Math 8 — Functions and Modeling

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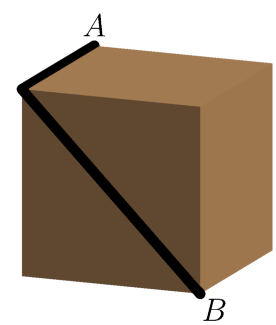
Introduction

This course aims to refresh and provide some conceptual foundation to several mathematical topics commonly encountered at grade-school level. The job of a teacher is often one of *selection*; choosing examples and explanations suited to the level and experience of your students. To do this effectively, you need to understand concepts at a higher level than you'll likely ever teach. Not all of our topics are central to the grade-school curriculum, and it is not our goal to teach you *how* to teach, though the ideas and approaches we'll explore are often suitable for a grade-school audience. The *mathematics* in this course shouldn't present much difficulty, requiring at most elementary calculus and a tiny bit of linear algebra; you should instead be considering how to *explain* the material, particularly to students with less mathematical knowledge than yourself.

We start with two motivational problems.[1]



1. You wish to travel across the surface of a cube between two opposite vertices so that your path is as short as possible.

   Should you follow the path indicated?

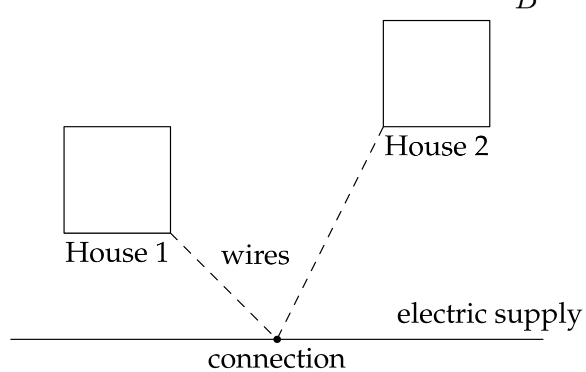
   If yes, explain why.

   If not, how should you find the shortest path?

2. Two houses are to be connected to the electricity supply using a single connection.

   How should we determine where to place the connection so as to minimize the required length of wire?

   What information do you need in order to find the connection point?



The goal isn't merely to find the right answer! Consider how you might discuss these problems with grade-school students of different ability levels. Why might calculus *not* be a sensible approach? Are there any similarities between the two problems? Brainstorm some strategies. . .

[1] We are grateful to materials from UT Austin's UTeach program for suggesting several of the examples in this course including these motivational problems.

1 Sets & Functions

1.1 Basic Definitions

Consider how central functions are to mathematics, and how long you've been using them. How would you *define* "function" to someone with limited mathematical knowledge? Would you use words like *rule, assign, element, domain, vertical line test,* etc.? How helpful are these to your audience?

Examples 1.1. How would you explain the idea that the following do or do not represent functions?

1. $y = x^2$
2. Mon: fish, Tue: pork, Wed: fajitas, Thur: carbonara, Fri: pizza, Sat: fish, Sun: pizza
3. $(3,5)$, $(2,6)$, $(4,2)$, $(3,1)$.
4. $x^2 = y^2$

After considering the example, perhaps you settle on a semi-formal definition:

A function f is rule which assigns to each input x exactly one output $f(x)$

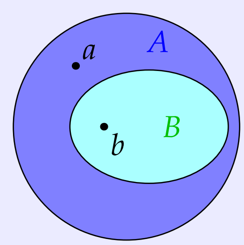
Is this a useful definition? In what ways is it imprecise? How much does this matter?

The answers to these questions depend on your audience! From a formal palate a teacher selects enough to convey something important without overburdening and intimidating their students. For a more complete picture, we begin by thinking about what we allow to be *inputs* and *outputs.*

Definition 1.2. A *set* A is a collection of objects, known as *elements.*[2] The notation $a \in A$ means that a is an element of A, sometimes read as 'a lies in A.' Sets are usually written with upper case letters and elements with lower.

A set B is a *subset* of a set A, written $B \subseteq A$, if every element of B is also an element of A: that is,

$$b \in B \implies b \in A$$



The picture indicates sets A, B and elements a, b for which $B \subseteq A$, $a \in A$, $b \in B$ and $a \notin B$.

Examples 1.3. 1. Suppose the elements of a set A are the numbers 1, 3, 5, 7 and 9. The simplest way to write this is using *roster notation*: list the elements (in any order!) between braces

$$A = \{1,3,5,7,9\}$$

Subsets are commonly expressed in *set-builder notation*: $\{a \in A : \text{condition on } a\}$. For example,

$$B = \{a \in A : 2 < a < 8\} \qquad \text{(the set of } a \text{ in } A \text{ such that } a \text{ lies strictly between 2 and 8)}$$

In roster notation, $B = \{3,5,7\}$: plainly B is a subset of A. Can you express B in other ways using set-builder notation?

[2]This is enough for our purposes, though a course in set theory will convince you that this definition has its own problems. Selection is always at work, at all levels of education. . .

2. You should be familiar with common sets of numbers: we summarize these using informal combinations of roster and set-builder notation.

   Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. For instance $5 \in \mathbb{N}$ but $-3 \notin \mathbb{N}$.

   Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$. For instance $-4 \in \mathbb{Z}$ but $\frac{4}{5} \notin \mathbb{Z}$.

   Rational numbers or fractions: $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N}\right\}$. For instance $-\frac{6}{7} \in \mathbb{Q}$; in this case $p = -6$ is an integer, and $q = 7$ a natural number.

   Real numbers $\mathbb{R}$: for instance $\sqrt{2} \in \mathbb{R}$. A formal definition is difficult, but you should be used to identifying the real line with a *ruler*. *Intervals* are particularly important subsets, e.g.

   $$[-4, \pi) = \{x \in \mathbb{R} : -4 \leq x < \pi\}$$

   is a half-open interval.

   You should also be familiar with the *Cartesian plane*: $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The notation $(3, 4) \in \mathbb{R}^2$ describes a *point* in the plane with *co-ordinates* $x = 3$, $y = 4$: don't confuse this with the *interval* $(3, 4) = \{x \in \mathbb{R} : 3 < x < 4\}$ which is a subset of $\mathbb{R}$!

   The subset relationships between these sets should be familiar:

   $$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

   You should also have informally encountered the notion of *irrationality*: for instance, $\sqrt{2}$ and π are real numbers but not rational numbers.

The reason we need this language when discussing functions is that the inputs and outputs of a function are *elements* of sets. Here is a *very* formal definition of "function."

