# SAMPLE 3A FINAL

**Problem 1.** (25 pts) Let $A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) (10 pts) Determine if $\vec{b}$ is in the span of the columns of $A$.

(b) (10 pts) Determine if the columns of $A$ are linearly independent. Compute the null space of $A$.

(c) (5 pts) What is the rank of $A$? What is the Nullity of $A$?

**Problem 2.** (20 pts) Let $T(x_1, x_2) = (x_1 + 2x_2, x_1 - 2x_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) (10 pts) Show that $T$ is a linear transformation.

(b) (10 pts) Compute the standard matrix for $T$. Use this matrix to show that $T$ is one-to-one and onto.

**Problem 3.** (20 pts) Let $A = \begin{bmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{bmatrix}$.

(a) (10 pts) Compute $det(A)$ (using any method applicable). Use this to determine whether the columns of $A$ are linearly independent.

(b) (5 pts) Compute $det(A^T A)$.

(c) (5 pts) Let $S$ be a region in $\mathbb{R}^4$ of "volume" 1. Let $A(S)$ be the region in $\mathbb{R}^4$ obtained by mapping $S$ via the map $\vec{x} \mapsto A\vec{x}$. Compute the volume of $A(S)$.

**Problem 4.** (30 pts) Let $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

(a) (15 pts) Compute the eigenvalues and bases for the eigenspaces of $A$.

(b) (5 pts) Determine if $A$ is diagonalizable. If yes, diagonalize it (i.e. find $P$ invertible and $D$ diagonal such that $A = PDP^{-1}$.)

(c) (10 pts) Find a nonzero $2 \times 2$ matrix $B$ that is invertible but not diagonalizable. Explain your answer.

**Problem 5.** (25 pts)

(a) (5 pts) Let $A = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$. Compute the eigenvalues of $A$. The transformation corresponding to $A$ (i.e. $\vec{x} \mapsto A\vec{x}$) is the composition of a rotation and a scaling. The the angle $\varphi$ of rotation $(-\pi < \varphi \leq \pi)$ and give the scale factor $r$.

(b) (10 pts) Let $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Find an invertible matrix $P$ and a matrix $C$ of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PC^{-1}P$.

(c) (10 pts) Solve the following initial-valued problem: $\vec{x}'(t) = A\vec{x}(t)$, where $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.