# FINITE DIFFERENCE METHODS FOR POISSON EQUATION

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The finite difference method replaces each derivative by a difference quotient in the classical formulation. It is simple to code and efficient to compute, offering a direct and intuitive approach to the numerical solution of partial differential equations. Its main limitation is flexibility: standard finite difference methods require higher regularity of the solution and uniform grids. Difficulties also arise when imposing boundary conditions.

## 1. Finite Difference Method in 1-D

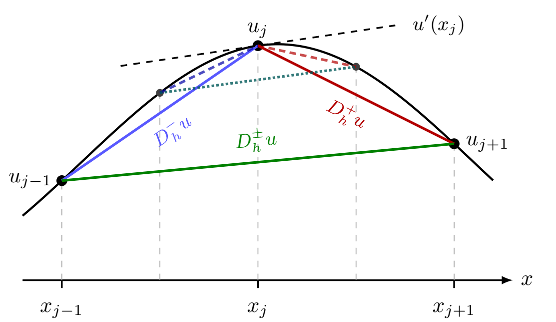
Consider Poisson's equation in 1-D: Given a function $f(x)$ and two constants $g_D$ and $g_N$, find $u(x)$ such that

$$\begin{aligned} -u''(x) &= f(x), \quad x \in \Omega := (0,1), \\ u'(0) &= g_N \quad \text{(Neumann boundary condition)}, \\ u(1) &= g_D \quad \text{(Dirichlet boundary condition)}. \end{aligned}$$

**Uniform grids.** For a positive integer $N$, consider a uniform grid $\mathcal{T}_h$ of the interval $[0,1]$:

$$0 = x_0 < x_1 < \cdots < x_N = 1, \quad x_j = jh, \ j = 0, \dots, N,$$

where $h = 1/(N+1)$ is the mesh size. In the finite difference method, we use the vector $\boldsymbol{u} = (u_0, \dots, u_N)^{\intercal}$, where $u_i \approx u(x_i)$ for $i = 1, \dots, N$.




**Difference formulas.** Popular difference formulas at an interior node $x_j$ for a discrete function are:

- Backward difference: $(D^- u)_j = \dfrac{u_j - u_{j-1}}{h}$;
- Forward difference: $(D^+ u)_j = \dfrac{u_{j+1} - u_j}{h}$;
- Central difference: $(D^\pm u)_j = \dfrac{u_{j+1} - u_{j-1}}{2h}$;

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- Second central difference: $(D^2u)_j = \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$.

By Taylor expansion, it follows that

$$(D^-u)_j - u'(x_j) = \mathcal{O}(h), \quad (D^+u)_j - u'(x_j) = \mathcal{O}(h),$$
$$(D^{\pm}u)_j - u'(x_j) = \mathcal{O}(h^2), \quad (D^2u)_j - u''(x_j) = \mathcal{O}(h^2).$$

The first-order $\mathcal{O}(h)$ result is straightforward. We verify the second-order $\mathcal{O}(h^2)$ for the central difference as follows

$$u(x_{j+1}) - u(x_j) = u'(x_j)h + \tfrac{1}{2}u''(x_j)h^2 + \tfrac{1}{6}u^{(3)}(x_j)h^3 + \mathcal{O}(h^4), \tag{1a}$$

$$u(x_{j-1}) - u(x_j) = -u'(x_j)h + \tfrac{1}{2}u''(x_j)h^2 - \tfrac{1}{6}u^{(3)}(x_j)h^3 + \mathcal{O}(h^4). \tag{1b}$$

Subtracting (1b) from (1a) yields

$$(D^{\pm}u)_j - u'(x_j) = \mathcal{O}(h^2),$$

while adding them gives

$$(D^2u)_j - u''(x_j) = \mathcal{O}(h^2).$$

So at interior nodes, we get the finit difference method for $-u'' = f$:

$$- u_{i-1} + 2u_i - u_{i+1} = f(x_i)h^2, \tag{2}$$

where we move the scaling $h^2$ to the right hand side.