Definition 1.4. The *Cartesian product* of sets A, B is the set of *ordered pairs*

$$A \times B = \big\{(a, b) : a \in A, b \in B\big\}$$

A *function* from A to B is a non-empty subset $f \subseteq A \times B$ which satisfies the *vertical line test*

$$\text{For each } a \in A \text{, there is a } \textit{unique } b \in B \text{ such that } (a, b) \in f \tag{$*$}$$

Instead of writing $f \subseteq A \times B$ and $(a, b) \in f$, we use the more familiar notation

$$f : A \to B \quad \text{and} \quad f(a) = b$$

To a function $f : A \to B$ are associated three sets:

- Domain: $\operatorname{dom} f = A$ is the set of *inputs.*
- Codomain: $\operatorname{codom} f = B$ is the set of *possible outputs.*
- Range: $\operatorname{range} f = \{b \in B : b = f(a) \text{ for some } a \in A\}$ is the set of *realized outputs.*

This probably isn't the definition you should give to 10th graders, or even to freshman calculus students! But what should you do? How much of this is helpful in a a given context?

Example (1.1.2 cont.). We consider our earlier food-based example in this formal setting. To do this properly, we have to carefully label the constituent sets. For instance:

$$A = \{\text{Mon, Tue, Wed, Thu, Fri}\}, \qquad B = \{\text{carbonara, fajitas, fish, pizza, pork}\},$$

$$f = \{(\text{Mon, fish}),\ (\text{Tue, pork}),\ (\text{Wed, fajitas}),$$
$$(\text{Thu, carbonara}),\ (\text{Fri, pizza}),\ (\text{Sat, fish}),\ (\text{Sun, pizza})\}$$

We made a choice with the codomain B: can you see how? What would be a different choice?

Try the other examples yourself.

Representing Functions

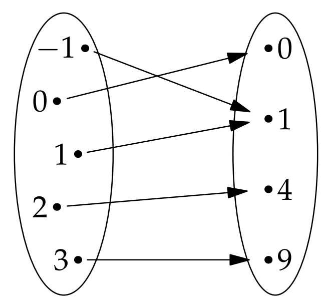
You should be familiar with several methods for representing a function.

Example 1.5. We consider the familiar *formula/rule* $f(x) = x^2$ in several contexts.

Table This presentation is most helpful when the domain is very small. The table shows the situation when $\operatorname{dom} f = \{-1, 0, 1, 2, 3\}$ and $\operatorname{range} f = \{0, 1, 4, 9\}$

x	-1	0	1	2	3
$f(x)$	1	0	1	4	9

Arrows A pictorial arrow diagram might also be useful for illustrating functions with small domains.



Graph This is simply the set of ordered pairs $\{(x, f(x)) : x \in \operatorname{dom} f\}$: in the context of Definition 1.4, *the graph is the function!*

For formulæ whose inputs and outputs are real numbers, two conventions are typically followed unless stated otherwise:

- The domain is *implied* to be all real numbers for which the formula makes sense.
- The codomain is taken to be the set of real numbers.

If no other information is provided, we'd assume the function defined by the formula $f(x) = x^2$ has both domain and codomain the entire set of real numbers: $f : \mathbb{R} \to \mathbb{R}$.

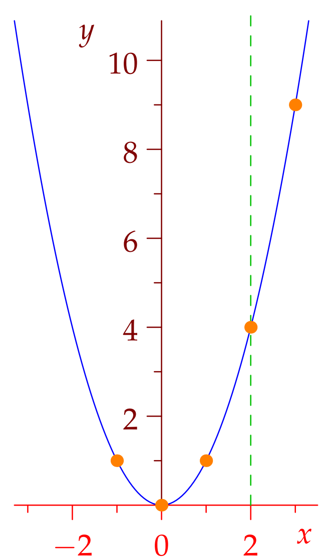
The range of the function is the set of possible outputs, in this case

$$\operatorname{range} f = \{x^2 \in \mathbb{R} : x \in \mathbb{R}\} = [0, \infty)$$

is the half-open interval of non-negative real numbers.

For 'calculus' functions like these, the vertical line test really involves vertical lines; every vertical line intersects the graph in precisely one point.

In the picture, the dots are the graph when the domain is the finite set $\{-1, 0, 1, 2, 3\}$ (as described in the table/arrow-diagram).



Can you think of other ways to represent a function? How might you decide which to use?

Exercises 1.1. 1. Let d represent the cost in millions of dollars to produce n cars, where n is measured in 1000s. As clearly as you can, explain what is meant by $d(25) = 431$.

2. Temperature readings T were recorded every two hours from midnight to noon. Time t was measured in hours from midnight.

t	0	2	4	6	8	10	12
T (°F)	82	75	74	75	84	90	93

   (a) Plot the readings and use them to sketch a rough graph of T as a function of t.

   (b) Use your graph to estimate the temperature at 10:30 a.m.

3. State parts 1, 3 and 4 of Example 1.1 using the formal language of Definition 1.4. If you have a function, state the domain and range and explain how you know you have a function. If you don't have a function, explain why not.

   (*Since insufficient information is provided, there is no single correct answer!*)

4. (a) Let $A = \{1, 3, 5, 7, 9\}$. Explain in words what is meant by the set

$$B = \{x \in A : x^2 > 10\}$$

   and state B in roster notation.

   (b) Find the set $C = \{x \in \mathbb{N} : (x-1)^2 < 16\}$ in roster notation.

   (c) Find the Cartesian product $B \times C$ in roster notation. Is it the same as $C \times B$?

5. Suppose that $f : \{-2, -1, 0, 1, 2\} \to \mathbb{R}$ is defined by the formula $f(x) = x^3 - 4x + 1$.

   Describe f using a table, an arrow diagram and a graph.

6. Find the implied domain and range for the functions defined by the following rules:

   (a) $f(x) = \frac{x^2-4}{x-2}$

   (b) $g(x) = \sqrt{x^2 - 16x}$

   (c) $h(x) = \frac{1}{x}\sqrt{4x - x^2}$

   (*What is the largest set of real numbers for which the formula makes sense*?)

7. The unit circle is often represented by the implicit equation $x^2 + y^2 = 1$.

   (a) Draw the circle and explain why the full circle isn't the graph of a function.

   (b) Describe *two* functions $f : [-1, 1] \to \mathbb{R}$ and $g : [-1, 1] \to \mathbb{R}$ whose graphs together comprise the circle. What are the ranges of the two functions?

