STATEMENT OF RESEARCH INTERESTS

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1. Introduction

My research in set theory deals with large cardinals, determinacy, generic absoluteness, and the connections between these topics. The rest of this section gives some background information and the following sections describe my past research and future research plans.

Generic absoluteness. Paul Cohen's work in the 1960s struck a blow against the idea of mathematics as absolute truth. Cohen showed that there is a natural mathematical statement, namely the Continuum Hypothesis, whose truth cannot be proved or disproved in ZFC. (Gödel sentences were discovered earlier but are arguably not natural mathematical statements.)

Cohen's method, called forcing, enlarges a given model of set theory by adjoining an ideal "generic" object, producing a new model that typically exhibits a different set theory. Applications of this method now constitute a major area of research. However, there are limits to what it can do. For example, forcing cannot change the truth of arithmetic statements simply because it cannot change the structure $(\mathbb{N}; 0, 1, +, \times)$ of arithmetic. Accordingly, we say that arithmetic statements are "generically absolute."

As we go up the complexity hierarchy, the nature of mathematical truth undergoes a transition from absolute (such as arithmetic) to relative (such as the Continuum Hypothesis.) For statements about real numbers, the question of generic absoluteness is quite subtle and is tied to the existence of tree representations for sets of real numbers.

A *tree representation* for a set A of "logician's reals" (integer sequences) is a continuous function assigning to every real x a tree T_x such that $x \in A$ if and only if T_x has an infinite branch. The existence of infinite branches is absolute, and the classical absoluteness theorems of Mostowski and Shoenfield can be proved using the existence of tree representations for Σ^1_1 and Σ^1_2 sets of reals respectively.

For more complex statements about real numbers, the connection between generic absoluteness phenomena and tree representations persists for a while, but now to get a complete picture we need to go beyond ZFC and introduce large cardinals. This is because while the trees for Σ^1_1 sets of reals are countable and the trees for Σ^1_2 sets of reals have size $\aleph_1$, the trees for more complicated sets typically have very large cardinality. For example, Σ^1_3 sets of reals are represented by trees of size equal to a measurable cardinal if one exists, and tree representations at higher levels of the projective hierarchy (Σ^1_4 *etc.*) can be obtained from Woodin cardinals if they exist.

For statements a bit more complex than the projective ones, generic absoluteness is not known to follow from *any* large cardinals. My research on generic absoluteness at this level, discussed in Section 2, suggests that it remains nevertheless intimately connected to the existence of tree representations and to the related notion of universally Baire sets of reals.

Large cardinals and determinacy. Large cardinal axioms are strengthenings of the axiom of infinity that postulate the existence of objects so large (loosely speaking) that their existence cannot be proved in ZFC. The interest of such objects lies not so much in their sheer size as in their capacity to tame incompleteness phenomena: large cardinals provide a coherent theory at the lower levels of the complexity hierarchy that is practically complete in the sense of being generically absolute.

The hierarchy of large cardinal axioms is intimately related to the hierarchy of determinacy axioms, beginning with the Axiom of Determinacy (AD) itself, which states that every two-player game of perfect information where the players alternate playing integers $n_0, n_1, n_2, \ldots$ is *determined*: one player or the other has a *winning strategy* ensuring that the resulting integer sequence belongs to the specified set of winning positions for that player.

Like the Axiom of Choice, AD can be motivated as an extrapolation of certain properties of finite sets to the realm of infinite sets. But AD contradicts AC, so we typically think of it as holding not in the universe V itself but in some alternative models. For example, it holds in the "derived models" that can be obtained from infinitely many Woodin cardinals. Derived models incorporate as many determined sets of reals as possible, so they are natural settings for the study of AD.

Section 3 is an outline of my past research and proposed future research on large cardinals and determinacy. I consider the influence of large cardinals on the the extent of derived models, both in terms of the strong determinacy axioms they satisfy and in terms of how many sets of reals they contain. I also consider how this influence is mediated by combinatorial compactness principles. The results I have obtained so far suggest the possibility that Jensen's square principle $\square_\lambda$ might play a role in the study of canonical determinacy models (*i.e.* derived models) analogous to the role it plays in the study of canonical models for large cardinals (*i.e.* core models.)

2. Universally Baire sets and generic absoluteness

A set of reals is *universally Baire* if the set and its complement have tree representations that continue to represent complementary sets in generic extensions (for a precise definition see Feng, Magidor, and Woodin [1].) This is a very strong regularity property that implies Lebesgue measurability and the property of Baire.

Even at the level of the projective sets, there are interesting open questions about the relationship between generic absoluteness and universal Baireness in ZFC. But large cardinals resolve these questions in a satisfactory manner by implying outright that projective properties are generically absolute and projective sets are universally Baire.