**Boundary conditions.** The Dirichlet boundary condition $u_N = u(1) = g_D$ is build into the equation by moving to the right hand side, i.e., at $x_{N-1}$, the equation becomes

$$- u_{N-2} + 2u_{N-1} = f(x_i)h^2 + g_D, \tag{3}$$

The Neumann boundary condition can be approximated by

$$g_N = u'(0) = u'(x_0) \approx \frac{u_1 - u_0}{h},$$

which is only first-order accurate. To improve, we introduce a ghost point $u_{-1}$ and apply a central difference

$$g_N = u'(0) = u'(x_0) \approx \frac{u_1 - u_{-1}}{2h}.$$

We add one more equation at the boundary node

$$-u_{-1} + 2u_0 - u_1 = f(0)h^2,$$

and solve $u_{-1} = u_1 - 2hg_N$ from the boundary condition. Finally, a scaling is introduced to preserve the symmetry of the matrix:

$$u_0 - u_1 = \frac{1}{2}f(0)h^2 - g_N h.$$

We write the corresponding linear algebraic equation as follows

$$A\boldsymbol{u} = \boldsymbol{b},$$

where

$$A_{N\times N} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \frac{1}{2}f(0)h^2 - g_N h \\ f_1 h^2 \\ f_2 h^2 \\ \vdots \\ f_{N-2}h^2 \\ f_{N-1}h^2 + g_D \end{bmatrix}.$$

## 2. Finite Difference Method in 2-D

In this section, for simplicity, we discuss the Poisson equation

$$-\Delta u = f$$

posed on the unit square $\Omega = (0,1)\times(0,1)$ with Dirichlet or Neumann boundary conditions. Recall that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Variable coefficients and more complex domains will be discussed in finite element methods. Furthermore we assume $u$ is smooth enough to enable us use Taylor expansion freely.

### 2.1. **Finite Difference Discretization.**

Given two integers $m, n \geq 2$, we construct a rectangular grid $\mathcal{T}_h$ by the tensor product of two uniform grids of $(0,1)$: $\{x_i = (i-1)h_x, i = 1, \cdots m, h_x = 1/(m-1)\}$ and $\{y_j = (j-1)h_y, j = 1, \cdots n, h_y = 1/(n-1)\}$. Let $h = \max\{h_x, h_y\}$ denote the size of $\mathcal{T}_h$. Denote by $\Omega_h = \{(x_i, y_j) \in \Omega\}$ and boundary $\Gamma_h = \{(x_i, y_j) \in \partial\Omega\}$.

We consider the discrete function space given by $\mathbb{V}_h = \{u_h(x_i, y_j), 1 \leq i \leq m, 1 \leq j \leq n\}$ which is isomorphism to $\mathbb{R}^N$ with $N = m \times n$. It is more convenient to use sub-index $(i,j)$ for the discrete function: $u_{i,j} := u_h(x_i, y_j)$. For a continuous function $u \in \mathcal{C}(\Omega)$, the interpolation operator $I_h : \mathcal{C}(\Omega) \to \mathbb{V}_h$ maps $u$ to a discrete function and will be denoted by $u_I$. By the definition $(u_I)_{i,j} = u(x_i, y_j)$. Note that the value of a discrete function is only defined at grid points. Values inside each cell can be obtained by the convex combination of values at grid points.

We shall use these difference formulation, especially the second central difference to approximate the Laplace operator at an interior node $(x_i, y_j)$:

$$\begin{aligned}(\Delta_h u)_{i,j} &= (D^2_{xx} u)_{i,j} + (D^2_{yy} u)_{i,j}\\ &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2}.\end{aligned}$$

It is called five point stencil since there are only five points involved. When $h_x = h_y$, it is simplified to

$$-(\Delta_h u)_{i,j} = \frac{4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2} \tag{4}$$

and can be denoted by the following stencil symbol

$$\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}.$$

For the right hand side, we simply take node values i.e. $f_{i,j} = (f_I)_{i,j} = f(x_i, y_j)$.

