TERNARY OPERATOR CATEGORIES

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Abstract. T*-categories are introduced as a ternary generalization of C*-categories. Their linking C*-categories are constructed and the Gelfand-Naimark representation theorems of Zettl for C*-ternary rings and for W*-ternary rings, are generalized to T*-categories. Biduals of C*-categories and of T*-categories are considered.

Dedicated to the memory of Ottmar Loos

Contents

1. Introduction and Preliminaries

1.1. **Introduction.** To provide motivation for this paper, consider a C*-algebra A and an idempotent linear map $P : A \to A$. If A is unital and P is completely positive and unital, then $P(A)$ is a C*-algebra with the product $a \cdot b = P(ab)$ for $a, b \in P(A)$, [3]. If instead P is just positive and unital, then $P(A)$ is a Jordan C*-algebra (=JB*-algebra) with the product $a \circ b = P((ab+ba)/2)$ for $a, b \in P(A)$, [8]. Finally, if P is just a contractive projection, then $P(A)$ is a JB*-triple with the triple product $\{abc\} = P((ab^*c + cb^*a)/2)$ for $a, b \in P(A)$, [10]. Thus, by removing a hypothesis on P, one is forced to consider a larger category than the category of

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C*-algebras and completely positive maps. In fact, the category of JB*-algebras is stable under the action of a positive projection and the category of JB*-triples is stable under the action of a contractive projection, [6, Theorem 3.3.1],[13, Theorem 14.4.1],[4, Theorem 5.6.59].

Now consider the notion of a C*-category, [11]. It consists of objects $X, Y, \dots$ and morphism sets $\mathrm{Hom}(X, Y)$ which satisfy a set of axioms relevant to C*-algebras. In particular, in a C*-category, $\mathrm{Hom}(X, X)$ is a C*-algebra. A W*-category was defined to be a C*-category with the additional requirement that $\mathrm{Hom}(X, Y)$ is the dual space of a Banach space. Sakai's theorem for W*-algebras was extended to W*-categories showing that $\mathrm{Hom}(X, Y)$, which is not necessarily a von Neumann algebra, has a unique predual. In [11], the Gelfand-Naimark theorem for C*-algebras was extended to C*-categories, and it showed that $\mathrm{Hom}(X, Y)$ is isomorphic to a ternary ring of operators (TRO). Although this fact was not explicitly mentioned, nevertheless, it was implicitly suggested by the following quote from [11, pp.79–80]:

> Naturally, the idea of using bounded linear mappings between different Hilbert spaces is such an obvious one that this paper may have many published and unpublished forerunners quite unknown to the authors. Indeed one of us (J. E. R.) has been toying with the idea of writing such a paper for many years but initially felt that the time was not yet ripe for such a development. In any case, the roots of this development go right back to the beginnings of the theory of operator algebras and perhaps the basic example of mappings between different Hilbert spaces are the intertwining operators of representation theory. The set of such intertwining operators forms a W*-category.

In addition, according to [11, p.79],

> There are at present many interesting directions of current research where W*-categories arise naturally: For example the representation theory of groupoids, the harmonic analysis of the action of non-Abelian groups on von Neumann algebras, the action of group duals on von Neumann algebras, and non-Abelian cohomology in an operator algebraic context.

It is also noteworthy that category theory is being used in physics, see for example [12]. It thus appears that, in order to take full advantage of the theory which has been developed for TROs and W*-TROs, it would be beneficial to extend these two concepts to the operator category setting. We begin that process in this paper, and are planning a sequel in some non-associative contexts.

We define a T*-category and a TW*-category, which are modeled on TROs and W*-TROs in much the same way that C*-categories and W*-categories are modeled on C*-algebras and W*-algebras. As an example, consider the Murray-von Neumann classification of W*-algebras into finite and infinite and types I, II, III. As of this writing, no such classification of W*-categories has been undertaken since the morphism sets (X, Y) are not necessarily W*-algebras. However, there is a Murray-von Neumann classification of W*-TROs [20], which can be used to decompose W*-categories in the same way that W*-algebras can be decomposed into types I,II,III, and finite, infinite (see Proposition 4.9). Thus, by extending the notion of W*-category to TW*-category, such a decomposition is possible without leaving the category (see Proposition 4.24).

More precisely, in this paper we give, in section 3, a purely algebraic definition of a "ternary category" (a concept which seems to have been overlooked in the literature) and construct its corresponding "linking category." Turning to "operator categories" we define, in section 4, a T*-category and show that its linking category is a C*-category. We extend the Gelfand-Naimark theorem for C*-ternary rings, which characterizes them in terms of TROs, [22, Theorem 3.1], to T*-categories. We also define a TW*-category and show that its linking category is a W*-category, and we extend the Gelfand-Naimark theorem for W*-ternary rings, which characterizes them in terms of W*-TROs, [22, Theorem 4.1], to TW*-categories. We also show, in section 5, that the bidual of a C*-category is a W*-category, and the bidual of a T*-category is a TW*-category.

1.2. **Associative triple systems.** The following construction is taken essentially verbatim from [17] (see also [18, pp. 28–30]) and is central to the paper. The complications due to taking direct sums, which were not necessary in [22], are unavoidable since the module actions defined in Lemma 1.2(ii) make essential use of the second component and are critical to the proof of the key Proposition 2.3(iv).

A vector space V with a trilinear map $m : V \times V \times V \to V$ with $m(x, y, z)$, called the triple product, and denoted by (x, y, z) is called an *associative triple system* if it satisfies

$$(x, y, (z, u, v)) = ((x, y, z), u, v) = (x, (u, z, y), v)$$

for all elements $x, y, z, u, v \in V$. Many examples will appear in this paper. If the base field is the complex numbers, the triple product is assumed to be conjugate linear in the middle variable.

Let M be an associative triple system with triple product denoted by $[hgf]$. Let

$$E(M) = \mathrm{End}\,(M) \oplus [\overline{\mathrm{End}\,(M)}]^{op},$$

where the notation $\overline{V}$ for a complex vector space means that the scalar multiplication in $\overline{V}$ is $(\lambda, v) \in \mathbb{C} \times V \mapsto \lambda \circ v = \overline{\lambda} v$.

We shall often denote the products in $E(M)^{op}$ and in $[\mathrm{End}\,(M)]^{op}$ by $X \circ Y = YX$. Explicitly, for $A = (A_1, A_2)$ and $A' = (A'_1, A'_2)$ belonging to $E(M)$,

$$AA' = (A_1, A_2)(A'_1, A'_2) = (A_1A'_1, A_2 \circ A'_2) = (A_1A'_1, A'_2A_2),$$

and for $B = (B_1, B_2)$ and $B' = (B'_1, B'_2)$ belonging to $E(M)^{op}$,

$$B \circ B' = (B_1, B_2) \circ (B'_1, B'_2) = (B'_1, B'_2)(B_1, B_2) = (B'_1B_1, B'_2 \circ B_2) = (B'_1B_1, B_2B'_2).$$

Involutions, that is, conjugate linear anti-isomorphisms of order 2, are defined on $E(M)$ by

$$A = (A_1, A_2) \mapsto \overline{A} = \overline{(A_1, A_2)} = (A_2, A_1),$$

so that $\overline{AA'} = \overline{A'}\,\overline{A}$, and

$$\overline{\lambda A} = \overline{(\lambda A_1, \lambda \circ A_2)} = \overline{(\lambda A_1, \overline{\lambda} A_2)} = (\overline{\lambda} A_2, \lambda A_1) = (\overline{\lambda} A_2, \overline{\lambda} \circ A_1) = \overline{\lambda}\ \overline{A};$$

and on $E(M)^{op}$ by

$$B = (B_1, B_2) \mapsto \overline{B} = \overline{(B_1, B_2)} = (B_2, B_1),$$

so that $\overline{B \circ B'} = \overline{B'} \circ \overline{B}$ and

$$\overline{\lambda B} = \overline{(\lambda B_1, \lambda \circ B_2)} = \overline{(\lambda B_1, \overline{\lambda} B_2)} = (\overline{\lambda} B_2, \lambda B_1) = (\overline{\lambda} B_2, \overline{\lambda} \circ B_1) = \overline{\lambda}\ \overline{B}.$$

For $g, h \in M$, define $L(g, h) = [gh\cdot], R(g, h) = [\cdot hg]$,

$$\ell(g, h) = (L(g, h), L(h, g)) = ([gh\cdot], [hg\cdot]) \in E(M)$$

and

$$r(g, h) = (R(h, g), R(g, h)) = ([\cdot gh], [\cdot hg]) \in E(M)^{op}.$$

Next, define

$$L = L(M) = \mathrm{span}\,\{\ell(g, h) : g, h \in M\} \subset E(M)$$

and

$$R = R(M) = \mathrm{span}\,\{r(g, h) : g, h \in M\} \subset E(M)^{op}.$$

The next three lemmas follow straightforwardly from the above construction so their proofs are left to the reader. Their statements have their origins in [17] and are reproduced in [18, pp. 28–30].

Lemma 1.1. *With the above notation*

(i): $R(f,g)L(h,k) = L(h,k)(R(f,g)$[1]

(ii): $\ell(g,h)\ell(g',h') = \ell([ghg'],h') = \ell(g,[h'g'h])$

(iii): $r(g,h)\circ r(g',h') = r(g,[hg'h']) = r([g'hg],h')$, *where, as indicated, the product on the left is taken in* $E(M)^{op}$.

(iv): L *is a* *-*subalgebra of* $E(M)$ *and* R *is a* *-*subalgebra of* $E(M)^{op}$.

Lemma 1.2. *Let* $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^{op}$, *and* $f \in M$. *Then*

(i): M *is a left* $E(M)$*-module via*

$$(A, f) \mapsto A \cdot f = A_1 f,$$

a right $E(M)^{op}$*-module via*

$$(f, B) \mapsto f \cdot B = B_1 f,$$

and an (L,R)*-bimodule.*

(ii): *Let* $\overline{M}$ *denote the vector space* M *with the element* f *denoted by* $\overline{f}$ *and with scalar multiplication defined by* $(\lambda, \overline{f}) \mapsto \lambda \circ \overline{f} = \overline{\overline{\lambda} f}$. *Then*

$\overline{M}$ *is a left* $E(M)^{op}$*-module via*

$$(B, \overline{f}) \mapsto B \cdot \overline{f} = \overline{B_2 f},$$

a right $E(M)$*-module via*

$$(\overline{f}, A) \mapsto \overline{f} \cdot A = \overline{A_2 f},$$

and an (R,L)*-bimodule.*

Thus we have

$$(AA')\cdot f = A\cdot(A'\cdot f) \text{ and } f\cdot(B\circ B') = (f\cdot B)\cdot B',$$
$$\overline{f}\cdot(AA') = (\overline{f}\cdot A)\cdot A' \text{ and } (B\circ B')\cdot\overline{f} = B\cdot(B'\cdot\overline{f}),$$
$$(A\cdot f)\cdot B = A\cdot(f\cdot B) \text{ and } (B\cdot\overline{f})\cdot A = B\cdot(\overline{f}\cdot A),$$

where the product $B\circ B'$ is taken in $E(M)^{op}$.

Given an associative triple system M, let

$$\mathcal{A} = \mathcal{A}(M) = L(M) \oplus M \oplus \overline{M} \oplus R(M)$$

and write the elements a of $\mathcal{A}$ as matrices

$$a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}, (A \in L(M), B \in R(M), f, g \in M).$$

Define multiplication and involution in $\mathcal{A}$ by

$$aa' = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + \ell(f,g') & A\cdot f' + f\cdot B' \\ \overline{g}\cdot A' + B\cdot\overline{g'} & r(g,f') + B\circ B' \end{bmatrix} \tag{1.1}$$

(the product $B\circ B'$ taken in $E(M)^{op}$) and

$$a^{\sharp} = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}^{\#} = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix}. \tag{1.2}$$

Lemma 1.3. $\mathcal{A}(M)$ *is an associative* *-*algebra and for* $f, g, h \in M$,

$$\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix}^{\#}\begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & [fgh] \\ 0 & 0 \end{bmatrix}.$$

[1] This is needed in the proof of the bimodule statements in Lemma 1.2.

Remark 1.4. The map $f \mapsto \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ is a triple isomorphism of M into $\mathcal{A}(M)$, the latter considered as an associative triple system with triple product $ab^{\#}c$, for $a, b, c \in \mathcal{A}(M)$, We refer to $\mathcal{A}(M)$ as the *standard embedding* of M. If the associative triple system M is a normed space, and $\|[hgf]\| \leq \|f\|\|g\|\|h\|$, then the *normed standard embedding* of M is defined in the same way but with R and L replaced by their closures in $B(M)$. In this case, the modules in Lemma 1.2 are continuous modules, and Banach modules if M is a Banach space.

In certain cases, the correspondence $M \to \mathcal{A}(M)$ will be a functor from the category of associative triple systems and triple homomorphisms to the category of associative *-algebras and *-homomorphisms. In the present context, we have the following lemma, whose straightforward but tedious proof is omitted.

Lemma 1.5. *Let $\varphi : M_1 \to M_2$ be a surjective triple homomorphism between associative triple systems M_1 and M_2. There is a *-homomorphism $\mathcal{A}(\varphi) : \mathcal{A}(M_1) \to \mathcal{A}(M_2)$ defined by*

$$\mathcal{A}(\varphi)\left(\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right) = \begin{bmatrix} \varphi_{11}(A) & \varphi(f) \\ \overline{\varphi(g)} & \varphi_{22}(B) \end{bmatrix},$$

where if $A = \sum_i([g_i h_i \cdot], [h_i g_i \cdot]) \in L(M_1)$,

$$\varphi_{11}(A) = \sum_i ([\varphi(g_i)\varphi(h_i)\cdot], [\varphi(h_i)\varphi(g_i)\cdot]) \in L(M_2),$$

and if $B = \sum_i([\cdot g_i h_i], [\cdot h_i g_i]) \in R(M_1)$,

$$\varphi_{22}(B) = \sum_i ([\cdot\varphi(g_i)\varphi(h_i)], [\cdot\varphi(h_i)\varphi(g_i)]) \in R(M_2).$$

1.3. **Ternary rings of operators.** Let H and K be complex Hilbert spaces. Denote by $\mathcal{B}(H, K)$ the set of all bounded linear operators from H to K, and write $\mathcal{B}(H)$ for $\mathcal{B}(H, H)$. Consider $\mathcal{B}(H, K)$ as a Banach space with the usual operator norm and additional algebraic structure given by *ternary product* $(x, y, z) \mapsto xy^*z$, so that for every $x, y, z \in \mathcal{B}(H, K)$ we have:

$$\|xy^*z\| \leq \|x\| \, \|y\| \, \|z\| \qquad \text{and} \qquad \|xx^*x\| = \|x\|^3 .$$

A Banach subspace X of $\mathcal{B}(H, K)$ is called a *TRO* (ternary ring of operators) if $xy^*z \in X$ for every choice of $x, y, z \in X$. A TRO $X \subseteq \mathcal{B}(H, K)$ is called a *W^*-TRO* if it is weak* closed (equivalently, weak operator closed, or strong operator closed) in $\mathcal{B}(H, K)$. A TRO that is dual as a Banach space is a W^*-TRO [9, Theorem 2.6], and every W^*-TRO has a unique Banach space predual, up to isometry [9, Proposition 2.4]. TROs are studied extensively in [2, §4.4, §8.3, 8.5.18], where we can find the following on page 351:

> Around 1999, interest in TROs picked up with the important paper [9]. As evidenced by the number of recent papers using them, it seems that TRO and C*-module methods are playing an increasingly central role in operator space theory at the present time.

