MATH 120A MIDTERM EXAM SOLUTIONS (YELLOW PAPER)

WINTER 2015

Problem 1 (10 points). Mark each statement 'T' for true (meaning always true) or 'F' for false (meaning sometimes false). You do NOT need to justify your answers to this problem.

T F "If G is a group whose operation is associative, then G is called an abelian group." False: a group is called abelian if its operation is commutative. (The operation of a group is always associative, by the definition of "group".)

T F "Let $(S, *)$ be a binary structure and let $A, B \subseteq S$. If A and B are closed under $*$, then their union $A \cup B$ is closed under $*$." False: for a counterexample with groups, let $(S, *) = (\mathbb{Z}_6, +_6)$, $A = \{0, 2, 4\}$, and $B = \{0, 3\}$.

T F "The set of all nonzero 3×3 matrices with real entries forms a group under the operation of matrix multiplication." False: nonzero matrices need not be invertible. Also, the product of two nonzero matrices can be the zero matrix (although this can only happen if neither one is invertible.)

T F "The group $(\mathbb{C}^*, \cdot)$ of nonzero complex numbers under multiplication has a subgroup that is isomorphic to $\mathbb{Z}_4$." True: i generates a cyclic subgroup that is isomorphic to $\mathbb{Z}_4$.

T F "If a group G has an element of order 2, then it also has a subgroup of order 2." True: if a is an element of order two, then $\langle a \rangle$ is a subgroup of order 2.

T F "Any two groups of order 2 are isomorphic to each other." True: the function sending the identity element of the first group to the identity element of the second group and sending the other element of the first group to the other element of the second group is an isomorphism.

T F "Being abelian is a structural property of groups." True. We proved this in class.

T F "Let G be a group and $a, b, c \in G$. If $aba = aca$, then $b = c$." True: multiply both sides on the left and on the right by a^{-1}.

T F "Every cyclic group is abelian." True. We proved this in class.

T Ⓕ "There is a bijection from $\mathbb{R}$ to $\mathbb{Q}$." False: $\mathbb{Q}$ is countable (see p. 5 of the textbook) but $\mathbb{R}$ is uncountable (see p. 6 of the textbook, or the e-mail that I sent on January 7.)

Problem 2 (4 points). Define the italicized terms:

(a) What is an *identity element* for a binary structure $(S, *)$?
(b) Assume that $(S, *)$ has an identity element and let $a \in S$.
What is an *inverse* of a in $(S, *)$?

Solution.

(a) An element $e \in S$ such that $a * e = a$ and $e * a = a$ for all $a \in S$.
(b) An element $b \in S$ such that $a * b = e_S$ and $b * a = e_S$ (where e denotes the identity element of S.)

Problem 3 (4 points). Let $n \in \mathbb{N}$ and recall that $GL_n(\mathbb{R})$ denotes the group of all invertible $n \times n$ matrices with real entries under the operation of matrix multiplication.
Let $\vec{x} \in \mathbb{R}^n$ and define the set

$$H = \{A \in GL_n(\mathbb{R}) : A\vec{x} = \vec{x}\},$$

where $A\vec{x}$ denotes the product of a matrix and a vector in the usual sense.
Prove that H is a subgroup of $GL_n(\mathbb{R})$.

Solution.

closure: Let $A, B \in H$. Then $(AB)\vec{x} = A(B\vec{x}) = A\vec{x} = \vec{x}$, so $AB \in H$.
identity: $I_n\vec{x} = \vec{x}$, so $I_n \in H$.
inverses: Let $A \in H$. Then $A\vec{x} = \vec{x}$, so $A^{-1}\vec{x} = A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} = I_n\vec{x} = \vec{x}$. Therefore $A^{-1} \in H$.

Remark. The equation $(AB)\vec{x} = A(B\vec{x})$ is not an instance of associativity of $GL_n(\mathbb{R})$, because $\vec{x}$ is a vector, not a matrix, so $\vec{x} \notin GL_n(\mathbb{R})$. Nevertheless, the equation is true. (This is an instance of a group action on a set, as discussed in Section 16 of the textbook, which we will probably not reach this term.)

