad H(P) = \frac{1}{2} \operatorname{tr} \operatorname{Hess} f(0,0)$$

It bears repeating that these expressions are only valid *at the origin* $O \in U$ (equivalently $P \in S$). Although the co-ordinates u, v will extend nearby on the surface, the first fundamental form need not be diagonal anywhere except at the origin.

Now suppose we rotate the (x,y)-plane so that the axes point in the principal directions. Then the Hessian is also diagonal ($f_{uv}(0,0) = 0$) and the principal curvatures at P are

$$k_1 = f_{uu}(0,0) \quad \text{and} \quad k_2 = f_{vv}(0,0)$$

Theorem 3.34. *If the graph of $z = f(x,y)$ is tangent to the (x,y)-plane at the origin O so that the axes are the curvature directions, then the Maclaurin approximation of the function $f(x,y)$ is*

$$\begin{aligned} f(x,y) &\approx f(O) + (x\ y)\ \nabla f\big|_O + \frac{1}{2}(x\ y) \operatorname{Hess} f(O) \begin{pmatrix} x \\ y \end{pmatrix} + \textit{ higher order terms} \\ &= \frac{1}{2}k_1(O)x^2 + \frac{1}{2}k_2(O)y^2 + \textit{ higher order terms} \end{aligned}$$

Example 3.35. Let $f(x,y) = x^2 - y^2$ (above picture). *At the origin,* $\mathbf{x}(u,v) = \big(u, v, u^2 - v^2\big)$ has

$$\mathrm{I} = \mathrm{d}u^2 + \mathrm{d}v^2, \quad \mathbb{I} = 2(\mathrm{d}u^2 - \mathrm{d}v^2), \quad k_1 = 2, \quad k_2 = -2, \quad K = -4, \quad H = 0$$

In this case the Maclaurin approximation is exact!

$$\frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2 = x^2 - y^2 = f(x,y)$$

[21] This amounts to applying a rigid motion (direct isometry) to the surface, which does nothing to the fundamental forms.

Level Curves: intersections with planes parallel to the tangent plane

If c is small, then the intersection of S with a plane $c\mathbf{n}_P + T_PS$ parallel to the tangent plane is a *level curve*; in our analysis, they correspond to level curves $f(x, y) = \text{constant}$. Theorem 3.34 tells us how level curves depend on the curvatures. For instance, if k_1, k_2 have opposite signs, then for small c,

$$k_1x^2 + k_2y^2 \approx 2c$$

is approximately a *hyperbola*.

Definition 3.36. Suppose k_1, k_2, K, H are the curvatures of a surface S at a point P. We say that P is:

Elliptic $\iff K > 0 \iff k_1, k_2 \neq 0$ and have the same sign.
Level curves near P are approximately *ellipses.*

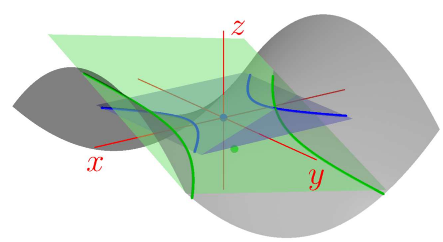
Hyperbolic $\iff K < 0 \iff k_1, k_2 \neq 0$ and have opposite signs.
Level curves near P are approximately *hyperbolæ.*

Parabolic $\iff K = 0$ and $H \neq 0 \iff$ exactly one of k_1, k_2 is zero.
Level curves near P are approximately a pair of *parallel lines*, e.g. $x = \pm c$.

Planar $\iff K = H = 0 \iff k_1 = k_2 = 0$.
The curvatures provide no data as to the level curves near P.

Example (3.35, mk. II). For the graph of $z = x^2 - y^2$, the level curve $x^2 - y^2 = c \neq 0$ is a hyperbola. In fact this is true everywhere on this surface: under the usual parametrization $\mathbf{x}(u, v) = \big(u, v, u^2 - v^2\big)$, we have

$$K = -\frac{4}{(1 + 4u^2 + 4v^2)^2} \quad \text{and} \quad H = \frac{4(v^2 - u^2)}{(1 + 4u^2 + 4v^2)^{3/2}}$$



Since $K < 0$ everywhere, all points are hyperbolic.

In the picture, shifted tangent planes $c\mathbf{n}_P + T_PS$ and their intersections with the surface are drawn for two points. In both cases the level curves are genuine hyperbolæ.

Normal Curvature: intersections with planes containing the normal vector

Theorem 3.34 is the surface analogy of Exercise 1.6.5: a regular curve in $\mathbb{E}^2$ passing through the origin horizontally at $t = 0$ has its graph given locally by

$$y = \frac{1}{2}\kappa(0)x^2 + \text{higher order terms} \tag{$*$}$$

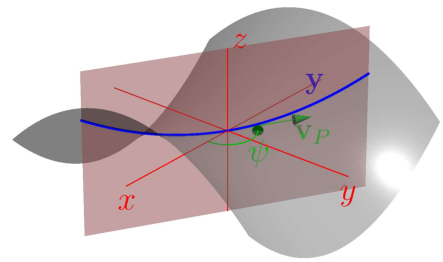
We put this to work by considering the curvature of curves passing through a point on a surface.

Definition 3.37. Let S be a surface and $\mathbf{v}_P \in T_PS$ a non-zero tangent vector.
The *normal curvature* $\nu(\mathbf{v}_P)$ is the curvature at P of the curve[22] defined by the intersection of the surface S and the normal plane $\operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$.
We say that $\mathbf{v}_P$ is *asymptotic* if $\nu(\mathbf{v}_P) = 0$.

[22]Strictly, the curve is the *connected component* of $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$ containing P.

Example (3.35, mk. III). Consider the hyperbolic paraboloid $z = x^2 - y^2$ at the origin $P = O$. Fix an angle ψ and let $\mathbf{v}_P = (\cos\psi, \sin\psi)$. The intersection curve $\mathbf{y} \subseteq S \cap \operatorname{Span}\{\mathbf{v}_O, \mathbf{n}_O\}$ may be parametrized using polar co-ordinates:

$$\mathbf{y}(r) = \big(r\cos\psi, r\sin\psi, r^2(\cos^2\psi - \sin^2\psi)\big)$$



which amounts to the graph of the function $g(r) = r^2\cos 2\psi$.
The normal curvature is the curvature at $r = 0$ of this curve:

$$\nu(\mathbf{v}_O) = \kappa(0) = \frac{g''(0)}{[1 + g'(0)^2]^{3/2}} = 2\cos 2\psi$$

Think about how the this corresponds to the picture and observe that

$$\mathbf{v}_O \text{ is asymptotic} \iff \cos 2\psi = 0 \iff \psi = \pm\tfrac{\pi}{4}$$

Our next result generalizes the method in the example.

Theorem 3.38 (Euler). *Suppose $\mathbf{v}_P$ makes angle ψ with the first principal curvature direction. Then*

$$\nu(\mathbf{v}_P) = k_1\cos^2\psi + k_2\sin^2\psi$$

In particular, the principal curvatures are the extremes of normal curvature: if $k_1 \le k_2$, then

$$k_1 \le \nu(\mathbf{v}_P) \le k_2$$

where the bounds are realized precisely when $\mathbf{v}_P$ points in a curvature direction.