Let X be a TRO contained in $\mathcal{B}(H, K)$. The left C^*-algebra of X, denoted by C is the C^*-subalgebra of $\mathcal{B}(K)$ generated by elements of the form xy^* with $x, y \in X$. Similarly, the right C^*-algebra of X, denoted by D, is the C^*-subalgebra of $\mathcal{B}(H)$ generated by elements of the form y^*z with $y, z \in X$ (C and D need not be unital algebras). The connection between C and D is made via the linking C^*-algebra of X, defined as $A_X = \left[\begin{smallmatrix} C & X \\ X^* & D \end{smallmatrix}\right]$, where $X^* = \{x^* : x \in X\}$ is the space of adjoints of elements of X. It is often convenient to make the identification

$$\mathcal{B}(K \oplus H) = \left[\begin{smallmatrix} \mathcal{B}(K) & \mathcal{B}(H,K) \\ \mathcal{B}(K,H) & \mathcal{B}(H) \end{smallmatrix}\right]$$

and regard A_X as a C^*-subalgebra of $\mathcal{B}(K \oplus H)$. The linking C^*-algebra A_X is uniquely determined by X and is independent of the Hilbert spaces H and K on which X is represented. Thus, for the most part, we may assume that TROs $X \subseteq \mathcal{B}(H,K)$ act non-degenerately on H and K (XH is norm dense in K and X^*K is norm dense in H). In this case, the C^*-algebras C and D act non-degenerately on K and H, respectively. It is clear that $CX \subseteq X$ and $XD \subseteq X$, so X is a C-D bimodule. In fact, $CX = X = XD$ and X can be regarded as a non-degenerate and faithful Hilbert C-D bimodule with inner products ${}_C\langle x, y\rangle = xy^*$ and $\langle x, y\rangle_D = x^*y$ defined on X.

Let X be a W^*-TRO contained in $\mathcal{B}(H,K)$. The left von Neumann algebra of X, denoted by M, is the von Neumann subalgebra of $\mathcal{B}(K)$ generated by elements of the form xy^* with $x, y \in X$. The right von Neumann algebra of X, denoted by N, is the von Neumann subalgebra of $\mathcal{B}(H)$ generated by elements of the form y^*z with $y, z \in X$. The linking von Neumann algebra of X is defined as $R_X = \left[\begin{smallmatrix} M & X \\ X^* & N \end{smallmatrix}\right]$ and it is viewed as a von Neumann subalgebra of $\mathcal{B}(K \oplus H)$. The weak* closure $\bar{X}$ of a TRO X is a W^*-TRO.

Example 1.6. If M is a TRO, then $\mathcal{A}(M)$ (see Remark 1.4) is a C*-algebra which is *-isomorphic to the linking algebra A_M of M via the map

$$A_M \ni \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} \mapsto \begin{bmatrix} \sum_i([x_i y_i \cdot], [y_i x_i \cdot]) & z \\ \overline{w} & \sum_j([\cdot u_j v_j], [\cdot v_j u_j]) \end{bmatrix} \in \mathcal{A}(M)$$

(cf. Example 4.15).

1.4. **Categories.** In this subsection, we record the basic definitions in category theory that we use. See, for example, [12, Chapter 0].

Definition 1.7. A *category* $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}),\ \mathrm{Mor}(\mathcal{C}),\ \circ)$ consists of the following entities.

(1) A class $\mathrm{Ob}(\mathcal{C})$ of *objects*.
(2) For each X, Y in $\mathrm{Ob}(\mathcal{C})$, a class $\mathrm{Hom}(X,Y)$ of *morphisms* (or *maps*) from X to Y, with f in $\mathrm{Hom}(X,Y)$ written $X \xrightarrow{f} Y$ or $f\colon X \to Y$. The class of all morphisms is denoted $\mathrm{Mor}(\mathcal{C})$, so $\mathrm{Hom}(X,Y) \subseteq \mathrm{Mor}(\mathcal{C})$.
(3) For each object X, there is a morphism $1_X \in \mathrm{Hom}(X,X)$ such that $1_Y \circ f = f \circ 1_X = f$ for each $f \in \mathrm{Hom}(X,Y)$.
(4) For each X, Y, Z in $\mathrm{Ob}(\mathcal{C})$, a function
$$\begin{array}{rcl} \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) & \to & \mathrm{Hom}(X,Z) \\ (f,g) & \mapsto & g \circ f = gf \end{array}$$
called *morphism composition* (or just *composition*), which is associative in the sense that $(hg)f = h(gf)$ for all composable morphisms in the category.

When convenient and not confusing, we shall often, as in [11], denote $\mathrm{Hom}(X,Y)$ simply by (X,Y).

Definition 1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (*covariant*) *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following entities.

(1) A function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ that associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$.
(2) For each X, Y in $\mathrm{Ob}(\mathcal{C})$, a function $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X), F(Y))$ that associates to each morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ a morphism $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathcal{D}$ such that
$$F(g \circ f) = F(g) \circ F(f) \quad \text{and} \quad F(1_X) = 1_{F(X)}$$
for all composable morphisms f, g in $\mathcal{C}$.

Definition 1.9. A subcategory of a category $\mathcal{C}$ is a category $\mathcal{S}$ whose objects are objects in $\mathcal{C}$ and whose morphisms are morphisms in $\mathcal{C}$ with the same identities and composition of morphisms. If X, Y are objects of $\mathcal{S}$, then the morphism set of $\mathcal{S}$ from X to Y is denoted $(X,Y)_{\mathcal{S}}$, and we have $(X,Y)_{\mathcal{S}} \subset (X,Y) := (X,Y)_{\mathcal{C}}$.

Definition 1.10. Let $\mathbb{K}$ be a field. A category $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}),\ \mathrm{Mor}(\mathcal{C}),\ \circ)$ is called a $\mathbb{K}$-*linear category* (or a $\mathbb{K}$-*algebroid*) if each $\mathrm{Hom}(X,Y) \subseteq \mathrm{Mor}(\mathcal{C})$ has the structure of a vector space over $\mathbb{K}$ and composition of morphisms

$$\begin{array}{rcl}\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) & \to & \mathrm{Hom}(X,Z) \\ (f,g) & \mapsto & g \circ f\end{array}$$

is $\mathbb{K}$-bilinear.

For any object X in a $\mathbb{K}$-linear category, (X,X) is a unital associative algebra. For any such associative algebra A, the category with A as its sole object, and A as its morphisms, is a $\mathbb{K}$-linear category with composition being the product in A.

Definition 1.11. Let $\mathcal{A}$ be a $\mathbb{K}$-linear category and $\mathcal{J}$ a subcategory. Then $\mathcal{J}$ is an ideal of $\mathcal{A}$ if for objects X, Y of $\mathcal{J}$, $(X,Y)_{\mathcal{J}}$ is a linear subspace of (X,Y) and objects X, Y, Z

$$(\textit{right ideal}) \quad (Y,Z)_{\mathcal{J}} \circ (X,Y) \subset (X,Z)_{\mathcal{J}}$$

and

$$(\textit{left ideal}) \quad (Y,Z) \circ (X,Y)_{\mathcal{J}} \subset (X,Z)_{\mathcal{J}}$$

(composition in $\mathcal{A}$).

If $\mathcal{J}$ is a two-sided ideal in $\mathcal{C}$, the quotient $\mathcal{C}/\mathcal{J}$ is the category with the same objects as $\mathcal{C}$ and with morphism sets the quotient spaces $(X,Y)/(X,Y)_{\mathcal{J}}$. There is a natural quotient functor from $\mathcal{C}$ to $\mathcal{C}/\mathcal{J}$ (see [19, section 4]).

Definition 1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{K}$-linear categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *linear functor* if the map $F : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$ is linear.

2. C*-ternary rings

Recall that a C*-ternary ring was introduced in [22] as a complex Banach space $(Z, \|\cdot\|)$ with a ternary operation $(\cdot,\cdot,\cdot) : Z \times Z \times Z \to Z$ which is linear in the outer variables and conjugate linear in the middle variable, associative in the sense that

$$(((v,w,x),y,z) = (v,(y,x,w),z) = (v,w,(x,y,z)),$$

and for which $\|(x,y,z)\| \le \|x\|\|y\|\|z\|$ and $\|(x,x,x)\| = \|x\|^3$. In addition, if Z is a dual Banach space, it is called a W*-ternary ring.

In order to prove our main results in this section (Theorems 4.11 and 4.14 below), we shall invoke the following Gelfand-Naimark theorem for C*-ternary rings, which uses the following terminology. A linear bijection $\varphi : Z_1 \to Z_2$ between two C*-ternary rings $(Z_1, (\cdot,\cdot,\cdot)_1)$ and $(Z_2, (\cdot,\cdot,\cdot)_2)$ is an *isomorphism* if $\varphi((x,y,z)_1) = (\varphi(x),\varphi(y),\varphi(z))_2$ and an *anti-isomorphism* if $\varphi((x,y,z)_1) = -(\varphi(x),\varphi(y),\varphi(z))_2$.

Theorem 2.1 (Theorem 3.1 in [22])**.** *Let Z be a C*-ternary ring.*

- **(i):** *Z is the direct sum of two C*-ternary subrings Z_+ and Z_-which are respectively isometrically isomorphic and isometrically anti-isomorphic to a ternary ring of operators (TRO).*
- **(ii):** *The decomposition is unique: if Z_1 and Z_2 are C*-ternary subrings of Z with $Z = Z_1 \oplus Z_2$, Z_1 isomorphic to a TRO, and Z_2 anti-isomorphic to a TRO, then $Z_+ = Z_1$ and $Z_- = Z_2$.*

(iii): *There exists one, and only one, operator $T : Z \to Z$ satisfying*
- $T^2 = I$;
- $T((x, y, z)) = (Tx, y, z) = (x, Ty, z) = (x, y, Tz)$ *for* $x, y, z \in Z$;
- $(Z, T \circ (x, y, z))$ *is a C*-ternary ring which is isomorphic to a TRO.*

Remark 2.2. Zettl shows ([22, Proposition 3.2 and p. 130]) that if a C*-ternary ring $(Z, (x, y, z))$ is a right Banach $\mathfrak{A}$-module for some C*-algebra $\mathfrak{A}$, and there is a conjugate bilinear form $\alpha : Z \times Z \to \mathfrak{A}$ with $\|\alpha\| \le 1$ satisfying

(i): $\alpha(x \cdot a, y) = \alpha(x, y)a$
(ii): $\alpha(x, y)^* = \alpha(y, x)$
(iii): $(x, y, z) = x \cdot \alpha(z, y)$
(iv): $\operatorname{span} \alpha(Z, Z)$ is dense in $\mathfrak{A}$,

then $Z_+ = \{z \in Z : \alpha(z, z) \ge 0\}$ and $Z_- = \{z \in Z : \alpha(z, z) \le 0\}$, and that Z_+ and Z_- are orthogonal, so that $\|(\alpha, \beta)\| = \max(\|\alpha\|, \|\beta\|)$ for $(\alpha, \beta) \in Z_+ \oplus Z_-$.

Let $(M, [\cdot, \cdot, \cdot])$ be a C*-ternary ring. Recall that, M being a normed associative triple system, it is, by Remark 1.4, a left $L(M)$-Banach module via $L(M) \times M \ni (A, f) \mapsto A \cdot f = A_1 f \in M$ and a right $R(M)^{op}$-Banach module via $M \times R(M) \ni (f, B) \mapsto f \cdot B = B_1 f \in M$, and that

$$\mathcal{A} = \{a = \begin{bmatrix} A & f \\ \bar{g} & B \end{bmatrix} : A \in L(M), B \in R(M), f, g \in M\},$$

is an algebra with multiplication (1.1) and involution (1.2).

We note that the C*-algebra $\mathfrak{A}$ in Remark 2.2 is the closed span of $\{[\cdot gh] : g, h \in V\}$ and it is *-isomorphic to $R(M)$ via the map $\mathfrak{A} \ni B_1 \mapsto \sigma(B_1) = (B_1, B_1^*) \in R(M)$. Similarly $\tau : \mathfrak{B} \to L(V)$ is the *-isomorphism $A_1 \mapsto (A_1, A_1^*)$, where $\mathfrak{B}$ is the close span of $\{[gh\cdot] : g, h \in V\}$. The C*-ternary ring M is thus both a Banach (L, R)-bimodule and a Banach $(\mathfrak{B}, \mathfrak{A})$-bimodule.

The following proposition is a key to the proof of Theorem 4.11. Because of the length of the proof of (iv), we defer it to section 6. Also, although the proof of (i) follows from the fact, just noted, that $R(M)$ is *-isomorphic to $\mathfrak{A}$, we give a direct proof in the present context.

Proposition 2.3. *Let M be a C*-ternary ring. With the above notation, we have*

(i): *$R(M)$ is a C*-algebra with the norm from $B(M)$.*
(ii): *M is a right Banach $R(M)^{op}$-module.*
(iii): *With $\langle f|g\rangle = \langle f|g\rangle_M : M \times M \to R(M)$ defined by $\langle f|g\rangle = r(g, f) = ([\cdot gf], [\cdot fg])$, we have*

$$\langle f \cdot B|g\rangle = \langle f|g\rangle \circ B.$$

(iv): *If M is a right $R(M)^{op}$-Hilbert module, then $\mathcal{A}$ can be normed to be a C*-algebra.*

Proof. (i) Recall that $R = R(M)$ is the closed span of

$$\{r(f, g) = ([\cdot fg], [\cdot gf]) : f, g \in M\} \subset B(M) \oplus [\overline{B(M)}]^{op}.$$

Let $U = \sum_i r(f_i, g_i) = (\sum_i [\cdot f_i g_i], \sum_i [\cdot g_i f_i]) = (U_1, U_2) \in R$, and recall that $\overline{U} = (U_2, U_1)$ is the involution[2] on R. We shall show, by mimicking the proof of [22, Proposition 3.2 (1)], that $\|U\|^2 = \|\overline{U}U\|$, proving that R is a C*-algebra.

[2]$\overline{U}$ is obviously well defined. However, that fact, proved in [22], that $\sum_i [\cdot f_i g_i] = 0$ implies that $\sum_i [\cdot g_i f_i] = 0$, requires proof using associativity of the ternary product. We don't need that argument here since it is obvious that if $(U_1, U_2) = (0, 0)$, then $\overline{(U_1, U_2)} = (U_2, U_1) = (0, 0)$.

Let $h = (h_1, h_2) \in M \oplus M$. We have

$$\begin{aligned} [Uh, Uh, Uh] &= [U_1h_1, U_1h_1, U_1h_1] \oplus [U_2h_2, U_2h_2, U_2h_2] \\ &= [\sum_i [h_1 f_i g_i], U_1h_1, U_1h_1] \oplus [\sum_i [h_2 g_i f_i], U_2h_2, U_2h_2] \\ &= \sum_i [h_1, [U_1h_1, g_i, f_i], U_1h_1] \oplus \sum_i [h_2, [U_2h_2, f_i, g_i], U_2h_2] \\ &= [h_1 \oplus h_2, (\sum_i [U_1h_1, g_i, f_i]) \oplus (\sum_i [U_2h_2, f_i, g_i]), U_1h_1 \oplus U_2h_2] \\ &= [h_1 \oplus h_2, U_2U_1h_1 \oplus U_1U_2h_2, U_1h_1 \oplus U_2h_2] \\ &= [h, \overline{U}Uh, Uh], \end{aligned}$$

so that

$$\|Uh\|^3 = \|[Uh, Uh, Uh]\| \leq \|h\|\|\overline{U}Uh\|\|Uh\|$$

and

$$\|Uh\|^2 \leq \|h\|\|\overline{U}Uh\| \leq \|h\|^2\|\overline{U}U\| \leq \|\overline{U}\|\|U\|\|h\|^2 = \|U\|^2\|h\|^2.$$

(ii) This is immediate from Lemma 1.2(i), as noted in Remark 1.4.

(iii) Let $B = (B_1, B_2) = r(h, k) = ([\cdot hk], [\cdot kh])$. Then

$$\langle f \cdot B|g\rangle = \langle B_1 f|g\rangle = r(g, B_1 f) = r(g, [fhk]) = ([\cdot g[fhk]], [\cdot[fhk]g])$$

and

$$\begin{aligned} \langle f|g\rangle \circ B &= r(g, f) \circ (B_1, B_2) \\ &= ([\cdot gf], [\cdot fg]) \circ ([\cdot hk], [\cdot kh]) \\ &= ([\cdot hk], [\cdot kh])([\cdot gf], [\cdot fg]) \\ &= ([\cdot hk][\cdot gf], [\cdot fg][\cdot kh]) \\ &= ([[\cdot gf]hk], [[\cdot kh]fg]), \end{aligned}$$

as required.