1.2 Linear Polynomials

Perhaps the simplest functions are the *linear polynomials*, whose graphs are straight lines;

$$y = f(x) = mx + c \quad \text{where } m, c \text{ are constants} \tag{$*$}$$

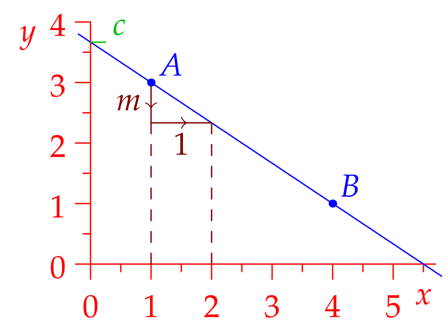
These functions make for very simple models: increase the input by Δx and the output changes by $\Delta y = m\Delta x$ *regardless* of the starting value x. Given some experimental data or a physical situation relating two quantities x and y, a *linear model* is an linear polynomial $(*)$ relating these variables. In practice, models are usually only *approximate*. Later in the course we'll consider how to find good linear models for approximately linear data.

Example 1.6. Find the equation of the straight line through the points $A = (1, 3)$ and $B = (4, 1)$.

Substitute both points into the equation and solve

$$\begin{cases} 3 = m + c \\ 1 = 4m + c \end{cases} \overset{(*)}{\Longrightarrow} -2 = 3m$$

$$\Longrightarrow m = -\frac{2}{3}, \quad c = 3 - m = \frac{11}{3}$$



There is some technique required in $(*)$: how would you explain this to a grade-school student?

The *gradient/slope* m represents how far one climbs/falls on travelling one unit to the right.

The *y-intercept* c is the intersection of the graph with the y-axis.

Some bookkeeping is required here: how do we know that every (non-vertical) line corresponds to such a linear function? This follows easily from a useful fact regarding parametrizations.

Theorem 1.7. *Given distinct points A, B, the set of points on the line through A and B is*

$$\ell_{A,B} = \{(1-t)A + tB : t \in \mathbb{R}\}$$

In this description of a line, $t = 0$ corresponds to A and $t = 1$ to B. In essence, we are laying a ruler onto the line whose units correspond to the distance $|AB|$.

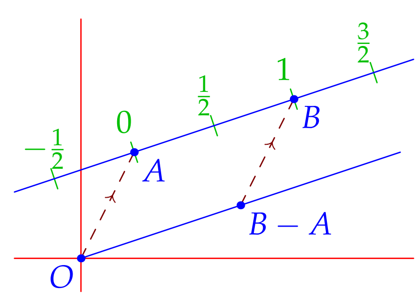
Proof. There are several ways to think about this; we use what is essentially *vector addition*.

The line through the origin and the point $B - A$ is described by the set of points

$$\ell_{O,B-A} = \{t(B-A) : t \in \mathbb{R}\}$$

We simply shift this line by A.

The points corresponding to various values of t are marked.



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Would you consider presenting an argument like this in a grade-school class? Is the argument helpful for understanding *how* to visualize/describe a line? Discuss…

Exercise 5 uses this result to show how any non-vertical line may be described by a linear polynomial. This is nothing more than a generalization of a simple example.

Example 1.8. The line through points $A = (3,6)$ and $B = (-1,4)$ may be parametrized by

$$(x,y) = (1-t)(3,6) + t(-1,4) = \big(3-4t, 6-2t\big)$$

By solving for t in terms of x, we see that this has equation

$$y = 6 - 2t = 6 - 2 \cdot \frac{3-x}{4} = \frac{1}{2}x - \frac{9}{2}$$

Exercises 1.2.

1. The cost of gasoline is \$4.20 per gallon on January 1^{st} and \$4.90 on March 1^{st}. State a *linear* function/model for how the cost of gasoline depends on time.

2. You have a choice of three different cell-phone plans.

   (a) No monthly charge and 10¢ per minute for all calls.

   (b) \$10 per month and 5¢ per minute for all calls.

   (c) \$30 per month, regardless of how many calls you make.

   How should you determine which of the plans to purchase?

3. Revisit Exercise 1.1.2. Find an approximate linear model $T(t) = mt + c$ for this data.

   (*There is no perfect answer*!)

4. Suppose $y = mx + c$ is the equation of a linear function. Choose any two points A, B on this line, and thus find an explicit parametrization in the style of Theorem 1.7.

5. Suppose $A = (x_0, y_0)$ and $B = (x_1, y_1)$ are given. If $x_1 \neq x_0$, find the equation $y = mx + c$ of the line through these points.

   (*You should recognize m as the familiar 'rise over run'*)

6. A straight line is sometimes described as the set of points $(x,y) \in \mathbb{R}^2$ satisfying an equation of the form

   $$ax + by = c$$

   for some constants a, b, c where a, b are not both zero. How does this approach differ from our use of linear polynomials?

7. Suppose that a linear polynomial $f(x) = mx + c$ is also a *linear function*:

   $$\text{For all } \lambda, x \in \mathbb{R}, \quad f(\lambda x) = \lambda f(x)$$

   What can you say about f?

   (*This is the meaning of* linear *you'll encounter in a linear algebra class*)

1.3 Quadratic Polynomials

Quadratic polynomials are functions of the form $y = f(x) = ax^2 + bx + c$ where $a \neq 0$. The simplest is $y = x^2$, the standard parabola opening upwards. Here are some commonly encountered activities:

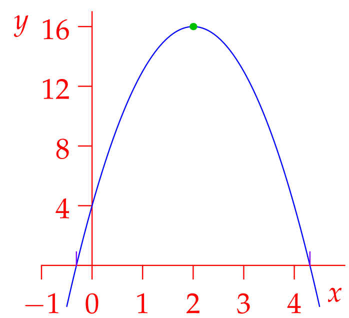
1. Find the *roots/zeros* of f, the solutions x to the equation $f(x) = 0$.
2. Sketch the *graph* of the function f.
3. Use quadratic functions to model a real-world problem.

You likely know two methods for finding zeros: factorizing and the quadratic formula, each of which has its problems. With experience it is easy to spot that

$$x^2 + 2x - 15 = (x-3)(x+5) = 0 \iff x = 3 \text{ or } x = -5$$

though the required creativity can make this difficult, particularly when coefficients are large. Students often prefer the quadratic formula since it always works, though at the cost of some intimidating algebra. We'll think about factorization shortly. First, we see how *completing the square* lies behind both the quadratic formula and the standard approach to graphing quadratic functions.

Example 1.9. Describe/graph the parabola $y = -3x^2 + 12x + 4$.



Pay attention to the x terms; $-3x^2 + 12x = -3(x^2 - 4x)$. Now

$$-3(x-2)^2 = -3(x^2 - 4x + 4) = -3x^2 + 12x - 12$$

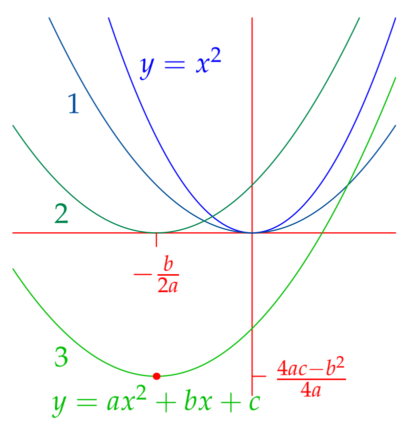
gives most of what we want: note how we *divided the x-coefficient by two.* To finish, just tidy everything up,

$$y = (-3x^2 + 12x - 12) + 16 = -3(x-2)^2 + 16$$

The parabola therefore opens downwards ($-3 < 0$) with its apex (maximum) at $(x, y) = (2, 16)$.