Proof. Choose axes so the curvature directions at P are $\mathbf{i}, \mathbf{j}$, and $\mathbf{n}_P = \mathbf{k}$. The surface is locally a graph $z = f(x,y)$. If (r,ψ) are polar co-ordinates in the (x,y)-plane, Theorem 3.34 says that

$$z = f(x,y) \approx \tfrac{1}{2}k_1(r\cos\psi)^2 + \tfrac{1}{2}k_2(r\sin\psi)^2 + \cdots = \frac{1}{2}(k_1\cos^2\psi + k_2\sin^2\psi)r^2 + \cdots$$

Fix ψ and let $\mathbf{v}_P = \left(\begin{smallmatrix}\cos\psi\\ \sin\psi\end{smallmatrix}\right)$ (assume unit length since only the direction matters). Our curve of interest $\mathbf{y} \subseteq S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$ may be parametrized

$$\mathbf{y}(r) = r\mathbf{v}_P + f(r\cos\psi, r\sin\psi)\,\mathbf{n}_P = \begin{pmatrix} r\cos\psi \\ r\sin\psi \\ f(r\cos\psi, r\sin\psi)\end{pmatrix} = \begin{pmatrix} r\cos\psi \\ r\sin\psi \\ \frac{1}{2}\nu r^2 + \cdots\end{pmatrix}$$

The last equality used observation $(*)$, where ν is the normal curvature. For the first result, simply compare the z-expressions in the displayed equations. For the final observation, note that

$$\nu(\mathbf{v}_P) = k_1(1 - \sin^2\psi) + k_2\sin^2\psi = k_1 + (k_2 - k_1)\sin^2\psi \in [k_1, k_2]$$

and that the bounds are achieved precisely when $\psi = 0, \frac{\pi}{2}$, when $\mathbf{v}_P$ is a curvature direction. ∎

Examples 3.39. 1. If P is a planar point ($k_1 = k_2 = 0$), all normal curvatures are zero and all directions are asymptotic.

2. (Example 3.30.1) All points of the paraboloid $z = r^2$ are elliptic (everywhere $k_1, k_2 > 0$). The surface has no asymptotic directions at any point, indeed the normal curvature in the direction $\mathbf{v}_P = (\cos\psi, \sin\psi)$ at $P = (r\cos\theta, r\sin\theta, r^2)$ is

$$\nu(\mathbf{v}_P) = k_1 \cos^2\psi + k_2 \sin^2\psi = \frac{2}{(1+4r^2)^{3/2}}\left[\cos^2\psi + (1+4r^2)\sin^2\psi\right] > 0$$

3. If $k_2 \neq 0$, then $\mathbf{v}_P = \left(\begin{smallmatrix}\cos\psi \\ \sin\psi\end{smallmatrix}\right)$ is asymptotic $\iff$ $\tan\psi = \pm\sqrt{-\frac{k_1}{k_2}}$.

The Second Fundamental Form and the Local Shape of a Surface

Our standard approach is to transfer calculations about surfaces back to the parametrization space. With this in mind, we consider special tangent vectors with respect to the second fundamental form.

Definition 3.40. Let $\mathbf{x} : U \to \mathbb{E}^3$ be an oriented surface and $\vec{w}_p \in T_p\mathbb{R}^2$.

1. A tangent vector $\vec{w}_p \neq \vec{0}$ is *asymptotic* if $\mathrm{II}(\vec{w}_p, \vec{w}_p) = 0$.

2. The *Dupin indicatrix* at $p \in U$ is the set of tangent vectors $\vec{w}_p$ such that $\mathrm{II}(\vec{w}_p, \vec{w}_p) = \pm 1$.

Theorem 3.41. *The notions of* asymptotic *in Definitions 3.37 & 3.40 coincide:*

$$\mathbf{v}_P = \mathrm{d}\mathbf{x}(\vec{w}_p) \in T_PS \text{ is asymptotic} \iff \vec{w}_p \in T_p\mathbb{R}^2 \text{ is asymptotic}$$

The proof is an exercise. Recalling Theorem 3.20, a direction is asymptotic if and only if the *normal acceleration* in said direction is zero.

The Dupin indicatrix turns out to precisely describe level curves near a point. To see this, write $\vec{w}_p = a\vec{s}_p + b\vec{t}_p$ where $\vec{s}_p, \vec{t}_p$ are orthonormal curvature directions;[23] the indicatrix at p has equation

$$\mathrm{II}(\vec{w}_p, \vec{w}_p) = (a\ b)\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = k_1 a^2 + k_2 b^2 = \pm 1$$

This defines a conic in the tangent space $T_p\mathbb{R}^2$ whose type depends on the signs of the principal curvatures. In essence, the Dupin indicatrix indicates the level curve obtained by taking the intersection $S \cap (c\mathbf{n}_P + T_PS)$ for *infinitesimal* c. We summarize all possibilities in a table using the point-types introduced in Definition 3.36:

type of point	# asymptotic directions	Dupin indicatrix
elliptic	0	ellipse
hyperbolic	2	two hyperbolæ
parabolic	1	two parallel lines
planar	∞	empty

[23] With respect to $\vec{s}_p, \vec{t}_p$, the matrices of the fundamental forms at p are $[\mathrm{I}_p] = \left(\begin{smallmatrix}1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ and $[\mathrm{II}_p] = \left(\begin{smallmatrix}k_1 & 0 \\ 0 & k_2\end{smallmatrix}\right)$

Examples 3.42. For a parametrized surface $\mathbf{x}$ at a given point $p = (u_0, v_0)$, write $\vec{w}_p = a\,\frac{\partial}{\partial u}\Big|_p + b\,\frac{\partial}{\partial v}\Big|_p$.

1. (Exercise 3.2.6) The tangent developable of the unit-speed helix has

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = -\frac{v_0}{4}\,\mathrm{d}u^2(\vec{w}_p, \vec{w}_p) = -\frac{v_0}{4}a^2$$

The single asymptotic direction is $\vec{w}_p = \frac{\partial}{\partial v}\Big|_p$. The Dupin indicatrix is a pair of parallel lines

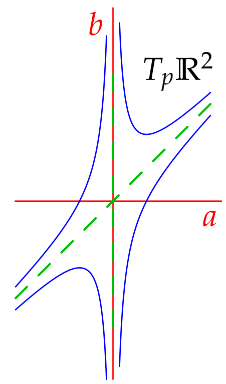
$$-\frac{v_0}{4}a^2 = \pm 1 \implies \vec{w}_p = \pm\frac{2}{\sqrt{|v_0|}}\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p \qquad (b \text{ is arbitrary!})$$

2. In its usual parametrization, the surface $z = x^2 y$ has

$$\mathbb{I} = \frac{2}{\sqrt{1 + 4u^2v^2 + u^4}}\left(v\,\mathrm{d}u^2 + 2u\,\mathrm{d}u\mathrm{d}v\right)$$

At $p = (-1, 2)$ (i.e., $\mathbf{x}(p) = (-1, 2, 2)$) we see that

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = \frac{2}{\sqrt{18}}\left(2a^2 - 2ab\right) = \frac{2\sqrt{2}}{3}a(a-b)$$



The point is hyperbolic with asymptotic directions $\frac{\partial}{\partial v}\Big|_p$ and $\frac{\partial}{\partial u}\Big|_p - \frac{\partial}{\partial v}\Big|_p$ ($a = 0$ and $a - b = 0$). The indicatrix comprises two hyperbolæ $a(a-b) = \pm\frac{3}{2\sqrt{2}}$.

Exercises 3.4. 1. Consider the graph of the function $z = x^2 - 3y^2 + 7xy^3 + 9y^4$.

   (a) Find the Gauss and mean curvatures *at the origin.*
   (*Hint: use Theorem 3.34*)

   (b) Find the normal curvature at the origin for the curve in the surface described by $x = y$.

2. As in Example 3.35, mk. III (page 70), find the asymptotic directions at the origin for the surface $z = y^2 - 3x^2$.

3. For the elliptic paraboloid $z = x^2 + y^2$, let $P = (1, 2, 5)$ be a fixed point.

   (a) Find the maximum and minimum values for the normal curvature at P.

   (b) Find the Dupin indicatrix at P.

