MATH 161 MIDTERM EXAM SOLUTIONS

(WHITE EXAM)

Problem 1 (3 points). State *either*

(1) Pasch's axiom, *or*
(2) the plane separation property.

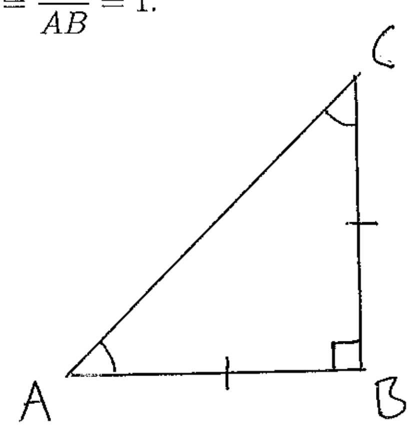
Indicate which one you are attempting to state by circling (1) or (2) above.

Solution. See book or notes.

Problem 2 (5 points). Prove that $\tan(45°) = 1$. (For the definition of $\tan(\alpha)$ you may use $\sin(\alpha)/\cos(\alpha)$ or you may use the "opposite over adjacent" definition.)

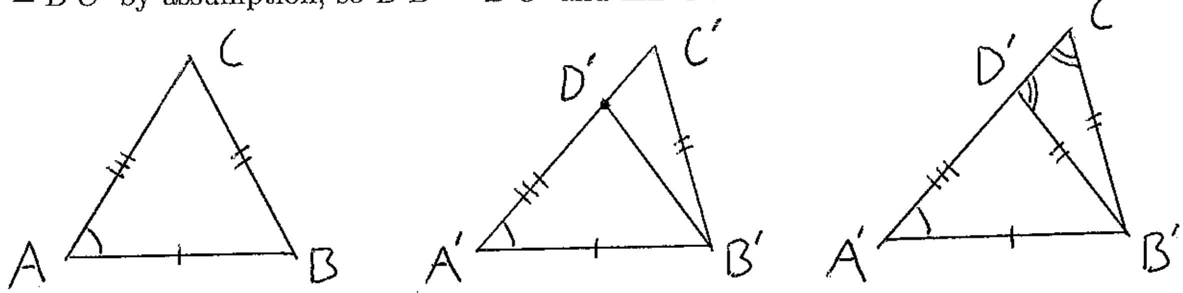
Solution. Take an isosceles right triangle $\triangle ABC$ with a right angle at B, so $CB = AB$. The base angles of any isosceles triangle are congruent and the three angles sum to $180°$, so $m\angle A = m\angle C = 45°$. Therefore

$$\tan(45°) = \tan(\angle A) = \frac{CB}{AB} = 1.$$



Problem 3 (5 points). Let $\triangle ABC$ and $\triangle A'B'C'$ be two acute triangles (all the internal angles are less than $90°$.) Assume that $\angle A \cong \angle A'$, $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$. Prove that $\triangle ABC \cong \triangle A'B'C'$.

Solution. We claim that the third sides are also congruent: $AC = A'C'$. Assme toward a contradiction that they are not: $AC \neq A'C'$. Then without loss of generality we may assume that $AC < A'C'$. Let D' be the point on $\overline{A'C'}$ such that $AC = A'D'$. Then $\triangle ABC \cong \triangle A'B'D'$ by the SAS congruence theorem. In particular $BC = B'D'$. We have $BC = B'C'$ by assumption, so $B'D' = B'C'$ and $\triangle B'C'D'$ is isosceles.



Therefore $\angle B'D'C'$ is the base angle of an isosceles triangle, so it is acute. On the other hand, the supplementary angle $\angle B'D'A'$ is congruent to $\angle BCA$, which by our hypothesis is also acute. Two acute angles cannot be supplementary, so we have reached a contradiction.

This proves the claim that $AC = A'C'$, and now we may apply the SAS congruence theorem (or SSS congruence theorem) to show that $\triangle ABC \cong \triangle A'B'C'$.

Remark. Note that this argument still works just assuming the angles are $\leq 90°$. The case of right triangles was one of your homework problems.

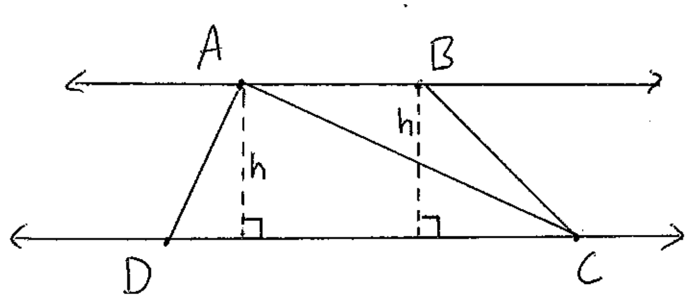
Problem 4 (5 points). Let $ABCD$ be a quadrilateral such that the sides $\overline{AB}$ and $\overline{CD}$ are parallel. The diagonal $\overline{AC}$ splits $ABCD$ into triangles $\triangle ABC$ and $\triangle CDA$. Show that

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle CDA)} = \frac{AB}{CD}.$$

You may use the usual formula for the area of a triangle.