Remarks on common errors: Note that "$\vec{x}^{-1}$" is meaningless. Also "$\vec{x}\vec{y}$" is meaningless in the context of this problem: although one can take the inner product $\vec{x} \cdot \vec{y}$ of vectors, it is defined as $\vec{x}^\top \vec{y}$. Anyway, there is no other vector $\vec{y}$ to consider in this problem besides the fixed vector $\vec{x}$.

Problem 4 (4 points). Let $(S, *)$ denote an arbitrary binary structure.
Prove that the property P saying

$$a * a = a \text{ for every } a \in S$$

is a structural property.

Solution. Assume that $(S, *)$ has the property P and let ϕ be an isomorphism from $(S, *)$ to another binary structure $(S', *')$. We want to show that $(S', *')$ has the property P. Let $a' \in S'$. Take $a \in S$ such that $\phi(a) = a'$, using surjectivity of ϕ. Because $a * a = a$ and ϕ is a homomorphism, we have $a' *' a' = \phi(a) *' \phi(a) = \phi(a * a) = \phi(a) = a'$, as desired.

Problem 5 (4 points). For each part, make sure to justify your answer:

(a) Let $\mathbb{R}^*$ denote the set of all nonzero real numbers.
Give an example of a nontrivial subgroup of $(\mathbb{R}^*, \cdot)$ that is finite.

(b) Give an example of a proper subgroup of $(\mathbb{Q}, +)$ that is NOT cyclic.

Solution.

(a) Let $G = \langle -1 \rangle = \{-1, 1\}$.[1] Clearly G is nontrivial and finite, and it is a subgroup because it is the cyclic subgroup generated by -1. (Or argue directly that $\{-1, 1\}$ is a subgroup because 1 is the identity element and $-1 \cdot -1 = 1$.)

(b) Define

$$G = \{a \in \mathbb{Q} : a = 2^n m \text{ for some } m, n \in \mathbb{Z}\}.$$

This is a subgroup: clearly it contains the identity element 0 and is closed under inverses (negatives), and it is closed under addition because if $m, n, m', n' \in \mathbb{Z}$ and we assume without loss of generality that $n \leq n'$, then $2^n m + 2^{n'} m' = 2^n(m + 2^{n'-n} m') \in G$.

It is proper because $1/3 \notin G$. It is not cyclic: if $a \in G$, say $a = 2^n m$ where $m, n \in \mathbb{Z}$, then 2^{n-1} is in G but not in $\langle a \rangle$.

This problem was tricky; if I remember correctly, no student solved it on the exam. To solve this problem, it would help to remember problem 5 from homework set 1.

[1] This is the only example.

Problem 6 (4 points). Consider the group $\mathbb{Z}_{12}$ with the operation of addition modulo 12.

(a) List all of the subgroups of $\mathbb{Z}_{12}$.
How do you know that your list is complete?

(b) Draw the subgroup diagram of $\mathbb{Z}_{12}$.

Solution.

(a) Because $\mathbb{Z}_{12}$ is cyclic and every subgroup of a cyclic group is cyclic, it suffices to list all of the cyclic subgroups of $\mathbb{Z}_{12}$:

$$\begin{aligned}
\langle 0\rangle &= \{0\}\\
\langle 1\rangle &= \mathbb{Z}_{12}\\
\langle 2\rangle &= \{0,2,4,6,8,10\}\\
\langle 3\rangle &= \{0,3,6,9\}\\
\langle 4\rangle &= \{0,4,8\}\\
\langle 5\rangle &= \{0,5,10,3,8,1,6,11,4,9,2,7\} = \mathbb{Z}_{12}\\
\langle 6\rangle &= \{0,6\}.
\end{aligned}$$

This is all of them because $\langle a\rangle = \langle 12-a\rangle$.

(b) Here is one way to draw the subgroup diagram:

$$\begin{tikzcd}
& \mathbb{Z}_{12} \arrow[dl, dash] \arrow[dr, dash] & & \\
\langle 3\rangle \arrow[dr, dash] & & \langle 2\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
& \langle 6\rangle \arrow[dr, dash] & & \langle 4\rangle \arrow[dl, dash] \\
& & \{0\} &
\end{tikzcd}$$