4. For the hyperbolic paraboloid $z = x^2 - y^2$, let $p = (u_0, v_0)$ and $P = \left(u_0, v_0, u_0^2 - v_0^2\right)$. If $c \neq 0$, prove that the intersection of the parallel plane $c\mathbf{n}_P + T_PS$ and the paraboloid may be expressed

$$(x - u_0)^2 - (y - v_0)^2 = \text{constant}, \quad z = x^2 - y^2$$

   That is, the level curves really are hyperbolæ.

5. Consider the graph of the surface $z = x^2 + y^4$.

   (a) Compute the Gauss curvature and classify all points according to Definition 3.36.

   (b) Sketch the level curves $z = 1$, $\frac{1}{100}$ and $\frac{1}{10000}$ and compare to the Dupin indicatrix at $(0, 0)$.

6. Prove Theorem 3.41 by considering the normal acceleration of the curve $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$.

3.5 Adaptive Frames & Gauss' Remarkable Theorem

In this section we repurpose the idea of a moving frame first encountered when studying curves.

Definition 3.43. Let $\mathbf{x} : U \to \mathbb{E}^3$ parametrize a surface S. A *moving frame* for S is a triple of smooth functions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ on U such that, for each $p \in U$,

$$\{\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p)\} \text{ is a positively oriented orthonormal basis of } T_{\mathbf{x}(p)}\mathbb{E}^3$$

When S is oriented, we say that a moving frame is *adaptive* if $\mathbf{e}_3 = \mathbf{n}$ is the unit normal field.

For an adaptive frame, the tangent plane at each point is $T_{\mathbf{x}(p)}S = \operatorname{Span}\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$.
We will often refer to the matrix-valued function $\mathcal{E} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} : U \to SO_3(\mathbb{R})$ as the frame.

Examples 3.44. We'll repeatedly analyze three examples through this section.

1. The parabolic cylinder $\mathbf{x}(u,v) = \left(u, v, \frac{1}{2}u^2\right)$ has an adaptive frame

$$\mathbf{e}_1 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}1\\0\\u\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}-u\\0\\1\end{pmatrix}$$

2. The sphere of radius R in spherical polar co-ordinates $\mathbf{x}(\psi, \phi)$ has an adaptive frame

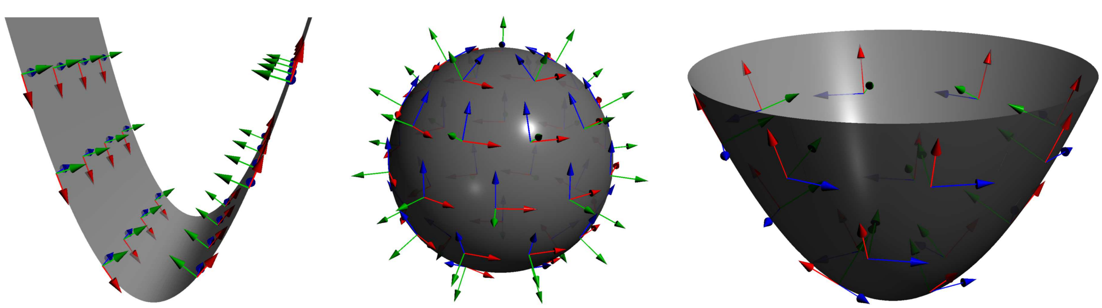
$$\mathbf{e}_1 = \begin{pmatrix}-\sin\psi\\ \cos\psi\\ 0\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}-\cos\psi\sin\phi\\ -\sin\psi\sin\phi\\ \cos\phi\end{pmatrix} \qquad \mathbf{e}_3 = \mathbf{x} = \begin{pmatrix}\cos\psi\cos\phi\\ \sin\psi\cos\phi\\ \sin\phi\end{pmatrix}$$

   We use ψ instead of θ since we'll need the latter for something else momentarily...

3. The paraboloid $\mathbf{x}(r,\psi) = \left(r\cos\psi, r\sin\psi, \frac{1}{2}r^2\right)$ has an adaptive frame

$$\mathbf{e}_1 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}\cos\psi\\ \sin\psi\\ r\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}-\sin\psi\\ \cos\psi\\ 0\end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}-r\cos\psi\\ -r\sin\psi\\ 1\end{pmatrix}$$

In the pictures we've reduced the lengths of the frame vectors for clarity.



In each case $\mathbf{e}_1, \mathbf{e}_2$ were obtained by differentiating with respect to the co-ordinates (and normalizing if necessary). This works because the co-ordinate systems for all three examples are *orthogonal.*

As with the Frenet frame approach to curves, our strategy is to analyse a surface $\mathbf{x} : U \to \mathbb{E}^3$ two stages:

1. Describe how $\mathbf{x}$ moves with respect to the frame $\mathcal{E}$.
2. Describe how the frame $\mathcal{E}$ moves (with respect to itself).

We describe infinitesimal changes using 1-forms, following an approach pioneered by Élie Cartan around 1899.

Definition 3.45. Let $\mathbf{x} : U \to \mathbb{E}^3$ be a smooth map and $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ a moving frame. The *metric forms* θ_j and *connection forms* ω_{jk} are the 1-forms on U defined by

$$\theta_j := \mathbf{e}_j \cdot \mathrm{d}\mathbf{x}, \quad \omega_{jk} = \mathbf{e}_j \cdot \mathrm{d}\mathbf{e}_k$$

where $j, k \in \{1, 2, 3\}$.

Since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthonormal, these forms are nothing more than the co-ordinates of $\mathrm{d}\mathbf{x}$, $\mathrm{d}\mathbf{e}_1$, $\mathrm{d}\mathbf{e}_2$ and $\mathrm{d}\mathbf{e}_3$ with respect to the moving frame:

$$\mathrm{d}\mathbf{x} = \sum_{j=1}^{3}(\mathbf{e}_j \cdot \mathrm{d}\mathbf{x})\mathbf{e}_j = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2 + \mathbf{e}_3\theta_3, \qquad \mathrm{d}\mathbf{e}_k = \sum_{j=1}^{3}\mathbf{e}_j\omega_{jk} \tag{$*$}$$

The frame is adaptive if and only if $\theta_3 = 0$. Moreover, as the next result shows, for any frame there are only three independent connection forms (compare this with Theorem 1.29).

Lemma 3.46. *For all j, k, we have $\omega_{jk} = -\omega_{kj}$. In particular $\omega_{jj} = 0$.*

Proof. Take the exterior derivative of the identity $\mathbf{e}_j \cdot \mathbf{e}_k = 0$ or 1, to obtain

$$0 = \mathrm{d}\mathbf{e}_j \cdot \mathbf{e}_k + \mathbf{e}_j \cdot \mathrm{d}\mathbf{e}_k = \omega_{kj} + \omega_{jk}$$

∎

If $(*)$ are arranged in matrix format, the subscripts follow the usual row/column convention:

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \mathcal{E}\Theta, \qquad \mathrm{d}\mathcal{E} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}0 & \omega_{12} & \omega_{13}\\ -\omega_{12} & 0 & \omega_{23}\\ -\omega_{13} & -\omega_{23} & 0\end{pmatrix} = \mathcal{E}\omega$$

The second expression should remind you of the Frenet–Serret equations for a curve! The metric forms get their name because they *measure* small changes on the surface. The connection forms tell us how nearby frames are related (*connected*): abusing notation a little, if $\vec{s}_p \in T_p\mathbb{R}^2$, then

$$\mathcal{E}(p + \vec{s}_p) - \mathcal{E}(p) \approx \mathrm{d}\mathcal{E}(\vec{s}_p) = \mathcal{E}(p)\omega(\vec{s}_p)$$

The fundamental forms of $\mathbf{x}$ can be written in terms of Θ and ω; in an adaptive frame this is particularly simple.