This is easy, if intimidating, to repeat in general:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \qquad (*) \end{aligned}$$



The graph is that of the standard parabola which has been:

1. Vertically scaled by a;
2. Shifted horizontally by $-\frac{b}{2a}$;
3. Shifted vertically by $\frac{4ac-b^2}{4a}$

By solving $(*)$ for x, we see that completing the square yields the *quadratic formula.*

Theorem 1.10. *If $a \neq 0$, then* $ax^2 + bx + c = 0 \iff x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example (1.9 cont). Our analysis suggests two methods for finding the roots.

1. Quadratic formula: with $a = -3$, $b = 12$, $c = 4$, we have

$$x = \frac{-12 \pm \sqrt{12^2 - 4(-3)\cdot 4}}{2(-3)} = \frac{-12 \pm 4\sqrt{3^2+3}}{-6} = 2 \pm \frac{\sqrt{12}}{3} = 2 \pm \frac{2\sqrt{3}}{3}$$

   While it is always tempting to jump for a formula, it often leads to difficult surd expressions. We simplified by noticing the common factor of 4^2 inside the square root. Without this, we'd be faced with $\sqrt{144+48} = \sqrt{192}$.

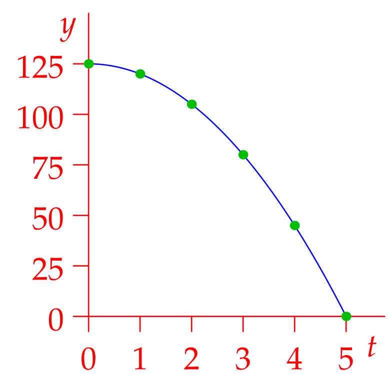
2. Use the fact that we've already completed the square:

$$-3(x-2)^2 + 16 = 0 \iff (x-2)^2 = \frac{16}{3} \iff x = 2 \pm \frac{4}{\sqrt{3}}$$

   In many cases it is simpler to complete the square than to use the quadratic formula—remember that they are equivalent!

Polynomials are often employed in modelling due to their simplicity and ease of evaluation. As you saw in calculus, the motion of a falling body, or of any projectile can be modelled using quadratic polynomials, an observation going back to at least to Galileo in the early 1600s: the distance travelled by a falling body is proportional to the *square* of the time taken $y(t) - y(0) \propto t^2$.

Example 1.11. A body is dropped from a height of 125 meters, taking exactly 5 seconds to reach the ground. Its height at time t seconds is given by $y(t) = 125 - 5t^2$ m.



This certainly fits Galileo's observation: $y(t) - y(0) = -5t^2$ is indeed proportional to t^2.

Over each interval of 1 s, we may ask how far the body falls; we summarize in a table.

t	0	1	2	3	4	5
$y(t)$	125	120	105	80	45	0
$y(t) - y(0)$	0	-5	-20	-45	-80	-125
Δy		-5	-15	-25	-35	-45

Since each interval has duration 1 s, each Δy is the *average speed* of the falling body over that interval.

You'll have seen problems like this in calculus; likely you want to *differentiate* to find the *velocity* $y(t) = -10t$ m/s and *acceleration* $y''(t) = -10$ m/s^2. However, historically and in introductory calculus, it is problems like these that *motivate the definition* of the derivative.[3]

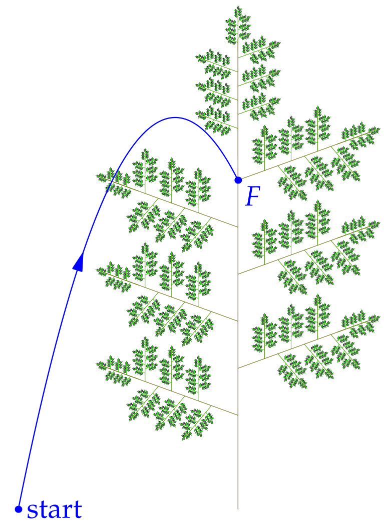
Armed with calculus, Galileo's observation is that the height $y(t)$ solves the differential equation

$$\frac{d^2y}{dt^2} = -g$$

where g is the constant acceleration due to gravity; approximately 32 ft/s^2 or 10 m/s^2. Unless you are explicitly teaching calculus or Newtonian physics, this is probably a bad place to start!

[3] The last line of the table really does suggest that speed is a linear function!

Example 1.12. Your frisbee is stuck 15 m up a tree. Standing 10 m from the base, you throw a ball with the intent of knocking the frisbee out of the tree.



The standard approach to modelling such problems involves considering the horizontal and vertical motions separately.

Horizontal $x(t) = pt + q$ is a *linear function* of time.

Vertical $y(t) = -10t^2 + rt + s$ is a *quadratic function* of time.

Substituting for t yields a quadratic function for the trajectory

$$y(x) = ax^2 + bx + c$$

We'll leave the details of the solution to Exercise 4. For the present, consider why there are *multiple answers*; can you explain why *without* explicitly solving the problem?

Exercises 1.3. 1. Complete the square for each quadratic function, use it to find the range and to graph the function.

   (a) $f(x) = x^2 - 6x + 5$
   (b) $f(x) = -x^2 + x + 1$
   (c) $f(x) = -3x^2 + 8x + 5$

2. For the quadratic function $y = 2x^2 - 5x + 7$, produce a table for $x \in \{0, 1, 2, 3, 4, 5, 6\}$ similarly to that in Example 1.11. What do you observe about Δy?

3. (a) Find the equations of all quadratic polynomial functions which pass through the points $(1, 3)$ and $(2, 4)$.
   (b) More generally, if $P = (a, b)$ and $Q = (c, d)$ are given, where $c \neq a$, find all quadratic functions whose graphs contain P and Q.

4. Consider the frisbee/tree problem (Example 1.12).

   (a) Assume that you're standing at the origin and the frisbee is at the point $(10, 15)$. Find all trajectories.
   (b) (Hard) Find a formula linking the initial speed and gradient of the parabola (the initial speed and direction in which you throw the ball).
      i. If you throw the ball in such a way that the initial *vertical* speed of the ball is twice its *horizontal* speed, find how fast you have to throw the ball in order to hit the frisbee.
      ii. What is the *minimum* speed at which you could throw the ball if you want to dislodge the frisbee?

