MATH CIRCLE LOGIC PROBLEMS

NAM TRANG

A	B	A $\wedge$ B	A$\vee$ B	if A then B	$\neg$ A
T	T	T	T	T	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	T	T	T

Typically, a statement one wants to prove is of the form: if A then B. Two often-used approaches to show if A then B is true are:

(i) Direct proof: we assume A is true. We try to show B is true.

(ii) Proof by contradiction: we assume A is true and we also assume B is FALSE. We then try to obtain a contradiction from this assumption.

Example. Prove (both by a direct proof and by a proof by contradiction) that for all integers n, if n^2 is odd, then n is odd.

1) The four-digit number 4365 has an interesting property. If we write it backwards, and add the original number to the reversed number, we get a string of 9s: 4365 + 5634 = 9999. a) Show that 2097 and 4185 each have the same property.

b) Find two more four-digit numbers that have this property.

c) Find a six-digit number that has this property (that is the sum of it and the number obtained from it by writing it backwards is 999999).

d) Find a three-digit number that has this property.

Solutions. (a) is easy. (b) 1818 and 2727 are such examples. (c) 234567 is an example. (d) This is impossible because there is no number of the form ABC such that B + B = 9.

2) (Lewis Carroll puzzle) Suppose:

1. A says B lies;
2. B says C lies;
3. C says A and B lie.

This set of exercises are taken from Camp Logic: A Week of Logic Games and Activities for Young People by Mark Saul and Sian Zelbo

Determine who lies and who tells he truth. Use both the direct method of exhausting all posibilities and the method of proof by contradiction.

Solutions. There are a few ways to approach this.

1. Students work individually to construct the blank table. It is a worthwhile activity in itself for students to list all 8 possibilities. (Because there are 3 people and there are 2 possibilities for each, there are $2 \times 2 \times 2$ total possibilities.) It is useful to have a "canonical" way of listing the possibilities. That is, once students understand that there are eight possibilities for truth-tellers and liars, suggest to them that they be listed in the order given above (or any other order you choose, so long as it is the same for all students).

2. Students use proof by contradiction to rule out possibilities and fill in the chart. Have students consider each possibility and see whether it results in a contradiction. As they reason, they can fit their thinking into the chart to verify that there is only one possible answer. Listing all the possibilities like this makes things much clearer, and shortens many arguments. For example: Assume that B is lying. B lies $\rightarrow$ C tells the truth $\rightarrow$ A lies $\rightarrow$ B tells the truth (Contradiction!) Now we can cross off any possibility on the table in which B is lying, and so on.

3) If exactly one of the following is true, which is it?

(a) Puppies are cuter than kittens.

(b) Puppies are cuter than bunnies and kittens.

(c) Puppies are by far the cutest animals.

Solution. If we assume (c) to be true, then (b) and (a) are true too. That contradicts the statement that only one is true. If we assume (b) to be true, then (a) is true too. That also contradicts the statement that only one is true. Therefore (a) is the only one that can be true. It is the only one that doesn't lead to a contradiction. This conclusion is enough for the problem to be worthwhile. But if a class is interested, we might go further. If (a) is the only statement that is true, then (c) must be false: there must be some animals cuter than puppies. And (b) must also be false, which means that bunnies are cuter than puppies. So we can infer a situation about animals, not just about statements: Bunnies are cuter than puppies, which are cuter than kittens.

The next 4 problems are applications of the pigeonhole principle.

Pigeonhole Principles. if more than n pigeons are placed into n pigeonholes, some pigeonhole must contain more than one pigeon.

4) Ian has a bowling party for his birthday and invites fifteen friends. Show that at least two of the friends must have knocked over the same number of pins on their first turn.

Solution. There are eleven ways to knock down pins (zero to ten pins) in one throw. These are the pigeonholes. The fifteen results of the friends? individual throws are the pigeons. Since there are more pigeons than holes, two of Ian?s friends must have had the same result.

5) Kira's sock drawer has twelve blue socks, thirteen red, and seven green. She reaches in without looking. How many socks must she grab to make sure that she has a matching pair? How many must she grab to be sure that she has three matching pairs?

Solution. The holes here are the colors of the socks, and there are three of them. The pigeons are the colors of the socks Kira draws. To guarantee two socks of the same color, Kira must draw one more than three socks, so four socks. To guarantee three matching pairs, she must draw eight. Kira can have as many as seven socks and still only two pairs: B, B, B, B, B, R, G. With the eighth sock she must have a third pair.

6) Alek has a bowling party for his birthday and invites fifteen friends. If everyone in the room has at least one friend at the party, then show that two people must have the same number of friends in attendance.