(iv) See section 6. □

Lemma 2.4. *If M is a C*-ternary ring with decomposition $M = M_+ \oplus M_-$, then M_+ is a right $R(M)^{op}$-Hilbert module.*

Proof. In Remark 2.2, with $\alpha(f, g) = \langle f|g\rangle = r(f, g)$, (i) holds by Proposition 2.3(iii), and (ii) and (iv) follow from the definition of α. To prove (iii) in Remark 2.2, it suffices to show that $[hgf] = h \cdot \langle f|g\rangle$. But $h \cdot \langle f|g\rangle = h \cdot ([\cdot gf], [\cdot fg]) = [hgf]$. Thus $M_+ = \{f \in M : \alpha(f, f) \geq 0\}$ and is therefore a right $R(M)^{op}$-Hilbert module. □

Lemma 2.5. *A surjective homomorphism between C*-ternary rings is contractive.*

Proof. Let $\phi : M \to N$ be a surjective homomorphism of C*-ternary rings $M = M_+ \oplus M_-$ and $N = N_+ \oplus N_-$. Then $N = \phi(M_+) \oplus \phi(M_-)$ is the sum of two orthogonal ideals. Also, $\phi(M_+) \simeq M_+/\ker \phi|_{M_+}$ which is isomorphic to a quotient of a TRO, which, by ([9, Proposition 2.2]) is a TRO. Similarly $\phi(M_-) \simeq M_-/\ker \phi|_{M_-}$ is anti-isomorphic to a TRO. So by uniqueness of the Zettl decomposition, $\phi(M_\pm) = N_\pm$. Note that a TRO homomorphism of TROs is contractive ([9, Proposition 2.1]), and since ϕ restricts to a homomorphism of $M_\pm$ onto $N_\pm$, $\phi|_{M_\pm}$ is contractive. For example, if ψ is an isomorphism of M_+ onto a TRO V, and ξ is an isomorphism of N_+ onto a TRO W, then $\xi \circ \phi \circ \psi^{-1}$ is a homomorphism from V to W, hence contractive, and $\|\phi(x_+)\| = \|\xi\phi(x_+)\| = \|(\xi \circ \phi \circ \psi^{-1})(\psi(x_+)\| \leq \|\psi(x_+)\| = \|x_+\|$. Thus, if $x = x_+ + x_- \in M$, $\|\phi(x)\| = \|\phi(x_+) + \phi(x_-)\| = \max(\|\phi(x_+)\|, \|\phi(x_-)\|) \leq \max(\|x_+\|, \|x_-\|) = \|x\|$. □

Lemma 2.6. *Let $\phi : M \to N$ be a surjective homomorphism between C*-ternary rings M and N. There is a *-homomorphism $\mathcal{A}(\phi) : \mathcal{A}(M) \to \mathcal{A}(N)$ defined by*

$$\mathcal{A}(\varphi)\left(\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right) = \begin{bmatrix} \varphi_{11}(A) & \varphi(f) \\ \overline{\varphi(g)} & \varphi_{22}(B) \end{bmatrix}, \tag{2.1}$$

where if $A = \sum_i([g_i h_i \cdot], [h_i g_i \cdot]) \in L(M)$,

$$\varphi_{11}(A) = \sum_i([\varphi(g_i)\varphi(h_i)\cdot], [\varphi(h_i)\varphi(g_i)\cdot]) \in L(N), \tag{2.2}$$

and if $B = \sum_i([\cdot g_i h_i], [\cdot h_i g_i]) \in R(M)$,

$$\varphi_{22}(B) = \sum_i([\cdot\varphi(g_i)\varphi(h_i)], [\cdot\varphi(h_i)\varphi(g_i)]) \in R(N).$$

Proof. It is enough to show that the mapping (2.1) is well-defined, which will follow from $\|\varphi_{11}(A)\| \leq \|A\|$ and $\|\varphi_{22}(B)\| \leq \|B\|$. The rest of the proof involves straightforward but tedious algebra. In fact the contractivity of ϕ_{11} and of ϕ_{22} follow by direct calculation from Lemma 2.5 and surjectivity of ϕ. □

Proposition 2.7. *If M is a C*-ternary ring, then $\mathcal{A}(M)$ is a C*-algebra.*

Proof. With $M = M_+ \oplus M_-$, $\mathcal{A}(M_+)$ is isomorphic to a C*-algebra by Lemma 2.4 and Proposition 2.3(iv). If M_- is anti-isomorphic to a TRO V, then by Lemma 2.6, $\mathcal{A}(M_-)$ is *-isomorphic to $\mathcal{A}(V)$, which by Example 1.6 is *-isomorphic to the linking algebra of V. Thus $\mathcal{A}(M) = \mathcal{A}(M_+) \oplus \mathcal{A}(M_-)$ is a C*-algebra. □

3. Ternary categories

3.1. **Ternary categories.** The following definition is basic to this paper.

Definition 3.1. A *ternary category* $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}),\ \mathrm{Mor}(\mathcal{C}),\ \circ)$ consists of the following entities.

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects*.
- For each X, Y in $\mathrm{Ob}(\mathcal{C})$, a class $\mathrm{Hom}(X, Y)$ of *morphisms* (or *maps*) from X to Y, with f in $\mathrm{Hom}(X, Y)$ written $X \xrightarrow{f} Y$ or $f\colon X \to Y$. The class of all morphisms is denoted $\mathrm{Mor}(\mathcal{C})$, so $\mathrm{Hom}(X, Y) \subseteq \mathrm{Mor}(\mathcal{C})$.
- For each X, Y, Z, W in $\mathrm{Ob}(\mathcal{C})$, a function
$$\begin{array}{rcl} \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Z,Y) \times \mathrm{Hom}(Z,W) & \to & \mathrm{Hom}(X,W) \\ (f,g,h) & \mapsto & h \circ g^* \circ f \ = \ hg^*f, \end{array}$$
called *morphism composition* (or just *composition*), which is associative for all composable morphisms in the category, namely
$$(lk^*h)g^*f = l(gh^*k)^*f = lk^*(hg^*f) \tag{3.1}$$
whenever
$$X \xrightarrow{f} Y \xleftarrow{g} Z \xrightarrow{h} W \xleftarrow{k} U \xrightarrow{\ell} V.$$
To be precise, because of the twist in the middle term, (3.1) is defined only if $U = Y$, that is, for $f \in (X, Y), k \in (Y, W), h \in (Z, W), g \in (Z, Y), \ell \in (Y, V)$.

Definition 3.1 differs from Definition 1.7 in two respects. First, there is no stipulation for the existence of an identity morphism for each object (however, see Remark 3.5). Second, composition of morphisms is a ternary operation as opposed to a binary one. By an abuse of language, we use the term 'ternary category' even though it may not be a category. In addition, by an abuse of notation, we warn that the notation g^* is symbolic, without independent meaning (An exception occurs, for example, in Remark 4.8).

A *linear ternary category* is a ternary category in which $\mathrm{Hom}(X, Y)$ is a linear space over a field $\mathbb{K}$ and composition is linear in the outer variables and conjugate linear in the middle variable. In a linear ternary category, $\mathrm{Hom}(X, Y)$ is an associative triple system (see subsection 1.2).

Example 3.2. If M is an associative triple system, then the category with M as its sole object, and M as its morphisms, is a $\mathbb{K}$-linear ternary category with composition being the triple product in M.

Example 3.3. The class of all sets (as objects) together with all binary relations between them (as morphisms), with composition of relations, forms a category which we denote by $\mathcal{R}$. Thus, if X, Y are sets, $\mathrm{Hom}(X, Y) = P(X \times Y)$ is the power set of $X \times Y$. If $F \subseteq X \times Y$ is a relation then $F^* = \{(y, x) : (x, y) \in F\} \subseteq Y \times X$ is its converse. We have $(F^*)^* = F$ and $(G \circ F)^* = F^* \circ G^*$ for all composable relations. It follows that $\mathcal{R}$ is a dagger category (i.e. category with involution). It can be viewed as a ternary category with the natural (ternary) composition of relations, $F \circ G^* \circ H$.

Definition 3.4. Let $\mathcal{C}$ be a $\mathbb{K}$-linear ternary category and $\mathcal{J}$ a subcategory. Then $\mathcal{J}$ is an (ternary) ideal of $\mathcal{C}$ if for objects X, Y, Z, W of $\mathcal{J}$, $(X, Y)_{\mathcal{J}}$ is a linear subspace of (X, Y) and

$$(Z, W)_{\mathcal{J}} \circ (Z, Y) \circ (X, Y) \subset (X, W)_{\mathcal{J}},$$
$$(Z, W) \circ (Z, Y)_{\mathcal{J}} \circ (X, Y) \subset (X, W)_{\mathcal{J}},$$
$$(Z, W) \circ (Z, Y) \circ (X, Y)_{\mathcal{J}} \subset (X, W)_{\mathcal{J}}.$$

If $\mathcal{J}$ is an ideal in $\mathcal{C}$, the quotient $\mathcal{C}/\mathcal{J}$ is the category with the same objects as $\mathcal{C}$ and with morphism sets the quotient spaces $[X, Y] := (X, Y)/(X, Y)_{\mathcal{J}}$. There is a natural quotient functor from $\mathcal{C}$ to $\mathcal{C}/\mathcal{J}$, given by $(X, Y) \ni f \mapsto \overline{f} = f + (X, Y)_{\mathcal{J}} \in [X, Y]$. The composition in $\mathcal{C}/\mathcal{J}$ is given by

$$(\overline{f}, \overline{g}, \overline{h}) \mapsto \overline{[hgf]},$$

and is easily seen to be well-defined and associative in the sense of Definition 3.1.

Remark 3.5.

- As in the case of non-unital categories (cf. [19, Definition 3.1]), there are no identity morphisms per se in ternary categories. However, one can call a family of morphisms $\{u_X : X \text{ object of } \mathcal{C}\}$ in a ternary category $\mathcal{C}$ which satisfy $u_Y u_Y^* f = f = f u_X^* u_X$ a "unitary", and more generally morphisms (u_X) which satisfy $u_X = u_X u_X^* u_X$ "tripotents," or "partial isometries".[3]
- A *unitary ternary category* is a ternary category $\mathcal{C}$ containing a unitary element (u_X). In this case, one has a (unital) category $\mathcal{A}$ with the same objects and the same morphisms as $\mathcal{C}$, but with composition given by $f \circ g = f u_Y^* g$ for $g \in (X, Y)$, $f \in (Y, Z)$.

Example 3.6. The class of all complex Hilbert spaces (as objects) together with all bounded linear operators between them (as morphisms), with morphism composition $(f, g, h) \mapsto hg^* f$, forms a unitary ternary category which we will denote by $\mathcal{H}_+$ (cf. Example 4.2 and Remark 4.13(iii)).

Definition 3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be ternary categories. A (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ (or ternary functor for emphasis) consists of the following entities.

[3]In this case, a maximal tripotent in a T*-category (Definition 4.6) would satisfy

$$[u_Y u_Y^* f] + [f u_X^* u_X] = f + [[u_Y u_Y^* f] u_X^* u_X]$$

for $f \in (X, Y)$. Tripotents and Peirce decompositions in T*-categories are worthy of further study.

- A function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ that associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$.
- For each X, Y in $\mathrm{Ob}(\mathcal{C})$, a function $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X), F(Y))$ that associates to each morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ a morphism $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathcal{D}$ such that[4]
$$F(h \circ g^* \circ f) = F(h) \circ F(g)^* \circ F(f)$$
for all composable morphisms f, g, h in $\mathcal{C}$.

3.2. **The linking category of a linear ternary category.** Let $\mathcal{C}$ be a $\mathbb{C}$-linear ternary category with objects $X, Y, Z, \ldots$ and morphisms $(X,Y), (Z,W), \ldots$. Denote the composition of morphisms $f \in (X,Y), g \in (Z,Y), h \in (Z,W)$ by $[hgf]$.

For a pair of objects X, Y, (X,Y) is an associative triple system, so all of the machinery in subsection 1.2 is available by replacing M there by (X,Y). Thus,
$$E(X,Y) := \mathrm{End}\,((X,Y)) \oplus [\overline{\mathrm{End}\,((X,Y))}]^{op},$$
and for $g, h \in (X,Y)$,
$$\ell(g,h) = (L(g,h), L(h,g)) = ([gh\cdot], [hg\cdot]) \in E(X,Y),$$
$$r(g,h) = (R(h,g), R(g,h)) = ([\cdot gh], [\cdot hg]) \in E(X,Y)^{op},$$
$$L = L(X,Y) = \mathrm{span}\,\{\ell(g,h) : g, h \in (X,Y)\} \subset E(X,Y), \text{ and}$$
$$R = R(X,Y) = \mathrm{span}\,\{r(g,h) : g, h \in (X,Y)\} \subset E(X,Y)^{op}.$$

Recall that there are two reverses of multiplication in the definition of $E(X,Y)^{op}$, and involutions are defined on $E(X,Y)$ and $E(X,Y)^{op}$ by $\overline{(A_1, A_2)} = (A_2, A_1$ and $\overline{(B_1, B_2)} = (B_2, B_1)$.

From Lemmas 1.1 and 1.2,

- L is a *-subalgebra of $E(X,Y)$ and R is a *-subalgebra of $E(X,Y)^{op}$.
- (X,Y) is a left $E(X,Y)$-module via $(A,f) \mapsto A \cdot f = A_1 f$,
 a right $E(X,Y)^{op}$-module via $(f,B) \mapsto f \cdot B = B_1 f$, and an (L,R)-bimodule;
- $\overline{(X,Y)}$ is a left $E(X,Y)^{op}$-module via $(B, \overline{f}) \mapsto B \cdot \overline{f} = \overline{B_2 f}$,
 a right $E(X,Y)$-module via $(\overline{f}, A) \mapsto \overline{f} \cdot A = \overline{A_2 f}$, and an an (R,L)-bimodule.

Given objects X, Y in a linear ternary category $\mathcal{C}$, let
$$\mathcal{A} = \mathcal{A}(X,Y) = L(X,Y) \oplus (X,Y) \oplus \overline{(X,Y)} \oplus R(X,Y)$$
and write the elements a of $\mathcal{A}$ as matrices
$$a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}, (A \in L(X,Y), B \in R(X,Y), f, g \in (X,Y)).$$

Define multiplication and involution in $\mathcal{A}$ by
$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + \ell(f, g') & A \cdot f' + f \cdot B' \\ \overline{g} \cdot A' + B \cdot \overline{g'} & r(g, f') + B \circ B' \end{bmatrix} \tag{3.2}$$
and
$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}^{\#} = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix}. \tag{3.3}$$

From Lemma 1.3, $\mathcal{A}(X,Y)$ is an associative *-algebra and for $f, g, h \in (X,Y)$,
$$\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix}^{\#} \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & [fgh] \\ 0 & 0 \end{bmatrix}.$$

[4]The reader is reminded that g^* and $F(g)^*$ are symbolic notations for the middle term of a ternary composition, so have no meaning if isolated.

Definition 3.8. Given a linear ternary category $\mathcal{C}$, the *linking category* $A_{\mathcal{C}}$ of $\mathcal{C}$ is as follows. The objects of the category $A_{\mathcal{C}}$ are the same[5] as the objects of $\mathcal{C}$. The morphism set $\mathrm{Hom}(X,Y)$ is defined to be $\{0\}$ if $X \neq Y$, and $\mathrm{Hom}(X,X) = \mathcal{A}(X,X)$, with composition as follows. If $a \in \mathrm{Hom}(X,Y)$ and $b \in \mathrm{Hom}(Y,Z)$, then $b \circ a$ must be $\{0\}$ unless $X = Y = Z$, in which case $b \circ a$ is defined to be the product ab in $\mathcal{A}(X,X)$.

In general, $A_{\mathcal{C}}$ is a non unital category. By adjoining the identity operator to L and to R, one can define a unital linking category, but this is not needed for our purposes.

