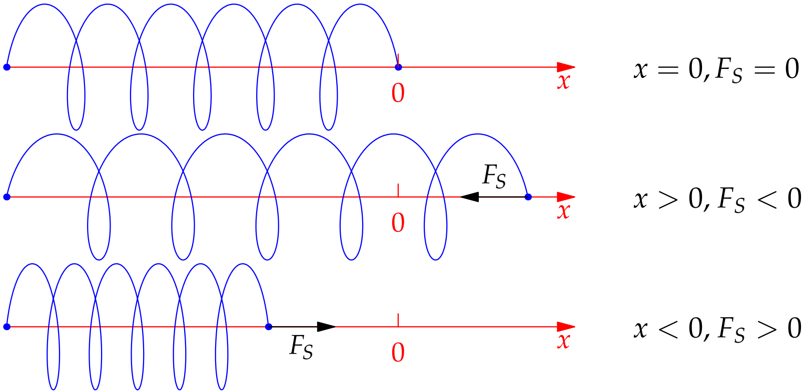
2.4 Mechanical Vibrations

Mass m attached to spring

$x =$ distance to right of equilibrium

$F_S =$ force on mass due to spring



Summing the Forces

Three forces act on mass

1. Spring force: $F_S = -kx$ ($k > 0$ constant = ‘stiffness’)
2. Resistive force: $F_R = -cx'$ ($c \geq 0$ constant)
3. External force: $F_E = F(t)$ (time-dependent)

Newton’s second law $\Longrightarrow mx'' = F_S + F_R + F_E$

$\rightsquigarrow$ 2nd-order, constant coeff ODE

$$mx'' + cx' + kx = F(t)$$

Motion can be:

- ‘Undamped’ $c = 0$, or
- ‘Damped’ $c > 0$
- ‘Free’: $F(t) \equiv 0$, or
- ‘Driven’: $F(t) \not\equiv 0$

Free undamped (simple harmonic) motion

$mx'' + kx = 0$ has general solution

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \gamma)$$

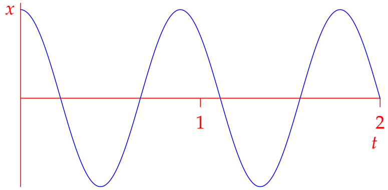
with[1] *Circular frequency* $\omega_0 = \sqrt{\frac{k}{m}}$ rad/s

Amplitude $C = \sqrt{A^2 + B^2}$ m

Phase angle $\gamma = \tan^{-1}\frac{B}{A}$ rad (or $\tan^{-1}\frac{B}{A} + \pi$)

Period $T = \frac{2\pi}{\omega_0}$ s

Frequency $f = \frac{\omega_0}{2\pi}$ Hz ($= \frac{1}{\text{s}}$)



[1] Units assume kg-m-s, etc.

Free damped motion ($c > 0$)

Write $\omega_0 = \sqrt{\frac{k}{m}}$ and $p = \frac{c}{2m} > 0$, then

$$mx'' + cx' + kx = 0 \Longrightarrow x'' + 2px' + \omega_0^2 x = 0$$

Cases depend on roots of characteristic equation

$$\lambda^2 + 2p\lambda + \omega_0^2 = 0 \Longrightarrow \lambda_1, \lambda_2 = -p \pm \sqrt{p^2 - \omega_0^2}$$

Solutions depend on sign of $p^2 - \omega_0^2$ (equiv $c^2 - 4km$)

Damping	$p^2 - \omega_0^2$	Roots λ_1, λ_2
Under-damping	< 0	Complex, real part < 0
Critical Damping	$= 0$	Repeated real, negative
Over-damping	> 0	Distinct real, negative

Underdamping: $p^2 - \omega_0^2 < 0$

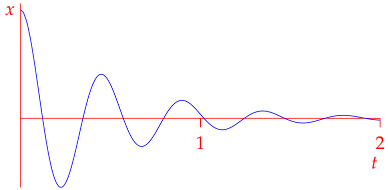
$c^2 < 4km$: damping force small

Roots $\lambda_1, \lambda_2 = -p \pm i\omega_1$ where $\omega_1 := \sqrt{\omega_0^2 - p^2} < \omega_0$

General solution

$$x(t) = e^{-pt}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) = Ce^{-pt}\cos(\omega_1 t - \gamma)$$

- *Lower frequency oscillations* $\omega_1 < \omega_0$ than undamped
- $x(t) \to 0$ as $t \to \infty$



Critical damping: $p^2 - \omega_0^2 = 0$

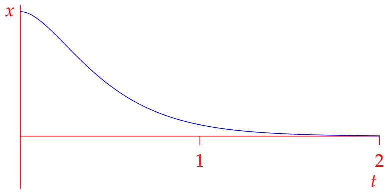
$c^2 = 4km$: damping perfectly matched to spring/mass

Roots *real, negative* and *repeated*: $\lambda_1 = \lambda_2 = -p$

General solution

$$x(t) = (A + Bt)e^{-pt}$$

- *No oscillations*
- $x(t) \to 0$ as $t \to \infty$



Overdamping: $p^2 - \omega_0^2 > 0$

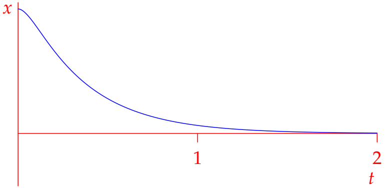
$c^2 > 4km$: damping large compared to spring stiffness/mass

Roots $\lambda_1, \lambda_2 = -p \pm \sqrt{p^2 - \omega_0^2}$ *real* and *negative*

General solution

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

- *No oscillations*
- $x(t) \to 0$ as $t \to \infty$



Changing c and k

The animations show what happens for various values of c and k, and the initial conditions $x(0) = 1, x'(0) = 0$

Increase k Faster response, becomes more bouncy/shaky

Increase c Slower response, becomes softer/smoother

Suspension Examples

Vehicle suspensions may be modeled by these equations:

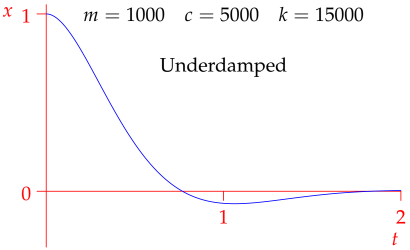
m is 1/2 or 1/4 the vehicle's mass (per wheel)

k is the sptiffness of each spring

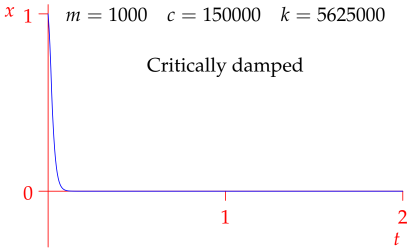
c comes from the hydraulic dampers

Choose k and c to fit application

Tractor/Semi-truck Usually very underdamped: c, k small, $c^2 < 4km$ for slow, relaxed response



Sports Car Close to critically damped:
c, k very large
Fast, stiff response



Family sedan Slightly underdamped:
c, k moderate
Smoother response