The finite difference method for solving the Poisson equation is simply

$$-(\Delta_h u)_{i,j} = f_{i,j}, \quad 1 \leq i \leq m, 1 \leq j \leq n, \tag{5}$$

with appropriate processing of different boundary conditions; see §2.2. Here in (5), we use (4) for all grid points including boundary points but simply drop terms involving grid points outside of the domain.

Let us give an ordering of $N = m \times n$ grids and use a single index $k = 1$ to $N$ for $u_k = u_{i(k),j(k)}$ which is called a linear indexing. For example, the index map $k \to$

$(i(k), j(k))$ can be easily written out for the lexicographical ordering. With any choice of linear indexing, (5) can be written as a linear algebraic equation:

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \tag{6}$$

where $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{f} \in \mathbb{R}^N$.

**Remark 2.1.** There exist different orderings for the grid points. Although they give equivalent matrixes up to permutations, different ordering does matter when solving linear algebraic equations.

2.2. **Boundary conditions.** In this section we shall discuss how to deal with boundary conditions in finite difference methods. The Dirichlet boundary condition is relatively easy and the Neumann boundary condition requires the ghost points.

**Dirichlet boundary condition.** For the Poisson equation with Dirichlet boundary condition

$$-\Delta u = f \ \text{ in } \Omega, \quad u = g \text{ on } \Gamma = \partial\Omega, \tag{7}$$

the value on the boundary is given by the boundary conditions. Namely $u_{i,j} = g(x_i, y_j)$ for $(x_i, y_j) \in \partial\Omega$ and thus these variables should be eliminated in the equation (6). There are several ways to impose the Dirichlet boundary condition.

One approach is to let $a_{ii} = 1, a_{ij} = 0, j \neq i$ and $f_i = g(x_i)$ for nodes $x_i \in \Gamma$. Note that this will destroy the symmetry of the corresponding matrix. To keep the symmetry, one keep the original matrix but add a large scaled identity matrix to the boundary nodes, e.g. $\mathrm{I}_\Gamma/\epsilon$ and the corresponding right hand side is also rescaled $g_\Gamma/\epsilon$. When $\epsilon \ll 1$, the boundary condition $u|_\Gamma \approx g_\Gamma$.

Another approach is to modify the right hand side at interior nodes and solve only equations at interior nodes. Let us consider a simple example with 9 nodes. The only unknown is $u_5$ using the lexicographical ordering. By the formula of the discrete Laplace operator at that node, we obtain the adjusted equation

$$\frac{4}{h^2}u_5 = f_5 + \frac{1}{h^2}(u_2 + u_4 + u_6 + u_8).$$

We use the following Matlab code to illustrate the implementation of Dirichlet boundary condition. Let `bdNode` be a logic array representing boundary nodes: `bdNode(k)=1` if $(x_k, y_k) \in \partial\Omega$ and `bdNode(k)=0` otherwise.

```
freeNode = ~bdNode;
u = zeros(N,1);
u(bdNode) = g(node(bdNode,:));
f = f-A*u;
u(freeNode) = A(freeNode,freeNode)\f(freeNode);
```

The matrix `A(freeNode,freeNode)` is symmetric and positive definite (SPD) (see Exercise 1) and thus ensure the existence of the inverse.

**Neumann boundary condition.** For the Poisson equation with Neumann boundary condition

$$-\Delta u = f \ \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma,$$

there is a compatible condition for $f$ and $g$:

$$-\int_\Omega f \, \mathrm{d}x = \int_\Omega \Delta u \, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}S = \int_{\partial\Omega} g \, \mathrm{d}S. \tag{8}$$

A natural approximation to the normal derivative is a one sided difference, for example:

$$\frac{\partial u}{\partial n}(x_1, y_j) = \frac{u_{1,j} - u_{2,j}}{h} + \mathcal{O}(h).$$

But this is only a first order approximation. To treat Neumann boundary condition more accurately, we introduce the ghost points outside of the domain and next to the boundary.