Solution. Consider AB as the base of $\triangle ABC$ and consider CD as the base of $\triangle CDA$. Then the heights of these triangles are equal because both are equal to the perpendicular distance h between the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. (We proved in class that the perpendicluar distance between parallel lines is well-defined.) So we have

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle CDA)} = \frac{\frac{1}{2}(AB)h}{\frac{1}{2}(CD)h} = \frac{AB}{CD}.$$

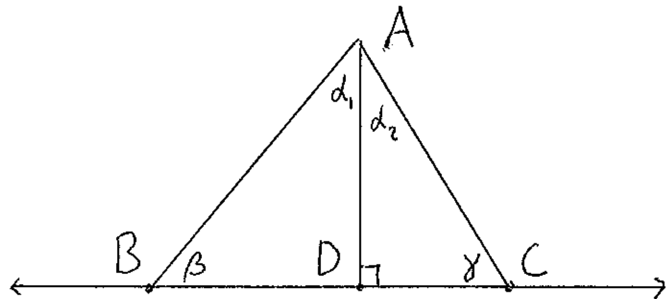


Problem 5 (5 points). For this problem, do *not* assume Euclid's fifth postulate (the parallel postulate) or any of its consequences such as Playfair's postulate, or the statement that the sum of the angles of a triangle is 180°, *etc.*

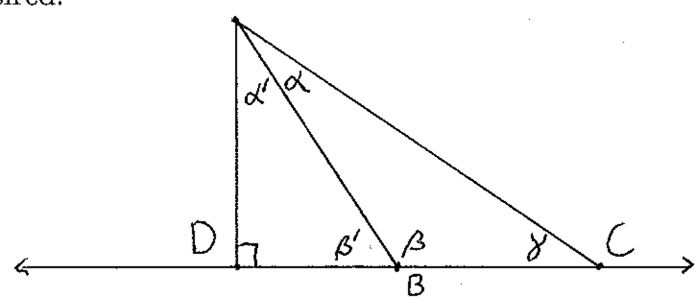
Assuming that the sum of the interior angles of every *right* triangle is 180°, prove that the sum of the interior angles of *every* triangle is 180°.

Remark. You are asked to prove that *all* triangles have a certain property, so you should begin with an arbitrary triangle ("let $\triangle ABC$ be a triangle") and end by proving that it has the desired property ("...so $m\angle A + m\angle B + m\angle C = 180°$.") At some point in the middle, you will need to use the hypothesis regarding right triangles. You can't assume that your triangle $\triangle ABC$ is itself a right triangle, so you should proceed by constructing some right triangles that are related to $\triangle ABC$ somehow, applying the hypothesis to these right triangles, and seeing if this tells you anything about $\triangle ABC$.

Solution. Let $\triangle ABC$ be a triangle. Take a point D on the line $\overleftrightarrow{BC}$ such that $\overline{AD} \perp \overleftrightarrow{BC}$. If D is equal to B or C then $\triangle ABC$ is a right triangle, so its angle sum is 180° and we're done. If not, we consider two cases.



Case 1: D is between B and C. Labeling the angles as shown in the picture, we have $\alpha_1 + \beta + 90° = 180°$ and $\alpha_2 + \gamma + 90° = 180°$ by our hypothesis. So $\alpha_1 + \beta = 90°$ and $\alpha_2 + \gamma = 90°$. Therefore $(\alpha_1 + \alpha_2) + \beta + \gamma = 90° + 90°$. In other words the angle sum of $\triangle ABC$ is 180°, as desired.



Case 2: D is not between B and C. Then we may assume without loss of generality that B is between D and C. Labeling the angles as shown in the picture, we have $\alpha' + \beta' + 90° = 180°$ and $(\alpha' + \alpha) + \gamma + 90° = 180°$ by our hypothesis. Moreover the angles β and β' are supplementary. So we have

$$\alpha' + \beta' = 90°$$
$$\alpha' + \alpha + \gamma = 90°$$
$$\beta' + \beta = 180°.$$

Subtracting the first equation from the sum of the other two, we have $\alpha + \beta + \gamma = 180°$. In other words the angle sum of $\triangle ABC$ is 180°, as desired.