Lemma 3.47. *In an adaptive frame*

$$\mathrm{I} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \theta_1^2 + \theta_2^2 \quad \textit{and} \quad \mathrm{II} = -\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{e}_3 = -\theta_1\omega_{13} - \theta_2\omega_{23}$$

Examples (3.44, mk. II). You needn't compute all exterior derivatives $\mathrm{d}\mathbf{e}_k$: use the skew-symmetry of ω to help; also consider which frame fields are easier to differentiate! The expressions for the fundamental forms should be a sanity check since we know how to compute them already.

1. The parabolic cylinder has

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}1\\0\\u\end{pmatrix}\mathrm{d}u + \begin{pmatrix}0\\1\\0\end{pmatrix}\mathrm{d}v = \sqrt{1+u^2}\mathbf{e}_1\,\mathrm{d}u + \mathbf{e}_2\,\mathrm{d}v \implies \theta_1 = \sqrt{1+u^2}\,\mathrm{d}u,\ \ \theta_2 = \mathrm{d}v$$
$$\implies \mathrm{I} = (1+u^2)\,\mathrm{d}u^2 + \mathrm{d}v^2$$

   Since $\mathbf{e}_2$ is constant, we have $\mathrm{d}\mathbf{e}_2 = \mathbf{0}$ from which

$$\omega_{12} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_2 = 0, \quad \omega_{23} = -\omega_{32} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_2 = 0$$

   The final connection form requires a derivative:

$$\omega_{13} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_3 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}1\\0\\u\end{pmatrix}\cdot\left[\frac{-u}{(1+u^2)^{3/2}}\begin{pmatrix}-u\\0\\1\end{pmatrix} + \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}-1\\0\\0\end{pmatrix}\right] = \frac{-1}{1+u^2}\mathrm{d}u$$

   Putting it together, we have

$$\omega = \frac{1}{1+u^2}\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\mathrm{d}u \quad\text{and}\quad \mathrm{I\!I} = -\theta_1\omega_{13} - \theta_2\omega_{23} = \mathrm{d}u^2$$

2. For the sphere of radius R, $\mathrm{d}\mathbf{x} = R\cos\phi\,\mathbf{e}_1\,\mathrm{d}\psi + R\mathbf{e}_2\,\mathrm{d}\phi$, whence

$$\theta_1 = R\cos\phi\,\mathrm{d}\psi,\ \ \theta_2 = R\mathrm{d}\phi \implies \mathrm{I} = R^2\big(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2\big)$$
$$\mathrm{d}\mathbf{e}_1 = \begin{pmatrix}-\cos\psi\\-\sin\psi\\0\end{pmatrix}\mathrm{d}\psi \implies \begin{cases}\omega_{12} = -\mathbf{e}_2\cdot\mathrm{d}\mathbf{e}_1 = -\sin\phi\,\mathrm{d}\psi\\ \omega_{13} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_1 = \cos\phi\,\mathrm{d}\psi\end{cases}$$
$$\omega_{23} = \mathbf{e}_2\cdot\mathrm{d}\mathbf{e}_3 = \begin{pmatrix}-\cos\psi\sin\phi\\-\sin\psi\sin\phi\\\cos\phi\end{pmatrix}\cdot\left[\begin{pmatrix}-\sin\psi\cos\phi\\\cos\psi\cos\phi\\0\end{pmatrix}\mathrm{d}\psi + \begin{pmatrix}-\cos\psi\sin\phi\\-\sin\psi\sin\phi\\\cos\phi\end{pmatrix}\mathrm{d}\phi\right] = \mathrm{d}\phi$$
$$\implies \mathrm{I\!I} = -\theta_1\omega_{13} - \theta_2\omega_{23} = -R\big(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2\big)$$

3. For the paraboloid,

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}\cos\psi\\\sin\psi\\r\end{pmatrix}\mathrm{d}r + r\begin{pmatrix}-\sin\psi\\\cos\phi\\0\end{pmatrix}\mathrm{d}\psi = \sqrt{1+r^2}\mathbf{e}_1\,\mathrm{d}r + r\mathbf{e}_2\,\mathrm{d}\psi$$
$$\implies \theta_1 = \sqrt{1+r^2}\,\mathrm{d}r, \quad \theta_2 = r\,\mathrm{d}\psi \implies \mathrm{I} = (1+r^2)\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2$$

   The connection forms are comparatively ugly. The low-hanging fruit is $\mathrm{d}\mathbf{e}_2 = \begin{pmatrix}-\cos\psi\\-\sin\psi\\0\end{pmatrix}\mathrm{d}\psi$, which quickly yields two of them:

$$\omega_{12} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_2 = -\frac{\mathrm{d}\psi}{\sqrt{1+r^2}}, \quad \omega_{23} = -\omega_{32} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_2 = \frac{-r\,\mathrm{d}\psi}{\sqrt{1+r^2}}$$

The last connection form requires a nastier differentiation, though only one of the three terms in $\mathrm{d}\mathbf{e}_3$ provides a non-zero result when dotted with $\mathbf{e}_1$:

$$\omega_{13} = \mathbf{e}_1 \cdot \mathrm{d}\mathbf{e}_3 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}\cos\psi\\ \sin\psi\\ r\end{pmatrix} \cdot \left[\cdots + \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}-\cos\psi\\ -\sin\psi\\ 0\end{pmatrix}\mathrm{d}r\right] = \frac{-\mathrm{d}r}{1+r^2}$$

We therefore obtain the connection form matrix

$$\omega = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix} 0 & -\mathrm{d}\psi & \frac{-1}{\sqrt{1+r^2}}\,\mathrm{d}r \\ \mathrm{d}\psi & 0 & -r\,\mathrm{d}\psi \\ \frac{1}{\sqrt{1+r^2}}\,\mathrm{d}r & r\,\mathrm{d}\psi & 0 \end{pmatrix}$$

and second fundamental form

$$\mathbb{I} = -\sqrt{1+r^2}\,\mathrm{d}r\frac{-\mathrm{d}r}{1+r^2} - r\,\mathrm{d}\psi\frac{-r\,\mathrm{d}\psi}{\sqrt{1+r^2}} = \frac{1}{\sqrt{1+r^2}}(\mathrm{d}r^2 + r^2\mathrm{d}\psi^2)$$

The Structure Equations for a Moving Frame

The metric and connection forms satisfy matrix equations $\mathrm{d}\mathbf{x} = \mathcal{E}\Theta$ and $\mathrm{d}\mathcal{E} = \mathcal{E}\omega$. Since $\mathrm{d}^2 = 0$, something nice happens when we take the exterior derivatives of these expressions:

$$\mathbf{0} = \mathrm{d}^2\mathbf{x} = \mathrm{d}(\mathrm{d}\mathbf{x}) = \mathrm{d}(\mathcal{E}\,\Theta) = \mathrm{d}\mathcal{E}\wedge\Theta + \mathcal{E}\,\mathrm{d}\Theta = \mathcal{E}(\omega\wedge\Theta + \mathrm{d}\Theta)$$
$$0 = \mathrm{d}^2\mathcal{E} = \mathrm{d}(\mathrm{d}\mathcal{E}) = \mathrm{d}(\mathcal{E}\omega) = \mathrm{d}\mathcal{E}\wedge\omega + \mathcal{E}\mathrm{d}\omega = \mathcal{E}(\omega\wedge\omega + \mathrm{d}\omega)$$

The notation $\omega\wedge\Theta$ means matrix multiplication using the wedge product of forms to evaluate each entry.[24] Since each $\mathcal{E}(p)$ is an invertible matrix, we conclude two identities.