We extend the lattice by allowing the index $0 \le i, j \le n+1$. Then we can use centeral difference scheme:

$$\frac{\partial u}{\partial n}(x_1, y_j) = \frac{u_{0,j} - u_{2,j}}{2h} + \mathcal{O}(h^2).$$

The value $u_{0,j}$ is not well defined. We need to eliminate it from the equation. This is possible since on the boundary point $(x_1, y_j)$, we have two equations:

(9a) $$4u_{1,j} - u_{2,j} - u_{0,j} - u_{1,j+1} - u_{1,j-1} = h^2 f_{1,j}$$

(9b) $$u_{0,j} - u_{2,j} = 2h\, g_{1,j}.$$

From (9b), we get $u_{0,j} = 2h\, g_{1,j} + u_{2,j}$. Substituting it into (9a) and scaling by a factor $1/2$, we get an equation at point $(x_1, y_j)$:

$$2u_{1,j} - u_{2,j} - 0.5\, u_{1,j+1} - 0.5\, u_{1,j-1} = 0.5\, h^2 f_{1,j} + h\, g_{1,j}.$$

The scaling is to preserve the symmetry of the matrix. We can deal with other boundary points by the same technique except the four corner points.

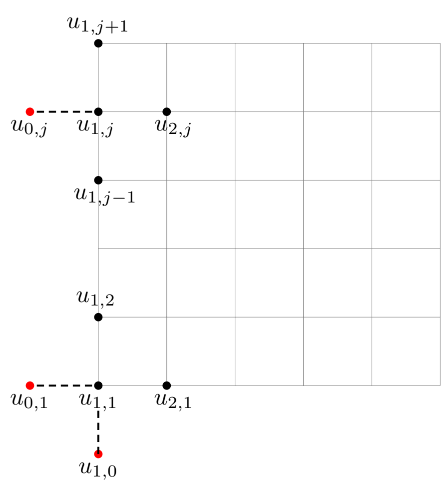



FIGURE 1. Ghost points for Neumann boundary conditions

At corner points, even the norm vector is not well defined. We will use average of two directional derivatives to get an approximation. Taking $(0,0)$ as an example, we have

(10a) $$4u_{1,1} - u_{2,1} - u_{0,1} - u_{1,1} - u_{1,0} = h^2 f_{1,1},$$

(10b) $$u_{0,1} - u_{2,1} = 2h\, g_{1,1},$$

(10c) $$u_{1,0} - u_{1,2} = 2h\, g_{1,1}.$$

So we can solve $u_{0,1}$ and $u_{1,0}$ from (10b) and (10c), and substitute them into (10a). Again to maintain the symmetry of the matrix, we multiply (10a) by $1/4$. This gives an equation for the corner point $(x_1, y_1)$

$$u_{1,1} - 0.5\, u_{2,1} - 0.5\, u_{1,2} = 0.25\, h^2 f_{1,1} + h\, g_{1,1}.$$

Similar techniques will be used to deal with other corner points. We then end with a linear algebraic equation

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}.$$

It can be shown that the corresponding matrix $\boldsymbol{A}$ is still symmetric but only semi-definite (see Exercise 2). The kernel of $\boldsymbol{A}$ consists of constant: $\boldsymbol{A}\boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = c$. This requires a discrete version of the compatible condition (8):

$$\sum_{i=1}^{N} f_i = 0 \tag{11}$$

and can be satisfied by the modification `f = f - mean(f)`.