Solution. There are sixteen people at Alek?s party: Alek plus his fifteen friends. (These are the pigeons.) Everyone must have from one to fifteen friends. (These are the holes.) There are more pigeons than holes, so two people must have the same number of friends.

7) A huge deck of cards numbered 1 through 2014 is shuffled. Michael removes cards from the deck, one by one. What is the largest number of cards he must remove to ensure that two of the cards have a difference that is a multiple of 5?

Solution. How many cards must we draw? Let's begin by considering the possible remainders when the numbers on the cards are divided by 5. There are five possible remainders: 0, 1, 2, 3, and 4. If two numbers have the same remainder when divided by 5, then they must differ by a multiple of 5. Since there are five possible remainders, we can choose as many as five cards, and it is possible that no two of them will differ by a multiple of 5. But if we draw a sixth card, its remainder must match with one of those we've already chosen. The answer is six cards.

8) The lattice below consists of five columns of five points each, all equally spaced. Lines are drawn connecting pairs of lattice points, but never two in the same row or column. (An example is shown.) How many lines must be drawn to ensure that two of the lines are parallel?

Solution. Two lines are parallel if they have the same slope (the same ratio of rise to run). The pigeons are the lines, and the holes are the possible slopes. There are twenty-two possible slopes. (Since we are not connecting points from the same row or column, we don't have horizontal or vertical lines, and we don't have to deal with 0's anywhere.)

1:1, 1:2, 1:3, 1:4, 2:1, 2:3, 3:1, 3:2, 3:4, 4:1, 4:3,

-1:1, -1:2, -1:3, -1:4, -2:1, -2:3, -3:1, -3:2, -3:4, -4:1, -4:3.

So if we draw in twenty-three lines, two of them must have the same slope. These two lines are parallel.

The last two are miscellaneous logic problems.

9) Two mathematicians, Albert and Isaac, chat. Isaac says he has three children who all have the same birthday (but who weren't necessarily born in the same year). Albert asks their ages. Isaac replies, "The product of the ages of my children is 72." Albert points out that this is not enough information to determine their ages. Isaac responds with another clue – he tells Albert the

sum of the ages of his children. But Albert again points out that there is not enough information. Finally Isaac says, "My youngest child is named Galileo." At last, Albert correctly determines the ages of Isaac's children. What are the ages?

Solution. If the product of his three children's ages is 72, there are the following possibilities:

1 * 1 * 72 = 72
1 * 2 * 36 = 72
1 * 3 * 24 = 72
1 * 4 * 18 = 72
1 * 6 * 12 = 72
1 * 8 * 9 = 72
2 * 2 * 18 = 72
2 * 3 * 12 = 72
2 * 4 * 9 = 72
2 * 6 * 6 = 72
3 * 3 * 8 = 72
3 * 4 * 6 = 72

Isaac later gives Albert the sum of their ages, but we don't know what number he says. We do, however, know that Albert can't figure it out from that information. So, we take the possibilities listed above and add them up:

1 + 1 + 72 = 74
1 + 2 + 36 = 39
1 + 3 + 24 = 28
1 + 4 + 18 = 23
1 + 6 + 12 = 19
1 + 8 + 9 = 18
2 + 2 + 18 = 22
2 + 3 + 12 = 17
2 + 4 + 9 = 15
2 + 6 + 6 = 14
3 + 3 + 8 = 14
3 + 4 + 6 = 13

The only way Albert wouldn't be able to figure out Isaac children's ages by knowing the sum is if the sum was 14, because there are two possibilities. So either the children's ages are 2, 6, and 6, or 3, 3, and 8. But Isaac points out that he has a youngest child. So the ages must be 2, 6, and 6.

10) A clock is observed. The hour hand is exactly at the minute mark, and the minute hand is six minutes ahead of it. Later, the clock is observed again. This time, the hour hand is exactly on a different minute mark, and the minute hand is seven minutes ahead of it. How much time elapsed between the first and second observations?

Solution. The hour hand is exactly on a minute mark five times per hour – on the hour, twelve minutes past the hour, twenty four minutes past, thirty six minutes past, and forty eight minutes past.

Let X be the number of hours, and Y be the number of minutes past the hour. When the hour hand is on a minute mark, the position of the hour hand is 5X + Y/12, and the position of the minute hand is Y. On the first occasion, Y = 5X + Y/12 + 6. This is equivalent to 60X = 11Y - 72. Since Y can only take one of the values in the set 0, 12, 24, 36, 48 , it can be determined that the only legal values for the equation are X = 1 and Y = 12. So the time is 1:12.

Similarly, the second occasion's equation is 60X = 11Y - 84. The only legal values here are X = 3 and Y = 24. So the time is 3:24.

Between 1:12 and 3:24, two hours and twelve minutes have elapsed.