Theorem 3.48. *The metric and connection forms satisfy the* structure equations*; each amounts to three separate equations after multiplying out the matrix expressions.*

1. $\mathrm{d}\Theta + \omega\wedge\Theta = \mathbf{0}$, *equivalently* $\mathrm{d}\theta_j + \sum_{k\neq j}\omega_{jk}\wedge\theta_k = 0$ *for each* $j = 1, 2, 3$
2. $\mathrm{d}\omega + \omega\wedge\omega = 0$, *equivalently* $\mathrm{d}\omega_{jk} + \omega_{ji}\wedge\omega_{ik} = 0$ *where* i, j, k *are distinct.*

These are easy to remember if you pay attention to the indices! In an adaptive frame ($\theta_3 = 0$), things are a little simpler and some of the equations get special names:

First structure equations	$\begin{cases}\mathrm{d}\theta_1 + \omega_{12}\wedge\theta_2 = 0\\ \mathrm{d}\theta_2 + \omega_{21}\wedge\theta_1 = 0\end{cases}$
Symmetry equation	$\omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = 0$
Gauss equation	$\mathrm{d}\omega_{12} + \omega_{13}\wedge\omega_{32} = 0$
Codazzi equations	$\begin{cases}\mathrm{d}\omega_{13} + \omega_{12}\wedge\omega_{23} = 0\\ \mathrm{d}\omega_{23} + \omega_{21}\wedge\omega_{13} = 0\end{cases}$

[24] Be careful not to reverse the order: $\Theta\wedge\omega$ makes no sense since the dimensions of the matrices are incompatible! Similarly, $\omega\wedge\omega$ is unlikely to be zero…

Examples (3.44, mk. III). 1. For the parabolic cylinder, $\Theta = \begin{pmatrix} \sqrt{1+u^2}\,\mathrm{d}u \\ \mathrm{d}v \\ 0 \end{pmatrix}$ and $\omega = \frac{1}{1+u^2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\mathrm{d}u$, so all the structure equations are trivial:

$$\mathrm{d}\Theta = \mathbf{0} = -\omega \wedge \Theta, \quad \mathrm{d}\omega = 0 = -\omega \wedge \omega$$

2. For the sphere, $\Theta = R\begin{pmatrix} \cos\phi\,\mathrm{d}\psi \\ \mathrm{d}\phi \\ 0 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & \sin\phi\,\mathrm{d}\psi & -\cos\phi\,\mathrm{d}\psi \\ -\sin\phi\,\mathrm{d}\psi & 0 & \mathrm{d}\phi \\ \cos\phi\,\mathrm{d}\psi & -\mathrm{d}\phi & 0 \end{pmatrix}$, from which

$$\mathrm{d}\Theta = R\begin{pmatrix} -\sin\phi \\ 0 \\ 0 \end{pmatrix}\mathrm{d}\phi \wedge \mathrm{d}\psi = -\omega \wedge \Theta$$

$$\mathrm{d}\omega = \begin{pmatrix} 0 & \cos\phi & \sin\phi \\ -\cos\phi & 0 & 0 \\ -\sin\phi & 0 & 0 \end{pmatrix}\mathrm{d}\phi \wedge \mathrm{d}\psi = -\omega \wedge \omega$$

3. For the paraboloid, $\Theta = \begin{pmatrix} \sqrt{1+r^2}\mathrm{d}r \\ r\,\mathrm{d}\psi \\ 0 \end{pmatrix}$ and $\omega = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix} 0 & -\mathrm{d}\psi & -\frac{\mathrm{d}r}{\sqrt{1+r^2}} \\ \mathrm{d}\psi & 0 & -r\,\mathrm{d}\psi \\ \frac{\mathrm{d}r}{\sqrt{1+r^2}} & r\,\mathrm{d}\psi & 0 \end{pmatrix}$.

The first equations aren't too bad to check:

$$\mathrm{d}\Theta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}r \wedge \mathrm{d}\psi = -\omega \wedge \Theta$$

The second are a little nastier: you should check that

$$\mathrm{d}\omega = \frac{1}{(1+r^2)^{3/2}}\begin{pmatrix} 0 & r & 0 \\ -r & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\mathrm{d}r \wedge \mathrm{d}\psi = -\omega \wedge \omega$$

Gauss' Remarkable Theorem

Suppose we have an adaptive frame for an oriented local surface $\mathbf{x}$. If θ_1, θ_2 were linearly dependent at p, then the differential $\mathrm{d}\mathbf{x} = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2 : T_p\mathbb{R}^2 \to T_{\mathbf{x}(p)}S = \operatorname{Span}\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$ would have rank ≤ 1 and thus not be a bijection. We conclude that $\{\theta_1, \theta_2\}$ forms a basis of the space of 1-forms at p, and that *any other 1-form may be written as a linear combination thereof...*

Lemma 3.49. *There exist unique functions a, b, c such that*

$$\omega_{13} = a\theta_1 + b\theta_2, \quad \omega_{23} = b\theta_1 + c\theta_2$$

With respect to these functions, the second fundamental form, Gauss and mean curvatures are

$$\mathbb{I} = -a\theta_1^2 - 2b\theta_1\theta_2 - c\theta_1^2, \qquad K = ac - b^2, \qquad H = -\frac{1}{2}(a+c)$$

Proof. That $\omega_{13} = a\theta_1 + b\theta_2$ and $\omega_{23} = \hat{b}\theta_1 + c\theta_2$ are linear combinations of θ_1, θ_2 is the above discussion. By the symmetry equation and the fact that $\theta_1 \wedge \theta_2 \neq 0$,

$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2 = (-b + \hat{b})\,\theta_1 \wedge \theta_2 \implies \hat{b} = b$$

The formula for $\mathrm{I\!I}$ follows from Lemma 3.47.

Moreover, if $\vec{w}_1$ and $\vec{w}_2$ are the dual vector fields to θ_1, θ_2, then the matrices of $\mathrm{I}, \mathrm{I\!I}$ with respect to these fields are the identity matrix[25] and $B = \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}$. The Gauss and mean curvatures are the determinant and half the trace of B (Theorem 3.29). ■

Now consider the final connection form ω_{12}. Since θ_1, θ_2 form a basis at each point, we may write

$$\omega_{12} = f\theta_1 + g\theta_2$$

for some functions $f, g : U \to \mathbb{R}$. Applying the 1^{st} structure equations,

$$\begin{aligned} d\theta_1 &= -\omega_{12} \wedge \theta_2 = -f\theta_1 \wedge \theta_2 \\ d\theta_2 &= -\omega_{21} \wedge \theta_1 = -\theta_1 \wedge \omega_{12} = -g\theta_1 \wedge \theta_2 \end{aligned}$$

whence f, g (and ω_{12}) are determined by θ_1, θ_2. This brings us to the capstone result of these notes.

Theorem 3.50 (Gauss' Theorem Egregium). *The Gauss curvature depends only on the first fundamental form.*

Proof. By the above discussion, ω_{12} (and thus $d\omega_{12}$) depends only on θ_1, θ_2, which may be recovered from $\mathrm{I} = \theta_1^2 + \theta_2$ by writing it as a sum of squares. But now the Gauss equation reads

$$d\omega_{12} = \omega_{13} \wedge \omega_{23} = (a\theta_1 + b\theta_2) \wedge (b\theta_1 + c\theta_2) = (ac - b^2)\theta_1 \wedge \theta_2 = K\theta_1 \wedge \theta_2$$

■

An explicit formula for K as a function of the coefficients E, F, G of I can be found; see Exercise 9.

Egregium (Latin for *remarkable/outstanding*) is the (modest!) term Gauss applied after proving his result in 1827. Why did he consider it so remarkable? The original definition of K relied on the normal field; an object *outside* the surface which helps describe its position/orientation in $\mathbb{E}^3$. Gauss' Theorem, however, says that K is *intrinsic* to the surface: it depends only on the *metric* (first fundamental form) which may be understood by an occupant of the surface with no ability to escape (travel outside the surface) in to view its shape. By contrast, the second fundamental form and the mean curvature depend on how a surface is embedded; these are *extrinsic* quantities.

As a nice side-effect, the result provides what is often a faster method for calculating K.