## 3. Error estimate

In order to analyze the error, we need to put functions into a normed space. A "natural" norm for the finite linear space $\mathbb{V}_h$ is the maximum norm: for $v \in \mathbb{V}_h$,

$$\|v\|_{\infty,\Omega_h} = \max_{\substack{1\leq i\leq n+1,\\ 1\leq j\leq m+1}} \{|v_{i,j}|\}.$$

The subscript $h$ indicates this norm depends on the triangulation, since for different $h$, we have different numbers of $v_{i,j}$. Note that this is the $l^\infty$ norm for $\mathbb{R}^N$.

Define $\Delta_h : \mathbb{V}_h \to \overset{\circ}{\mathbb{V}}_h$ as the discrete Laplace operator. That is given a function $v \in \mathbb{V}_h$, $-\Delta_h v$ only gives values at the interior grid points using five point stencil $(4, -1, -1, -1, -1)$. We first introduce the discrete maximal principal and barrier functions.

**Theorem 3.1** (Discrete Maximum Principle)**.** *Let $v \in \mathbb{V}_h$ satisfy*

$$\Delta_h v \geq 0.$$

*Then*

$$\max_{\Omega_h} v \leq \max_{\Gamma_h} v,$$

*and the equality holds if and only if $v$ is constant.*

*Proof.* Suppose $\max_{\Omega_h} v > \max_{\Gamma_h} v$. Then we can take an interior node $x_0$ where the maximum is achieved. Let $x_1, x_2, x_3$, and $x_4$ be the four neighbors used in the stencil. Then

$$4v(x_0) = \sum_{i=1}^{4} v(x_i) - h^2 \Delta_h v(x_0) \leq \sum_{i=1}^{4} v(x_i) \leq 4v(x_0).$$

Thus equality holds throughout and $v$ achieves its maximum at all the nearest neighbors of $x_0$ as well. Applying the same argument to the neighbors in the interior, and then to their neighbors, etc, we conclude that $v$ is constant which contradicts to the assumption $\max_{\Omega_h} v > \max_{\Gamma_h} v$. The second statement can be proved easily by a similar argument. □

**Theorem 3.2.** *Let $u_h$ be the solution of*

$$-\Delta_h u_h = f_I \quad \text{at } \Omega_h \setminus \Gamma_h, \qquad u_h = g_I \quad \text{at } \Gamma_h. \tag{12}$$

*Then*

$$\|u_h\|_{\infty,\Omega_h} \leq \tfrac{1}{8}\|f_I\|_{\infty,\Omega_h\setminus\Gamma_h} + \|g_I\|_{\infty,\Gamma_h}. \tag{13}$$

*Proof.* Let

$$\phi = \tfrac{1}{4}\big[(x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\big],$$

which satisfies $\Delta_h \phi_I = 1$ on $\Omega_h \setminus \Gamma_h$ and $0 \leq \phi \leq \frac{1}{8}$ on $\overline{\Omega}$. Set $M = \|f_I\|_{\infty,\Omega_h\setminus\Gamma_h}$.

**Upper bound.** Define

$$v := u_h + M\phi_I.$$

Then in $\Omega_h \setminus \Gamma_h$,

$$\Delta_h v = \Delta_h u_h + M = -f_I + M \geq 0,$$

and on $\Gamma_h$,

$$v = g_I + M\phi_I \leq g_I + \tfrac{M}{8}.$$

By the discrete maximum principle,

$$\max_{\Omega_h} v \leq \max_{\Gamma_h} v \leq \max_{\Gamma_h} g_I + \tfrac{M}{8}.$$

Hence,

$$\max_{\Omega_h} u_h \leq \max_{\Omega_h} v \leq \max_{\Gamma_h} g_I + \tfrac{M}{8}.$$