The natural next step in the complexity hierarchy beyond projective would be Σ^2_1. But this is too much of a leap: if we are allowed an existential quantifier ranging over arbitrary sets of reals, then we can express the continuum hypothesis, which cannot be generically absolute even assuming large cardinals.

There are two ways to avoid this problem. The first way is to only ask for generic absoluteness between models where the CH is true (see Larson [3] for an exposition of Woodin's work in this area.) The second way is to restrict our existential quantifier to sets of reals that are "nice" in some way, for example universally Baire.

We follow the second way and consider this restricted version of Σ^2_1, denoted by $(\Sigma^2_1)^{\mathrm{uB}}$. Generic absoluteness for $(\Sigma^2_1)^{\mathrm{uB}}$ statements follows from the existence of a proper class of Woodin cardinals (due to Woodin; see Steel [7].) By contrast, generic absoluteness for the slightly larger pointclass $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}}$ obtained from it by complementation and projection is not known to follow from any large cardinal axioms whatsoever.

I am investigating generic absoluteness for the pointclass $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}}$ in relation to the extent of the universally Baire sets, assuming large cardinals. A preliminary result along these lines supports the idea that tree representations are inherent in generic absoluteness phenomena:

Theorem 2.1 (Wilson [9]). *If κ is a measurable limit of Woodin cardinals, then the following statements are equivalent modulo small forcing:*

- *In V_κ, two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness holds.*
- *In V_κ, every $(\Sigma^2_1)^{\mathrm{uB}}$ set of reals is universally Baire.*

I would like to get rid of the measurable cardinal and generalize this result to the more natural setting of a proper class of Woodin cardinals. In this setting, Grigor Sargsyan and I obtained the following equiconsistency result:

Theorem 2.2 (Sargsyan and Wilson [5]). *The following theories are equiconsistent modulo the base theory* ZFC + *"there is a proper class of Woodin cardinals":*

(1) Two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness holds.
(2) Every $(\Sigma^2_1)^{\mathrm{uB}}$ set of reals is universally Baire.

Our method does not seem to show equivalence, even modulo forcing. However, the implication (2) $\Rightarrow$ (1) does hold (Wilson [9, Proposition 4.6]) and I hope to strengthen Theorem 2.2 to an equivalence by proving the converse implication:

Question 2.3. Assume that there is a proper class of Woodin cardinals. If two-step $\exists^{\mathbb{R}}(\mathbf{\Pi}^2_1)^{\mathrm{uB}}$ generic absoluteness holds, then must every $(\Sigma^2_1)^{\mathrm{uB}}$ set of reals be universally Baire?

A related question deals with the relationship between large cardinals and tree representations:

Question 2.4. Are the following theories equiconsistent?

(1) There is a proper class of Woodin cardinals and a strong cardinal.
(2) There is a proper class of Woodin cardinals and every $(\Sigma^2_1)^{\mathrm{uB}}$ set of reals is universally Baire.

Woodin proved the forward direction of Question 2.4 by forcing with $\mathrm{Col}(\omega, 2^{2^\kappa})$ where κ is a strong cardinal. This result was subsequently improved by Wilson [11] to $\mathrm{Col}(\omega, \kappa^+)$, which is optimal. The reverse direction would most likely require an adaptation of methods used by Woodin (unpublished) and Steel [6] to calculate the consistency strength of "there is a limit λ of Woodin cardinals and $<\lambda$-strong cardinals" in terms of determinacy.

3. Derived models, covering, and infinitary combinatorics

The Axiom of Determinacy (AD) is equiconsistent with the existence of infinitely many Woodin cardinals. More specifically, given a limit λ of Woodin cardinals and letting $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter, Woodin showed (using work of Martin and Steel) that AD holds in the model $L(\mathbb{R}^*)$ where $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]}$. Often one can build a larger model of determinacy that contains more subsets of $\mathbb{R}^*$ than just the constructible ones.

The *derived model of* V *at* λ, denoted $D(V, \lambda)$, is essentially the largest determinacy model (in terms of inclusion) that can be built in the symmetric extension $V(\mathbb{R}^*)$. See Woodin [13, Theorem 31] for a precise definition. Note that it is usually harmless to speak of "the" derived model $D(V, \lambda)$ without specifying the generic filter $G \subset \mathrm{Col}(\omega, {<}\lambda)$ because the Levy collapse forcing is almost homogeneous.

