MATH 120A SAMPLE MIDTERM EXAM SOLUTIONS

WINTER 2015

Problem 1 (10 points).

T F "The operation $*$ on $\mathbb{R}$ defined by $a * b = \max(a, b)$ is associative." True: $(a * b) * c$ and $a * (b * c)$ are both equal to whichever of a, b, and c is largest.

T F "The set of all nonzero integers forms a group under multiplication." False: it has an identity element 1, but the element 2 has no inverse because there is no nonzero integer a such that $2a = 1$.

T F "Any two groups of order 4 are isomorphic to each other." False. You saw a counterexample in homework set 3.

T F "Every element of the group $(\mathbb{Z}, +)$ has infinite order." False: the identity element, 0, has order 1.

T F "Let A be a set. If there is a surjection from $\mathbb{N}$ to A, but there is no bijection from $\mathbb{N}$ to A, then A is finite." True.[1]

T F "Let $(S, *)$ be a binary operation and let $A, B \subseteq S$. If A and B are closed under $*$, then their intersection $A \cap B$ is closed under $*$." True: let $x, y \in A \cap B$. Then $x * y \in A$ because $x, y \in A$ and A is closed under $*$. Similarly $x * y \in B$ because $x, y \in B$ and B is closed under $*$. Therefore $x * y \in A \cap B$.

T F "If a and b are identity elements of a binary structure $(S, *)$, then $a = b$." True: the expression $a * b$ evaluates to a and also to b.

T F "The group $(\mathbb{R}^*, \cdot)$ of nonzero real numbers under multiplication has a subgroup that is isomorphic to $\mathbb{Z}_4$." False: if it did, then the element of $\mathbb{R}^*$ corresponding to the element 1 of $\mathbb{Z}_4$ would have order 4. But every element a of $(\mathbb{R}^*, \cdot)$ either has order 1 (if $a = 1$,) order 2 (if $a = -1$,) or infinite order (if $a \neq \pm 1$.)

T F "If G is a cyclic group, then G has no subgroups other than $\{e_G\}$ and G." False: for example, $\mathbb{Z}$ is cyclic and has nontrivial proper subgroups of the form $n\mathbb{Z}$.

T F "Let G_1 and G_2 be subgroups of $GL_2(\mathbb{R})$ under matrix multiplication. If every element of G_1 is diagonal, and $G_1 \cong G_2$, then every element of G_2 is diagonal." False: this is not a structural property. For a counterexample, let $G_1 = \{I, -I\}$ and $G_2 = \{I, A\}$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[1] Let me outline a proof even though it is outside the scope of this course. If there is a surjection f from $\mathbb{N}$ to A then there is an injection g from A to $\mathbb{N}$ given by defining $g(a)$ to be the least $n \in \mathbb{N}$ such that $f(n) = a$. If the range of g is unbounded in $\mathbb{N}$, then one can define a bijection from $\mathbb{N}$ to A. Otherwise, A is finite.

Problem 2 (4 points). Let S be a set.

(a) What is a *binary operation* on S?

(b) If $*$ is a binary operation on S and A is a subset of S, what does it mean for A to be *closed* under $*$?

Solution.

(a) A binary operation on a set S is a function from $S \times S$ to S.

(b) It means that $x * y \in A$ for all $x, y \in A$.

Problem 3 (4 points). Let G be a group and let $a \in G$. Define the function $\phi : G \to G$ by $\phi(x) = axa^{-1}$. Prove that ϕ is an isomorphism from G to G.

Solution. To see that ϕ is injective, let $x, y \in G$ and assume that $\phi(x) = \phi(y)$, which means that $axa^{-1} = aya^{-1}$. Multiplying both sides on the left by a^{-1} and on the right by a, we get $x = y$.

To see that ϕ is surjective, let $y \in G$ and note that $\phi(a^{-1}ya) = aa^{-1}yaa^{-1} = y$.

To see that ϕ is a homomorphism, let $x, y \in G$ and note that $\phi(x)\phi(y) = axa^{-1}aya^{-1} = axya^{-1} = \phi(xy)$.

Problem 4 (4 points). Let G denote an arbitrary group. Prove that the property P saying

$$a^2 = e_G \text{ for every } a \in G$$

is a structural property.

Solution. Let G and G' be groups and let ϕ be an isomorphism from G to G'. Assume that G has property P. We want to show that G' has property P. Let $a' \in G'$. Take $a \in G$ such that $\phi(a) = a'$. We have $a^2 = e_G$, and applying ϕ to both sides gives $\phi(a^2) = \phi(e_G)$. Because ϕ is a homomorphism we have $\phi(a^2) = \phi(a)^2 = (a')^2$ and $\phi(e_G) = e_{G'}$, so $(a')^2 = e_{G'}$ as desired.

Problem 5 (4 points). For each part, make sure to justify your answer:

(a) Give an example of a set S and an operation $*$ on S that is associative but not commutative.

(b) Give an example of a set S and an operation $*$ on S that is commutative but not associative.

Solution.

(a) One example is where S is $M_2(\mathbb{R})$ and $*$ is matrix multiplication. We saw in class that this operation is associative but not commutative.

(b) One example is where S is $\mathbb{R}$ and $*$ is defined by $a * b = (a+b)/2$. We saw in class that this operation is commutative but not associative.

Remark. This is not a very good problem that I wrote here. On the real exam I will try to avoid the situation where a problem says "justify your answer" and a sufficient justification is just "we saw the whole thing in class." But if there is a problem like this it would be a good idea to add some further argument if you have time.

Problem 6 (4 points). Consider the set of matrices $G = \{I, A, B, C\}$ where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then G forms a group under matrix multiplication (you may assume this.)

(a) Compute the order of each element of this group G.

(b) Draw a subgroup diagram for G.

Solution. (a) The order of I is 1 because it is the identity element. The order of A is 2 because $A \neq I$ and $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. The order of B is 2 because $B \neq I$ and $B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. The order of C is 2 because $C \neq I$ and $C^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

(b)

$$\begin{tikzcd}[arrows=dash]
 & G \arrow[dl] \arrow[d] \arrow[dr] & \\
\{I, A\} \arrow[dr] & \{I, B\} \arrow[d] & \{I, C\} \arrow[dl] \\
 & \{I\} &
\end{tikzcd}$$