**Lower bound.** Similarly, define

$$w := -u_h + M\phi_I.$$

Then in $\Omega_h \setminus \Gamma_h$,

$$\Delta_h w = -\Delta_h u_h + M = f_I + M \geq 0,$$

and on $\Gamma_h$,

$$w = -g_I + M\phi_I \leq -g_I + \tfrac{M}{8}.$$

Applying the discrete maximum principle again gives

$$\max_{\Omega_h} w \leq \max_{\Gamma_h} w \leq -\min_{\Gamma_h} g_I + \tfrac{M}{8},$$

that is,

$$-\min_{\Omega_h} u_h \leq -\min_{\Gamma_h} g_I + \tfrac{M}{8} \quad \Longrightarrow \quad \min_{\Omega_h} u_h \geq \min_{\Gamma_h} g_I - \tfrac{M}{8}.$$

Combining the two estimates yields

$$\|u_h\|_{\infty,\Omega_h} \leq \tfrac{1}{8}\|f_I\|_{\infty,\Omega_h\setminus\Gamma_h} + \|g_I\|_{\infty,\Gamma_h},$$

which proves the desired stability bound. □

**Corollary 3.3.** *Let $u$ be the solution of the Dirichlet problem (7) and $u_h$ the solution of the discrete problem (12). Then*

$$\|u_I - u_h\|_{\infty,\Omega_h} \leq \frac{1}{8}\|\Delta_h u_I - (\Delta u)_I\|_{\infty,\Omega_h\setminus\Gamma_h}.$$

The next step is to study the consistence error $\|\Delta_h u_I - (\Delta u)_I\|_{h,\infty}$. The following Lemma can be easily proved by Taylor expansion.

**Lemma 3.4.** *If $u \in \mathcal{C}^4(\Omega)$, then*

$$\|\Delta_h u_I - (\Delta u)_I\|_{\infty,\Omega_h\backslash\Gamma_h} \leq \frac{h^2}{6}\max\left\{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\infty,\Omega}, \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\infty}\right\}.$$

We summarize the convergence results on the finite difference methods in the following theorem.

**Theorem 3.5.** *Let $u$ be the solution of the Dirichlet problem (7) and $u_h$ the solution of the discrete problem (12). If $u \in \mathcal{C}^4(\Omega)$, then*

$$\|u_I - u_h\|_{\infty,\Omega_h} \leq Ch^2,$$

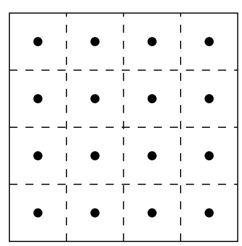
*with constant*

$$C = \frac{1}{48}\max\left\{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\infty,\Omega}, \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\infty}\right\}.$$

In practice, the second order of convergence can be observed even the solution $u$ is less smooth than $\mathcal{C}^4(\Omega)$, i.e. the requirement $u \in \mathcal{C}^4(\Omega)$. This restriction comes from the point-wise estimate. In finite element method, we shall use integral norms to find the right setting of function spaces.

## 4. Cell Centered Finite Difference Methods

In some applications, notably in computational fluid dynamics (CFD), the Poisson equation is solved on slightly different grids. In this section, we consider the finite difference method for the Poisson equation discretized at cell centers; see the figure below.



At interior nodes, the standard stencil $(4, -1, -1, -1, -1)$ can still be used, but boundary conditions are treated differently. The distance between interior nodes remains $h$, while the near-boundary nodes (centers of cells adjacent to the boundary) are $h/2$ away from the boundary. One can verify that for Neumann boundary conditions, the stencil for near-boundary nodes is $(3, -1, -1, -1)$, and for corner cells it is $(2, -1, -1)$. The boundary condition values are evaluated and moved to the right-hand side.

The Dirichlet boundary condition is more subtle for cell-centered differences. We can still introduce ghost grid points and apply the standard $(4, -1, -1, -1)$ stencil for near-boundary nodes, even though no grid points lie on the boundary. The ghost values can be eliminated by linear extrapolation, requiring $(u_{0,j} + u_{1,j})/2 = g(0, y_j) := g_{1/2,j}$.

$$\frac{5u_{1,j} - u_{2,j} - u_{1,j-1} - u_{1,j+1}}{h^2} = f_{1,j} + \frac{2g_{1/2,j}}{h^2}. \tag{14}$$

The stencil becomes $(5, -1, -1, -1, -2)$ for near-boundary nodes and $(6, -1, -1, -2, -2)$ for corner nodes, where the last entry represents the coefficient of the boundary condition. The symmetry of the resulting matrix is still preserved.