   (*Hint: You'll need some calculus! In the language of the original problem, the initial slope is* $m = \frac{r}{p}$ *and speed* $v = \sqrt{p^2 + r^2}$*; why?*)

1.4 Polynomials, Factorization & the Rational Roots Theorem

Recall our simple example of factorization in the previous section

$$x^2 + 2x - 15 = (x-3)(x+5) = 0 \iff x = 3 \text{ or } x = -5$$

That this approach provides *all* roots depends on several familiar algebraic facts:

1. Factor Theorem: $f(c) = 0 \iff x - c$ is a *factor* of $f(x)$.
2. No zero-divisors: $pq = 0 \iff p = 0$ or $q = 0$.
3. A quadratic has *at most two* distinct roots.

We'll examine this more closely at the end of this section. For students first learning factorization, it isn't the *why* that's the challenge, it's the *how.* Multiplying out $(x-3)(x+5)$ is mechanical, but factorizing requires some creativity; we can't really factor without somehow knowing that 3 and -5 are roots! Beyond making a lucky guess, how do we go about this?

Example 1.13. Let's re-examine $f(x) = x^2 + 2x - 15 = 0$ in a couple of stages.

Integer solutions The simplest type of root would be an *integer* n. If $f(n) = 0$, observe that

$$\begin{aligned} n^2 + 2n - 15 = 0 &\implies n(n+2) = 15 \implies 15 \text{ is divisible by } n \\ &\implies n = \pm 1, \pm 3, \pm 5, \pm 15 \end{aligned}$$

There are only *eight possible candidates.* It doesn't take long to test all of them:

n	1	-1	3	-3	5	-5	15	-15
$f(n)$	-12	-17	0	-12	20	0	240	180

The two integer solutions are therefore $x = 3$ and $x = -5$.

Rational Solutions If you believe that a quadratic polynomial has *at most* two solutions, then you're done. The next simplest possibility, however, is that a solution be a *rational number* $x = \frac{p}{q}$ where we may assume this is in *simplest terms*.[4] Substituting into the polynomial, we see that

$$\frac{p^2}{q^2} + 2\frac{p}{q} - 15 = 0 \iff p^2 + 2pq - 15q^2 = 0$$

Remembering that p, q are *integers*, we rearrange this equation in two ways:

$p(p+2q) = 15q^2$ Since the left side is a multiple of p, so also is the *right.* Since p, q have no common factors, it follows that p divides into 15 (15 is a multiple of p).

$p^2 = q(15q - 2p)$ Since the right side is a multiple of q, so also is the *left.* Since p, q have no common factors, we conclude that $q = 1$.

The upshot is that the only rational solutions to $f(x) = 0$ are the two *integers* we've already found!

[4] I.e., $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no common factors: $\gcd(p, q) = 1$.

Definition 1.14. A *degree n polynomial* is any function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the *coefficients* a_k are constants with $a_n \neq 0$.

A quadratic polynomial has degree 2 and a linear polynomial $mx + c$ degree one[5] (if $m \neq 0$).
Our analysis in Example 1.13 is easily generalized in a famous result.

Theorem 1.15 (Rational Roots). *Suppose $f(x) = a_n x^n + \cdots + a_0$ has* integer *coefficients where a_n and a_0 are non-zero. If $x = \frac{p}{q}$ is a rational root in simplest terms, then q divides into a_n and p divides into a_0.*

Proof. Substitute into the function and multiply by q^n to obtain an equation where everything is an *integer*

$$\underbrace{a_n p^n + \overbrace{a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1}}^{\text{divisible by } q}}_{\text{divisible by } p} \overbrace{}^{} + a_0 q^n = 0$$

By considering the braced terms and recalling that p, q have no common factors, we conclude that a_n is divisible by q and a_0 by p. ∎

Examples 1.16. 1. If $x = \frac{p}{q}$ is a rational root in lowest terms of $f(x) = 2x^2 - x - 3$, then $q = 1$ or 2 and $p = \pm 1$ or ± 3. The possibilities are therefore

$$x \in \left\{\pm 1, \pm 3, \pm\tfrac{1}{2}, \pm\tfrac{3}{2}\right\}$$

all of which are easily checked:

x	1	-1	3	-3	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$
$f(x)$	-2	0	12	18	-3	-2	0	3

The two roots are indicated and the polynomial can be factorized $f(x) = (2x-3)(x+1)$.

2. If the cubic polynomial $f(x) = x^3 - 2x^2 + 5$ had any rational roots, the only possibilities would be $\pm 1, \pm 5$. However, none of these work,

$$f(1) = 4, \quad f(-1) = 2, \quad f(5) = 80, \quad f(-5) = -170$$

whence $f(x) = 0$ has no rational roots.

Unless there are very few candidates, it can be time-consuming to check them all by hand. Moreover, unless you find n distinct rational solutions, you still don't know that you've found everything. The rational roots theorem is therefore typically used together with factorization; it really just gives you some options for where to start. This still isn't easy, as the next example shows.

[5]A non-zero constant polynomial has degree zero. Convention is for the *zero polynomial* $y \equiv 0$ to have degree $-\infty$, so that the theorem $\deg fg = \deg f + \deg g$ holds for all polynomials.

Example 1.17. Consider the cubic function $f(x) = x^3 - x^2 - 7x + 10$. The rational roots theorem gives us eight candidates for rational roots: $x = \pm 1, \pm 2, \pm 5, \pm 10$. It is not difficult to check the first few of these in your head, for instance,

$$f(2) = 8 - 4 - 14 + 10 = 0$$

By the factor theorem, $x - 2$ must be a factor of $f(x)$. The factorization can be performed in various ways. Here are three options, though all are essentially versions of the same process.

Long/synthetic division You should have practiced this in high-school.

$$\begin{array}{r l}
 & \phantom{x^3 -{}} x^2 + x - 5 \\ \cline{2-2}
x - 2 \big) & x^3 - x^2 - 7x + 10 \\
 & -x^3 + 2x^2 \\ \cline{2-2}
 & \phantom{-x^3 +{}} x^2 - 7x \\
 & - x^2 + 2x \\ \cline{2-2}
 & - 5x + 10 \\
 & \phantom{-x^3 - x^2 -{}} 5x - 10 \\ \cline{2-2}
 & 0
\end{array}
\implies x^3 - x^2 - 7x + 10 = (x-2)(x^2 + x - 5)$$

Multiply out and solve We know that $f(x) = (x-2)q(x)$ where $q(x)$ is some quadratic polynomial. Thus let $q(x) = ax^2 + bx + c$ and multiply out:

$$x^3 - x^2 - 7x + 10 = (x-2)(ax^2 + bx + c) = ax^3 + (b - 2a)x^2 + (c - 2b)x - 2c$$

Equating coefficients, we obtain the same factorization as before,

$$a = 1, \quad b = -1 + 2a = 1, \quad c = \frac{10}{-2} = -5$$

Term-by-term factorization With practice you can factorize in one line with no working!