Remark 3.9. The category $A_{\mathcal{C}}$ can be considered as a ternary category under the composition $[abc] = ab^{\#}c$ and, by Lemma 1.3, we obtain a linear ternary functor F from $\mathcal{C}$ to $A_{\mathcal{C}}$ by associating the object X of $\mathcal{C}$ to the object X of $A_{\mathcal{C}}$ and the morphism $f \in (X,Y)$ to 0 if $X \neq Y$ and otherwise to the morphism $\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \in \mathcal{A}(X,X)$.

It is possible to define the morphism sets of $A_{\mathcal{C}}$ as $(X,Y)_{A_{\mathcal{C}}} = \mathcal{A}(X,Y)$ even if $X \neq Y$. In that case, we could then define $b \circ a$ to be 0 unless $X = Y = Z$ and otherwise to be the product ab in $\mathcal{A}(X,X)$. In either case, the price paid is that the linear ternary functor of $\mathcal{C}$ into $A_{\mathcal{C}}$ in Remark 3.9 is not faithful (see Theorem 4.11 below).

Example 3.10. If M is an associative triple system, $A(M)$ its standard embedding (see Remark 1.4) and $\mathcal{C}$ is the category with M as its sole object and M as its morphisms with composition given by the triple product in M, then by [17, Satz 1] (reproduced in [18, Theorem 2, p.30]), $A_{\mathcal{C}}$ is the category with $A(M)$ as its only object and the elements of $A(M)$ as morphisms with composition being multiplication in the associative algebra $A(M)$.

Definition 3.11. If the linear ternary category $\mathcal{C}$ is normed, that is, if (X,Y) is a normed space and $\|f \circ g^* \circ h\| \leq \|f\|\|g\|\|h\|$, then the *normed* linking category of $\mathcal{C}$ is defined in the same way but with R and L replaced by their closures in $B((X,Y))$. In this case, the modules in Lemma 1.2 are continuous modules, and Banach modules if (X,Y) is a Banach space.

4. Operator categories

4.1. **C*-categories.** In this subsection we recall the notion of C*-category from [11] and [19].

Definition 4.1. A *C*-category* is a $\mathbb{C}$-linear category $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}),\ \mathrm{Mor}(\mathcal{C}),\ \circ)$ with the following additional properties.

- **(i):** (X,Y) is a complex Banach space.
- **(ii):** Composition is bilinear.[6]
- **(iii):** There is an involution, that is, a collection of maps $(X,Y) \ni f \mapsto f^* \in (Y,X)$ which are conjugate linear, involutive, and satisfy $(g \circ f)^* = f^* \circ g^*$ for $f \in (X,Y),\ g \in (Y,Z)$.
- **(iv):** $\|g \circ f\| \leq \|f\|\|g\|$ for $f \in (X,Y),\ g \in (Y,Z)$.
- **(v):** $\|h\|^2 = \|h^*h\|$ for $h \in (X,Y)$.
- **(vi):** For all $h \in \mathrm{Hom}(X,Y)$, $h^*h = g^*g$ for some $g \in \mathrm{Hom}(X,X)$.

A *C*-functor* is a linear functor between C*-categories which satisfies $F(f^*) = F(f)^*$.

[5] Or could be considered as the same as the objects of $\mathcal{C}$, for example, in one to one correspondence with the objects of $\mathcal{C}$.

[6] This is part of the definition of $\mathbb{C}$-linear, but included here for easy reference.

For any object X in a C*-category, (X, X) is a C*-algebra, and as a consequence of Theorem 4.5 and Remark 4.8, (X, Y) is a C*-ternary ring which is isomorphic to a TRO. For any C*-algebra A, the category with A as its only object, and A as it morphisms, with composition and involution being multiplication and involution in A, is a C*-category.

Example 4.2. The class of all complex Hilbert spaces (as objects) together with all bounded linear operators between them (as morphisms), with morphism composition $(f, g) \mapsto f \circ g$, forms a C*-category which we denote by $\mathcal{H}$.

Example 4.3. Let A be a C^*-algebra and let (H, ρ), (K, σ) be a pair of *-representations of A on Hilbert spaces H, K. An operator $t \in \mathcal{B}(H, K)$ with $t\rho(a) = \sigma(a)t$, for all $a \in A$, is called an *intertwiner*, the collection of all intertwiners between ρ and σ is denoted by $\mathrm{Hom}(\rho, \sigma)$. If $t \in \mathrm{Hom}(\rho, \sigma)$ then $t^* \in \mathrm{Hom}(\sigma, \rho)$, it follows that $\mathrm{Hom}(\rho, \sigma)$ is a weakly closed TRO contained in $\mathcal{B}(H, K)$ and $\mathrm{Hom}(\rho, \rho)$ is a C^*-subalgebra of $\mathcal{B}(H)$. The class of all *-representations of A on Hilbert spaces (as objects) together with intertwiners of these representations (as morphisms) is a C^*-category, denoted $\mathrm{Rep}(A)$, which can be viewed as a T^*-category with the natural (ternary) composition of intertwiners.

Example 4.4. Let Γ be a countable discrete group and let (H, ρ), (K, σ) be a pair of unitary representations of Γ on Hilbert spaces H, K. The collection $\mathrm{Hom}(\rho, \sigma)$ of intertwining operators between ρ and σ, i.e. operators $t \in \mathcal{B}(H, K)$ with $t\rho(g) = \sigma(g)t$ for all $g \in \Gamma$, is a weakly closed TRO contained in $\mathcal{B}(H, K)$, and $\mathrm{Hom}(\rho, \rho)$ is a von Neumann subalgebra of $\mathcal{B}(H)$. The class of all unitary representations of Γ on Hilbert spaces (as objects) together with intertwiners of these representations (as morphisms) is a C^*-category, denoted $\mathrm{Rep}(\Gamma)$, which can be viewed as a T^*-category (see Definition 4.6) with the natural (ternary) composition of intertwiners.

It is shown in [11] that every C^*-category $\mathcal{C}$ can be realized as a "concrete" C^*-sub-category of $\mathcal{H}$ (see also [19], which among other things, gives the proof of Theorem 4.5 below in more detail). Theorem 4.5 can be viewed as a generalization of the celebrated Gelfand-Naimark representation theorem which says that every abstract C^*-algebra can be realized as a concrete C^*-subalgebra of some $B(H)$, and it serves as the motivation for the results in this section.

Theorem 4.5 (Proposition 1.14 in [11])**.** *For every C^*-category $\mathcal{A}$, there is a faithful C^*-functor from $\mathcal{A}$ to $\mathcal{H}$.*

4.2. T*-categories.

Definition 4.6. A *T^*-category* is a ternary category $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}),\ \mathrm{Mor}(\mathcal{C}),\ \circ)$ with the following additional properties[7].

- **(i):** (X, Y) is a complex Banach space.
- **(ii):** For each X, Y, Z, W in $\mathrm{Ob}(\mathcal{C})$, a function
$$\begin{array}{rcl} \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Z,Y) \times \mathrm{Hom}(Z,W) & \to & \mathrm{Hom}(X,W) \\ (f,g,h) & \mapsto & [h \circ g^* \circ f] \end{array}$$
called *morphism composition* (or just *composition*), which is associative in the sense that $[[(lk^*h)]g^*f] = [l[gh^*k]^*f] = [lk^*[hg^*f]]$, whenever the compositions are defined (see Definition 3.1).
- **(iii):** Composition is linear in the outer variables and conjugate linear in the middle variable.
- **(iv):** $\|[gh^*f]\| \leq \|h\|\|f\|\|g\|$ for $f \in (X, Y), h \in (Z, Y), g \in (Z, W)$.

[7]Items (ii) and (iii) are parts of the definition of ternary category, but included here for easy reference.

(v): $\|hh^*h\| = \|h\|^3$, for $h \in (X,Y)$.

A *T^*-functor* is a linear functor between T*-categories. A T*-category is a *TW*-category* if each morphism set is a dual space.

For any objects X, Y in a T*-category (resp. TW*-category), (X,Y) is a C*-ternary ring (resp. W*-ternary ring). A *C^**-ternary ring X, or its concrete analogue, a ternary ring of operators (TRO), can be viewed as a T*-category with one object and the elements of X themselves as morphisms, with morphism composition given by the ternary operation in X.

As mentioned in subsection 1.1, Zhong-Jin Ruan [20] presented a classification scheme and proved various structure theorems for weakly closed ternary rings of operators (W*-TROs) of particular types. A W*-TRO V of type I, II, or III was defined according to the Murray-von Neumann type of its linking von Neumann algebra R_V, defined in subsection 1.3.

A W*-TRO V is of type I, II, or III according as R_V is a von Neumann algebra of the corresponding type. A W*-TRO of type II is said to be of type $II_{\epsilon,\delta}$, where $\epsilon, \delta \in \{1, \infty\}$, if $M(V)$ is of type II_ϵ and $N(V)$ is of type II_δ. A sample result is that a W*-TRO of type I is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha, B(K_\alpha, H_\alpha))$ ([20, theorem 4.1]).

Definition 4.7. A TW*-category is of type I (resp. II,III) if each morphism set (i.e. C*-ternary ring) (X,Y) is isomorphic as a C*-ternary ring (for example isomorphic or anti-isomorphic) to a W*-TRO of type I (resp. II,III).

In connection with Theorem 4.5, it is also proved ([11, Proposition 2.13]) that for every W*-category there is a faithful normal C*-functor into $\mathcal{H}$, which is obviously a W*-category. What is missing however, as mentioned in subsection 1.1, is a type classification of W*-categories into W*-categories of types I, II, and III. We remedy this in Propositions 4.9 and 4.24.

Remark 4.8. A C*-category (resp. W*-category) becomes a T*-category (resp. TW*-category) with ternary product $[hgf] = f \circ g^* \circ h$, and by Theorem 4.5 (resp. [11, Proposition 2.13]), each morphism set (X,Y) in a C*-category (resp. W*-category) is isomorphic to a TRO (resp. W*-TRO).

Proposition 4.9. *Each W*-category $\mathcal{C}$, considered as a TW*-category, is the direct sum $\mathcal{C}_I \oplus \mathcal{C}_{II} \oplus \mathcal{C}_{III}$, where $\mathcal{C}_i$, $i = I, II, III$, is a TW*-category of type i.*

Proof. By [11, Proposition 2.13], in a W*-category, each morphism space (X,Y) is isomorphic to a W*-TRO. By Ruan's classification $(X,Y) = (X,Y)_I \oplus (X,Y)_{II} \oplus (X,Y)_{III}$ and it suffices to take $\mathcal{C}_i$ to be the T*-category with morphism sets $(X,Y)_i$. □

Let $\mathcal{C}$ be a T*-category. Since $\mathcal{A}(X,Y)$ plays no role in what follows if $X \neq Y$, we will focus on the morphism sets (X,X) and for notation's sake, denote (X,X) by $\tilde{X}$, $L(X,X)$ by L, $R(X,X)$ by R, and $\mathcal{A}(X,X)$ by $\mathcal{A}$. Recall that $\tilde{X}$ is a left L-module via $L \times \tilde{X} \ni (A,f) \mapsto A \cdot f = A_1 f \in \tilde{X}$ and a right R^{op}-module via $\tilde{X} \times R \ni (f,B) \mapsto f \cdot B = B_1 f \in \tilde{X}$, and that

$$\mathcal{A} = \{a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} : A \in L, B \in R, f, g \in \tilde{X}\},$$

is an algebra with multiplication (3.2) and involution (3.3).

Remark 4.10. If $\mathcal{C}$ is a T*-category and $\mathcal{J}$ is a closed ideal (meaning $(X,Y)_{\mathcal{J}}$ is a closed subspace of the Banach space (X,Y)), then $\mathcal{C}/\mathcal{J}$ is a T*-category.

Proof. Items (i)-(iii) in Definition 4.6 are clear. To prove (iv), let $\overline{f} \in [X,Y], \overline{g} \in [Z,Y], \overline{h} \in [Z,W]$ and choose $f' \in (X,Y)_{\mathcal{J}}$, $g' \in (Z,Y)_{\mathcal{J}}$, $h' \in (Z,W)_{\mathcal{J}}$ such that $\|f + f'\| \leq \|\overline{f}\| + \epsilon$,

$\|g+g'\| \le \|\overline{g}\| + \epsilon$, $\|h+h'\| \le \|\overline{h}\| + \epsilon$. Then

$$\begin{aligned} \|[\overline{h}\overline{g}\overline{f}]\| &= \|[\overline{h+h'}, \overline{g+g'}, \overline{f+f'}]\| = \|\overline{[h+h', g+g', f+f']}\| \\ &\le \|[h+h', g+g', f+f']\| \le \|[h+h'\|\|g+g'\|\|f+f'\| \\ &\le \|\overline{h}\|\|\overline{g}\|\|\overline{f}\| + O(\epsilon). \end{aligned}$$

As for (v), if $\overline{h} \in [X,Y]$, then

$$\begin{aligned} \|\overline{h}\|^3 &= \inf_{k\in(X,Y)_{\mathcal{J}}} \|h+k\|^3 = \inf_{k\in(X,Y)_{\mathcal{J}}} \|[h+k, h+k, h+k]\| \\ &= \inf_{k\in(X,Y)_{\mathcal{J}}} \|[hhh] + \text{an element of } (X,Y)_{\mathcal{J}}\| \\ &\ge \inf_{k\in(X,Y)_{\mathcal{J}}} \|[hhh]+k\| = \|\overline{[hhh]}\| = \|[\overline{h},\overline{h},\overline{h}]\|. \end{aligned}$$

□

The following theorem is the first main result of this paper.

Theorem 4.11. *If $\mathcal{C}$ is a T*-category then $A_{\mathcal{C}}$ is a C*-category and there is an ideal $\mathcal{K} \neq \mathcal{C}$ of $\mathcal{C}$ and a faithful T*-functor from $\mathcal{C}/\mathcal{K}$ to $A_{\mathcal{C}}$, the latter considered as a T*-category.*

Proof. It is clear that $A_{\mathcal{C}}$, as defined in Definition 3.8, is a linear non-unital category which, when considered as a ternary category, satisfies (ii), (iii) and (vi) in Definition 4.1. Items (i), (iv), and (v) in Definition 4.1 are tantamount to the morphism sets of $A_{\mathcal{C}}$ of the form $\mathcal{A}(X,X)$ being normed as C*-algebras. This fact is immediate from Proposition 2.7.

The ideal $\mathcal{K}$ of $\mathcal{C}$ defined by $(X,Y)_{\mathcal{K}} = (X,Y)_{\mathcal{C}}$ if $X \neq Y$ and $(X,X)_{\mathcal{K}} = 0$, is the kernel of the functor F given by Remark 3.9, so it induces a faithful functor $\tilde{F} = F/\mathcal{K}$ from $\mathcal{C}/\mathcal{K}$ to $A_{\mathcal{C}}$. □

The following is the category analog of the Hamana extension of a TRO homomorphism to a *-homomorphism of the linking C*-algebras, [2, 8.3.5].

Remark 4.12. Let ρ be a T*-functor from a T*-category $\mathcal{C}$ to a T*-category $\mathcal{D}$. Then there is a C*-functor $\widehat{\rho}$ from $A_{\mathcal{C}}$ to $A_{\mathcal{D}}$ which extends ρ.

Proof. For each object X of $A_{\mathcal{C}}$, set $\widehat{\rho}(X) = \rho(X)$, which is an object of $\mathcal{D}$ and hence of $A_{\mathcal{D}}$. If $X \neq Y$, then $(X,Y)_{A_{\mathcal{C}}} = 0$, so set $\widehat{\rho}((X,Y)_{A_{\mathcal{C}}}) = 0$. For $\phi = \rho|_{(X,X)_{A_{\mathcal{C}}}}$ where $(X,X)_{A_{\mathcal{C}}} = \mathcal{A}(X,X)$, let $\widehat{\rho}(\phi)$ be the element $\mathcal{A}(\phi) \in (\rho(X),\rho(X))_{A_{\mathcal{D}}} = \mathcal{A}(\rho(X),\rho(X))$ given by Lemma 2.6. □

Remark 4.13. Direct sums of categories were defined in [19, Definition 3.8]. The same definition can be made for ternary categories.