1. Compute the first fundamental form $\mathrm{I} = d\mathbf{x} \cdot d\mathbf{x}$ and express it as a sum of squares $\mathrm{I} = \theta_1^2 + \theta_2^2$.
2. Write $\omega_{12} = f\theta_1 + g\theta_2$ and compute f, g using the 1^{st} structure equations.
3. Use the Gauss equation to find K.

We need only calculate 1-forms $\theta_1, \theta_2, \omega_{12}$ that are related to the *tangent* part of the moving frame. The unit normal $\mathbf{e}_3$ needn't be considered or calculated.

---

[25] $\theta_j(\vec{w}_k) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ implies that $d\mathbf{x}(\vec{w}_1) = \mathbf{e}_1$ and $d\mathbf{x}(\vec{w}_2) = \mathbf{e}_2$ are orthonormal.

Examples (3.44, mk. IV). We return to our examples one last time. Even though we've already calculated the connection forms, the goal is to see that $\omega_{12} = f\theta_1 + g\theta_2$ and thus K may be found directly from I.

1. The parabolic cylinder has $\mathrm{I} = (1+u^2)\,\mathrm{d}u^2 + \mathrm{d}v^2$ so the natural choice is

   $$\theta_1 = \sqrt{1+u^2}\,\mathrm{d}u \quad \text{and} \quad \theta_2 = \mathrm{d}v$$

   Since $\mathrm{d}\theta_1 = 0 = \mathrm{d}\theta_2$ we see that $f = g = 0$. We conclude that

   $$\omega_{12} = 0 \implies \mathrm{d}\omega_{12} = 0 \implies K = 0$$

2. For the sphere $\mathrm{I} = R^2(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2)$ so we choose $\theta_1 = R\cos\phi\,\mathrm{d}\psi$ and $\theta_2 = R\,\mathrm{d}\phi$. Certainly $0 = \mathrm{d}\theta_2 = -g\theta_1 \wedge \theta_2 \implies g = 0$. Moreover,

   $$\mathrm{d}\theta_1 = -f\theta_1 \wedge \theta_2 \implies R\sin\phi\,\mathrm{d}\psi \wedge \mathrm{d}\phi = -fR^2\cos\phi\,\mathrm{d}\psi \wedge \mathrm{d}\phi \implies f = -R^{-1}\tan\phi$$

   We conclude that $\omega_{12} = -R^{-1}\tan\phi\,\theta_1 = -\sin\phi\,\mathrm{d}\psi$, from which

   $$\mathrm{d}\omega_{12} = \cos\phi\,\mathrm{d}\psi \wedge \mathrm{d}\phi = \frac{1}{R^2}\theta_1 \wedge \theta_2 \implies K = \frac{1}{R^2}$$

3. For the paraboloid, $\mathrm{I} = (1+r^2)\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2$ so we choose $\theta_1 = \sqrt{1+r^2}\,\mathrm{d}r$ and $\theta_2 = r\,\mathrm{d}\psi$. This time $\mathrm{d}\theta_1 = 0 \implies f = 0$ and

   $$\mathrm{d}\theta_2 = -g\theta_1 \wedge \theta_2 \implies \mathrm{d}r \wedge \mathrm{d}\psi = -gr\sqrt{1+r^2}\,\mathrm{d}r \wedge \mathrm{d}\psi \implies g = -\frac{1}{r\sqrt{1+r^2}}$$

   We conclude that $\omega_{12} = -\frac{1}{r\sqrt{1+r^2}}\,\theta_2 = -\frac{1}{\sqrt{1+r^2}}\,\mathrm{d}\psi$, from which

   $$\mathrm{d}\omega_{12} = \frac{r}{(1+r^2)^{3/2}}\,\mathrm{d}r \wedge \mathrm{d}\psi = \frac{1}{(1+r^2)^2}\theta_1 \wedge \theta_2 \implies K = \frac{1}{(1+r^2)^2}$$

Since K depends only on the metric, it is invariant under *isometric* transformations of the surface. This helps explain why the Gauss curvature of a cylinder and a cone are both zero: both may constructed by rolling up a flat plane without other distortion.

The contrapositive of Gauss' Theorem is also important: surfaces with distinct Gauss curvatures cannot be isometric. Since the metric I determines how we measure angle and length, this explains why a perfect flat map ($K = 0$) of any part of the Earth ($K = \frac{1}{R^2}$) is impossible to achieve. The holy grail of map-making would be a map free of direction, angle and length/area distortion:

1. Straight lines on the map should correspond to paths of shortest distance on the Earth.

2. Angles on the map should equal corresponding angles on the Earth's surface.

3. Areas on the map and the Earth should be in constant ratio.

Gauss' Theorem implies that you cannot have all these properties in one map. In fact, at most one of these properties is possible in a single map.

Riemannian Geometry

We can even employ the method when there is no surface! The idea is to equip a domain with an abstract first fundamental form and use it to compute lengths, angles, area, geodesics, curvature, etc.

Example 3.51. The Poincaré disk model of hyperbolic space is the disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ equipped with the metric (first fundamental form)

$$\mathrm{I} = \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - x^2 - y^2)^2} = \frac{4(\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2)}{(1 - r^2)^2}$$

As one approaches the boundary of the disk, the idea is that measured distance gets larger: the boundary circle is in fact *infinitely far* from any point inside the disk. To express I as a sum of squares, a natural choice is $\theta_1 = \frac{2\mathrm{d}r}{1-r^2}$ and $\theta_2 = \frac{2r\,\mathrm{d}\psi}{1-r^2}$, from which $\mathrm{d}\theta_1 = 0 \implies f = 0$ and

$$\mathrm{d}\theta_2 = -g\theta_1 \wedge \theta_2 \implies \frac{2(1+r^2)}{(1-r^2)^2}\mathrm{d}r \wedge \mathrm{d}\psi = -\frac{4gr}{(1-r^2)^2}\,\mathrm{d}r \wedge \mathrm{d}\psi \implies g = -\frac{1+r^2}{2r}$$

from which

$$\mathrm{d}\omega_{12} = \mathrm{d}(g\theta_2) = -\mathrm{d}\left(\frac{1+r^2}{1-r^2}\right) \wedge \mathrm{d}\psi = \frac{-4r}{(1-r^2)^2}\,\mathrm{d}r \wedge \mathrm{d}\psi = -\theta_1 \wedge \theta_2 \implies K = -1$$

Hyperbolic space is the canonical example of a *negatively curved geometry*. There is no surface here, no second fundamental form, and no mean curvature! Since there is no surface, it is harder to visualize what K means in this context (e.g. Section 3.4).[26]

The Gauss curvature of a surface is the simplest avatar of a more general object called the *Riemann curvature tensor.* As an example of how this is applied, in general relativity[27] mass is construed as changing the metric of spacetime (i.e. I); it can be seen that this metric is compatible with unique connection (essentially ω) from which the curvature ($\mathrm{d}\omega + \omega \wedge \omega$) may be computed. When a physicist says *spacetime is curved,* this is what they mean: there is no *exterior* to spacetime from which we can measure curvature, so everything is computed intrinsically.

The Fundamental Theorem of Surfaces

Recall the equivalence of spacecurves up to rigid motions (Theorem 1.38) and the Fundamental Theorem of Biregular Spacecurves (Corollary 1.42). A similar discussion is available for surfaces once we replace curvature and torsion with the fundamental forms $\mathrm{I}, \mathrm{I\!I}$.

