MATH 161 MIDTERM EXAM SOLUTIONS

(YELLOW EXAM)

Problem 1 (3 points). State *either*

(1) Euclid's fifth postulate (also known as the parallel postulate), *or*
(2) Playfair's postulate.

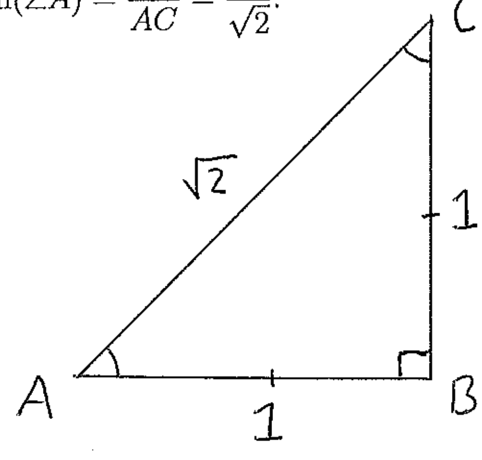
Indicate which axiom you are attempting to state by circling (1) or (2) above.

Solution. See book or notes.

Problem 2 (5 points). Prove that $\sin(45°) = 1/\sqrt{2}$.

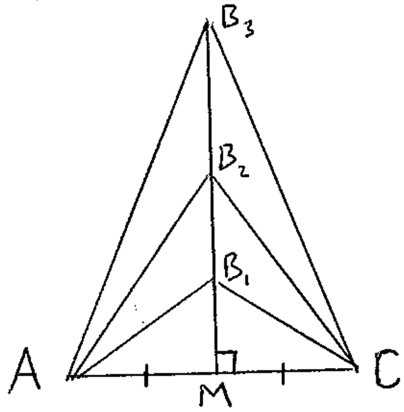
Solution. Take an isosceles right triangle $\triangle ABC$ with a right angle at B, and $CB = AB = 1$. The base angles of any isosceles triangle are congruent and the three angles sum to 180°, so $m\angle A = m\angle C = 45°$. By Pythagoras's theorem, $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$, so

$$\sin(45°) = \sin(\angle A) = \frac{CB}{AC} = \frac{1}{\sqrt{2}}.$$



Problem 3 (5 points). Let $\triangle AB_1C$, $\triangle AB_2C$, and $\triangle AB_3C$ be isosceles triangles with the same base $\overline{AC}$. Prove that the points B_1, B_2, and B_3 are collinear (in other words, there is a line going through all three points.)

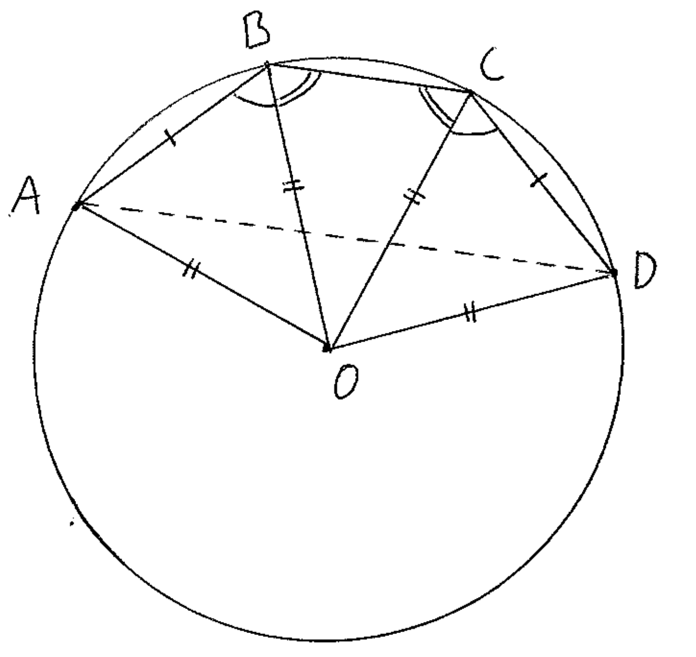
Solution. Let M be the midpoint of $\overline{AC}$. Then $\triangle AMB_i \cong \triangle CMB_i$ for $i = 1, 2, 3$ by the SSS congruence theorem. In particular the angles $\angle AMB_i$ and $\angle CMB_i$ are congruent, and because they are supplementary, they are both right angles. Therefore the points B_1, B_2, and B_3 all lie on the perpendicular bisector of the segment $\overline{AC}$, so they are collinear.



Problem 4 (5 points). Suppose that the quadrilateral $ABCD$ is inscribed in a circle (in other words, there is a circle containing the points A, B, C, and D.) Suppose that $\overline{AB} \cong \overline{CD}$. Prove that $\angle ABC \cong \angle BCD$.

Solution. Let O denote the center of the circle. Then $AO = BO = CO = DO = r$ where r is the radius of the circle. Therefore $\triangle ABO \cong \triangle DCO$ by the SSS congruence theorem, and in particular $\angle ABO \cong \angle DCO$. Moreover the triangle $\triangle BCO$ is isosceles, so its base angles are congruent: $\angle CBO \cong \angle BCO$. Therefore we have

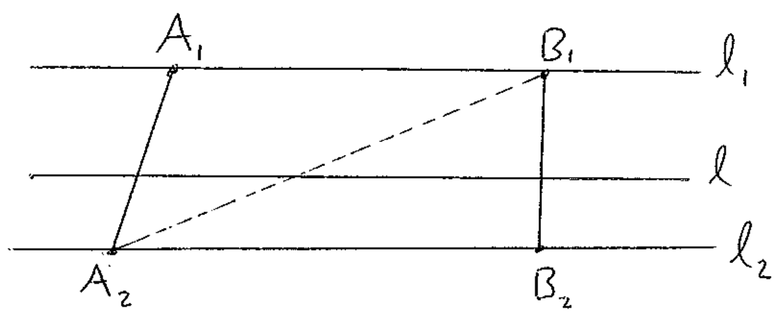
$$m\angle ABC = m\angle ABO + m\angle CBO = m\angle DCO + m\angle BCO = m\angle BCD.$$



Problem 5 (5 points). Let ℓ, ℓ_1, and ℓ_2 be distinct lines that are all parallel to each other (no two of them intersect.) Let A_1 and B_1 be points on ℓ_1 and let A_2 and B_2 be points on ℓ_2. Prove that if the segment $\overline{A_1A_2}$ intersects ℓ, then the segment $\overline{B_1B_2}$ also intersects ℓ.

Solution. Assume that the segment $\overline{A_1A_2}$ intersects ℓ. So the points A_1 and A_2 are on opposite sides of ℓ by definition. The segment $\overline{A_1B_1}$ is contained in ℓ_1, which is parallel to ℓ, so it doesn't intersect ℓ. So the points A_1 and B_1 are on the same side of ℓ by definition. Therefore B_1 and A_2 are on opposite sides of ℓ by the plane separation property.

The segment $\overline{A_2B_2}$ is contained in ℓ_2, which is parallel to ℓ, so it doesn't intersect ℓ. So the points A_2 and B_2 are on the same side of ℓ by definition. Therefore B_1 and B_2 are on opposite sides of ℓ by the plane separation property. In other words, the segment $\overline{B_1B_2}$ intersects ℓ, as desired.



Remark. We can also prove this using Pasch's axiom instead of the plane separation property (indeed, this must be the case, because they are equivalent.) To prove it this way, consider the triangles $\triangle A_1B_1A_2$ and $\triangle B_1A_2B_2$.