(i) If $\mathcal{C}$ and $\mathcal{D}$ are categories (resp. ternary categories) whose objects may be considered identical, then the direct sum $\mathcal{C} \oplus \mathcal{D}$ is defined as the category (resp. ternary category) whose objects are identified with the objects of $\mathcal{C}$ or $\mathcal{D}$, and with morphism sets

$$\text{Hom}(X,Y)_{\mathcal{C}\oplus\mathcal{D}} = \text{Hom}(X,Y)_{\mathcal{C}} \oplus \text{Hom}(X,Y)_{\mathcal{D}},$$

and composition defined coordinatewise.

(ii) If $\mathcal{C}$ is a T*-category, T*-subcategories $\mathcal{C}_{\pm}$ are defined as follows. Recall that by Theorem 2.1, we have $(X,Y) = (X,Y)_+ \oplus (X,Y)_-$ for each pair of objects X, Y of $\mathcal{C}$. The objects of $\mathcal{C}_{\pm}$ are the same as the objects of $\mathcal{C}$, and for such objects X, Y,

$$(X,Y)_{\mathcal{C}_{\pm}} := (X,Y)_{\pm}.$$

It is clear that $\mathcal{C}$ is isomorphic to $\mathcal{C}_+ \oplus \mathcal{C}_-$ and that $A_{\mathcal{C}}$ is isomorphic to $\mathcal{A}_{C_+} \oplus \mathcal{A}_{C_-}$.

(iii) The category $\mathcal{H}$ consisting of complex Hilbert spaces $H, K, L, \dots$ as objects and bounded linear maps $B(H,K)$ as morphisms is a C*-category with composition ST for $S \in B(K,L)$ and $T \in B(H,K)$ (See Definition 4.2). It is a T*-category with composition RS^*T. We shall now denote this T*-category $\mathcal{H}$ by $\mathcal{H}_+$ and let $\mathcal{H}_-$ denote $\mathcal{H}$ as a T*-category with composition $-RS^*T$.

The following may be called the Gelfand-Naimark theorem for T*-categories.

Theorem 4.14. *Let $\mathcal{C}$ be a T*-category. Then there is an ideal $\mathcal{K} \neq \mathcal{C}$ in $\mathcal{C}$ and a faithful T*-functor from $\mathcal{C}/\mathcal{K}$ to the T*-category $\mathcal{H}_+ \oplus \mathcal{H}_-$.*

Proof. By Theorem 4.5, there is a faithful C*-functor $G_\pm$ from $A_{\mathcal{C}_\pm}$ to $\mathcal{H}$. With $A_{\mathcal{C}_\pm}$ considered as a T*-category, we have that $G_\pm$ is a T*-functor from $A_{\mathcal{C}_\pm}$ to $\mathcal{H}_\pm$. By Theorem 4.11, there is a T*-functor $F_\pm$ from $\mathcal{C}_\pm$ to $A_{\mathcal{C}_\pm}$, and it suffices to consider $H/\mathcal{K}$, where $\mathcal{K}$ is the ideal in Theorem 4.11, and $H = (G_+ \circ F_+) \oplus (G_- \circ F_-)$. □

We close this subsection with some examples of linking C*-categories.

Example 4.15. (Cf. Example 1.6) If X is a TRO, and $\mathcal{C}$ is the T*-category $(\{X\}, X)$ with X as its sole object and the elements of X as its morphisms from X to X, and with composition $[zyx] = xy^*z$, then $A_{\mathcal{C}}$ is the category $(\{X\}, \mathcal{A}(X,X))$ with X as its sole object and the elements of $\mathcal{A}(X,X)$ as its morphisms from X to X, and with composition being the multiplication in $\mathcal{A}(X,X)$. As expected, the C*-algebra $\mathcal{A}(X,X)$ is *-isomorphic to the linking algebra A_X of the TRO X under the map

$$A_X \ni \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} \mapsto \begin{bmatrix} \sum_i([x_i y_i \cdot], [y_i x_i \cdot]) & z \\ \overline{w} & \sum_j([\cdot u_j v_j], [\cdot v_j u_j]) \end{bmatrix} \in \mathcal{A}(X,X).$$

Example 4.16. If $(Z, (\cdot,\cdot,\cdot))$ is a C*-ternary ring, and $\mathcal{C}$ is the T*-category $(\{Z\}, Z)$ with Z as its sole object and the elements of Z as its morphisms from Z to Z, and with composition $[zyx] = (x,y,z)$, then $A_{\mathcal{C}}$ is the category $(\{Z\}, \mathcal{A}(Z,Z))$ with Z as its sole object and the elements of $\mathcal{A}(Z,Z)$ as its morphisms from Z to Z, and with composition being the multiplication in $\mathcal{A}(Z,Z)$.

The C*-algebra $\mathcal{A}(Z,Z) = \mathcal{A}(Z_+) \oplus \mathcal{A}(Z_-)$ is *-isomorphic to $A_X \oplus \mathcal{B}$ where X is a TRO which is isomorphic to Z_+, and $\mathcal{B}$ is a C*-algebra which is related to a TRO Y which is anti-isomorphic to Z_-. Precisely, $\mathcal{A}(Z_+)$ is isomorphic to A_X, which is the closure of

$$\left\{ \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} : x_i, y_i, u_j, v_j, z, w \in X \right\}$$

with multiplication

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix} \begin{bmatrix} \alpha' & z' \\ w'^* & \beta' \end{bmatrix} = \begin{bmatrix} \alpha\alpha' + xy'^* & \alpha x' + x\beta' \\ y^*\alpha' + \beta y'^* & y^*\alpha' + \beta\beta' \end{bmatrix}$$

and $\mathcal{A}(Z_-)$ is isomorphic to $\mathcal{B}$, which is the closure of

$$\left\{ \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} : x_i, y_i, u_j, v_j, z, w \in Y \right\}$$

with multiplication

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix} \begin{bmatrix} \alpha' & z' \\ w'^* & \beta' \end{bmatrix} = \begin{bmatrix} \alpha\alpha' - xy'^* & -\alpha x' - x\beta' \\ -y^*\alpha' - \beta y'^* & -y^*\alpha' + \beta\beta' \end{bmatrix}.$$

Example 4.17. If $\mathcal{C}$ is the T*-category $\mathcal{H}_+$ of Hilbert spaces and bounded linear maps (see Remark 4.13(iii)), then $A_{\mathcal{H}_+}$ is the C*-category with the same objects as $\mathcal{H}_+$, and for each such object (Hilbert space) H, $(H,H)_{\mathcal{H}_+} = B(H)$ and $(H,H)_{A_{\mathcal{H}_+}} = \mathcal{A}(H,H) = B(H) \oplus B(H) = M_2(B(H))$.

Example 4.18. If $\mathcal{C}$ is any C*-category, considered as a T*-category then $A_{\mathcal{C}}$ is the C*-category with the same objects as $\mathcal{C}$, and for each such object X, $(X,X)_{A_{\mathcal{C}}}$ is a C*-algebra and $(X,X)_{A_{\mathcal{C}}} = \mathcal{A}(X,X) = M_2((X,X))$.

Example 4.19. Let A be a C^*-algebra. Then the class $\mathcal{C}$ of all Hilbert C^*-modules over A (as objects) together with all bounded A-linear and adjointable operators (as morphisms), with morphism composition $(f,g,h) \mapsto hg^*f$, forms a T^*-category. In this case $\mathcal{A}(X,X)$ is isomorphic to the linking algebra as defined in [2, 8.1.17, pp. 303–304] and $A_{\mathcal{C}}$ is therefore a subcategory of the C*-category of C*-Hilbert A-modules and bounded A-linear maps.

4.3. **The linking W*-category of a TW*-category.** The proofs of the main results in this section (Theorems 4.23 and 4.25 below) are based on the tools leading up to the following Gelfand-Naimark theorem for W*-ternary rings, which recall are C*-ternary rings with a predual.

Theorem 4.20 (Theorem 4.1 in [22])**.** *A W*-ternary ring Z is the direct sum of two W*-ternary subrings Z_+ and Z_- which are respectively normally isometrically isomorphic and normally isometrically anti-isomorphic to a W*-TRO. Normally means the isomorphism and anti-isomorphic are weak*-continuous.*

Let V be a C*-ternary ring with triple product denoted by $[hgf]$. By Proposition 2.3, V is the off-diagonal corner of a C*-algebra $\mathcal{A}(V)$, where

$$\mathcal{A}(V) = \begin{bmatrix} L & V \\ \overline{V} & R \end{bmatrix},$$

and $L = L(V)$ and $R = R(V)$ are C*-algebras. Consider

$$\tilde{A}(V) = \begin{bmatrix} M(L) & V \\ \overline{V} & M(R) \end{bmatrix}, \tag{4.1}$$

where $M(L)$ and $M(R)$ are the multiplier algebras of L and of R.

Proposition 4.21. *If the C*-ternary ring V is a dual space, then $M(R(V))$ and $M(L(V))$ are W*-algebras, and therefore V is the off-diagonal corner of a W*-algebra.*

Proof. In order to use the results of [22, section 4], we recall that the C*-algebra $\mathfrak{A}$ in Remark 2.2 is the closed span of $\{[\cdot gh] : g,h \in V\}$ and it is *-isomorphic to $R(V)$ via the map $\mathfrak{A} \ni B_1 \mapsto \sigma(B_1) = (B_1, B_1^*) \in R(V)$. Similarly $\tau : \mathfrak{B} \to L(V)$ is the *-isomorphism $A_1 \mapsto (A_1, A_1^*)$, where $\mathfrak{B}$ is the close span of $\{[gh\cdot] : g,h \in V\}$. The C*-ternary ring V is thus both a Banach (L,R)-bimodule and a Banach $(\mathfrak{B},\mathfrak{A})$-bimodule.

Using only the assumption that V is a right Hilbert $\mathfrak{A}$-module, it is proved in [22, section 4], that $M(\mathfrak{A})$ is a W*-algebra. It follows that, provided V is a right Hilbert R^{op}-module, $M(R)$ is a W*-algebra, and by a parallel argument, that $M(L)$ is also a W*-algebra. The reduction to V being a Hilbert module is obtained by considering V with a triple product modified by the operator T in Theorem 2.1(iii). It follows that

$$\tilde{A}(V) = M(L) \oplus V \oplus \overline{V} \oplus M(R)$$

is the dual of

$$\tilde{A}(V)_* = M(L)_* \oplus V_* \oplus \overline{V}_* \oplus M(R)_*,$$

where V_* is the predual of V, so $\tilde{A}(V)$ is a W*-algebra. □

Recall that a TW*-category is a T*-category in which each morphism set is a dual space. A W*-ternary ring was introduced in [22] as a C*-ternary ring which is a dual space. For any objects X, Y in a TW*-category, (X, Y) is a W*-ternary ring. A W^*-ternary ring X, or its concrete analogue, a weakly closed ternary ring of operators (W*-TRO), can be viewed as a TW*-category with one object X and the elements of X themselves as morphisms, with morphism composition given by the ternary operation in X.

Definition 4.22. Given a TW*-category $\mathcal{C}$, the *linking W*-category* $\tilde{A}_{\mathcal{C}}$ of $\mathcal{C}$ is as follows. The objects of the category $A_{\mathcal{C}}$ are the same as the objects of $\mathcal{C}$. The morphism set $\mathrm{Hom}(X, Y)$ is defined to be $\{0\}$ if $X \neq Y$, and $\mathrm{Hom}(X, X) = \tilde{\mathcal{A}}(X, X)$, as in (4.1) with $V = (X, X)$, and with composition as follows. If $a \in \mathrm{Hom}(X, Y)$ and $b \in \mathrm{Hom}(Y, Z)$, then $b \circ a$ must be $\{0\}$ unless $X = Y = Z$, in which case $b \circ a$ is defined to be the product ab in $\tilde{\mathcal{A}}(X, X)$.

The following is the W*-version of Theorem 4.11.

Theorem 4.23. *If $\mathcal{C}$ is a TW*-category then $\tilde{A}_{\mathcal{C}}$ is a W*-category and there is an ideal $\mathcal{K} \neq \mathcal{C}$ of $\mathcal{C}$ and a faithful TW*-functor from $\mathcal{C}/\mathcal{K}$ to $\tilde{A}_{\mathcal{C}}$, the latter considered as a TW*-category.*

Proof. It is clear that $\tilde{A}_{\mathcal{C}}$, as defined in Definition 4.22, is a linear non-unital category which, when considered as a ternary category, satisfies (ii), (iii) and (vi) in Definition 4.1. Items (i), (iv), and (v) in Definition 4.1 are tantamount to the morphism sets of $\tilde{A}_{\mathcal{C}}$ of the form $\tilde{\mathcal{A}}(X, X)$ being W*-algebras. This fact is immediate from Proposition 4.21.

The ideal $\mathcal{K}$ of $\mathcal{C}$ defined by $(X, Y)_{\mathcal{K}} = (X, Y)_{\mathcal{C}}$ if $X \neq Y$ and $(X, X)_{\mathcal{K}} = 0$, is the kernel of the functor F given by Remark 3.9, so it induces a faithful functor $\tilde{F} = F/\mathcal{K}$ from $\mathcal{C}/\mathcal{K}$ to $\tilde{A}_{\mathcal{C}}$. □

The following is the appropriate version of Proposition 4.9.

Proposition 4.24. *Each TW*-category $\mathcal{C}$ is the direct sum $\mathcal{C}_I \oplus \mathcal{C}_{II} \oplus \mathcal{C}_{III}$, where $\mathcal{C}_i$, $i = I, II, III$, is a TW*-category of type i.*

Proof. In a TW*-category, each morphism space (X, Y) is isomorphic as a W*-ternary ring to a W*-TRO. By Ruan's classification

$$(X, Y)_\pm = ((X, Y)_\pm)_I \oplus ((X, Y_\pm))_{II} \oplus ((X, Y)_\pm)_{III}$$

and it suffices to take $\mathcal{C}_i$ to be the T*-category with morphism sets

$$(X, Y)_i = ((X, Y)_+)_i \oplus ((X, Y)_-)_i.$$

□

The following is the W*-version of Theorem 4.14.

Theorem 4.25. *Let $\mathcal{C}$ be a TW*-category. Then there is an ideal $\mathcal{K} \neq \mathcal{C}$ in $\mathcal{C}$ and a faithful TW*-functor H from $\mathcal{C}/\mathcal{K}$ to the TW*-category $\mathcal{H}_+ \oplus \mathcal{H}_-$.*

Proof. By [11, Proposition 2.13], there is a faithful W*-functor $G_\pm$ from $\tilde{A}_{\mathcal{C}_\pm}$ to $\mathcal{H}$. With $\tilde{A}_{\mathcal{C}_\pm}$ considered as a TW*-category, we have that $G_\pm$ is a TW*-functor from $\tilde{A}_{\mathcal{C}_\pm}$ to $\mathcal{H}_\pm$. By Theorem 4.23, there is a TW*-functor $F_\pm$ from $\mathcal{C}_\pm$ to $\tilde{A}_{\mathcal{C}_\pm}$, and it suffices to consider $H/\mathcal{K}$, where $\mathcal{K}$ is the ideal in Theorem 4.23, and $H = (G_+ \circ F_+) \oplus (G_- \circ F_-)$. □

5. Bidual categories

5.1. **The bidual of a C*-category.** We begin by reviewing the well-known and celebrated Arens multiplications. If A is an algebra with algebraic dual A' and bidual A'', the following two multiplications on A'' were defined in [1], and are referred to as the first and second Arens products, denoted by FG and $F \cdot G$ respectively for $F, G \in A''$. Each product extends the product in A when A is identified with its canonical image in A''.

domain	First Arens product FG		Second Arens product $F \cdot G$	
$A \times A$	$(a,b) \mapsto ba \in A$	(product in A)	$(a,b) \mapsto ab \in A$	(product in A)
$A' \times A$	$(f,b) \mapsto bf \in A'$	$\langle bf, a\rangle = \langle f, ba\rangle$	$(f,b) \mapsto fb \in A'$	$\langle fb, a\rangle = \langle f, ab\rangle$
$A'' \times A'$	$(F,f) \mapsto fF \in A'$	$\langle fF, b\rangle = \langle F, bf\rangle$	$(F,f) \mapsto Ff \in A'$	$\langle Ff, b\rangle = \langle F, fb\rangle$
$A'' \times A''$	$(F,G) \mapsto FG \in A''$	$\langle FG, f\rangle = \langle F, fG\rangle$	$(F,G) \mapsto F \cdot G \in A''$	$\langle F \cdot G, f\rangle = \langle G, Ff\rangle$

If $\varphi : A \to B$ is an algebra homomorphism, then $\varphi'' : A'' \to B''$ is an algebra homomorphism in either product extending φ. When the two products coincide, the algebra A is called Arens regular. If A is a *-algebra, its involution extends to a mapping on A'' via $\langle F^*, f\rangle = \overline{\langle F, f^*\rangle}$ and $\langle f^*, a\rangle = \overline{\langle f, a^*\rangle}$ for $f \in A', a \in A$. However, since $(FG)^* = G^* \cdot F^*$, $F \mapsto F^*$ is not an involution unless A is Arens regular.