The equivalence problem is almost identical. Suppose $\mathbf{x} : U \to \mathbb{E}^3$ is an oriented surface, and that $A \in O_3(\mathbb{R})$ and $\mathbf{b} = \mathbb{E}^3$ are constants. Then $\mathbf{y} := A\mathbf{x} + \mathbf{b}$ is a new surface, the result of applying an isometry to $\mathbf{x}$. A moving frame for $\mathbf{x}$ is transformed to a frame for $\mathbf{y}$ via

$$\mathcal{E}_{\mathbf{y}} = \begin{pmatrix} A\mathbf{e}_1 & A\mathbf{e}_2 & \pm A\mathbf{e}_3 \end{pmatrix} \quad \text{where } \pm 1 = \det A$$

[26] The most famous consequence concerns angle-sums of geodesic triangles: $A + B + C = \pi + \int_\triangle K$. If $K < 0$, the angle-sum of a geodesic triangle is *less than* 180°. When $K > 0$ (e.g., a sphere), the angle sum is greater than 180°. This topic, the related *Gauss–Bonnet Theorem*, and other consequences, are a matter for another course.

[27] Really this is *pseudo*-Riemannian geometry, since I is not positive-definite.

The upshot is that $\mathbf{n_y} = (\det A)A\mathbf{n_x}$, and $\mathrm{I}, \mathbb{I}$ transform exactly as κ, τ:

$$\mathrm{I}_\mathbf{y} = \mathrm{d}\mathbf{y}\cdot\mathrm{d}\mathbf{y} = (A\mathrm{d}\mathbf{x})\cdot(A\mathrm{d}\mathbf{x}) = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{I}_\mathbf{x}$$
$$\mathbb{I}_\mathbf{y} = -\mathrm{d}\mathbf{y}\cdot\mathrm{d}\mathbf{n_y} = -(\det A)(A\mathrm{d}\mathbf{x})\cdot(A\mathrm{d}\mathbf{n_x}) = (\det A)\mathbb{I}_\mathbf{x}$$

As with curves, we may ask the question in reverse. If we know the fundamental forms, can we also recover the surface up to a rigid motion? The answer is yes, though with a caveat: unlike κ, τ for spacecurves, the fundamental forms cannot be chosen independently.

Theorem 3.52 (Bonnet). *Suppose* I *and* $\mathbb{I}$ *are symmetric bilinear forms where* I *is positive-definite. Provided the Gauss–Codazzi equations are satisfied, there exists a local parametrized surface with these fundamental forms, which is moreover unique up to rigid motions.*

Everything ultimately depends on a generalization of the existence/uniqueness theorem for ODE (another version of the *Frobenius Theorem* (3.33)). Here is a rough sketch of how the process works.

1. Suppose we are given $\mathrm{I}, \mathbb{I}$ on U, and initial conditions at some $p \in U$ (for the surface $\mathbf{x}(p) = \mathbf{x}_0$ and frame $\mathcal{E}(p) = \mathcal{E}_0$).
2. Since I is positive-definite, it may be written $\mathrm{I} = \theta_1^2 + \theta_2^2$.
3. The first structure equations determine ω_{12} and $\mathbb{I}$ determines ω_{13} and ω_{23} (Lemma 3.49).
4. The Frobenius Theorem shows that the initial value problem
$$\mathrm{d}\mathcal{E} = \mathcal{E}\omega \qquad \mathcal{E}(p) = \mathcal{E}_0 \tag{$*$}$$
has a unique local solution provided the Gauss–Codazzi equations $\mathrm{d}\omega + \omega\wedge\omega = 0$ are satisfied. The solution $\mathcal{E}$ is $SO_3(\mathbb{R})$-valued and supplies an adapted frame (compare Corollary 1.41).
5. To find the surface, solve a second initial value problem
$$\mathrm{d}\mathbf{x} = \mathcal{E}\Theta \qquad \mathbf{x}(p) = \mathbf{x}_0$$
Frobenius says this has a unique solution provided $\mathrm{d}\Theta + \omega\wedge\Theta = 0$. Since this is precisely what we used to determine ω in step 2, we don't need to check this condition.
6. Any different choice of metric forms in step 2 merely rotates $\mathcal{E}$ around $\mathbf{n} = \mathbf{e}_3$ and does not affect the resulting surface.

It is a little easier to understand the integrability condition when written in co-ordinates: $(*)$ is a linear system of eighteen PDE in nine unknowns

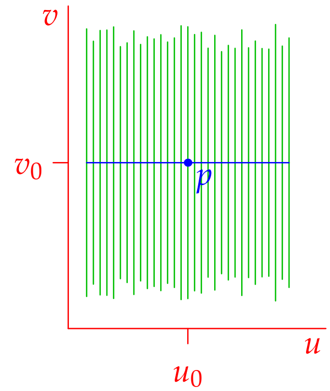
$$\begin{cases} \frac{\partial\mathcal{E}}{\partial u} = \mathcal{E}P \\ \frac{\partial\mathcal{E}}{\partial v} = \mathcal{E}Q \end{cases} \quad \text{where } P = \omega\left(\tfrac{\partial}{\partial u}\right),\ Q = \omega\left(\tfrac{\partial}{\partial v}\right) \text{ are skew-symmetric matrix functions}$$

The Gauss–Codazzi equations are essentially the fact that mixed partial derivatives commute:[28]

$$0 = \mathcal{E}_{uv} - \mathcal{E}_{vu} = \mathcal{E}_v P + \mathcal{E}P_v - \mathcal{E}_u Q + \mathcal{E}Q_u = \mathcal{E}\big(P_v - Q_u - [P,Q]\big)$$
$$P_v - Q_u - [P,Q] = \tfrac{\partial}{\partial v}\omega\left(\tfrac{\partial}{\partial u}\right) - \tfrac{\partial}{\partial u}\omega\left(\tfrac{\partial}{\partial v}\right) - \left[\omega\left(\tfrac{\partial}{\partial u}\right), \omega\left(\tfrac{\partial}{\partial v}\right)\right] = (\mathrm{d}\omega + \omega\wedge\omega)\left(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial u}\right)$$

[28] $[P,Q] = PQ - QP$ and $\mathrm{d}\omega$ is evaluated as in Exercise 2.3.10.

The part that requires some proof is that the integrability condition ($P_v - Q_u = [P, Q]$) is *sufficient* for a solution. This is not as hard as it sounds; here is another sketch:



1. If $p = (u_0, v_0)$, use Picard's ODE existence/uniqueness theorem to solve an initial value problem on the horizontal line $v = v_0$:
$$\frac{d\widetilde{\mathcal{E}}}{du} = \widetilde{\mathcal{E}}P(u, v_0), \qquad \widetilde{\mathcal{E}}(u_0, v_0) = \mathcal{E}_O$$
2. For *each* u_1, apply the ODE theorem to solve another IVP on the vertical line $u = u_1$:
$$\frac{d\mathcal{E}}{dv} = \mathcal{E}Q(u_1, v), \qquad \mathcal{E}(u_1, v_0) = \widetilde{\mathcal{E}}(u_1, v_0)$$
3. Finally, one shows that the resulting $\mathcal{E}$ is differentiable with respect to u, and uses the integrability condition to check that $\mathcal{E}_u = \mathcal{E}P$ as required.

The first two steps may be accomplished approximately using a numerical method to desired accuracy, so this amounts to an algorithm for the approximation of $\mathcal{E}$. The same approach can then be followed to approximate the surface.