(a) To create x^3, the first term of the quadratic factor must be x^2

$$x^3 - x^2 - 7x + 10 = (x-2)(x^2 + \cdots) = x^3 - 2x^2 + \cdots$$

(b) To correct the x^2 term, add x (i.e., $x^2 - 2x^2 = -x^2$):

$$(x-2)(x^2 + x + \cdots) = x^3 - x^2 - 2x + \cdots$$

(c) To correct the x term, subtract 5:

$$(x-2)(x^2 + x - 5) = x^3 - x^2 - 7x + 10$$

(d) Since the last term 10 is correct, the factorization worked!

You might have seen other approaches involving arranging the coefficients in a table. Regardless, the calculations required to complete these methods are exactly those seen above; all these methods are versions of the same thing.

Why Does Factorization Work?

The theory of factorization relies on some algebra. Here is a *brief* treatment.

Theorem 1.18 (Factor Theorem). *Suppose $f(x)$ is a degree n polynomial. Then:*

1. *A value c is a root if and only if $f(x) = (x-c)q(x)$ for some (degree $n-1$) polynomial $q(x)$.*
2. *The polynomial has* at most *n distinct roots.*

Proof. 1. ($\Leftarrow$) This is essentially trivial: $f(x) = (x-c)q(x) \implies f(c) = (c-c)q(c) = 0$.

($\Rightarrow$) This relies on the *division algorithm for polynomials*: if f, g are polynomials, then there are unique polynomials q, r with[6]

$$f(x) = g(x)q(x) + r(x) \quad \text{and} \quad \deg r < \deg g$$

In the special case where $g(x) = x - c$ is linear, then $r(x)$ must be a constant and so

$$f(x) = (x-c)q(x) + f(c)$$

2. Suppose $c_1, \ldots, c_n$ are distinct real roots. By part 1, $f(x) = (x-c_1)q_1(x)$. Since

$$0 = f(c_2) = (c_2 - c_1)q_1(c_2) \implies q_1(c_2) = 0$$

we may factor $x - c_2$ from $q_1(x)$ to obtain

$$f(x) = (x-c_1)(x-c_2)q_2(x), \qquad \deg q_2 = n-2$$

Repeat this process to factor out all n linear polynomials $x - c_k$:

$$f(x) = (x-c_1)\cdots(x-c_n)q_n, \qquad \deg q_n = n - n = 0$$

It follows that $q_n \neq 0$ is *constant*. Plainly $f(c) = (c-c_1)\cdots(c-c_n)q_n = 0 \implies c = c_j$ for some j, so there are no other roots. ∎

Example (1.17 cont). We know that $f(x) = x^3 - x^2 - 7x + 10 = (x-2)(x^2+x-5)$. But then

$$f(x) = 0 \iff x - 2 = 0 \text{ or } x^2 + x - 5 = 0$$

The former gives the root $x = 2$, and the latter can be attacked via the quadratic formula or completing the square; the polynomial therefore has exactly three real roots

$$x = 2, \frac{-1 \pm \sqrt{21}}{2}$$

[6] For a given example, q, r may be found by synthetic division. This is similar (and may be demonstrated similarly) to the more familiar division algorithm for integers: if m, n are integers, then there are unique integers q, r for which

$$m = qn + r \text{ and } 0 \leq r < |n|$$

In elementary school, this is typically written $m \div n = q \,\mathrm{r}\, r$ (q remainder r); e.g., $23 \div 4 = 5 \,\mathrm{r}\, 3$ corresponds to $23 = 5 \times 4 + 3$.

Example 1.19. We finish with a quick example of how long division (or any other factorization method as in Example 1.17) computes the ingredients in the division algorithm.

If $f(x) = x^3 + 7x^2 - 2$ and $g(x) = x^2 - 2$, then

$$
\begin{array}{r|rrr}
 & x & +7 & \\
\hline
x^2-2 & x^3 + 7x^2 & & -2 \\
 & -x^3 & +2x & \\
\hline
 & 7x^2 & +2x & -2 \\
 & -7x^2 & & +14 \\
\hline
 & & 2x & +12
\end{array}
$$

$\implies x^3 + 7x^2 - 2 = (x^2 - 2)(x + 7) + (2x + 12)$

Otherwise said, $f(x) = g(x)q(x) + r(x)$, where

$q(x) = x + 7, \ r(x) = 2x + 12$ and $\deg r = 1 < 2 = \deg g$.

Exercises 1.4.

1. Apply the rational root theorem to the polynomial $x^3 + 2x^2 - x - 2$ and use it to factorize the polynomial.
2. Repeat the previous question for the polynomial $6x^2 + x - 2$.
3. Use the rational roots theorem to prove that the polynomial $2x^5 - 3x + 7$ has no rational roots.
4. Factorize the following polynomials and thereby find their (real) roots. Explain your steps carefully.
   (a) $f(x) = x^3 + 2x^2 - 3x$
   (b) $f(x) = x^4 - 13x^2 + 36$
   (c) $f(x) = x^3 - 7x - 6$
5. Show that the polynomial $f(x) = x^6 - 2x^5 - x^4 - 4x^3 - 4x^2 - 4x - 6$ has exactly two real roots by factorizing it.
6. The polynomial $f(x) = 2x^4 - 3x^3 + 2x^2 + 3x - 9$ has only one rational root. Find it and factorize the polynomial as $f(x) = g(x)q(x)$ where $\deg g = 1$.
7. Find unique polynomials $q(x)$ and $r(x)$ for which $f(x) = g(x)q(x) + r(x)$ and $\deg r < \deg g$.
   (a) $f(x) = x^3 + 1$ and $g(x) = x + 2$.
   (b) $f(x) = x^4 + x^3 - 2$ and $g(x) = x^2 + 1$.
8. Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial. 'Complete the cube' by finding a constant k such that
   $$f(x) = a(x-k)^3 + p(x-k) + q$$
   has no $(x-k)^2$ term (here p, q are constants).
   (*Hint: evaluate* $f(x+k)$)
9. Suppose that $\deg f = k$ and $\deg g = l$.
   (a) Show that $\deg(fg) = kl$.
   (b) Is it always the case that $\deg(f+g) = \max(k, l)$? Why/why not?

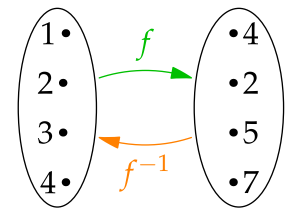
1.5 Inverse Functions & the Horizontal Line Test

The informal idea of an inverse function is that f^{-1} takes the *output* of f and returns its *input* (and vice versa).