In the following analog of the Arens construction for categories, there is essentially only one Arens multiplication. This simplification is due to the fact that in the morphism spaces (X,Y), $a \circ b$ and $b \circ a$ are simultaneously defined only if $X = Y$. We shall therefore only use the first Arens product, with the understanding that morphism spaces (X,X) might not be Arens regular. Also, because the composition in categories is more akin to composition of functions, we shall use the notation $G \circ F$ for the analog of FG. For all other products, including composition, for notation's sake, we shall just use juxtaposition.

Definition 5.1. Let $\mathcal{C}$ be a linear category with objects $X, Y, Z, \ldots$, morphism spaces $(X,Y) = \{a, b, c, \ldots\}$, dual spaces $(X,Y)' = \{f, g, h, \ldots\}$, and bidual spaces $(X,Y)'' = \{F, G, H, \ldots\}$. For objects X, Y, Z a composition defined on $(X,Y)'' \times (Y,Z)'' \to (X,Z)''$ and given by the Arens construction is as follows.

$(X,Y) \times (Y,Z) \ni (a,b) \mapsto ba := b \circ a \in (X,Z)$	(composition in $\mathcal{C}$)
$(X,Z)' \times (Y,Z) \ni (f,b) \mapsto bf \in (X,Y)'$	$\langle bf, a\rangle = \langle f, ba\rangle$, $a \in (X,Y)$
$(X,Y)'' \times (X,Z)' \ni (F,f) \mapsto fF \in (Y,Z)'$	$\langle fF, b\rangle = \langle F, bf\rangle$, $b \in (Y,Z)$
$(X,Y)'' \times (Y,Z)'' \ni (F,G) \mapsto G \circ F \in (X,Z)''$	$\langle G \circ F, f\rangle = \langle G, fF\rangle$, $f \in (X,Z)'$

The composition

$$G \circ F : (X,Y)'' \times (Y,Z)'' \to (X,Z)''$$

is an extension of the composition

$$b \circ a : (X,Y) \times (Y,Z) \to (X,Z)$$

in $\mathcal{C}$. That is, if $a \mapsto \widehat{a}$ denotes the canonical inclusion of (X,Y) into $(X,Y)''$, then for $(a,b) \in (X,Y) \times (Y,Z)$,

$$\widehat{b} \circ \widehat{a} = \widehat{ba}.$$

The following lemma is a straightforward consequence of Definition 5.1 and justifies Definition 5.3. We include the proof for completeness.

Lemma 5.2. *For $F \in (X,Y)'', G \in (Y,Z)'', H \in (Z,W)''$, we have $G \circ F \in (X,Z)''$, $H \circ G \in (Y,W)''$, and $(H \circ G) \circ F = H \circ (G \circ F)$.*

Proof. For $f \in (X,Z)'$,

$$\langle H \circ (G \circ F), f\rangle = \langle H, f(G \circ F)\rangle,$$

and

$$\langle (H \circ G) \circ F, f\rangle = \langle H \circ G, fF\rangle = \langle H, (fF)G\rangle.$$

Thus it suffices to prove

$$f(G \circ F) = (fF)G.$$

For $a \in (Z,W)$,

$$\langle f(G \circ F), a\rangle = \langle G \circ F, af\rangle = \langle G, (af)F\rangle,$$

and

$$\langle (fF)G, a\rangle = \langle G, a(fF)\rangle,$$

so it suffices to prove

$$a(fF) = (af)F.$$

For $b \in (Y,Z)$,

$$\langle a(fF), b\rangle = \langle fF, ab\rangle = \langle F, (ab)f\rangle,$$

and

$$\langle (af)F, b\rangle = \langle F, b(af)\rangle,$$

so it suffices to prove

$$(ab)f = b(af).$$

For $c \in (X,Y)$,

$$\langle (ab)f, c\rangle = \langle f, (ab)c\rangle,$$

and

$$\langle b(af), c\rangle = \langle af, bc\rangle = \langle f, a(bc)\rangle,$$

completing the proof. □

The composition $G \circ F$ is weak*-continuous in its first variable G, for if $G_\alpha \to G$, then for $f \in (Y,Z)'$, $\langle G_\alpha, f\rangle \to \langle G, f\rangle$ and so if $g \in (X,Z)'$, $\langle G_\alpha \circ F, g\rangle = \langle G_\alpha, gF\rangle \to \langle G, gF\rangle = \langle G \circ F, g\rangle$.

Definition 5.3. The *Arens bidual* of a linear category $\mathcal{C}$, denoted by $\mathcal{C}''$, or by $(\mathcal{C}'', \text{Arens})$, is the linear category having the same objects as $\mathcal{C}$, morphism sets $\mathrm{Hom}(X,Y) = (X,Y)''$ and composition given by the Arens construction in Definition 5.1. The category $\mathcal{C}$ is said to be *Arens regular* if the composition $G \circ F$ is separately weak*-continuous, that is, it is also weak*-continuous in the second variable.

Lemma 5.4. *If $\mathcal{C}$ is an Arens regular *-linear category with involutions $(X,Y) \ni a \mapsto a^* \in (Y,X)$, then the linear involution defined as*

$(X,Y)'' \ni F \mapsto F^* \in (Y,X)''$ *with* $\langle F^*, f\rangle = \overline{\langle F, f^*\rangle}$, $f \in (Y,X)'$,
$(Y,X)' \ni f \mapsto f^* \in (X,Y)'$ *with* $\langle f^*, a\rangle = \overline{\langle f, a^*\rangle}$, $a \in (X,Y)$,

*is an algebra involution. Hence, the Arens bidual $(\mathcal{C}'', Arens)$ is a *-linear category.*

Proof. Let $F \in (X,Y)''$, $G \in (Y,Z)''$, and by Arens regularity, we may assume that $F = \hat{a}$, $G = \hat{b}$ for $a \in (X,Y), b \in (Y,Z)$. Then

$$\begin{aligned}\langle (G \circ F)^*, f\rangle &= \overline{\langle G \circ F, f^*\rangle} = \overline{\langle G, f^*F\rangle} = \overline{\langle f^*F, b\rangle} = \overline{\langle F, bf^*\rangle}\\ &= \overline{\langle bf^*, a\rangle} = \overline{\langle f^*, ba\rangle} = \langle f, (ba)^*\rangle = \langle f, a^*b^*\rangle\end{aligned}$$

and

$$\begin{aligned}\langle F^*\circ G^*, f\rangle &= \langle F^*, fG^*\rangle = \overline{\langle F, (fG^*)^*\rangle} = \overline{\langle (fG^*)^*, a\rangle}\\ &= \langle fG^*, a^*\rangle = \langle G^*, a^*f\rangle = \overline{\langle G, (a^*f)^*\rangle}\\ &= \overline{\langle (a^*f)^*, b\rangle} = \langle a^*f, b^*\rangle = \langle f, a^*b^*\rangle.\end{aligned}$$

□

Lemma 5.5. *A C*-category is Arens regular.*

Proof. [8] Let ρ be a faithful C*-functor from a C*-category $\mathcal{C}$ to $\mathcal{H}$, and for objects X and Y of $\mathcal{C}$, consider the following commutative diagram:

$$\begin{array}{ccccc} & & A & & \\ & & \uparrow\subset & & \\ (X,Y) & \xrightarrow{\rho} & \mathcal{R} & \xrightarrow{\pi} & \pi(\mathcal{R}) \\ \downarrow\kappa & & \downarrow\kappa & & \downarrow\subset \\ (X,Y)'' & \xrightarrow{\rho''} & \mathcal{R}'' & \xrightarrow{\pi''} & \mathcal{S}\subset B(H_\pi), \end{array}$$

where by Remark 4.8, $\rho = \rho_{X,Y}$ is, in particular, a C*-ternary ring isomorphism onto a TRO $\mathcal{R} = \mathcal{R}(X,Y) = \rho((X,Y))$, $\pi = \pi_{X,Y}$ is the restriction to $\mathcal{R}$ of the universal representation of the C*-algebra A generated by $\mathcal{R}$ on the Hilbert space H_π, $\mathcal{S} = \mathcal{S}(X,Y)$ is the weak operator closure of the TRO $\pi(\mathcal{R})$, and κ denotes the canonical inclusion of a Banach space into its bidual. By [16, Lemma], the bi-adjoint π'' of π is, as well as a TRO-isomorphism of $\mathcal{R}''$ onto $\mathcal{S}$, a homeomorphism of $\mathcal{R}''$ with its weak*-topology and $\mathcal{S}$ with its the weak operator topology from $B(H_\pi)$ (which coincides with its weak*-topology). It follows that composition at the level of the $\mathcal{R}(X,Y)''$ spaces is separately weak*-continuous, and since each $\rho''_{X,Y}$ is a weak*-weak* homeomorphism, the same holds for composition at the level of the $(X,Y)''$. □

Proposition 5.6. *If $\mathcal{C}$ is a C*-category, then it's Arens bidual $(\mathcal{C}'', Arens)$ is a C*-category.*

Proof. Items (i), (ii), and (iv) in Definition 4.1 are immediate. Item (iii) holds by Lemmas 5.4 and 5.5. By Remark 4.8, (X,Y) is a C*-ternary ring which is isomorphic to a TRO M. The bidual M'' of M is also isomorphic to a TRO, by [16, Lemma], from which items (v) and (vi) in Definition 4.1 follow. Indeed, for (v), since the faithful C*-functor ρ from $\mathcal{C}$ to $\mathcal{H}$ satisfies $\rho(c\circ b^*\circ a) = \rho(c)\rho(b)^*\rho(a)$, we have (by a familiar argument)

$$\begin{aligned}\|H\|^3 &= \|\rho''(H)\|^3 = \|\rho''(H)\rho''(H)^*\rho''(H)\| \le \|\rho''(H)\|\|\rho''(H)^*\rho''(H)\|\\ &= \|H\|\|\rho''(H^*\circ H)\| = \|H\|\|H^*\circ H\| \le \|H^3\|,\end{aligned}$$

so $\|H\|^2 = \|H^*\circ H\|$. As for (vi), if $H \in (X,Y)''$, observe that $\rho''(H)^*\rho''(H)$ is a positive operator in the C*-algebra $\rho''(X,X)$ on the Hilbert space $\rho(X)$. □

Let us now consider a different approach to the definition of the bidual of a C*-category which is based on the Sherman-Takeda proof that the bidual of a C*-algebra is a C*-algebra [21, III.2.4], [14, 10.1.12], [7, 12.1.3].

In the proof of Lemma 5.5, set $\sigma := \pi\circ\rho$, which is also a faithful C*-functor mapping (X,Y) onto $\pi(\mathcal{R})$, and set $\tau = \tau_{X,Y} := \sigma'' = \pi''\circ\rho''$, which is a homeomorphism of $(X,Y)''$ with the weak*-topology onto $\mathcal{S}$ with the weak operator topology. Then, for $F \in (X,Y)''$ and

[8] After giving this proof, we discovered that this Lemma follows as in the proof of Proposition 5.13. We are including this proof since, besides its intrinsic interest, it is needed in the definition of the Sherman-Takeda bidual of a C*-category (Definition 5.7).

$G \in (Y,Z)''$, note that $\tau(G) \in \mathcal{S}(Y,Z)$, $\tau(F) \in \mathcal{S}(X,Y)$, so that for $G = $w*-$\lim_\beta \widehat{b_\beta}$, $b_\beta \in (Y,Z)$ and $F = $w*-$\lim_\alpha \widehat{a_\alpha}$, $a_\alpha \in (X,Y)$, we have

$$\begin{aligned}\tau(G)\tau(F) &= \tau(\text{w*-}\lim_\beta \widehat{b_\beta})\tau(\text{w*-}\lim_\alpha \widehat{a_\alpha}) \\ &= (\text{W-}\lim_\beta \sigma(b_\beta))(\text{W-}\lim_\alpha \sigma(a_\alpha)) \\ &= \text{W-}\lim_\alpha(\text{W-}\lim_\beta \sigma(b_\beta))\sigma(a_\alpha) \\ &= \text{W-}\lim_\alpha \text{W-}\lim_\beta \sigma(b_\beta a_\alpha) \in \mathcal{S}(X,Z).\end{aligned}$$

Thus we can define $G \bullet F \in (X,Z)''$, by

$$G \bullet F = \tau^{-1}(\tau(G)\tau(F)), \tag{5.1}$$

more precisely,

$$G \bullet F = \tau^{-1}_{X,Z}(\tau_{Y,Z}(G)\tau_{X,Y}(F)),$$

and we have, for $F \in (X,Y)'', G \in (Y,Z)'', H \in (Z,W)''$,

$$\begin{aligned}H \bullet (G \bullet F) &= \tau^{-1}(\tau(H)\tau(G \bullet F)) = \tau^{-1}(\tau(H)\tau(G)\tau(F)) \\ &= \tau^{-1}(\tau(H \bullet G)(\tau(F)) = (H \bullet G) \bullet F.\end{aligned}$$

Moreover, since $\tau(G \bullet F) = \tau(G)\tau(F)$, and π'' is a *-isomorphism, in the sense that

$$\pi''_{X,Z}(\rho''_{Y,Z}(G)\rho''_{X,Y}(F)) = \pi''_{Y,Z}(\rho''_{Y,Z}(G))\pi''_{X,Y}(\rho''_{X,Y}(F)),$$

we have

$$\rho''(G \bullet F) = \rho''(G)\rho''(F),$$

that is,

$$\rho''_{X,Z}(G \bullet F) = \rho''_{Y,Z}(G)\rho''_{X,Y}(F).$$

Also, for $a \in (X,Y), b \in (Y,Z)$,

$$\hat{b} \bullet \hat{a} = \tau^{-1}(\tau(\hat{b})\tau(\hat{a})) = (\rho'')^{-1}(\rho''(\hat{b})\rho''(\hat{a}))$$

so that

$$\rho''(\hat{b} \bullet \hat{a}) = \rho''(\hat{b})\rho''(\hat{a}) = \widehat{\rho(b)}\widehat{\rho(a)}.$$

Definition 5.7. The *Sherman-Takeda bidual* of a C*-category $\mathcal{C}$, denoted by $\mathcal{C}''$, or $(\mathcal{C}'', \text{S-T})$, is the linear category having the same objects as $\mathcal{C}$, morphism sets $\text{Hom}(X,Y) = (X,Y)''$ and composition defined on $(X,Y)'' \times (Y,Z)'' \to (X,Z)''$ given by (5.1).