Given more large cardinals in V one can show that the derived model satisfies stronger axioms of determinacy. For example, Woodin showed that if λ is a limit of $<\lambda$-strong cardinals as well as a limit of Woodin cardinals, then the derived model of V at λ satisfies $\mathsf{AD}_{\mathbb{R}}$, a strengthening of AD that postulates determinacy for games played on the reals rather than on the integers. (Under the assumption of AD, the axiom $\mathsf{AD}_{\mathbb{R}}$ is equivalent to the existence of tree representations for every set of reals; this is the characterization that will be relevant to us.)

I would like to obtain strong determinacy axioms in derived models from hypotheses whose strength is farther from the surface than that of large cardinal axioms, such as forcing axioms, strong compactness,[1] and combinatorial compactness principles. (I still assume the existence of a limit

[1]Although often considered as a large cardinal property, strong compactness resists classification in the usual large cardinal hierarchy.

of Woodin cardinals in order to speak of the derived model. But in some cases, as in my thesis [8], it may be possible to adapt these methods to a determinacy model obtained by a core model induction instead.)

Here is a preliminary result along these lines:

Theorem 3.1 (Wilson [10, Theorem 9.6]). *Let λ be a limit of Woodin cardinals. If λ is λ^+-strongly compact, then $D(V,\lambda)$ satisfies $\mathsf{AD}_{\mathbb{R}}$.*

To obtain sharper results, I plan to use a two-part method. The first part consists of showing that if the derived model at λ is not large in the sense of satisfying strong axioms of determinacy, then it must instead be large in the sense of being close to V. We can consider such results to be "covering theorems" for derived models.

As in covering theorems for canonical inner models, a model being "close to V" is taken to mean roughly that it computes certain successor cardinals correctly. The relevant cardinal in determinacy models is Θ, the successor of the continuum in the sense of surjections. The following result is an example of a covering theorem for derived models:

Theorem 3.2 (Wilson [12]; see also Wilson [10, Lemma 9.4]). *Let λ be a limit of Woodin cardinals. If λ is weakly compact and $\Theta^{D(V,\lambda)} < \lambda^+$, then $D(V,\lambda)$ satisfies $\mathsf{AD}_{\mathbb{R}}$.*

By analogy with covering theorems in inner model theory, we might expect a similar result in the singular case:

Question 3.3. Let λ be a limit of Woodin cardinals. If λ is singular and $\Theta^{D(V,\lambda)} < \lambda^+$, then must $D(V,\lambda)$ satisfy $\mathsf{AD}_{\mathbb{R}}$?

The second part of the proposed method consists of showing that if the derived model at λ is close to V in the sense that $\Theta^{D(V,\lambda)} = \lambda^+$, then some canonical structure supplied by determinacy gives rise to combinatorial incompactness phenomena in V. An analogy with inner model theory is that if λ^+ is computed correctly by a canonical inner model such as L, then Jensen's square principle $\square_\lambda$ holds. Here is a preliminary result in this direction:

Theorem 3.4 (Wilson, unpublished). *Let λ be a limit of Woodin cardinals. If Jensen's weak square principle $\square^*_\lambda$ fails, then either*

(1) $\Theta^{D(V,\lambda)} < \lambda^+$, *or*
(2) $D(V,\lambda) \models \mathsf{LSA}$.[2]

I hope to strengthen the conclusion of Theorem 3.4 along the lines of the analogy with inner model theory:

[2]The *Largest Suslin Axiom*, LSA, is a strong axiom of determinacy saying that there is a largest Suslin cardinal and this largest Suslin cardinal is a member of the Solovay sequence. I don't know if alternative (2) can be eliminated from the theorem; perhaps the hypothesis implies that (1) always holds.

Question 3.5. Let λ be a limit of Woodin cardinals. If $\Theta^{D(V,\lambda)} = \lambda^+$, then must $\Box_\lambda$ hold?

Ultimately I would like to combine positive answers to Questions 3.3 and 3.5 to obtain a positive answer to the following question, which asks for a strengthening of Theorem 3.1 of a combinatorial nature:

Question 3.6. Let λ be a limit of Woodin cardinals. Assume that $\Box_\lambda$ fails and λ is either weakly compact or singular. Must the derived model of V at λ satisfy $\mathsf{AD}_\mathbb{R}$?

Remark 3.7. Under the hypothesis of Question 3.6, I was able to show that some *initial segment* of the derived model satisfies $\theta_0 < \Theta$ (the first significant step toward $\mathsf{AD}_\mathbb{R}$) by using a mouse capturing theorem of Steel [6]. But the associated relative consistency result is not new (it is implied by work of Jensen *et al.* [2] in the weakly compact case and Sargsyan [4] in the singular case) and is not an entirely satisfying substitute for an implication anyway.

References

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