This treatment, however, is of low order (see Exercise 3). To achieve a smaller truncation error, we can use quadratic extrapolation. Using $u_{1/2,j}, u_{1,j}, u_{2,j}$ to fit a quadratic function and evaluating at $u_{0,j}$ gives

$$u_{0,j} = -2u_{1,j} + \tfrac{1}{3}u_{2,j} + \tfrac{8}{3}u_{1/2,j},$$

which leads to the modified boundary scheme:

$$\frac{6u_{1,j} - \frac{4}{3}u_{2,j} - u_{1,j-1} - u_{1,j+1}}{h^2} = f_{1,j} + \frac{\frac{8}{3}g_{1/2,j}}{h^2}. \tag{15}$$

We denote the near-boundary stencil by $(6, -\frac{4}{3}, -1, -1, -\frac{8}{3})$. The quadratic extrapolation yields a better convergence rate since the truncation error is reduced, but the symmetry of the matrix is lost.

For the Poisson equation, there is a way to retain both second-order accuracy and matrix symmetry. For simplicity, consider the homogeneous Dirichlet boundary condition, i.e., $u|_{\partial\Omega} = 0$. Then the tangential derivatives along the boundary vanish, in particular $\partial_t^2 u = 0$. On the boundary, the Laplacian can be written as $\Delta = \partial_t^2 + \partial_n^2$, and thus $\partial_n^2 u = \pm f$ on $\partial\Omega$, where the sign depends on whether the outward normal direction coincides with the coordinate axis direction.

Using $u_1$, $u_{1/2} = 0$, and $\partial_n^2 u = f$, we can fit a quadratic function and extrapolate to obtain an equation for the ghost point:

$$u_{1,j} + u_{0,j} = \frac{h^2}{4} f_{1/2,j}.$$

This leads to the modified boundary stencil:

$$\frac{5u_{1,j} - u_{2,j} - u_{1,j-1} - u_{1,j+1}}{h^2} = f_{1,j} + \frac{1}{4} f_{1/2,j}. \tag{16}$$

That is, we still use the same stencil but with a correction from the source term $f$.

## 5. Exercises

(1) Prove the following properties of the matrix $A$ formed in the finite difference methods for Poisson equation with Dirichlet boundary condition:
   (a) it is symmetric: $a_{ij} = a_{ji}$;
   (b) it is diagonally dominant: $a_{ii} \geq -\sum_{j=1, j\neq i}^{N} a_{ij}$;
   (c) it is positive definite: $u^\intercal A u \geq 0$ for any $u \in \mathbb{R}^N$ and $u^\intercal A u = 0$ if and only if $u = 0$. (*Hint: consider the quadratic form and complete perfect squares.*)

(2) Consider discrete Poisson matrix with Neumann boundary condition.
   (a) Write out the $9 \times 9$ matrix $A$ for $h = 1/2$.
   (b) Prove that in general the matrix corresponding to Neumann boundary condition is only semi-positive definite.
   (c) Show that the kernel of $A$ consists of constant vectors:
$$Au = 0 \iff u = c.$$

(3) Check the truncation error of schemes (14), (15) and (16) for different treatments of Dirichlet boundary condition in the cell centered finite difference methods.

(4) Consider the discrete Poisson matrix $A$ for Dirichlet problem.
   (a) Estimate the range of the spectrum of $A$.
   (b) Numerically show the spectrum of $A$ is symmetric with respect to 4.
   (c) Prove the spectrum of $A$ is symmetric with respect to 4. (*Hint: consider the block matrix using the red-black ordering.*)