Example 1.20. Define a simple function using a table or an arrow diagram

x	1	2	3	4
$f(x)$	4	2	5	7

y	4	2	5	7
$f^{-1}(y)$	1	2	3	4



The inverse f^{-1} is the function obtained by *reversing the arrows* or flipping the table upside-down.

Definition 1.21. A function $f : A \to B$ is *invertible* if it has an *inverse*: a function $f^{-1} : B \to A$ for which

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y \tag{$*$}$$

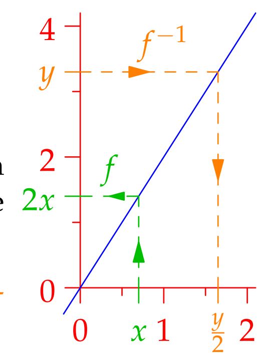
for all possible inputs $x \in A$ and $y \in B$.

Certainly Example 1.20 satisfies the input–output properties $(*)$. Our concerns are identifying when a function is invertible, how to make it so if not, and how to compute an inverse.

Examples 1.22. 1. The function $f(x) = 2x$ has inverse $f^{-1}(y) = \frac{y}{2}$.

The input–output conditions $(*)$ are certainly satisfied.

The graph admits an interpretation of f^{-1} similar to the arrow diagram.

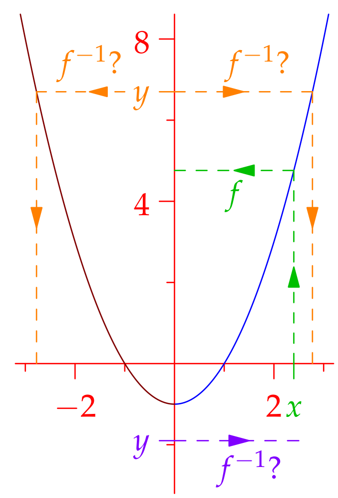


- The function f takes an input x, moves it vertically to the graph, then projects to the y-axis. This interpretation is precisely the vertical line test (Definition 1.4)!
- The inverse function *reverses the arrows*: transport an input y horizontally to the graph, then project to the x-axis.

2. Consider $f(x) = x^2 - 1$. This time, when attempting to move a real number y horizontally to the graph, we usually encounter one of two problems:

   (a) If $y > -1$, there are two choices of x (two intersections).

   (b) If $y < -1$, there is no intersection with the graph.



The naïve approach of *reversing the arrows* is insufficient to define an inverse. However, a simple remedy arises by staring at the graph:

- Problem (a) goes away if we delete the left half of the graph. Equivalently, we *restrict the domain* of f to $[0, \infty)$.
- Problem (b) disappears if we insist that $y \geq -1$. Equivalently, we *restrict the codomain* of f to its *range* $[-1, \infty)$.

After making these restrictions so that $f : [0, \infty) \to [-1, \infty)$, it is easily checked that

$$f^{-1}(y) = \sqrt{y+1}, \quad f^{-1} : [-1, \infty) \to [0, \infty)$$

satisfies the input–output conditions $(*)$ and is therefore the inverse of f:

$$x \in [0, \infty) \implies f^{-1}(f(x)) = \sqrt{(x^2 - 1) + 1} = x$$
$$y \in [-1, \infty) \implies f(f^{-1}(y)) = \left(\sqrt{y+1}\right)^2 - 1 = y$$

What makes a function invertible? The fixes in the last example can be rephrased succinctly:

Horizontal line test: every horizontal line must intersect the graph *exactly once*

This unpacks to two conditions, each of which addresses one of the problems seen in the example.

Definition 1.23. Let $f : A \to B$ be a function. We say that f is:

(a) *1–1/one-to-one* if distinct inputs $x_1 \neq x_2 \in A$ have distinct outputs $f(x_1) \neq f(x_2)$. Equivalently,

Given $x_1, x_2 \in A$, we have $f(x_1) = f(x_2) \implies x_1 = x_2$

If A, B are sets of real numbers, each horizontal line intersects the graph at most once.

(b) *Onto* if range $f = B$. Equivalently,

Given $y \in B$, there is some $x \in A$ for which $y = f(x)$

If $A, B \subseteq \mathbb{R}$, the horizontal line through $y \in B$ intersects the graph at least once.

Putting these ideas together, a function is both 1–1 and onto precisely when every $y \in B$ corresponds to a *unique* $x \in A$ for which $y = f(x)$. In summary:

Theorem 1.24. *$f : A \to B$ is invertible if and only if it is both 1–1 and onto. Its inverse is the function $f^{-1} : B \to A$ such that $f^{-1}(y) = x$ whenever $y = f(x)$.*

Example (1.22.2, mk. II). Consider the two properties in the context of the example $f(x) = x^2 - 1$:

(a) $f(x_1) = f(x_2) \implies x_1^2 - 1 = x_2^2 - 1 \implies x_1^2 = x_2^2 \implies x_1 = \pm x_2$.
To force f to be 1–1, it is enough to *restrict the domain* so that all x have the same sign: the obvious choice is dom $f = [0, \infty)$.

(b) range $f = \{x^2 - 1 : x \in [0, \infty)\} = [-1, \infty)$. We force f to be onto by *restricting its codomain* to $[-1, \infty)$.

The inverse function is obtained by solving $y = x^2 - 1$ for x:

$$x^2 = y + 1 \implies x = f^{-1}(y) = \sqrt{y+1}$$

The *non-negative square root* is used since $x \in \operatorname{dom} f = [0, \infty)$.

An algorithm for inverting functions Our discussion provides an algorithmic process for making a function $f : A \to B$ invertible and finding an inverse.

(a) Check that f is 1–1. If not, *restrict the domain* until it is.

(b) Check that f is onto. If not, *redefine* $B = \operatorname{range} f$.

(c) Solve $y = f(x)$ for $x = f^{-1}(y)$.

Since x is typically preferred as an input, it is common to *switch* x, y at the end of step 3 and write $y = f^{-1}(x)$. If $A, B \subseteq \mathbb{R}$, switching $x \leftrightarrow y$ is equivalent to *reflecting the graph* in the line $y = x$.
Note also that step (a) likely involves a *choice*; depending on how you restrict the domain, you can find multiple inverse functions! To see this in action, we return once more to our example.