Proposition 5.8. *If $\mathcal{C}$ is a C*-category, then it's Sherman-Takeda bidual $(\mathcal{C}'', S\text{-}T)$ is a C*-category.*

Proof. Items (i)-(iii) in Definition 4.1 are immediate. As for (iv) and (v), for $F \in (X,Y)''$ and $G \in (Y,Z)''$,

$$\|G \bullet F\| = \|\tau(G)\tau(F)\| \le \|\tau(G)\|\|\tau(F)\| = \|G\|\|F\|$$

and since $\tau_{X,X}$ is a *-isomorphism,

$$\|F^* \bullet F\| = \|\tau(F^*)\tau(F)\| = \|\tau(F)^*\tau(F)\| = \|\tau(F)\|^2 = \|F\|^2.$$

Finally, since $\tau(F)^*\tau(F)$ is a positive operator in $\mathcal{S}(X,X)$, it equals A^*A for some $A \in \mathcal{S}(X,X)$, and $A = \tau(G)$ for some $G \in (X,X)''$, so that $F^* \bullet F = \tau^{-1}(\tau(G)^*\tau(G)) = \tau^{-1}(\tau(G^*)\tau(G)) = G^* \bullet G$, proving (vi) in Definition 4.1. ∎

Proposition 5.9. *If $\mathcal{C}$ is a C*-category, then $G \circ F = G \bullet F$, that is, the Arens bidual coincides with the Sherman-Takeda bidual.*

Proof. For $f \in (X,Z)'$, $\langle G \circ F, f\rangle = \langle G, fF\rangle$ and [9]

$$\begin{aligned}
\langle G \bullet F, f\rangle &= \langle \tau^{-1}(\tau(G)\tau(F)), f\rangle \\
&= \langle (\rho'')^{-1} \circ (\pi'')^{-1}(\tau(G)\tau(F)), f\rangle \\
&= \langle (\rho'')^{-1}(\rho''(G)\rho''(F)), f\rangle \\
&= \langle \rho''(G)\rho''(F), (\rho')^{-1} f\rangle \\
&= \langle \rho''(G), R'_{\rho''(F)} \circ (\rho')^{-1} f\rangle \\
&= \langle G, \rho' \circ R'_{\rho''(F)} \circ (\rho')^{-1} f\rangle.
\end{aligned}$$

Hence $\langle G \circ F, f\rangle = \langle G \bullet F, f\rangle$ if and only if

$$fF = \rho' \circ R'_{\rho''(F)} \circ (\rho')^{-1} f. \tag{5.2}$$

For $b \in (Y,Z)$, $\langle fF, b\rangle = \langle F, bf\rangle$ and

$$\begin{aligned}
\langle \rho' \circ R'_{\rho''(F)} \circ (\rho')^{-1} f, b\rangle &= \langle R'_{\rho''(F)} \circ (\rho')^{-1} f, \rho(b)\rangle \\
&= \langle \widehat{\rho(b)}, R'_{\rho''(F)} \circ (\rho')^{-1} f\rangle \\
&= \langle \rho''(F), L'_{\widehat{\rho(b)}} \circ (\rho')^{-1} f\rangle \\
&= \langle F, \rho' \circ L'_{\widehat{\rho(b)}} \circ (\rho')^{-1} f\rangle,
\end{aligned}$$

so (5.2) is equivalent to

$$bf = \rho' \circ L'_{\widehat{\rho(b)}} \circ (\rho')^{-1} f. \tag{5.3}$$

For $a \in (X,Y)$, $\langle bf, a\rangle = \langle f, ba\rangle = \langle \widehat{ba}, f\rangle$ and

$$\begin{aligned}
\langle \rho' \circ L'_{\widehat{\rho(b)}} \circ (\rho')^{-1} f, a\rangle &= \langle L'_{\widehat{\rho(b)}} \circ (\rho')^{-1} f, \rho(a)\rangle \\
&= \langle \widehat{\rho(a)}, L'_{\widehat{\rho(b)}} \circ (\rho')^{-1} f\rangle \\
&= \langle \widehat{\rho(b)}\widehat{\rho(a)}, (\rho')^{-1} f\rangle \\
&= \langle (\rho'')^{-1}(\widehat{\rho(b)}\widehat{\rho(a)}), f\rangle,
\end{aligned}$$

so (5.3) is equivalent to

$$\rho''(\widehat{ba}) = \widehat{\rho(b)}\widehat{\rho(a)}, \tag{5.4}$$

which is equivalent to

$$\rho(ba) = \rho(b)\rho(a), \tag{5.5}$$

which holds since ρ is a C*-functor.

For completeness, we give details of the last stated equivalence. First note that (5.4) is the same as

$$\widehat{\rho(ba)} = \widehat{\rho(b)}\widehat{\rho(a)}. \tag{5.6}$$

By (5.6) we have

$$\langle f, \rho(ba)\rangle = \langle L'_{\widehat{\rho(b)}} f, \rho(a)\rangle = \langle L'_{\rho(b)} f, \rho(a)\rangle = \langle f, \rho(b)\rho(a)\rangle.$$

[9] The elements $R_{\rho''(F)} : \mathcal{R}(Y,Z)'' \to \mathcal{R}(X,Z)''$ and $L_{\widehat{\rho(b)}} : \mathcal{R}(X,Y)'' \to \mathcal{R}(Y,Z)''$ which appear below are the operators of right multiplication and left multiplication respectively and are each weak*-continuous.

Hence (5.6) implies (5.5). Conversely, by (5.5),

$$\begin{aligned}\langle\widehat{\rho(ba)}, f\rangle &= \langle f, \rho(ba)\rangle = \langle f, \rho(b)\rho(a)\rangle \\ &= \langle L'_{\rho(b)} f, \rho(a)\rangle = \langle\widehat{\rho(a)}, L'_{\rho(b)} f\rangle \\ &= \langle\widehat{\rho(a)}, L'_{\widehat{\rho(b)}} f\rangle = \langle\widehat{\rho(b)}\widehat{\rho(a)}, f\rangle.\end{aligned}$$

Hence (5.5) implies (5.6). □

5.2. **The bidual of a T*-category.**

Definition 5.10. Let $\mathcal{C}$ be a linear ternary category with objects $X, Y, Z, \ldots$, morphism spaces $(X, Y) = \{a, b, c, \ldots\}$, dual spaces $(X, Y)' = \{f, g, h, \ldots\}$, and bidual spaces $(X, Y)'' = \{F, G, H, \ldots\}$. For objects X, Y, Z, W a composition defined on $(X, Y)'' \times (Z, Y)'' \times (Z, W)'' \to (X, W)''$, denoted by

$$(X, Y)'' \times (Z, Y)'' \times (Z, W)'' \ni (F, G, H) \mapsto [HGF] \in (X, W)''$$

and given by the Arens construction is as follows.

(1) $(X, Y) \times (Y, Z) \times (Z, W) \ni (a, b, c) \mapsto [cba] \in (X, W)$
(composition in $\mathcal{C}$)

(2) $(X, W)' \times (X, Y) \times (Z, Y) \ni (f, a, b) \mapsto \mu_0(f, a, b) \in (Z, W)'$
$\langle\mu_0(f, a, b), c\rangle = \langle f, [cba]\rangle$, $c \in (Z, W)$

(3) $(Z, W)'' \times (X, W)' \times (X, Y) \ni (F, f, a) \mapsto \mu_1(F, f, a) \in (Z, Y)'$
$\langle\mu_1(F, f, a), b\rangle = \overline{\langle F, \mu_0(f, a, b)\rangle}$, $b \in (Z, Y)$

(4) $(Z, Y)'' \times (Z, W)'' \times (X, W)' \ni (F, G, f) \mapsto \mu_2(F, G, f) \in (X, Y)'$
$\langle\mu_2(F, G, f), a\rangle = \overline{\langle F, \mu_1(G, f, a)\rangle}$, $a \in (X, Y)$

(5) $(X, Y)'' \times (Z, Y)'' \times (Z, W)'' \ni (F, G, H) \mapsto [HGF] \in (X, W)''$
$\langle [HGF], f\rangle = \langle F, \mu_2(G, H, f)\rangle$, $f \in (X, W)'$

We note that $[HGF]$ is linear in the outer variables and conjugate linear in the middle variable, and in the case of normed ternary categories, is weak*-continuous in the right variable F.

The following lemma is a straightforward consequence of Definition 5.10. Again, for completeness, we include the proof.

Lemma 5.11. *Let $\mathcal{C}$ be a linear normed ternary category. Then for $F \in (X, Y)''$, $G \in (Z, Y)''$, $H \in (Z, W)''$, $K \in (U, W)''$ and $L \in (U, V)''$, we have*

(i): $[LK[HGF]] = [[LKH]GF]$.

(ii): *Assume that the triple product $[HGF]$ is separately weak*-continuous, that is, is also weak*-continuous in the left variable H and the middle variable G. Then*

$$[LK[HGF]] = [L[GHK]F] = [[LKH]GF].$$

Proof. (i) For $f \in (X, V)'$,

$$\langle [LK[HGF]], f\rangle = \langle [HGF], \mu_2(K, L, f)\rangle = \langle F, \mu_2(G, H, \mu_2(K, L, f))\rangle,$$

and

$$\langle [[LKH]GF, f\rangle = \langle F, \mu_2(G, [LKH], f)\rangle, \tag{5.7}$$

so it suffices to prove

$$\mu_2(G, H, \mu_2(K, L, f)) = \mu_2(G, [LKH], f).$$

For $a \in (X, Y)$,

$$\langle \mu_2(G, H, \mu_2(K, L, f)), a\rangle = \overline{\langle G, \mu_1(H, \mu_2(K, L, f), a)\rangle},$$

and

$$\langle \mu_2(G, [LKH], f), a\rangle = \overline{\langle G, \mu_1([LKH], f, a)\rangle}, \tag{5.8}$$

so it suffices to prove

$$\mu_1(H, \mu_2(K, L, f), a) = \mu_1([LKH], f, a).$$

For $b \in (Z, Y)$,

$$\langle \mu_1(H, \mu_2(K, L, f), a), b\rangle = \overline{\langle H, \mu_0(\mu_2(K, L, f), a, b)\rangle},$$

and

$$\langle \mu_1([LKH], f, a), b\rangle = \overline{\langle [LKH], \mu_0(f, a, b)\rangle} = \overline{\langle H, \mu_2(K, L, \mu_0(f, a, b))\rangle}, \tag{5.9}$$

so it suffices to prove

$$\mu_0(\mu_2(K, L, f), a, b) = \mu_2(K, L, \mu_0(f, a, b)).$$

For $c \in (Z, W)$,

$$\langle \mu_0(\mu_2(K, L, f), a, b), c\rangle = \langle \mu_2(K, L, f), [cba]\rangle = \overline{\langle K, \mu_1(L, f, [cba])\rangle},$$

and

$$\langle \mu_2(K, L, \mu_0(f, a, b), c\rangle = \overline{\langle K, \mu_1(L, \mu_0(f, a, b), c)\rangle}, \tag{5.10}$$

so it suffices to prove

$$\mu_1(L, f, [cba]) = \mu_1(L, \mu_0(f, a, b), c).$$

For $d \in (U, W)$,

$$\langle \mu_1(L, f, [cba]), d\rangle = \overline{\langle L, \mu_0(f, [cba], d)\rangle},$$

and

$$\langle \mu_1(L, \mu_0(f, a, b), c), d\rangle = \overline{\langle L, \mu_0(\mu_0(f, a, b), c, d)\rangle}, \tag{5.11}$$

so it suffices to prove

$$\mu_0(f, [cba], d) = \mu_0(\mu_0(f, a, b), c, d).$$

For $e \in (U, V)$,

$$\langle \mu_0(f, [cba], d), e\rangle = \langle f, [ed[cba]]\rangle,$$

and

$$\langle \mu_0(\mu_0(f, a, b), c, d), e\rangle = \langle \mu_0(f, a, b), [edc]\rangle = \langle f, [[edc]ba]\rangle. \tag{5.12}$$

This proves $[LK[HGF]] = [[LKH]GF]$.

(ii) For $f \in (X, V)'$,

$$\langle [L[GHK]F], f\rangle = \langle F, \mu_2([GHK], L, f)\rangle, \tag{5.13}$$

so by (5.7) and (5.13), it suffices to prove

$$\mu_2([GHK], L, f)) = \mu_2(G, [LKH], f).$$

For $a \in (X, Y)$,

$$\begin{aligned}
\langle \mu_2([GHK], L, f)), a\rangle &= \overline{\langle [GHK], \mu_1(L, f, a)\rangle} \\
&= \overline{\langle R_{K,H} G, \mu_1(L, f, a)\rangle} \\
&= \overline{\langle G, R'_{K,H}(\mu_1(L, f, a))\rangle}, \qquad (5.14)
\end{aligned}$$

where $R_{K,H}$ is, by assumption, the weak*-continuous operator sending G to $[GHK]$, so by (5.8) and (5.14), it suffices to prove

$$R'_{K,H}(\mu_1(L,f,a)) = \mu_1([LKH],f,a).$$

For $b \in (Z,Y)$,

$$\begin{aligned} \langle R'_{K,H}(\mu_1(L,f,a)),b\rangle &= \langle \widehat{b}, R'_{K,H}(\mu_1(L,f,a))\rangle \\ &= \langle [\widehat{b}HK], \mu_1(L,f,a)\rangle \\ &= \langle H, Q'_{\widehat{b},K}(\mu_1(L,f,a))\rangle \end{aligned} \tag{5.15}$$

where $Q_{H,K}$ is the conjugate linear and weak*-continuous operator sending G to $[HGK]$, and by (5.9) and (5.15), it suffices to prove

$$\overline{\langle H, \mu_2(K,L,\mu_0(f,a,b))\rangle} = \langle H, Q'_{\widehat{b},K}(\mu_1(L,f,a))\rangle. \tag{5.16}$$

We shall complete the proof by verifying (5.16), and we may assume that $H = \widehat{c}$ for some $c \in (Z,W)$. We may also assume, by weak*-continuity in the right variable, that $K = \widehat{d}$. Thus

$$\begin{aligned} \langle H, Q'_{\widehat{b},K}(\mu_1(L,f,a))\rangle &= \langle [\widehat{b}\widehat{c}K], \mu_1(L,f,a)\rangle \\ &= \langle K, L'_{\widehat{b}\widehat{c}}(\mu_1(L,f,a))\rangle \\ &= \overline{\langle \mu_1(L,f,a), [bcd]\rangle} \\ &= \overline{\langle L, \mu_0(f,a,[bcd])\rangle} \end{aligned}$$

and

$$\begin{aligned} \overline{\langle H, \mu_2(K,L,\mu_0(f,a,b))\rangle} &= \overline{\langle \mu_2(K,L,\mu_0(f,a,b)),c\rangle} \\ &= \langle K, \mu_1(L,\mu_0(f,a,b),c)\rangle \\ &= \langle \mu_1(L,\mu_0(f,a,b),c),d\rangle \\ &= \overline{\langle L, \mu_0(\mu_0(f,a,b),c,d))\rangle}. \end{aligned}$$

Thus (5.16) is equivalent to

$$\mu_0(\mu_0(f,a,b),c,d)) = \mu_0(f,a,[bcd])\rangle. \tag{5.17}$$

Take $e \in (U,V)$. Then

$$\langle \mu_0(\mu_0(f,a,b),c,d)),e\rangle = \langle \mu_0(f,a,b),[edc]\rangle = \langle f,[[edc]ba]\rangle$$

and

$$\langle \mu_0(f,a,[bcd]),e\rangle = \langle f,[e[bcd]a]\rangle$$

thus proving (5.17) and $[LK[HGF]] = [L[GHK]F]$. □

Definition 5.12. The *Arens bidual* of a linear ternary category $\mathcal{C}$, denoted $\mathcal{C}''$, or $(\mathcal{C}'', \text{Arens})$, is the linear category having the same objects as $\mathcal{C}$, morphism sets $\text{Hom}(X,Y) = (X,Y)''$ and composition given by the Arens construction in Definition 5.10. The category $\mathcal{C}$ is said to be *Arens regular* if the composition $[HGF]$ is separately weak*-continuous.

Proposition 5.13. *A T*-category $\mathcal{C}$ is Arens regular, and hence its Arens bidual $\mathcal{C}''$ is a T*-category.*

Proof. As stated in [15, Remark 2.10], every multilinear map $f : X_1 \times \cdots \times X_n \to Y$ from Banach spaces X_i satisfying Pelczynski's property V to a Banach space Y admits a unique separately weak*-continuous extension from $X_1'' \times \cdots \times X_n''$ to Y''. C*-ternary rings are JB*-triples and JB*-triples satisfy Pelczynski's property V ([5]). As stated in [15, Remark 2.3], if

$f : X_1 \times X_2 \times X_3 \to Y$ admits a norm preserving extension $F : X_1'' \times X_2'' \times X_3'' \to Y''$ (produced by any method) which is separately weak*-continuous, then f is Aron-Berner regular and therefore the extension given by Definition 5.10 (denoted by $f^{*\overline{***}}$ in [15]) is separately weak*-continuous. Thus a T*-category is Arens regular.

