SOME PROPERTIES RELATED TO THE DAUGAVET PROPERTY

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Abstract. A recent paper of Kadets, Martín, and Merí introduced some properties related to the well-known Daugavet Property of Banach spaces. In this short note, we further investigate the spaces possessing such properties.

Suppose X is a real Banach space, and ε stands for $+$ or $-$. We say that X has the *εSquare Daugavet Property* (*εSDP*, for short) if, for any rank one $T \in B(X)$, $\|I + \varepsilon T^2\| = 1 + \|T^2\|$. These properties were introduced in [3]. This short note complements some results of that paper.

The authors of [3] were investigating various generalizations of the Daugavet Property. Recall that a Banach space X is said to have the *Daugavet Property* (*DP*, for short) if, for every rank one $T \in B(X)$, we have

$$\|I + T\| = 1 + \|T\|. \tag{1}$$

Banach spaces with the DP have been studied extensively in the last 10-15 years (see e.g. Chapter 11 of [1] for information). For instance, if X is a Banach space with the DP, and $T \in B(X)$ doesn't fix a copy of ℓ_1, then (1) holds (Theorem 11.62 of [1]). Moreover, the unit balls of Banach spaces with the DP have no strongly exposed points (Lemma 11.46 of [1]).

The DP can be reformulated in terms of slices of the unit ball (see e.g. [4]). Similarly, slices are important for studying $\pm$SDP. For a Banach space X, we denote by S_X and B_X the unit sphere and the (closed) unit ball of X, respectively. For $x^* \in S_{X^*}$ and $\varepsilon \in (0, 1)$, we define the *slice*

$$S(x^*, \varepsilon) = \{x \in B_X \mid \langle x^*, x\rangle > 1 - \varepsilon\}.$$

Below we establish some geometric properties of Banach spaces with $\pm$SDP. Our main result is:

Theorem 1. *(a) Suppose X is a Banach space with the $-$Square Daugavet Property. Then any slice of B_X has diameter 2. Consequently, B_X has no strongly exposed points. Therefore, X fails the Radon-Nikodým Property.*

(b) If a Banach space X has the $+$Square Daugavet Property, then B_X has no strongly exposed points.

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Proof of Theorem 1(a). Consider $x^* \in S_{X^*}$, $\varepsilon \in (0,1)$, and $x \in S_X \cap S(x^*, \varepsilon)$. By Theorem 4.9(b) of [3], there exists $y \in S_X \cap S(x^*, \varepsilon)$ s.t. $\|(-x) + y\| > 2 - \varepsilon$. This shows that $\operatorname{diam} S(x^*, \varepsilon) = 2$, and therefore, B_X has no strongly exposed points. The last statement now follows from Section VII.3 of [2]. ∎

To prove Theorem 1(b), we need

Lemma 2. *Suppose a Banach space X has the +Square Daugavet Property, and x is a strongly exposed point of B_X, with an exposing functional x^*. Then X is isometric to $\ker x^* \oplus_1 \mathbb{R}$.*

Proof. We have to show that, for any $y \in \ker x^*$, $\|x + y\| = 1 + \|y\|$. First suppose $\|y\| = 1$. Then, by Theorem 4.9(a) of [3], for every $n \in \mathbb{N}$ there exists $x_n \in S(x^*, 1/n)$, s.t. $\|x_n + y\| > 2 - 1/n$. But x is a strongly exposed point, hence $\lim_n x_n = x$. Therefore, $\|x + y\| = 2$.

For the general case, consider $y \in S_X \cap \ker x^*$, and a function $\phi : [0, \infty) \to [0, \infty) : t \mapsto \|x + ty\|$. Clearly, $\phi(0) = 1$, ϕ is convex, and $\phi(t) \leqslant 1 + t$ for any t. On the other hand, $\phi(t) \geqslant 1 + t$ for $t \in [0,1]$. Indeed,

$$\phi(t) \geqslant \|x + y\| - (1 - t)\|y\| = 2 - (1 - t) = 1 + t.$$

Taken together, these facts show that $\phi(t) = 1 + t$, which is what we need. ∎

Proof of Theorem 1(b). Suppose, for the sake of contradiction, that B_X has a strongly exposed point x. Let $x^* \in S_{X^*}$ be an exposing functional, and set $X_0 = \ker x^*$. By Lemma 2, X is isometric to $X_0 \oplus_1 \mathbb{R}$. Find $x_0^* \in X_0^*$ and $x_0 \in X_0$ s.t. $\|x_0\| = 2/3$, $\|x_0^*\| = 1/2$, and $\langle x_0^*, x_0 \rangle = 1/3$. Then, for $x = x_0 \oplus (-1/3) \in S_X$ and $x^* = x_0^* \oplus 1 \in S_{X^*}$, $\langle x^*, x \rangle = 0$.

If X has the +SDP, then, by Theorem 4.9(a) of [3], there exists $y = y_0 \oplus \gamma \in S_X$, such that

$$\langle x^*, y \rangle = \langle x_0^*, y_0 \rangle + \gamma > 11/12, \tag{2}$$

and

$$\|x + y\| = \|x_0 + y_0\| + |\gamma - 1/3| > 23/12. \tag{3}$$

We shall show that these two inequalities cannot hold simultaneously.

Recall that $\|y_0\| = 1 - |\gamma|$, hence $|\langle x_0^*, y_0 \rangle| \leqslant \|y_0\|/2 \leqslant (1 - |\gamma|)/2$. By (2), $(1 - |\gamma|)/2 + \gamma > 11/12$, and therefore, $\gamma > 5/6$. Then

$$\|x_0 + y_0\| + |\gamma - 1/3| \leqslant \|x_0\| + \|y_0\| + \gamma - 1/3 = \|x_0\| + 2/3 \leqslant 5/3,$$

which contradicts (3). ∎

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