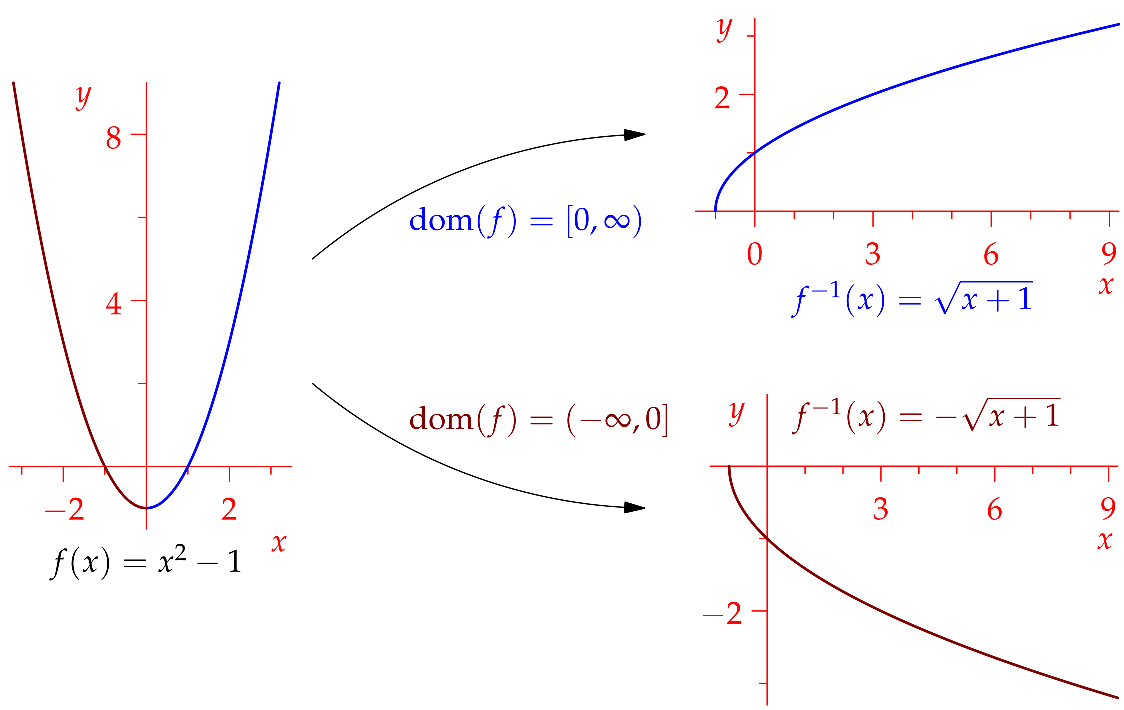
Example (1.22.2, mk. III). Recall that if $f(x) = x^2 - 1$, then

$$f(x_1) = f(x_2) \implies x_1 = \pm x_2$$

Instead of restricting the domain to $[0, \infty)$, we can instead force f to be 1–1 by taking the other half of the graph; by *choosing* $\operatorname{dom} f = (-\infty, 0]$. The range/codomain remains $[-1, \infty)$, but the inverse function is now different:

$$x^2 = y + 1 \implies x = -\sqrt{y+1} \in (-\infty, 0] = \operatorname{dom} f \implies f^{-1}(x) = -\sqrt{x+1}$$

This time the new domain for f forced us to use the *negative square root.*

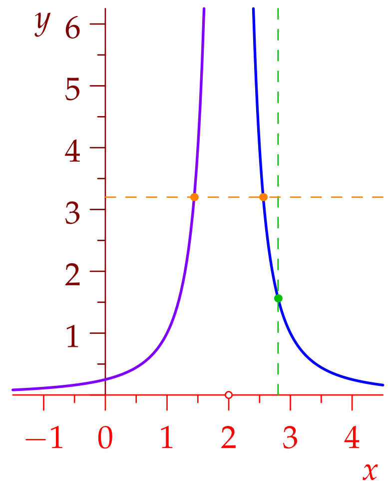


We could choose other domains on which f is 1–1, but these are the most natural choices.

The moral is that you cannot invert a function unless you are precise about its domain and range!

We finish with an algebraically tougher example: you may feel that more detail is justified!

Example 1.25. Let $y = f(x) = \frac{1}{(x-2)^2}$. Its implied *domain* consists of all real numbers except 2.



The *vertical line test* is clearly visible on the graph: every vertical line $x = a$, except $x = 2$, intersects the graph exactly once.

The *range* is the interval $\mathbb{R}^+ = (0, \infty)$ as can be seen by solving

$$f(x) = y \iff \frac{1}{x-2} = \pm\sqrt{y} \iff x = 2 \pm \frac{1}{\sqrt{y}}$$

Any positive output y may be obtained via $y = f\big(2 + \frac{1}{\sqrt{y}}\big)$. The $\pm$-term shows that f fails the *horizontal line test*: it isn't 1–1.

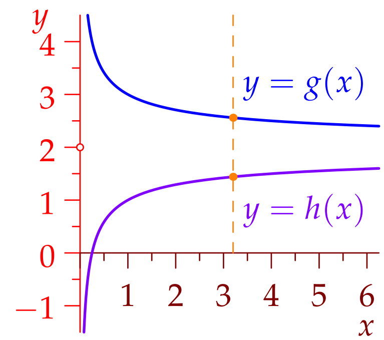
There are two natural choices for an inverse:

(a) Choose $\operatorname{dom} f = (2, \infty)$, then $\pm\sqrt{y} = \frac{1}{x-2}$ is *positive*. We take the *positive* square root and obtain the inverse function

$$g : (0, \infty) \to (2, \infty), \quad g(x) = 2 + \frac{1}{\sqrt{y}}$$

(b) Choose $\operatorname{dom} f = (-\infty, 2)$, then $\pm\sqrt{y} = \frac{1}{x-2}$ is *negative* and we obtain a second inverse function

$$h : (0, \infty) \to (-\infty, 2), \quad h(x) = 2 - \frac{1}{\sqrt{y}}$$



Exercises 1.5. 1. If $\operatorname{dom} f = \mathbb{R}$, check that $f(x) = x^3 + 8$ passes the horizontal line test. Find f^{-1}.

2. Consider $f(x) = x^2 + 2x - 3$. Similarly to Example 1.22, find *two inverses* of f.

3. Sketch the graph of the following function

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } 1 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 3 \end{cases}$$

Find *three* domains on which f is 1–1 and thus compute three distinct inverses.

4. Show that the following function $f : \mathbb{R} \to (\frac{3}{2}, \infty)$ is 1–1 and onto, sketch its graph and find f^{-1}.

$$f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \le 2 \\ 2 - \frac{1}{x} & \text{if } x > 2 \end{cases}$$

5. (Hard) Find the implied domain and range of $f(x) = \frac{x+1}{1+\frac{1}{x+1}}$. Now find an interval on which f is 1–1 and compute its inverse.

6. An astute student observes that Definition 1.21 only describes the properties satisfied by *an* inverse and asks why we keep referring to *the* inverse. How would you respond?