The Gauss–Codazzi equations in curvature-line co-ordinates Suppose (u, v) are curvature-line co-ordinates. Then the fundamental forms are

$$\mathrm{I} = E\,du^2 + G\,dv^2, \qquad \mathrm{II} = k_1 E\,du^2 + k_2 G dv^2 \tag{†}$$

where E, G are positive functions and k_1, k_2 are the principal curvatures. We therefore choose metric forms $\theta_1 = \sqrt{E}\,du$ and $\theta_2 = \sqrt{G}\,dv$. In the language of Lemma 3.49,

$$a = -k_1, \quad b = 0, \quad c = -k_2, \quad \omega_{13} = -k_1\sqrt{E}\,du, \quad \omega_{23} = -k_2\sqrt{G}\,dv$$

The first structure equations determine

$$\omega_{12} = \frac{1}{2\sqrt{EG}}(E_v\,du - G_u\,dv)$$

(see Exercise 9). Moreover, the Gauss–Codazzi equations are equivalent to

$$\begin{aligned}
d\omega_{12} + \omega_{21} \wedge \omega_{13} = 0 &\iff \left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v = -2k_1k_2\sqrt{EG} \\
d\omega_{13} + \omega_{12} \wedge \omega_{23} = 0 &\iff 2(k_1)_v E = (k_2 - k_1)E_v \\
d\omega_{23} + \omega_{21} \wedge \omega_{13} = 0 &\iff 2(k_2)_u G = (k_1 - k_2)G_u
\end{aligned}$$

These equations show the relationship between I and II: we cannot independently choose the metric (E, G) and the curvatures (k_1, k_2). However, if E, G, k_1, k_2 satisfy these equations, Bonnet's theorem guarantees the existence of a surface with fundamental forms (†), unique up to rigid motions.

While I, II cannot be chosen independently, Bonnet's result is considered the best description of the *minimal data* for a surface. You might suspect/hope that knowledge of K, H would be enough to determine a surface up to rigid motions, but Exercise 10 shows such to be vain!

Exercises 3.5. 1. The unit cylinder $\mathbf{x}(\phi, v) = (\cos\phi, \sin\phi, v)$ has adaptive frame

$$\mathbf{e}_1 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

(a) Directly compute the metric forms θ_j and connection forms ω_{jk}.

(b) That the six structure equations are satisfied should be obvious from your answers to (a): why?

(c) Why is it completely obvious from your answer to (a) that $K \equiv 0$?

2. For a general regular surface, explain why we cannot, in general, find co-ordinates u, v for which $\mathrm{I} = \mathrm{d}u^2 + \mathrm{d}v^2$.

3. For the paraboloid example (3.44.3) verify the Gauss–Codazzi equations $\mathrm{d}\omega + \omega \wedge \omega = 0$.
(*Hint: this is easier if you treat the three equations separately!*)

4. Verify that the metric $\mathrm{I} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ on the upper half-plane $y > 0$ has curvature $K = -1$.
(*Hint: Recall Example 3.51 and Exercise 3.2.9*)

5. Consider the catenoid $\mathbf{x}(u, v) = (\cos u \cosh v, \sin u \cosh v, v)$ obtained by revolving the catenary $x = \cosh z$ around the z-axis.

(a) Show that there exists a moving frame for which the metric forms are

$$\theta_1 = \cosh v \,\mathrm{d}u, \quad \theta_2 = \cosh v \,\mathrm{d}v$$

(b) Show that $\omega_{12} = \tanh v \,\mathrm{d}u = \frac{\sinh v}{\cosh v}\,\mathrm{d}u$ and use it to prove that the Gaussian curvature of the catenoid is

$$K = -\frac{1}{\cosh^4 v}$$

6. We re-prove Exercise 3.3.12 using our new language.

(a) Suppose a surface $\mathbf{x}$ is totally umbilic: $\mathbb{I} = \lambda \mathrm{I}$, where λ is some function. Explain why $\omega_{13} = -\lambda\theta_1$ and $\omega_{23} = -\lambda\theta_2$.

(b) Use the 1st structure equations and the Codazzi equations to prove that $\mathrm{d}\lambda = 0$.

(c) If $a = 0$, what is $\mathbf{x}$?

(d) If $a \neq 0$, define $\mathbf{c} := \mathbf{x} - \frac{1}{a}\mathbf{e}_3$. Prove that $\mathrm{d}\mathbf{c} = \mathbf{0}$ and hence conclude that the surface is (part of a) round sphere.

7. Suppose $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ is an adaptive frame for a surface. Any other adaptive frame (with the same orientation) is obtained by *rotating* around $\mathbf{e}_3$: that is $\hat{\mathcal{E}} = (\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \mathbf{e}_3)$ where

$$\hat{\mathbf{e}}_1 = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2, \quad \hat{\mathbf{e}}_2 = -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2$$

for some smooth function $\varphi : U \to \mathbb{R}$.

(a) Compute θ_1, θ_2 in terms of $\hat{\theta}_1, \hat{\theta}_2$ and conclude that $\hat{\theta}_1 \wedge \hat{\theta}_2 = \theta_1 \wedge \theta_2$.

(b) Use Definition 3.45 to compute $\hat{\omega}_{12}$ in terms of ω_{12} and φ. Verify that $\mathrm{d}\hat{\omega}_{12} = \mathrm{d}\omega_{12}$ so that the Gauss equation is identical for the new moving frame.

8. Suppose I is the 1st fundamental form of a surface. Suppose $\mathrm{I} = \theta_1^2 + \theta_2^2$ for some 1-forms θ_1, θ_2. Prove that there exists a moving frame $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ for which $\mathrm{d}\mathbf{x} = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2$.

   (*Hint: consider the dual vector fields to* θ_1, θ_2)

9. Suppose u, v are orthogonal co-ordinates so that $\theta_1 = \sqrt{E}\,\mathrm{d}u$ and $\theta_2 = \sqrt{G}\,\mathrm{d}v$.

   (a) Use the structure equations to prove that

   $$\omega_{12} = \frac{1}{2\sqrt{EG}}\left(E_v\,\mathrm{d}u - G_u\,\mathrm{d}v\right)$$

   (b) Hence deduce an explicit formula for the Gauss curvature in terms of the coefficients of the 1st fundamental form:

   $$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right)$$

   This can be multiplied out to remove the square roots, though you'll get more terms. A nastier expression (the Brioshi formula) may be found for general co-ordinates with $F \neq 0$.

10. In Exercise 3.3.5 we saw that the tangent developable $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ of a unit-speed curve has curvatures $K = 0$, $H = -\frac{\tau}{2v\kappa}$. Use this to describe *two* surfaces with the same curvature functions which are not related by a direct isometry.

11. Show that the surfaces parametrized by

    $$\mathbf{x}(u,v) = \big(u\cos\phi, u\sin\phi, \ln u\big), \quad \mathbf{y}(u,v) = \big(u\cos\phi, u\sin\phi, \phi\big)$$

    have the same Gauss curvature but distinct first fundamental forms $\mathrm{I}_\mathbf{x} \neq \mathrm{I}_\mathbf{y}$. To do this properly, you should argue that there is no reparametrization of $\mathbf{y}$ so that $K_\mathbf{x} = K_\mathbf{y}$ and $\mathrm{I}_\mathbf{x} = \mathrm{I}_\mathbf{y}$.

    (*Gauss' Theorem isn't biconditional: surfaces can have the same K without being locally isometric*)

12. Consider the family of surfaces

    $$\mathbf{x}^t(u,v) = \cos t\begin{pmatrix}\sin u\sinh v\\ -\cos u\sinh v\\ u\end{pmatrix} + \sin t\begin{pmatrix}\cos u\cosh v\\ \sin u\cosh v\\ v\end{pmatrix}, \qquad t \in [0, \tfrac{\pi}{2}]$$

    When $t = 0$ this is a *helicoid.* When $t = \frac{\pi}{2}$ this is the *catenoid* from Exercise 5.

    (a) Compute the first fundamental form of $\mathbf{x}^t$ and show that it is independent of t (the family $\mathbf{x}^t$ is therefore *isometric*).

    (b) Show that the unit normal of $\mathbf{x}^t$ is also independent of t:

    $$\mathbf{n}^t = \frac{1}{\cosh v}\begin{pmatrix}\cos u\\ \sin u\\ -\sinh v\end{pmatrix}$$

    Hence compute the second fundamental form of $\mathbf{x}^t$ for each t.

    (c) Find the Gauss and mean curvatures of all surfaces $\mathbf{x}^t$. What is special about this family? Relate this to Gauss' Theorem.