Items (i), (iii), and (iv) in Definition 4.6 are immediate. Item (ii) holds by Lemma 5.11. By Remark 4.8, (X, Y) is a C*-ternary ring. The bidual $(X, Y)''$ of (X, Y) is also a C*-ternary ring, by [16, Theorem 2], from which item (v) in Definition 4.6 follows. □

6. Proof of Proposition 2.3(iv)

Let M be a C*-ternary ring. Recall that, M being a normed associative triple system, it is, by Remark 1.4, a left $L(M)$-Banach module via $L(M) \times M \ni (A, f) \mapsto A \cdot f = A_1 f \in M$ and a right $R(M)^{op}$-Banach module via $M \times R(M) \ni (f, B) \mapsto f \cdot B = B_1 f \in M$, and that

$$\mathcal{A} = \{a = \begin{bmatrix} A & f \\ g & B \end{bmatrix} : A \in L(M), B \in R(M), f, g \in M\},$$

is an algebra with multiplication (1.1) and involution (1.2).

Proposition 6.1 (Restatement of Proposition 2.3)**.** *With the above notation, we have*

- **(i):** *$R(M)$ is a C*-algebra with the norm from $B(M)$.*
- **(ii):** *M is a right Banach $R(M)^{op}$-module.*
- **(iii):** *With $\langle f|g\rangle = \langle f|g\rangle_M : M \times M \to R(M)$ defined by $\langle f|g\rangle = r(g, f) = ([\cdot gf], [\cdot fg])$, we have*

$$\langle f \cdot B|g\rangle = \langle f|g\rangle \circ B.$$

- **(iv):** *If M is a right $R(M)^{op}$-Hilbert module, then $\mathcal{A}$ can be normed to be a C*-algebra.*

Proof. (i)-(iii) have been proved in section 2.

(iv) We mimic the proof in [2, 8.1.17, p. 303] by showing that the map $\pi : \mathcal{A} \to B(M \oplus R)$ to the bounded operators on the right R^{op}-Hilbert module $M \oplus R$ defined, for $a = \begin{bmatrix} A & f \\ g & B \end{bmatrix} \in \mathcal{A}$ by

$$\pi(a) \begin{bmatrix} f' \\ B' \end{bmatrix} = \begin{bmatrix} A & f \\ g & B \end{bmatrix} \begin{bmatrix} f' \\ B' \end{bmatrix} = \begin{bmatrix} A \cdot f' + f \cdot B' \\ r(g, f') + B \circ B' \end{bmatrix}, \tag{6.1}$$

is an injective *-homorphism. Letting $\|a\| = \|\pi(a)\|$ turns $\mathcal{A}$ into a C*-algebra.

We will use the facts that R is a right R^{op}-Hilbert module, via

$$R \times R^{op} \ni (B', B) \mapsto B \cdot B' = B' \circ B \in R,$$

and that $M \oplus R$ is a right R^{op}-Hilbert module, via

$$(M \oplus R) \times R^{op} \ni ((f, B'), B) \mapsto (f, B') \cdot B = (f \cdot B, B' \circ B) \in M \oplus R.$$

Thus for $b' = \begin{bmatrix} f' \\ B' \end{bmatrix}$, and $b'' = \begin{bmatrix} f'' \\ B'' \end{bmatrix}$ in $M \oplus R$,

$$\langle b', b''\rangle_{M \oplus R} = \langle f', f''\rangle_M + \langle B', B''\rangle_R,$$

where $\langle f, g\rangle_M := r(g, f) = ([\cdot gf], [\cdot fg])$ and $\langle B', B\rangle_R = \overline{B} \circ B'$.

First, with $a = \begin{bmatrix} A & f \\ g & B \end{bmatrix} \in \mathcal{A}$, $b' = \begin{bmatrix} f' \\ B' \end{bmatrix}$, and $b'' = \begin{bmatrix} f'' \\ B'' \end{bmatrix}$ in $M \oplus R$, we have

$$\begin{aligned}
\langle \pi(a^{\#})b', b''\rangle &= \langle \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix} \begin{bmatrix} f' \\ B' \end{bmatrix}, \begin{bmatrix} f'' \\ B'' \end{bmatrix} \rangle = \begin{bmatrix} \overline{A} \cdot f' + g \cdot B' \\ r(f, f') + \overline{B} \circ B' \end{bmatrix}, \begin{bmatrix} f'' \\ B'' \end{bmatrix} \rangle \\
&= \langle \overline{A} \cdot f' + g \cdot B', f'' \rangle_M + \langle r(f, f) + \overline{B} \circ B', B'' \rangle_R \\
&= \langle \overline{A} \cdot f', f'' \rangle_M + \langle g \cdot B', f'' \rangle_M + \langle r(f, f'), B'' \rangle_R + \langle \overline{B} \circ B', B'' \rangle_R
\end{aligned}$$

and

$$\begin{aligned}
\langle \pi(a)^* b', b'' \rangle &= \langle \begin{bmatrix} f' \\ B' \end{bmatrix}, \pi(a) \begin{bmatrix} f'' \\ B'' \end{bmatrix} \rangle = \langle \begin{bmatrix} f' \\ B' \end{bmatrix}, \begin{bmatrix} A \cdot f'' + f \cdot B'' \\ r(g, f'') + \overline{B} \circ B'' \end{bmatrix} \rangle \\
&= \langle f', A \cdot f'' \rangle_M + f \cdot B'' \rangle_M + \langle B', r(g, f'') + \overline{B} \circ B'' \rangle_R. \\
&= \langle f', A \cdot f'' \rangle_M + \langle f', f \cdot B'' \rangle_M + \langle B', r(g, f'') \rangle_R + \langle B', \overline{B} \circ B'' \rangle_R.
\end{aligned}$$

The fact that $\pi(a^{\#}) = \pi(a)^*$ now follows from the following four identities,

$$\langle \overline{A} \cdot f', f'' \rangle_M = \langle f', A \cdot f'' \rangle_M \tag{6.2}$$

$$\langle g \cdot B', f'' \rangle_M = \langle B', r(g, f'') \rangle_R \tag{6.3}$$

$$\langle r(f, f'), B'' \rangle_R = \langle f', f \cdot B'' \rangle_M \tag{6.4}$$

$$\langle \overline{B} \circ B', B'' \rangle_R = \langle B', \overline{B} \circ B'' \rangle_R. \tag{6.5}$$

To prove (6.2), we may assume that $A = \ell(h, k) = ([hk\cdot], [kh\cdot])$. Then

$$\begin{aligned}
\langle \overline{A} \cdot f', f'' \rangle_M &= \langle ([kh\cdot], [hk\cdot]) \cdot f', f'' \rangle_M \\
&= \langle [khf'], f'' \rangle_M \\
&= r(f'', [khf']) \\
&= ([\cdot f''[khf']], [\cdot[khf']f'']),
\end{aligned}$$

and

$$\begin{aligned}
\langle f', A \cdot f'' \rangle_M &= \langle f', [hkf''] \rangle_M \\
&= r([hkf''], f') \\
&= ([\cdot[hkf'']f'], [\cdot f'[hkf'']]).
\end{aligned}$$

To prove (6.3), we may assume that $B' = r(h, k) = (\cdot[hk], [\cdot kh])$. Then

$$\begin{aligned}
\langle g \cdot B', f'' \rangle_M &= \langle [ghk], f'' \rangle_M \\
&= r(f'', [ghk]) \\
&= ([\cdot f''[ghk]], [\cdot[ghk]f'']),
\end{aligned}$$

and

$$\begin{aligned}
\langle B', r(g, f'') \rangle_R &= \overline{r(g, f'')} \circ ([\cdot hk], [\cdot kh]) \\
&= ([\cdot f''g], [\cdot gf'']) \circ ([\cdot hk], [\cdot kh]) \\
&= ([\cdot hk], [\cdot kh])([\cdot f''g], [\cdot gf'']) \\
&= ([\cdot hk][\cdot f''g], [\cdot gf''][\cdot kh]) \\
&= ([[\cdot f''g]hk], [[\cdot kh]gf'']).
\end{aligned}$$

To prove (6.4), we may assume that $B'' = r(h,k) = (\cdot[hk], [\cdot kh])$. Then

$$\begin{aligned}\langle r(f,f'), B''\rangle_R &= \langle r(f,f'), r(h,k)\rangle_R \\ &= \overline{r(h,k)} \circ r(f,f') \\ &= ([\cdot ff'], [\cdot f'f])([\cdot kh], [\cdot hk]) \\ &= ([\cdot ff'][\cdot kh], [\cdot hk][\cdot f'f]) \\ &= ([[\cdot kh]ff'], [[\cdot f'f]hk])\end{aligned}$$

and

$$\begin{aligned}\langle f', f\cdot B''\rangle_M &= \langle f', [fhk]\rangle_M \\ &= r([fhk], f') \\ &= ([\cdot[fhk]f'], [\cdot f'[fhk]]).\end{aligned}$$

Finally to prove (6.5), we have

$$\langle \overline{B} \circ B', B''\rangle_R = \overline{B''} \circ (B \circ B') = (B \circ B')\overline{B''} = B'B\overline{B''}$$

and

$$\langle B', \overline{B} \circ B''\rangle_R = \overline{\overline{B} \circ B''} \circ B' = B'(\overline{B''\overline{B}}) = B'B\overline{B''}.$$

Next, we show that with $a, a'' \in \mathcal{A}$ and $b' = \begin{bmatrix} f' \\ B' \end{bmatrix}$, we have

$$\pi(a'')\pi(a)b' = \pi(a''a)b',$$

so that π is a homorphism.

We have

$$\begin{aligned}\pi(a'')\pi(a)b' &= \begin{bmatrix} A'' & f'' \\ \overline{g''} & B'' \end{bmatrix} \begin{bmatrix} A\cdot f' + f\cdot B' \\ r(g,f') + B\circ B' \end{bmatrix} \\ &= \begin{bmatrix} A''\cdot(A\cdot f') + A''\cdot(f\cdot B') + f''\cdot r(g,f') + f''\cdot(B\circ B') \\ r(g'', A\cdot f') + B''\circ r(g,f') + r(g'', f\cdot B') + B''\circ(B\circ B') \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\pi(a''a)b' &= \begin{bmatrix} A''A + \ell(f'',g) & A''\cdot f + f''\cdot B \\ \overline{g''}\cdot A + B''\cdot \overline{g} & r(g'',f) + B''\circ B \end{bmatrix} \begin{bmatrix} f' \\ B' \end{bmatrix} \\ &= \begin{bmatrix} (A''A)\cdot f' + \ell(f'',g)\cdot f' + (A''\cdot f)\cdot B' + (f''\cdot B)\cdot B' \\ r(\overline{g''}\cdot A, f) + r(B''\cdot\overline{g}, f') + r(g'',f)\circ B' + (B''\circ B)\circ B' \end{bmatrix}.\end{aligned}$$

The first components of $\pi(a'')\pi(a)b'$ and $\pi(a''a)b'$ are equal by the module properties and

$$\ell(f'',g)\cdot f' = f''\cdot r(g,f'), \tag{6.6}$$

and the second components are equal because of the three identities

$$r(\overline{g''}\cdot A, f') = r(g'', A\cdot f') \tag{6.7}$$

$$r(B''\cdot\overline{g}, f') = B''\circ r(g,f') \tag{6.8}$$

$$r(g'',f)\circ B' = r(g'', f\cdot B'). \tag{6.9}$$

To prove (6.6), we have

$$\ell(f'',g)\cdot f' = ([f''g\cdot], [gf''\cdot])\cdot f' = [f''gf']$$

and

$$f''\cdot r(g,f') = f''\cdot([\cdot gf'], [\cdot f'g]) = [f''gf'].$$

To prove (6.7), we may assume that $A = \ell(h,k) = ([hk\cdot],[kh\cdot])$. Then

$$r(\overline{g''}\cdot A, f') = r(\overline{g''}\cdot([hk\cdot],[kh\cdot]), f') = r([khg''], f') = ([\cdot[khg'']f'],[\cdot f'[khg'']])$$

and

$$r(g'', A\cdot f') = r(g'', ([hk\cdot],[kh\cdot])\cdot f') = r(g'', [hkf']) = ([\cdot g''[hkf'],[\cdot[hkf']g'']).$$

To prove (6.8), we may assume that $B'' = r(h,k) = (\cdot[hk],[\cdot kh])$. Then

$$r(B''\cdot\overline{g}, f') = r([gkh], f') = ([\cdot[gkh]f'],[\cdot f'[gkh])$$

and

$$\begin{aligned} B''\circ r(g,f') &= r(g,f')(\cdot[hk],[\cdot kh]) \\ &= ([\cdot gf'],[\cdot f'g])(\cdot[hk],[\cdot kh]) \\ &= ([\cdot gf'],[\cdot hk])(\cdot[kh],[\cdot f'g]) \\ &= ([[\cdot hk]gf'],[[\cdot f'g]kh]). \end{aligned}$$

To prove (6.9), we may assume that $B' = r(h,k) = (\cdot[hk],[\cdot kh])$. Then

$$\begin{aligned} r(g'',f)\circ B' &= ([\cdot g''f],[\cdot fg''])\circ([\cdot hk],[\cdot kh]) \\ &= (\cdot[hk],[\cdot kh])([\cdot g''f],[\cdot fg'']) \\ &= ([\cdot hk][\cdot g''f],[\cdot fg''][\cdot kh]) \\ &= ([[\cdot g''f]hk],[[\cdot kh]fg'']) \end{aligned}$$

and

$$r(g'', f\cdot B') = r(g'',[fhk]) = ([\cdot g''[fhk]],[\cdot[fhk]g'']).$$

Let us now show that π is injective. For $a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \mathcal{A}$, if $\pi(a) = 0$, then by (6.1)

$$A\cdot f' + f\cdot B' = 0 \text{ and } r(g,f') + B\circ B' = 0$$

for all $f' \in M, B' \in R$, and in particular,

$$A\cdot f' = 0 \text{ and } f\cdot B' = 0, \tag{6.10}$$

and

$$r(g,f') = 0 \text{ and } B\circ B' = 0. \tag{6.11}$$

From (6.10) with $B' = r(f,f)$, $[fff] = 0$ so $f = 0$. From (6.11), $B^*B = 0$ and $r(g,g) = 0$, so $B = 0$ and $g = 0$.

It remains to show that $A = 0$. Since $\pi(a^\#) = 0$, we have

$$\overline{A}\cdot f' = 0 \text{ for all } f' \in M. \tag{6.12}$$

Suppose first that $A = \ell(g,h)$. Then by (6.10) and (6.12), $[f'gh] = 0$ and $[f'hg] = 0$ so that $A = ([\cdot gh],[\cdot hg]) = 0$. By the same argument, if $A = \sum_i \ell(g_i,h_i)$, then $A = 0$.

Now suppose $A \in L$, let $\epsilon > 0$ and choose $A' = \sum_i \ell(g_i,h_i)$ with $\|A - A'\| < \epsilon$. Then $\|A'\cdot f'\| = \|(A-A')\cdot f'\| \le \epsilon\|f'\|$, so that $\|A\| \le \|A - A'\| + \|A'\| < 2\epsilon$, and $A = 0$.

It remains to show that $\pi(\mathcal{A})$ is a C*-algebra, that is, complete, and for this it is enough to show that the range of π is closed. For $T = [t_{ij}] \in B(M\oplus R)$, we have $\|t_{ij}\| \le \|T\|$. Thus if $T \in \overline{\pi(\mathcal{A})}$, then $t_{11} \in \overline{L} = L$, $t_{12} \in \overline{M} = M$, ..., and so $T \in \pi(\mathcal{A})$. □

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